Toroidal and Simple Orbifold Compactification

PHY 250

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Analytic and Anti-Analytic

Analytic function is complex differentiable:

$$f(x,y) = u(x,y) + iv(x,y)$$
$$z = x + iy$$
$$dz = dx + idy$$

Cauchy-Reimann conditions are:

$$\frac{du}{dx} = \frac{dv}{dy} \qquad \frac{dv}{dx} = -\frac{du}{dy}$$

Anti-analytic function has $\frac{df}{dz} = 0$

Using

$$\frac{df}{dz} = \frac{df}{dx}\frac{dx}{dz} + \frac{df}{dy}\frac{dy}{dz}$$

The Anti-Analytic version of the Cauchy-Reimann conditions are:

$$\frac{du}{dx} = -\frac{dv}{dy} \qquad \frac{dv}{dx} = \frac{du}{dy}$$

Hence the name Anti-Analytic!

Generalized Toroidal Compactification

On 5/26 we did simple compactification to $S^1 \times M^{25}$

Generalized compactification: d = 26 – k = number of non-compact dimensions

$$X^m \cong X^m + 2\pi R, d \le m \le 25$$

Space-time is $M^d \times T^k$, a k-Torus

Geometry of Torus depends on internal metric G_{mn}

k>1 so antisymmetric tensor has scalar components B_{mn} , Kaluza-Klein gauge bosons $A_\mu{}^m$ and antisymmetric gauge bosons $B_{m\,\mu}$

The Graviton-Dilaton action is:

$$S_{G-D} = \frac{(2\pi R)^k}{2k_0{}^2} \int d^d x \sqrt{-G_d}\, e^{-2\Phi_d} [R_d + 4\partial_\mu \Phi_d \partial^\mu \Phi_d - \frac{1}{4} G^{mn} G^{pq} \{\partial_\mu G_{mp} \partial^\mu G_{nq} + \partial_\mu B_{mp} \partial^\mu B_{nq}\} - \frac{1}{4} G_{mn} F_{\mu\nu}{}^m F^{\mu\nu n} - \frac{1}{4} G^{mn} H_{m\mu\nu} H_n{}^{\mu\nu} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda}]$$

Where k_0 is a normalized field constant rescaled by redefinition of Φ, $\Phi_d = \Phi - \frac{1}{4}\ln(\det(G_{mn}))$

$H_{m\mu\nu}$=generalized field strength, F=Faraday tensor, R_d comes from $G_{\mu\nu}$

Toroidal Compactification, cont.

Now we have a contribution from the antisymmetric tensor background B_{mn} which contributes to the world-sheet Lagrangian:

$$B_{mn}\partial_a(\varepsilon^{ab}X^m\partial_b X^n)$$

Since this is a total derivative for B_{mn} it has no effect locally, so the world-sheet is still a Conformal Field Theory. Focusing on the zero-mode contribution, and inserting into the world-sheet action:

$$X^m(\sigma^1,\sigma^2)=x^m(\sigma^2)+w^m R\sigma^1$$

Yields (a dot indicates derivative with respect to world-sheet time σ^2)

$$L=\frac{1}{2\alpha'}G_{mn}(\dot{x}^m\dot{x}^n+w^m w^n R^2)+\frac{i}{\alpha'}B_{mn}\dot{x}^m w^n R$$

$$p_m=-\frac{\partial L}{\partial v^m}=\frac{1}{\alpha'}(G_{mn}v^n-B_{mn}w^n R)$$ Canonical momenta

Where $v^m=i\dot{x}^m$ because we are using Euclidean signature metric

Results of Toroidal Compactification

Using Minkowski time,

$$v^m = \partial_0 x^m$$
$$p_m = \frac{n_m}{R}$$
$$v_m = \alpha' \frac{n_m}{R} + B_{mn} w^n R$$

The zero-mode contribution to the world-sheet Hamiltonian is:

$$\frac{1}{2\alpha'} G_{mn} (v^m v^n + w^m w^n R^2)$$

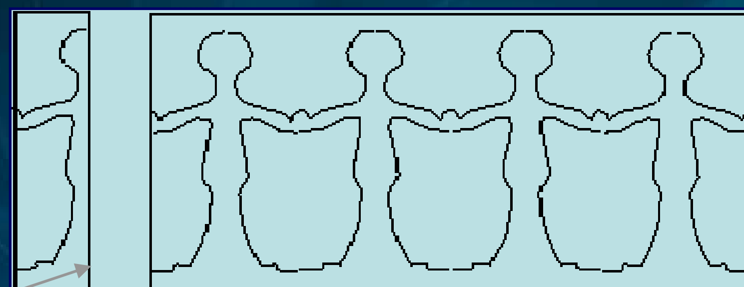
And the closed string mass (sum condition) is:

$$m^2 = \frac{1}{2\alpha'^2} G_{mn} (v_L^m v_L^n + v_R^m v_R^n) + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$
$$v_{L,R}^m = v^m \pm w^m R$$

And the level matching condition becomes

$$0 = G_{mn}(v_L^m v_L^n - v_R^m v_R^n) + 4\alpha'(N - \tilde{N}) = 4\alpha'(n_m w^m + N - \tilde{N})$$

What is an Orbifold?



Fixed Line

Mathematically

- A generalization of a manifold that allows the presence of the points whose neighborhood is diffeomorphic to a coset of R^n such as R^n/Γ (where Γ is a finite group). J.H. Conway defines 17 types

Physically

- A coset M/G where M is a manifold (or theory) and G is a group of its isometries (or symmetries) – not necessarily all of them. In string theory, these symmetries do not have to have a geometric interpretation. [1]

Example Orbifolds

Instead of periodic identification of X^m, consider points under reflection:

$$X^{25} \cong -X^{25}$$

Or more generally:

$$X^m \cong -X^m, 26-k \leq m \leq 25$$

There is a space of fixed points at $X^{26-k}=\ldots=X^{25}=0$.
The noncompact space identified in this manner is the quotient space R^k/Z_2
We can also use the same toroidal compactification as previously to form T^k/Z_2

For the noncompact case, there is 1 fixed point identified by:

$$X^{25} \cong X^{25} + 2\pi R$$

For the compact case, there are 2^k fixed points with each $X^m=0$ or πR
These singular spaces are orbifolds.

An orbifold is thus a generic manifold defined as a quotient space M/G where M is a manifold (or theory) and G is the (not necessarily complete) group of isometries/symmetries of the manifold/theory. Intuitively, it can be thought of as a mapping rotated about a point, line, or surface (i.e. manifold), where these points become singular points defining a "twisted sector":

$$X^{25}(\sigma^1 + 2\pi) = -X^{25}(\sigma^1)$$

Simple Orbifold Compactification

For the simple, compact 1-d orbifold S^1/Z_2, the untwisted sector produces the spectrum:

$$|N,\tilde{N};k^{\mu},n,w> \to (-1)^{N^{25}+\tilde{N}^{25}}|N,\tilde{N};k^{\mu},-n,-w>$$

So this reverses compact winding and momentum. To form linear combinations invariant under this, the states that are massless at generic R have n=w=0, so the number of 25-excitations is even. The spacetime graviton, antisymmetric tensor, dilaton, and tachyon survive. However, the Kaluza-Klein gauge bosons are no longer in the spectrum.

In the twisted sector:

$$X^{25}(z,\bar{z}) = i\left(\frac{\alpha'}{2}\right)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \frac{1}{m+\frac{1}{2}}\left(\frac{\alpha^{25}_{m+\frac{1}{2}}}{z^{m+\frac{1}{2}}} + \frac{\tilde{\alpha}^{25}_{m+\frac{1}{2}}}{\bar{z}^{m+\frac{1}{2}}}\right)$$

This antiperiodicity forbitds any center-of-mass coordinate or momentum, so the string cannot move away from the $X^{25}=0$ fixed point. A similar mode expansion occurs for the fixed point:

$$X^{25}(\sigma^1 + 2\pi) = 2\pi R - X^{25}(\sigma^1)$$

The expression is the same, with an additional constant term of πR. The mass-shell and level-matching conditions for the twisted sector is:

$$m^2 = \frac{4}{\alpha}(N - \frac{15}{16}), N = \tilde{N}$$

Conclusion

In general, this twisting can be thought of as gauging the discrete group H. Under T-duality, at specific radii, a toroidal theory at $R=2\alpha'^{1/2}$ can be shown to be equivalent to an orbifold theory $R=\alpha'^{1/2}$

The Euler characteristic for an orbifold can be defined as:

$$\mathrm{X} = V - E + F$$

Where V=vertices, E = edges, and F = faces

It turns out the the number of generations in a particle theory is equal to X/2

In general, we want to compactify to $M^4 x V^6$ where V is the Calabi-Yau space, which is defined as a manifold with nonvanishing harmonic spinors.

References

1. "Orbifold". From Physics Daily. http://www.physicsdaily.com/physics/Orbifold
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