Newtonian approximation in Causal Dynamical Triangulations

Adam Getchell

Contents

1 Motivation

1.1 Newton's Law of Gravitation from General Relativity

- Can we recover $F = -\frac{Gm_1m_2}{r^2}$ from CDT?
- Do we have a sensible notion of "mass" in Causal Dynamical Triangulations?
- Semi-classical approximations not yet completely convincing [6] – we would like direct results

1.2 Previous Work

- Separation between two objects $\gg$ Schwarzschild radius
- Self-fields not excluded
- Cylindrical symmetry
- Object size $\ll$ separation

We will see that a "strut" holds objects apart.

Following Chou[2], the most generally cylindrically symmetric static metric is:

$$ds^2 = g_{00}dt^2 - \left(g_{11}\left(dx^1\right)^2 + 2g_{12}dx^1dx^2 + g_{22}\left(dx^2\right)^2\right) - g_{33}d\phi^2 \tag{1}$$

where x^1,x^2 are any two coordinates in the meridianal (vertical) plane containing the z-axis.

We furthermore assume that $g_{\mu\nu} = f\left(x^1, x^2\right)$

For positive definite quadratic differential forms of two variables such as explicit values of g_{11},g_{12}, and g_{22} from the parenthetical part of Equation (1), one can make a real, single-valued, continuous transformation from x^1 and x^2 to u and v by:

$$x^1 = x^1(u,v), x^2 = x^2(u,v) \tag{2}$$

where $J = \left[\partial(x^1,x^2)/\partial(u,v)\right] \neq 0$ such that:

$$g_{11}\left(dx^1\right)^2 + 2g_{12}dx^1dx^2 + g_{22}\left(dx^2\right)^2 = e^{2m}\left(du^2 + dv^2\right) \tag{3}$$

Equation (1) becomes:

$$ds^2 = e^{2\nu}dt^2 - e^{2m}(du^2 + dv^2) - e^{2n}d\phi^2 \tag{4}$$

Explicitly, we then have the metric:

$$g_{\mu\nu} = \begin{pmatrix} e^{2\nu}dt^2 & 0 & 0 & 0 \\ 0 & -e^{2m}du^2 & 0 & 0 \\ 0 & 0 & -e^{2m}dv^2 & 0 \\ 0 & 0 & 0 & -e^{2n}d\phi^2 \end{pmatrix} \tag{5}$$

Using [1]:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{6}$$

We can get the non-zero Christoffel connections as:

$$\begin{array}{cc} \Gamma^1_{00} = e^{2(\nu-m)}\partial_u\nu & \Gamma^2_{22} = \partial_v m \\ \Gamma^1_{33} = -e^{2(n-m)}\partial_u n & \Gamma^2_{21} = \partial_u m \\ \Gamma^1_{22} = -\partial_u m & \Gamma^2_{11} = -\partial_v m \\ \Gamma^1_{21} = \partial_v m & \Gamma^3_{32} = \partial_v n \\ \Gamma^1_{11} = \partial_u m & \Gamma^3_{31} = \partial_u n \\ \Gamma^2_{00} = e^{2(\nu-m)}\partial_v\nu & \Gamma^0_{02} = \partial_v\nu \\ \Gamma^2_{33} = -e^{2(n-m)}\partial_v n & \Gamma^0_{01} = \partial_u\nu \end{array} \tag{7}$$

Using:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu\Gamma^{\rho}_{\nu\sigma} - \partial_\nu\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{8}$$

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} \tag{9}$$

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{10}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{11}$$

And solving for the Einstein Field Equations in a vacuum (i.e. $G_{\mu\nu} = 0$) we get:

$$G_{11} = -\partial_v m \partial_v n + (\partial_v n)^2 - \partial_v m \partial_v \nu + \partial_v n \partial_v \nu + (\partial_v \nu)^2 + \partial_v^2 n + \partial_v^2 \nu + \partial_u m \partial_u n + \partial_u m \partial_u \nu + \partial_u n \partial_u \nu = 0 \tag{12}$$

$$G_{21} = \partial_v n \partial_u m + \partial_v \nu \partial_u m + \partial_v m \partial_u n - \partial_v n \partial_u n + \partial_v m \partial_u \nu - \partial_v \nu \partial_u \nu - \partial_u \partial_v n - \partial_u \partial_v \nu = 0 \tag{13}$$

$$G_{22} = \partial_v m \partial_v n + \partial_v m \partial_v \nu + \partial_v n \partial_v \nu - \partial_u m \partial_u n + \partial_u m \partial_u \nu + (\partial_u n)^2 - \partial_u m \partial_u \nu + \partial_u n \partial_u \nu + (\partial_u \nu)^2 + \partial_u^2 n + \partial_u^2 \nu = 0 \tag{14}$$

$$G_{33} = e^{2(n-m)} \left((\partial_v \nu)^2 + \partial_v^2 m + \partial_v^2 \nu + (\partial_u \nu)^2 + \partial_u^2 m + \partial_u^2 \nu \right) = 0 \tag{15}$$

$$G_{00} = -e^{2(\nu-m)} \left((\partial_v n)^2 + \partial_v^2 m + \partial_v^2 n + (\partial_u n)^2 + \partial_u^2 m + \partial_u^2 n \right) = 0 \tag{16}$$

Let:

$$\chi = n + \nu \tag{17}$$

Adding together Eqns. (15) and (16) gives:

$$\partial_u^2 \chi + \partial_v^2 \chi + (\partial_u \chi)^2 + (\partial_v \chi)^2 = 0 \tag{18}$$

Setting:

$$\Phi = e^{\chi} = e^{n+\nu} \tag{19}$$

We recover Laplace's equation in the *uv*-plane:

$$\partial_u^2 \Phi + \partial_v^2 \Phi = 0 \tag{20}$$

The remaining equations are used for boundary conditions on Laplace's equation. In general, for a metric of the form:

$$ds^2 = e^{2\psi} dt^2 - e^{-2\psi} \left[e^{2\omega} \left(dr^2 + dz^2 \right) + r^2 d\phi^2 \right] \tag{21}$$

We have general solutions:

$$\nabla^2 \psi = \partial_r^2 \psi + \frac{\partial_r \psi}{r} + \partial_z^2 \psi \tag{22}$$

$$d\omega\left[\psi\right]=r\left[\left(\left(\partial_r\psi\right)^2-\left(\partial_z\psi\right)^2\right)dr+2\partial_r\psi\partial_z\psi dz\right] \tag{23}$$

Note that Eq(4) can be recovered from Eq(21) by substituting $\psi=\nu$,$m=\omega-\psi$, and $e^{2n}=r^2e^{-2\psi}$.

The solution of Eq(22) and Eq(23)for a point particle of mass m at $z=z_0$ is given by (explain "point" in Schwarzschild solution, check for singularities):

$$\psi=-\frac{m}{R} \tag{24}$$

$$\omega=-\frac{m^2r^2}{2R^4} \tag{25}$$

$$R=\sqrt{r^2+\left(z-z_0\right)^2} \tag{26}$$

What is meant by "point" particle? To find out, let's transform to the Schwarzschild equation:

$$ds^2=\left(1-\frac{2GM}{r}\right)dt^2-\frac{1}{\left(1-\frac{2GM}{r}\right)}dr^2-r^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \tag{27}$$

using: (TODO: fill in transforms)

For n point particles we have [5]:

$$\psi=-\sum_{j=1}^{N}\frac{m_j}{R_j} \tag{28}$$

$$\omega=-\frac{r^2}{2}\sum_j\frac{m_j^2}{R_j^4}+\sum_{j\neq k}\frac{m_jm_k}{\left(z_j-z_k\right)^2}\left[\frac{r^2+\left(z-z_j\right)\left(z-z_k\right)}{R_jR_k}-1\right] \tag{29}$$

$$R=\sqrt{r^2+\left(z-z_j\right)^2} \tag{30}$$

2 Applications to Causal Dynamical Triangulations

2.1 Preliminaries

A simplex is a generalization of a triangle to arbitrary dimension. For example, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a 4-simplex is a pentachoron.

An n-dimensional simplex has $n+1$ points or *vertices*. A convex hull, or minimal convex set of these points is the *m-face* of the *n-simplex*. Thus, a vertex is a *0-face*, and an edge between two vertices is the *1-face*. We can extend this notation to *2-faces* (triangles), *3-faces* (tetrahedrons), *4-faces* (pentachorons). We will not, at present, consider simplices of dimension higher than $n=4$, but this generalization gives us a useful way to reason about higher dimensional spaces.

The number of *m-faces* on our *n-simplex* is given by the binomial coefficient as:

$$\begin{pmatrix} n+1 \\ m+1 \end{pmatrix} \tag{31}$$

Thus, our pentachoron has 5 vertices, 10 edges, 10 faces (triangles), 5 cells (tetrahedrons), and 1 4-face, itself.

A given face can be shared by another simplex. By requiring that [7]:

- Every face of a simplex K is in K, and
- The intersection of any two simplices of K is a face of each of them

We build up a useful structure called a simplicial complex.

2.2 Code Correctness

Implementing CDT in computer code is non-trivial. As a first step, an independent implementation of CDT has given similar results. [4] However, we would like to do better by providing an implementation using Literate Programming coupled with Test Driven Development. This provides for the codebase to be better understood by researchers wishing to replicate results, and provides inherent integrity checks apart from the overall "it produced what we expected". Such methodology will be critical to expanding the performance of the code by using such techniques as parallel processing and highly optimized algorithms. The adage of "Make it work, then make it faster" applies.

The first building block of the code are the simplexes themselves. Using the known properties of simplicial complexes, we can provide for a series of checks that validate that simplices are being constructed correctly. Such checks will provide useful test cases when the underlying implementation of simplex data structures and moves are changed.

2.3 Issues

- Extrinsic Curvature *(To Do)*
- Imposing conditions of separation
- Checking that separation >> Schwarzschild radius
- Imposing cylindrical symmetry

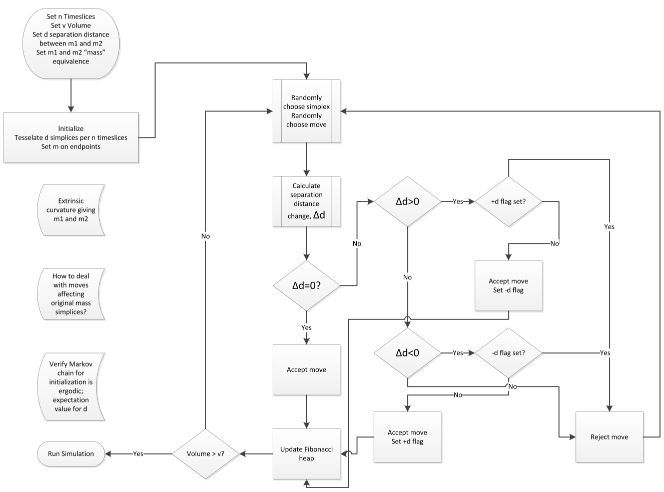
2.4 CDT Algorithm

(To Do: insert graphics)

- [(2,8): (1,4) + (4,1) $\rightarrow$8 simplices] + inverse = +2 moves
- [(4,6): ()+()+()+()$\rightarrow$6 simplices] + inverse = +2 moves
- [(2,4): two varieties of ()+()$\rightarrow$4 simplices], self-inverse = +2 moves

- [(3,3): two varieties of ()+()+()→3 simplices] + inverse = +4 moves

10 moves in all (*Check!*)



Dijkstra's Algorithm [3]

Solves single-source shortest-path problems on weighted, directed graph G=(V,E) of non-negative edge lengths

- Greedy algorithm
- Proven to be correct
- Complexity
  - $O(V^2)$ naively using adjacency list
  - $O(E \lg V)$ using priority queue iff all vertices reachable from source
  - $O(V \lg V + E)$ using Fibonacci heap (more relaxation calls than extract-min calls)
- Issue: confine edge length algorithm to particular time-slice
- Solution: Store Fibonacci heap of simplices per time-slice
  - Each simplex has 5 neighbors, so more compact than adjacency matrix
  - How to deal with moves affecting original "mass" simplices
  - How to create a 4d cylinder of height z=d
  - Verify Markov chain for initialization is ergodic
  - Calculate <d>

3 Summary

- Insert mass equivalence via extrinsic curvature
- Insert strut by enforcing separation distance
- Filter moves which alter separation distance via Markov chain
- Outlook
  - Write code!
  - Check Extrinsic Curvature
  - Compare results

References

[1] Sean Carroll. *Spacetime and Geometry: An Introduction to General Relativity*. Benjamin Cummings, September 2003.

[2] P. Y. Chou. The gravitational field of a body with rotational symmetry in einstein's theory of gravitation. *American Journal of Mathematics*, 53(2):289–308, April 1931. ArticleType: research-article / Full publication date: Apr., 1931 / Copyright © 1931 The Johns Hopkins University Press.

[3] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms, Second Edition*. The MIT Press, 2nd edition, September 2001.

[4] Rajesh Kommu. A validation of causal dynamical triangulations. *arXiv:1110.6875*, October 2011.

[5] Patricio. S Letelier and Samuel R Oliveira. Superposition of weyl solutions: The equilibrium forces. *arXiv:gr-qc/9710122*, October 1997. Class.Quant.Grav. 15 (1998) 421-433.

[6] T. Trzesniewski. Analysis of the semiclassical solution of CDT. *arXiv:1102.4643*, February 2011.

[7] Eric W. Weisstein. Simplicial complex. *MathWorld - A Wolfram Web Resource*.