Newtonian approximation in Causal Dynamical Triangulations

Adam Getchell

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1 Motivation

1.1 Newton's Law of Gravitation from General Relativity

- Can we recover $F = -\frac{Gm_1 m_2}{r^2}$ from CDT?
- Do we have a sensible notion of "mass" in Causal Dynamical Triangulations?
- Semi-classical approximations not yet completely convincing [5] – we would like direct results

1.2 Previous Work

- Separation between two objects $\gg$ Schwarzschild radius
- Self-fields not excluded
- Cylindrical symmetry
- Object size $\ll$ separation

We will see that a "strut" holds objects apart.

Following Chou[2], the most generally cylindrically symmetric static metric is:

$$ds^2 = g_{00}dt^2 - \left(g_{11}\left(dx^1\right)^2 + 2g_{12}dx^1dx^2 + g_{22}\left(dx^2\right)^2\right) - g_{33}d\phi^2 \tag{1}$$

where x^1,x^2 are any two coordinates in the meridianal (vertical) plane containing the z-axis.

We furthermore assume that $g_{\mu\nu} = f\left(x^1, x^2\right)$

For positive definite quadratic differential forms of two variables such as explicit values of g_{11},g_{12}, and g_{22} from the parenthetical part of Equation (1), one can make a real, single-valued, continuous transformation from x^1 and x^2 to u and v by:

$$x^1 = x^1(u,v),\, x^2 = x^2(u,v) \tag{2}$$

where $J = \left[{}^{\partial(x^1,x^2)}/_{\partial(u,v)}\right] \neq 0$ such that:

$$g_{11}\left(dx^1\right)^2 + 2g_{12}dx^1dx^2 + g_{22}\left(dx^2\right)^2 = e^{2m}\left(du^2 + dv^2\right) \tag{3}$$

Equation (1) becomes:

$$ds^2 = e^{2\nu}dt^2 - e^{2m}(du^2 + dv^2) - e^{2n}d\phi^2 \tag{4}$$

Explicitly, we then have the metric:

$$g_{\mu\nu} = \begin{pmatrix} e^{2\nu}dt^2 & 0 & 0 & 0 \\ 0 & -e^{2m}du^2 & 0 & 0 \\ 0 & 0 & -e^{2m}dv^2 & 0 \\ 0 & 0 & 0 & -e^{2n}d\phi^2 \end{pmatrix} \tag{5}$$

Using [1]:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \tag{6}$$

We can get the non-zero Christoffel connections as:

$$\begin{array}{cc} \Gamma^1_{00} = e^{2(\nu-m)}\partial_u\nu & \Gamma^2_{22} = \partial_v m \\ \Gamma^1_{33} = -e^{2(n-m)}\partial_u n & \Gamma^2_{21} = \partial_u m \\ \Gamma^1_{22} = -\partial_u m & \Gamma^2_{11} = -\partial_v m \\ \Gamma^1_{21} = \partial_v m & \Gamma^3_{32} = \partial_v n \\ \Gamma^1_{11} = \partial_u m & \Gamma^3_{31} = \partial_u n \\ \Gamma^2_{00} = e^{2(\nu-m)}\partial_v\nu & \Gamma^0_{02} = \partial_v\nu \\ \Gamma^2_{33} = -e^{2(n-m)}\partial_v n & \Gamma^0_{01} = \partial_u\nu \end{array} \tag{7}$$

Using:

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{8}$$

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \tag{9}$$

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{10}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{11}$$

And solving for the Einstein Field Equations in a vacuum (i.e. $G_{\mu\nu} = 0$) we get:

$$G_{11} = -\partial_v m \partial_v n + (\partial_v n)^2 - \partial_v m \partial_v \nu + \partial_v n \partial_v \nu + (\partial_v \nu)^2 + \partial_v^2 n + \partial_v^2 \nu + \partial_u m \partial_u n + \partial_u m \partial_u \nu + \partial_u n \partial_u \nu = 0 \tag{12}$$

$$G_{21} = \partial_v n \partial_u m + \partial_v \nu \partial_u m + \partial_v m \partial_u n - \partial_v n \partial_u n + \partial_v m \partial_u \nu - \partial_v \nu \partial_u \nu - \partial_u \partial_v n - \partial_u \partial_v \nu = 0 \tag{13}$$

$$G_{22} = \partial_v m \partial_v n + \partial_v m \partial_v \nu + \partial_v n \partial_v \nu - \partial_u m \partial_u n + \partial_u m \partial_u \nu + (\partial_u n)^2 - \partial_u m \partial_u \nu + \partial_u n \partial_u \nu + (\partial_u \nu)^2 + \partial_u^2 n + \partial_u^2 \nu = 0 \tag{14}$$

$$G_{33} = e^{2(n-m)} \left((\partial_v \nu)^2 + \partial_v^2 m + \partial_v^2 \nu + (\partial_u \nu)^2 + \partial_u^2 m + \partial_u^2 \nu \right) = 0 \tag{15}$$

$$G_{00} = -e^{2(\nu-m)} \left((\partial_v n)^2 + \partial_v^2 m + \partial_v^2 n + (\partial_u n)^2 + \partial_u^2 m + \partial_u^2 n \right) = 0 \tag{16}$$

Let:

$$\chi = n + \nu \tag{17}$$

Adding together Eqns. (15) and (16) gives:

$$\partial_u^2 \chi + \partial_v^2 \chi + (\partial_u \chi)^2 + (\partial_v \chi)^2 = 0 \tag{18}$$

Setting:

$$\Phi = e^{\chi} = e^{n+\nu} \tag{19}$$

We recover Laplace's equation in the *uv*-plane:

$$\partial_u^2 \Phi + \partial_v^2 \Phi = 0 \tag{20}$$

The remaining equations are used for boundary conditions on Laplace's equation. In general, for a metric of the form:

$$ds^2 = e^{2\psi} dt^2 - e^{-2\psi} \left[e^{2\omega} \left(dr^2 + dz^2 \right) + r^2 d\phi^2 \right] \tag{21}$$

We have general solutions:

$$\nabla^2 \psi = \partial_r^2 \psi + \frac{\partial_r \psi}{r} + \partial_z^2 \psi \tag{22}$$

$$d\omega[\psi] = r\left[\left((\partial_r\psi)^2 - (\partial_z\psi)^2\right)dr + 2\partial_r\psi\partial_z\psi dz\right] \tag{23}$$

Note that Eq(4) can be recovered from Eq(21) by substituting $\psi = \nu, m = \omega - \psi$, and $e^{2n} = r^2 e^{-2\psi}$.

The solution of Eq(22) and Eq(23)for a point particle of mass m at $z = z_0$ is given by (explain "point" in Schwarzschild solution, check for singularities):

$$\psi = -\frac{m}{R} \tag{24}$$

$$\omega = -\frac{m^2 r^2}{2R^4} \tag{25}$$

$$R = \sqrt{r^2 + (z - z_0)^2} \tag{26}$$

What is meant by "point" particle? To find out, lets transform to the Schwarzschild equation:

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{2GM}{r}\right)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{27}$$

using: (TODO: fill in transforms)

For n point particles we have [4]:

$$\psi = -\sum_{j=1}^{N}\frac{m_j}{R_j} \tag{28}$$

$$\omega = -\frac{r^2}{2}\sum_j \frac{m_j^2}{R_j^4} + \sum_{j\neq k}\frac{m_j m_k}{(z_j - z_k)^2}\left[\frac{r^2 + (z - z_j)(z - z_k)}{R_j R_k} - 1\right] \tag{29}$$

$$R = \sqrt{r^2 + (z - z_j)^2} \tag{30}$$

2 Applications to Causal Dynamical Triangulations

2.1 Preliminaries

A simplex is a generalization of a triangle to arbitrary dimension. For example, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a 4-simplex is a pentachoron.

An n-dimensional simplex has $n+1$ points or *vertices*. A convex hull, or minimal convex set of these points is the *m-face* of the *n-simplex*. Thus, a vertex is a *0-face*, and an edge between two vertices is the *1-face*. We can extend this notation to *2-faces* (triangles), *3-faces* (tetrahedrons), *4-faces* (pentachorons). We will not, at present, consider simplices of dimension higher than $n = 4$, but this generalization gives us a useful way to reason about higher dimensional spaces.

The number of *m-faces* on our *n-simplex* is given by the binomial coefficient as:

$$\binom{n+1}{m+1} \tag{31}$$

Thus, our pentachoron has 5 vertices, 10 edges, 10 faces (triangles), 5 cells (tetrahedrons), and 1 4-face, itself.

A given face can be shared by another simplex. By requiring that [6]:

- Every face of a simplex K is in K, and
- The intersection of any two simplices of K is a face of each of them

We build up a useful structure called a simplicial complex.

2.2 Issues

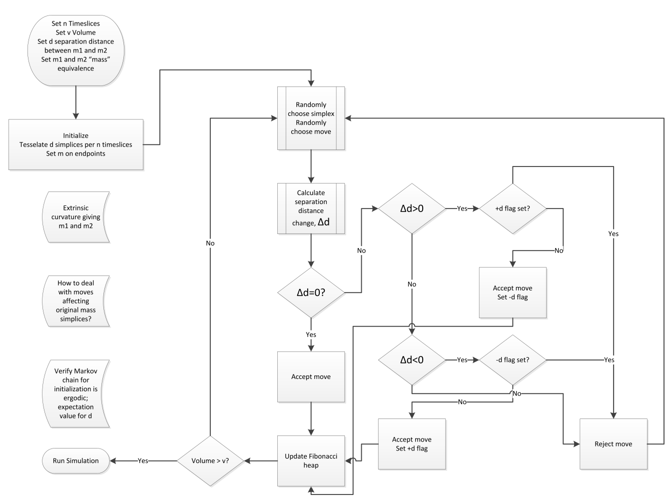
- Extrinsic Curvature *(To Do)*
- Imposing conditions of separation
- Checking that separation >> Schwarzschild radius
- Imposing cylindrical symmetry

2.3 CDT Algorithm

(To Do: insert graphics)

- [(2,8): (1,4) + (4,1) $\rightarrow$8 simplices] + inverse = +2 moves
- [(4,6): ()+()+()+()$\rightarrow$6 simplices] + inverse = +2 moves
- [(2,4): two varieties of ()+()$\rightarrow$4 simplices], self-inverse = +2 moves
- [(3,3): two varieties of ()+()+()$\rightarrow$3 simplices] + inverse = +4 moves

10 moves in all (*Check!)*



Dijkstra's Algorithm [3]

Solves single-source shortest-path problems on weighted, directed graph G=(V,E) of non-negative edge lengths

- Greedy algorithm
- Proven to be correct
- Complexity
  - $O(V^2)$ naively using adjacency list
  - $O(E \lg V)$ using priority queue iff all vertices reachable from source
  - $O(V \lg V + E)$ using Fibonacci heap (more relaxation calls than extract-min calls)
- Issue: confine edge length algorithm to particular time-slice
- Solution: Store Fibonacci heap of simplices per time-slice
  - Each simplex has 5 neighbors, so more compact than adjacency matrix
  - How to deal with moves affecting original "mass" simplices
  - How to create a 4d cylinder of height z=d
  - Verify Markov chain for initialization is ergodic
  - Calculate <d>

3 Summary

- Insert mass equivalence via extrinsic curvature
- Insert strut by enforcing separation distance
- Filter moves which alter separation distance via Markov chain
- Outlook
  - Write code!
  - Check Extrinsic Curvature
  - Compare results

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