CHAPTER

1 Economic Questions and Data

Ask a half dozen econometricians what econometrics is, and you could get a half dozen different answers. One might tell you that econometrics is the science of testing economic theories. A second might tell you that econometrics is the set of tools used for forecasting future values of economic variables, such as a firm's sales, the overall growth of the economy, or stock prices. Another might say that econometrics is the process of fitting mathematical economic models to real-world data. A fourth might tell you that it is the science and art of using historical data to make numerical, or quantitative, policy recommendations in government and business.

In fact, all these answers are right. At a broad level, econometrics is the science and art of using economic theory and statistical techniques to analyze economic data. Econometric methods are used in many branches of economics, including finance, labor economics, macroeconomics, microeconomics, marketing, and economic policy. Econometric methods are also commonly used in other social sciences, including political science and sociology.

This book introduces you to the core set of methods used by econometricians. We will use these methods to answer a variety of specific, quantitative questions from the worlds of business and government policy. This chapter poses four of those questions and discusses, in general terms, the econometric approach to answering them. The chapter concludes with a survey of the main types of data available to econometricians for answering these and other quantitative economic questions.

1.1 Economic Questions We Examine

Many decisions in economics, business, and government hinge on understanding relationships among variables in the world around us. These decisions require quantitative answers to quantitative questions.

This book examines several quantitative questions taken from current issues in economics. Four of these questions concern education policy, racial bias in mortgage lending, cigarette consumption, and macroeconomic forecasting.

Question #1: Does Reducing Class Size Improve Elementary School Education?

Proposals for reform of the U.S. public education system generate heated debate. Many of the proposals concern the youngest students, those in elementary schools. Elementary school education has various objectives, such as developing social skills, but for many parents and educators, the most important objective is basic academic learning: reading, writing, and basic mathematics. One prominent proposal for improving basic learning is to reduce class sizes at elementary schools. With fewer students in the classroom, the argument goes, each student gets more of the teacher's attention, there are fewer class disruptions, learning is enhanced, and grades improve.

But what, precisely, is the effect on elementary school education of reducing class size? Reducing class size costs money: It requires hiring more teachers and, if the school is already at capacity, building more classrooms. A decision maker contemplating hiring more teachers must weigh these costs against the benefits. To weigh costs and benefits, however, the decision maker must have a precise quantitative understanding of the likely benefits. Is the beneficial effect on basic learning of smaller classes large or small? Is it possible that smaller class size actually has no effect on basic learning?

Although common sense and everyday experience may suggest that more learning occurs when there are fewer students, common sense cannot provide a quantitative answer to the question of what exactly is the effect on basic learning of reducing class size. To provide such an answer, we must examine empirical evidence—that is, evidence based on data—relating class size to basic learning in elementary schools.

In this book, we examine the relationship between class size and basic learning, using data gathered from 420 California school districts in 1999. In the California data, students in districts with small class sizes tend to perform better on standardized tests than students in districts with larger classes. While this fact is consistent with the idea that smaller classes produce better test scores, it might simply reflect many other advantages that students in districts with small classes have over their counterparts in districts with large classes. For example, districts with small class sizes tend to have wealthier residents than districts with large classes, so students in small-class districts could have more opportunities for learning outside the classroom. It could be these extra learning opportunities that lead to higher test scores, not smaller class sizes. In Part II, we use multiple regression analysis to isolate the effect of changes in class size from changes in other factors, such as the economic background of the students.

Question #2: Is There Racial Discrimination in the Market for Home Loans?

Most people buy their homes with the help of a mortgage, a large loan secured by the value of the home. By law, U.S. lending institutions cannot take race into account when deciding to grant or deny a request for a mortgage: Applicants who are identical in all ways except their race should be equally likely to have their mortgage applications approved. In theory, then, there should be no racial bias in mortgage lending.

In contrast to this theoretical conclusion, researchers at the Federal Reserve Bank of Boston found (using data from the early 1990s) that 28% of black applicants are denied mortgages, while only 9% of white applicants are denied. Do these data indicate that, in practice, there is racial bias in mortgage lending? If so, how large is it?

The fact that more black than white applicants are denied in the Boston Fed data does not by itself provide evidence of discrimination by mortgage lenders because the black and white applicants differ in many ways other than their race. Before concluding that there is bias in the mortgage market, these data must be examined more closely to see if there is a difference in the probability of being denied for *otherwise identical* applicants and, if so, whether this difference is large or small. To do so, in Chapter 11 we introduce econometric methods that make it possible to quantify the effect of race on the chance of obtaining a mortgage, *holding constant* other applicant characteristics, notably their ability to repay the loan.

Question #3: How Much Do Cigarette Taxes Reduce Smoking?

Cigarette smoking is a major public health concern worldwide. Many of the costs of smoking, such as the medical expenses of caring for those made sick by smoking and the less quantifiable costs to nonsmokers who prefer not to breathe secondhand cigarette smoke, are borne by other members of society. Because these costs are borne by people other than the smoker, there is a role for government intervention in reducing cigarette consumption. One of the most flexible tools for cutting consumption is to increase taxes on cigarettes.

Basic economics says that if cigarette prices go up, consumption will go down. But by how much? If the sales price goes up by 1%, by what percentage will the quantity of cigarettes sold decrease? The percentage change in the quantity demanded resulting from a 1% increase in price is the *price elasticity of demand*.

If we want to reduce smoking by a certain amount, say 20%, by raising taxes, then we need to know the price elasticity of demand to calculate the price increase necessary to achieve this reduction in consumption. But what is the price elasticity of demand for cigarettes?

Although economic theory provides us with the concepts that help us answer this question, it does not tell us the numerical value of the price elasticity of demand. To learn the elasticity, we must examine empirical evidence about the behavior of smokers and potential smokers; in other words, we need to analyze data on cigarette consumption and prices.

The data we examine are cigarette sales, prices, taxes, and personal income for U.S. states in the 1980s and 1990s. In these data, states with low taxes, and thus low cigarette prices, have high smoking rates, and states with high prices have low smoking rates. However, the analysis of these data is complicated because causality runs both ways: Low taxes lead to high demand, but if there are many smokers in the state, then local politicians might try to keep cigarette taxes low to satisfy their smoking constituents. In Chapter 12, we study methods for handling this "simultaneous causality" and use those methods to estimate the price elasticity of cigarette demand.

Question #4: By How Much Will U.S. GDP Grow Next Year?

It seems that people always want a sneak preview of the future. What will sales be next year at a firm that is considering investing in new equipment? Will the stock market go up next month, and, if it does, by how much? Will city tax receipts next year cover planned expenditures on city services? Will your microeconomics exam next week focus on externalities or monopolies? Will Saturday be a nice day to go to the beach?

One aspect of the future in which macroeconomists are particularly interested is the growth of real economic activity, as measured by real gross domestic product (GDP), during the next year. A management consulting firm might advise a manufacturing client to expand its capacity based on an upbeat forecast of economic growth. Economists at the Federal Reserve Board in Washington, D.C., are mandated to set policy to keep real GDP near its potential in order to maximize employment. If they forecast anemic GDP growth over the next year, they might expand liquidity in the economy by reducing interest rates or other measures, in an attempt to boost economic activity.

Professional economists who rely on precise numerical forecasts use econometric models to make those forecasts. A forecaster's job is to predict the future

by using the past, and econometricians do this by using economic theory and statistical techniques to quantify relationships in historical data.

The data we use to forecast the growth rate of GDP are past values of GDP and the "term spread" in the United States. The *term spread* is the difference between long-term and short-term interest rates. It measures, among other things, whether investors expect short-term interest rates to rise or fall in the future. The term spread is usually positive, but it tends to fall sharply before the onset of a recession. One of the GDP growth rate forecasts we develop and evaluate in Chapter 14 is based on the term spread.

Quantitative Questions, Quantitative Answers

Each of these four questions requires a numerical answer. Economic theory provides clues about that answer—for example, cigarette consumption ought to go down when the price goes up—but the actual value of the number must be learned empirically, that is, by analyzing data. Because we use data to answer quantitative questions, our answers always have some uncertainty: A different set of data would produce a different numerical answer. Therefore, the conceptual framework for the analysis needs to provide both a numerical answer to the question and a measure of how precise the answer is.

The conceptual framework used in this book is the multiple regression model, the mainstay of econometrics. This model, introduced in Part II, provides a mathematical way to quantify how a change in one variable affects another variable, holding other things constant. For example, what effect does a change in class size have on test scores, *holding constant* or *controlling for* student characteristics (such as family income) that a school district administrator cannot control? What effect does your race have on your chances of having a mortgage application granted, *holding constant* other factors such as your ability to repay the loan? What effect does a 1% increase in the price of cigarettes have on cigarette consumption, *holding constant* the income of smokers and potential smokers? The multiple regression model and its extensions provide a framework for answering these questions using data and for quantifying the uncertainty associated with those answers.

1.2 Causal Effects and Idealized Experiments

Like many other questions encountered in econometrics, the first three questions in Section 1.1 concern causal relationships among variables. In common usage, an action is said to cause an outcome if the outcome is the direct result, or consequence,

of that action. Touching a hot stove causes you to get burned; drinking water causes you to be less thirsty; putting air in your tires causes them to inflate; putting fertilizer on your tomato plants causes them to produce more tomatoes. Causality means that a specific action (applying fertilizer) leads to a specific, measurable consequence (more tomatoes).

Estimation of Causal Effects

How best might we measure the causal effect on tomato yield (measured in kilograms) of applying a certain amount of fertilizer, say 100 grams of fertilizer per square meter?

One way to measure this causal effect is to conduct an experiment. In that experiment, a horticultural researcher plants many plots of tomatoes. Each plot is tended identically, with one exception: Some plots get 100 grams of fertilizer per square meter, while the rest get none. Moreover, whether a plot is fertilized or not is determined randomly by a computer, ensuring that any other differences between the plots are unrelated to whether they receive fertilizer. At the end of the growing season, the horticulturalist weighs the harvest from each plot. The difference between the average yield per square meter of the treated and untreated plots is the effect on tomato production of the fertilizer treatment.

This is an example of a **randomized controlled experiment**. It is controlled in the sense that there are both a **control group** that receives no treatment (no fertilizer) and a **treatment group** that receives the treatment (100 g/m^2 of fertilizer). It is randomized in the sense that the treatment is assigned randomly. This random assignment eliminates the possibility of a systematic relationship between, for example, how sunny the plot is and whether it receives fertilizer so that the only systematic difference between the treatment and control groups is the treatment. If this experiment is properly implemented on a large enough scale, then it will yield an estimate of the causal effect on the outcome of interest (tomato production) of the treatment (applying 100 g/m^2 of fertilizer).

In this book, the **causal effect** is defined to be the effect on an outcome of a given action or treatment, as measured in an ideal randomized controlled experiment. In such an experiment, the only systematic reason for differences in outcomes between the treatment and control groups is the treatment itself.

It is possible to imagine an ideal randomized controlled experiment to answer each of the first three questions in Section 1.1. For example, to study class size, one can imagine randomly assigning "treatments" of different class sizes to different groups of students. If the experiment is designed and executed so that the only systematic difference between the groups of students is their class size, then in

theory this experiment would estimate the effect on test scores of reducing class size, holding all else constant.

The concept of an ideal randomized controlled experiment is useful because it gives a definition of a causal effect. In practice, however, it is not possible to perform ideal experiments. In fact, experiments are relatively rare in econometrics because often they are unethical, impossible to execute satisfactorily, or prohibitively expensive. The concept of the ideal randomized controlled experiment does, however, provide a theoretical benchmark for an econometric analysis of causal effects using actual data.

Forecasting and Causality

Although the first three questions in Section 1.1 concern causal effects, the fourth—forecasting the growth rate of GDP—does not. You do not need to know a causal relationship to make a good forecast. A good way to "forecast" whether it is raining is to observe whether pedestrians are using umbrellas, but the act of using an umbrella does not cause it to rain.

Even though forecasting need not involve causal relationships, economic theory suggests patterns and relationships that might be useful for forecasting. As we see in Chapter 14, multiple regression analysis allows us to quantify historical relationships suggested by economic theory, to check whether those relationships have been stable over time, to make quantitative forecasts about the future, and to assess the accuracy of those forecasts.

1.3 Data: Sources and Types

In econometrics, data come from one of two sources: experiments or nonexperimental observations of the world. This book examines both experimental and nonexperimental data sets.

Experimental Versus Observational Data

Experimental data come from experiments designed to evaluate a treatment or policy or to investigate a causal effect. For example, the state of Tennessee financed a large randomized controlled experiment examining class size in the 1980s. In that experiment, which we examine in Chapter 13, thousands of students were randomly assigned to classes of different sizes for several years and were given standardized tests annually.

The Tennessee class size experiment cost millions of dollars and required the ongoing cooperation of many administrators, parents, and teachers over several years. Because real-world experiments with human subjects are difficult to administer and to control, they have flaws relative to ideal randomized controlled experiments. Moreover, in some circumstances, experiments are not only expensive and difficult to administer but also unethical. (Would it be ethical to offer randomly selected teenagers inexpensive cigarettes to see how many they buy?) Because of these financial, practical, and ethical problems, experiments in economics are relatively rare. Instead, most economic data are obtained by observing real-world behavior.

Data obtained by observing actual behavior outside an experimental setting are called **observational data**. Observational data are collected using surveys, such as telephone surveys of consumers, and administrative records, such as historical records on mortgage applications maintained by lending institutions.

Observational data pose major challenges to econometric attempts to estimate causal effects, and the tools of econometrics are designed to tackle these challenges. In the real world, levels of "treatment" (the amount of fertilizer in the tomato example, the student–teacher ratio in the class size example) are not assigned at random, so it is difficult to sort out the effect of the "treatment" from other relevant factors. Much of econometrics, and much of this book, is devoted to methods for meeting the challenges encountered when real-world data are used to estimate causal effects.

Whether the data are experimental or observational, data sets come in three main types: cross-sectional data, time series data, and panel data. In this book, you will encounter all three types.

Cross-Sectional Data

Data on different entities—workers, consumers, firms, governmental units, and so forth—for a single time period are called **cross-sectional data**. For example, the data on test scores in California school districts are cross sectional. Those data are for 420 entities (school districts) for a single time period (1999). In general, the number of entities on which we have observations is denoted n; so, for example, in the California data set, $n = 420$.

The California test score data set contains measurements of several different variables for each district. Some of these data are tabulated in Table 1.1. Each row lists data for a different district. For example, the average test score for the first district ("district #1") is 690.8; this is the average of the math and science test scores for all fifth graders in that district in 1999 on a standardized test (the Stanford

TABLE 1.1 Selected Observations on Test Scores and Other Variables for California School Districts in 1999

Observation (District) Number	District Average Test Score (fifth grade)	Student–Teacher Ratio	Expenditure per Pupil ($)	Percentage of Students Learning English
1	690.8	17.89	$6385	0.0%
2	661.2	21.52	5099	4.6
3	643.6	18.70	5502	30.0
4	647.7	17.36	7102	0.0
5	640.8	18.67	5236	13.9
. . .	. . .	. . .	. . .	. . .
418	645.0	21.89	4403	24.3
419	672.2	20.20	4776	3.0
420	655.8	19.04	5993	5.0

Note: The California test score data set is described in Appendix 4.1.

Achievement Test). The average student–teacher ratio in that district is 17.89; that is, the number of students in district #1 divided by the number of classroom teachers in district #1 is 17.89. Average expenditure per pupil in district #1 is $6385. The percentage of students in that district still learning English—that is, the percentage of students for whom English is a second language and who are not yet proficient in English—is 0%.

The remaining rows present data for other districts. The order of the rows is arbitrary, and the number of the district, which is called the **observation number**, is an arbitrarily assigned number that organizes the data. As you can see in the table, all the variables listed vary considerably.

With cross-sectional data, we can learn about relationships among variables by studying differences across people, firms, or other economic entities during a single time period.

Time Series Data

Time series data are data for a single entity (person, firm, country) collected at multiple time periods. Our data set on the growth rate of GDP and the term spread in the United States is an example of a time series data set. The data set

TABLE 1.2 Selected Observations on the Growth Rate of GDP and the Term Spread in the United States: Quarterly Data, 1960:Q1–2013:Q1

Observation Number	Date (year:quarter)	GDP Growth Rate (% at an annual rate)	Term Spread (% per year)
1	1960:Q1	8.8%	0.6%
2	1960:Q2	–1.5	1.3
3	1960:Q3	1.0	1.5
4	1960:Q4	–4.9	1.6
5	1961:Q1	2.7	1.4
. . .	. . .	. . .	. . .
211	2012:Q3	2.7	1.5
212	2012:Q4	0.1	1.6
213	2013:Q1	1.1	1.9

Note: The United States GDP and term spread data set is described in Appendix 14.1.

contains observations on two variables (the growth rate of GDP and the term spread) for a single entity (the United States) for 213 time periods. Each time period in this data set is a quarter of a year (the first quarter is January, February, and March; the second quarter is April, May, and June; and so forth). The observations in this data set begin in the first quarter of 1960, which is denoted 1960:Q1, and end in the first quarter of 2013 (2013:Q1). The number of observations (that is, time periods) in a time series data set is denoted T. Because there are 213 quarters from 1960:Q1 to 2013:Q1, this data set contains $T = 213$ observations.

Some observations in this data set are listed in Table 1.2. The data in each row correspond to a different time period (year and quarter). In the first quarter of 1960, for example, GDP grew 8.8% at an annual rate. In other words, if GDP had continued growing for four quarters at its rate during the first quarter of 1960, the level of GDP would have increased by 8.8%. In the first quarter of 1960, the long-term interest rate was 4.5%, the short-term interest rate was 3.9%, so their difference, the term spread, was 0.6%.

By tracking a single entity over time, time series data can be used to study the evolution of variables over time and to forecast future values of those variables.

TABLE 1.3 Selected Observations on Cigarette Sales, Prices, and Taxes, by State and Year for U.S. States, 1985–1995

Observation Number	State	Year	Cigarette Sales (packs per capita)	Average Price per Pack (including taxes)	Total Taxes (cigarette excise tax + sales tax)
1	Alabama	1985	116.5	$1.022	$0.333
2	Arkansas	1985	128.5	1.015	0.370
3	Arizona	1985	104.5	1.086	0.362
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
47	West Virginia	1985	112.8	1.089	0.382
48	Wyoming	1985	129.4	0.935	0.240
49	Alabama	1986	117.2	1.080	0.334
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
96	Wyoming	1986	127.8	1.007	0.240
97	Alabama	1987	115.8	1.135	0.335
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
528	Wyoming	1995	112.2	1.585	0.360

Note: The cigarette consumption data set is described in Appendix 12.1.

Panel Data

Panel data, also called **longitudinal data**, are data for multiple entities in which each entity is observed at two or more time periods. Our data on cigarette consumption and prices are an example of a panel data set, and selected variables and observations in that data set are listed in Table 1.3. The number of entities in a panel data set is denoted n, and the number of time periods is denoted T. In the cigarette data set, we have observations on $n = 48$ continental U.S. states (entities) for $T = 11$ years (time periods) from 1985 to 1995. Thus there is a total of $n \times T = 48 \times 11 = 528$ observations.

KEY CONCEPT 1.1

Cross-Sectional, Time Series, and Panel Data

- Cross-sectional data consist of multiple entities observed at a single time period.
- Time series data consist of a single entity observed at multiple time periods.
- Panel data (also known as longitudinal data) consist of multiple entities, where each entity is observed at two or more time periods.

Some data from the cigarette consumption data set are listed in Table 1.3. The first block of 48 observations lists the data for each state in 1985, organized alphabetically from Alabama to Wyoming. The next block of 48 observations lists the data for 1986, and so forth, through 1995. For example, in 1985, cigarette sales in Arkansas were 128.5 packs per capita (the total number of packs of cigarettes sold in Arkansas in 1985 divided by the total population of Arkansas in 1985 equals 128.5). The average price of a pack of cigarettes in Arkansas in 1985, including tax, was \$1.015, of which 37¢ went to federal, state, and local taxes.

Panel data can be used to learn about economic relationships from the experiences of the many different entities in the data set and from the evolution over time of the variables for each entity.

The definitions of cross-sectional data, time series data, and panel data are summarized in Key Concept 1.1.

Summary

1. Many decisions in business and economics require quantitative estimates of how a change in one variable affects another variable.
2. Conceptually, the way to estimate a causal effect is in an ideal randomized controlled experiment, but performing such experiments in economic applications is usually unethical, impractical, or too expensive.
3. Econometrics provides tools for estimating causal effects using either observational (nonexperimental) data or data from real-world, imperfect experiments.
4. Cross-sectional data are gathered by observing multiple entities at a single point in time; time series data are gathered by observing a single entity at multiple points in time; and panel data are gathered by observing multiple entities, each of which is observed at multiple points in time.

Key Terms

randomized controlled experiment (52)
control group (52)
treatment group (52)
causal effect (52)
experimental data (53)
observational data (54)
cross-sectional data (54)
observation number (55)
time series data (55)
panel data (57)
longitudinal data (57)

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Review the Concepts

1.1 Design a hypothetical ideal randomized controlled experiment to study the effect of reading on the improvement of a person's vocabulary. Suggest some impediments to implementing this experiment in practice.

1.2 Design a hypothetical ideal randomized controlled experiment to study the effect of the consumption of alcohol on long-term memory loss. Suggest some impediments to implementing this experiment in practice.

1.3 You are asked to study the causal effect of hours spent in remedial classes at schools by students who are struggling in mathematics on their final test scores and performance in the subject. Describe:

a. an ideal randomized controlled experiment to measure this causal effect;

b. an observational cross-sectional data set with which you could study this effect;

c. an observational time series data set for studying this effect; and

d. an observational panel data set for studying this effect.

CHAPTER

2 Review of Probability

This chapter reviews the core ideas of the theory of probability that are needed to understand regression analysis and econometrics. We assume that you have taken an introductory course in probability and statistics. If your knowledge of probability is stale, you should refresh it by reading this chapter. If you feel confident with the material, you still should skim the chapter and the terms and concepts at the end to make sure you are familiar with the ideas and notation.

Most aspects of the world around us have an element of randomness. The theory of probability provides mathematical tools for quantifying and describing this randomness. Section 2.1 reviews probability distributions for a single random variable, and Section 2.2 covers the mathematical expectation, mean, and variance of a single random variable. Most of the interesting problems in economics involve more than one variable, and Section 2.3 introduces the basic elements of probability theory for two random variables. Section 2.4 discusses three special probability distributions that play a central role in statistics and econometrics: the normal, chi-squared, and *F* distributions.

The final two sections of this chapter focus on a specific source of randomness of central importance in econometrics: the randomness that arises by randomly drawing a sample of data from a larger population. For example, suppose you survey ten recent college graduates selected at random, record (or "observe") their earnings, and compute the average earnings using these ten data points (or "observations"). Because you chose the sample at random, you could have chosen ten different graduates by pure random chance; had you done so, you would have observed ten different earnings and you would have computed a different sample average. Because the average earnings vary from one randomly chosen sample to the next, the sample average is itself a random variable. Therefore, the sample average has a probability distribution, which is referred to as its sampling distribution because this distribution describes the different possible values of the sample average that might have occurred had a different sample been drawn.

Section 2.5 discusses random sampling and the sampling distribution of the sample average. This sampling distribution is, in general, complicated. When the

sample size is sufficiently large, however, the sampling distribution of the sample average is approximately normal, a result known as the central limit theorem, which is discussed in Section 2.6.

2.1 Random Variables and Probability Distributions

Probabilities, the Sample Space, and Random Variables

Probabilities and outcomes. The gender of the next new person you meet, your grade on an exam, and the number of times your computer will crash while you are writing a term paper all have an element of chance or randomness. In each of these examples, there is something not yet known that is eventually revealed.

The mutually exclusive potential results of a random process are called the **outcomes**. For example, your computer might never crash, it might crash once, it might crash twice, and so on. Only one of these outcomes will actually occur (the outcomes are mutually exclusive), and the outcomes need not be equally likely.

The **probability** of an outcome is the proportion of the time that the outcome occurs in the long run. If the probability of your computer not crashing while you are writing a term paper is 80%, then over the course of writing many term papers you will complete 80% without a crash.

The sample space and events. The set of all possible outcomes is called the **sample space**. An **event** is a subset of the sample space, that is, an event is a set of one or more outcomes. The event "my computer will crash no more than once" is the set consisting of two outcomes: "no crashes" and "one crash."

Random variables. A random variable is a numerical summary of a random outcome. The number of times your computer crashes while you are writing a term paper is random and takes on a numerical value, so it is a random variable.

Some random variables are discrete and some are continuous. As their names suggest, a **discrete random variable** takes on only a discrete set of values, like 0, 1, 2, . . . , whereas a **continuous random variable** takes on a continuum of possible values.

Probability Distribution of a Discrete Random Variable

Probability distribution. The **probability distribution** of a discrete random variable is the list of all possible values of the variable and the probability that each value will occur. These probabilities sum to 1.

For example, let M be the number of times your computer crashes while you are writing a term paper. The probability distribution of the random variable M is the list of probabilities of each possible outcome: The probability that $M = 0$, denoted $\Pr(M = 0)$, is the probability of no computer crashes; $\Pr(M = 1)$ is the probability of a single computer crash; and so forth. An example of a probability distribution for M is given in the second row of Table 2.1; in this distribution, if your computer crashes four times, you will quit and write the paper by hand. According to this distribution, the probability of no crashes is 80%; the probability of one crash is 10%; and the probability of two, three, or four crashes is, respectively, 6%, 3%, and 1%. These probabilities sum to 100%. This probability distribution is plotted in Figure 2.1.

Probabilities of events. The probability of an event can be computed from the probability distribution. For example, the probability of the event of one or two crashes is the sum of the probabilities of the constituent outcomes. That is, $\Pr(M = 1 \text{ or } M = 2) = \Pr(M = 1) + \Pr(M = 2) = 0.10 + 0.06 = 0.16$, or 16%.

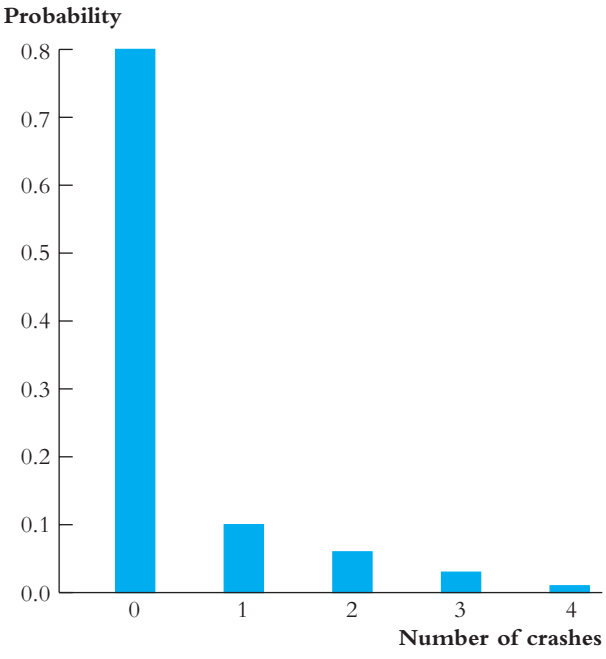
Cumulative probability distribution. The **cumulative probability distribution** is the probability that the random variable is less than or equal to a particular value. The last row of Table 2.1 gives the cumulative probability distribution of the random variable M. For example, the probability of at most one crash, $\Pr(M \leq 1)$, is 90%, which is the sum of the probabilities of no crashes (80%) and of one crash (10%).

TABLE 2.1 Probability of Your Computer Crashing *M* Times

	Outcome (number of crashes)				
	0	1	2	3	4
Probability distribution	0.80	0.10	0.06	0.03	0.01
Cumulative probability distribution	0.80	0.90	0.96	0.99	1.00

FIGURE 2.1 Probability Distribution of the Number of Computer Crashes

The height of each bar is the probability that the computer crashes the indicated number of times. The height of the first bar is 0.8, so the probability of 0 computer crashes is 80%. The height of the second bar is 0.1, so the probability of 1 computer crash is 10%, and so forth for the other bars.



A cumulative probability distribution is also referred to as a **cumulative distribution function**, a **c.d.f.**, or a **cumulative distribution**.

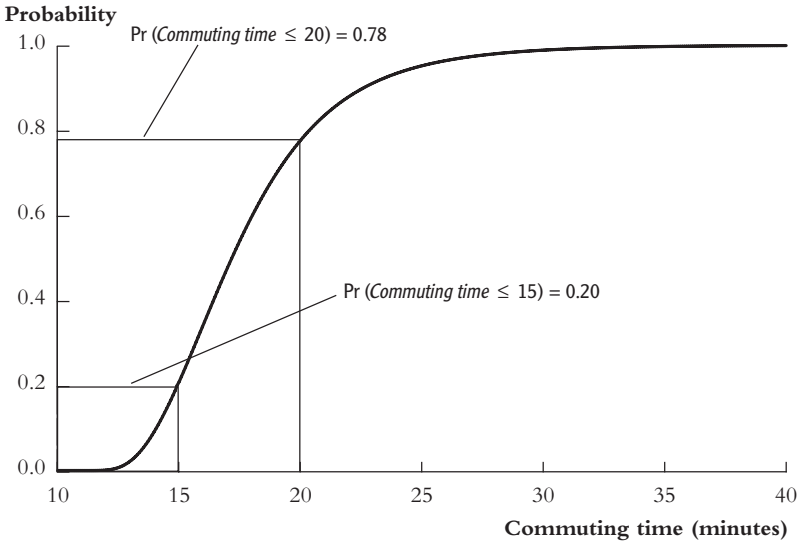
The Bernoulli distribution. An important special case of a discrete random variable is when the random variable is binary, that is, the outcomes are 0 or 1. A binary random variable is called a **Bernoulli random variable** (in honor of the seventeenth-century Swiss mathematician and scientist Jacob Bernoulli), and its probability distribution is called the **Bernoulli distribution**.

For example, let G be the gender of the next new person you meet, where $G = 0$ indicates that the person is male and $G = 1$ indicates that she is female. The outcomes of G and their probabilities thus are

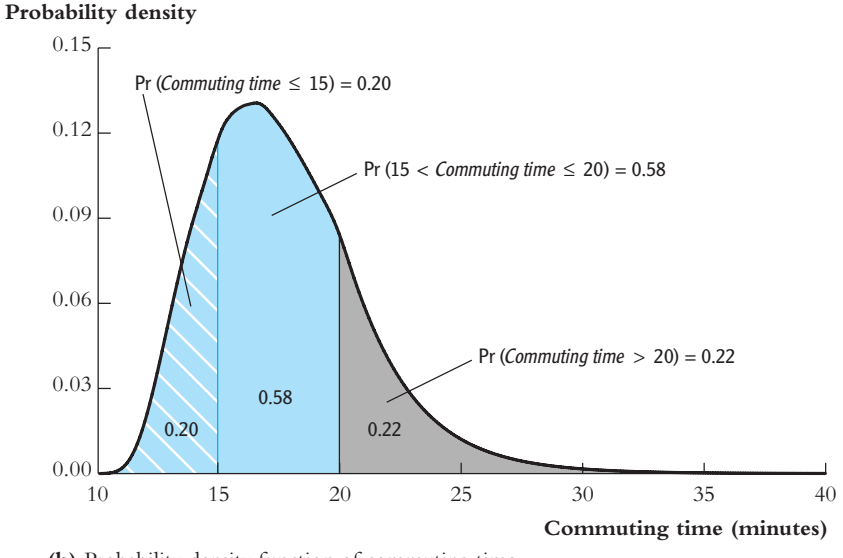
$$G = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1 - p, \end{cases} \tag{2.1}$$

where p is the probability of the next new person you meet being a woman. The probability distribution in Equation (2.1) is the Bernoulli distribution.

FIGURE 2.2 Cumulative Distribution and Probability Density Functions of Commuting Time



(a) Cumulative distribution function of commuting time



(b) Probability density function of commuting time

Figure 2.2a shows the cumulative probability distribution (or c.d.f.) of commuting times. The probability that a commuting time is less than 15 minutes is 0.20 (or 20%), and the probability that it is less than 20 minutes is 0.78 (78%). Figure 2.2b shows the probability density function (or p.d.f.) of commuting times. Probabilities are given by areas under the p.d.f. The probability that a commuting time is between 15 and 20 minutes is 0.58 (58%) and is given by the area under the curve between 15 and 20 minutes.

Probability Distribution of a Continuous Random Variable

Cumulative probability distribution. The cumulative probability distribution for a continuous variable is defined just as it is for a discrete random variable. That is, the cumulative probability distribution of a continuous random variable is the probability that the random variable is less than or equal to a particular value.

For example, consider a student who drives from home to school. This student's commuting time can take on a continuum of values and, because it depends on random factors such as the weather and traffic conditions, it is natural to treat it as a continuous random variable. Figure 2.2a plots a hypothetical cumulative distribution of commuting times. For example, the probability that the commute takes less than 15 minutes is 20% and the probability that it takes less than 20 minutes is 78%.

Probability density function. Because a continuous random variable can take on a continuum of possible values, the probability distribution used for discrete variables, which lists the probability of each possible value of the random variable, is not suitable for continuous variables. Instead, the probability is summarized by the **probability density function**. The area under the probability density function between any two points is the probability that the random variable falls between those two points. A probability density function is also called a **p.d.f.**, a **density function**, or simply a **density**.

Figure 2.2b plots the probability density function of commuting times corresponding to the cumulative distribution in Figure 2.2a. The probability that the commute takes between 15 and 20 minutes is given by the area under the p.d.f. between 15 minutes and 20 minutes, which is 0.58, or 58%. Equivalently, this probability can be seen on the cumulative distribution in Figure 2.2a as the difference between the probability that the commute is less than 20 minutes (78%) and the probability that it is less than 15 minutes (20%). Thus the probability density function and the cumulative probability distribution show the same information in different formats.

2.2 Expected Values, Mean, and Variance

The Expected Value of a Random Variable

Expected value. The **expected value** of a random variable Y, denoted $E(Y)$, is the long-run average value of the random variable over many repeated trials or occurrences. The expected value of a discrete random variable is computed as a weighted average of the possible outcomes of that random variable, where the weights are the probabilities of that outcome. The expected value of Y is also called the **expectation** of Y or the **mean** of Y and is denoted μ_Y.

For example, suppose you loan a friend \$100 at 10% interest. If the loan is repaid, you get \$110 (the principal of \$100 plus interest of \$10), but there is a risk of 1% that your friend will default and you will get nothing at all. Thus the amount you are repaid is a random variable that equals \$110 with probability 0.99 and equals \$0 with probability 0.01. Over many such loans, 99% of the time you would be paid back \$110, but 1% of the time you would get nothing, so on average you would be repaid $\$110 \times 0.99 + \$0 \times 0.01 = \$108.90$. Thus the expected value of your repayment (or the "mean repayment") is \$108.90.

As a second example, consider the number of computer crashes M with the probability distribution given in Table 2.1. The expected value of M is the average number of crashes over many term papers, weighted by the frequency with which a crash of a given size occurs. Accordingly,

$$E(M) = 0 \times 0.80 + 1 \times 0.10 + 2 \times 0.06 + 3 \times 0.03 + 4 \times 0.01 = 0.35. \quad (2.2)$$

That is, the expected number of computer crashes while writing a term paper is 0.35. Of course, the actual number of crashes must always be an integer; it makes no sense to say that the computer crashed 0.35 times while writing a particular term paper! Rather, the calculation in Equation (2.2) means that the average number of crashes over many such term papers is 0.35.

The formula for the expected value of a discrete random variable Y that can take on k different values is given as Key Concept 2.1. (Key Concept 2.1 uses "summation notation," which is reviewed in Exercise 2.25.)

KEY CONCEPT 2.1

Expected Value and the Mean

Suppose the random variable Y takes on k possible values, $y_1, \ldots, y_k$, where y_1 denotes the first value, y_2 denotes the second value, and so forth, and that the probability that Y takes on y_1 is p_1, the probability that Y takes on y_2 is p_2, and so forth. The expected value of Y, denoted $E(Y)$, is

$$E(Y) = y_1 p_1 + y_2 p_2 + \cdots + y_k p_k = \sum_{i=1}^{k} y_i p_i, \quad (2.3)$$

where the notation $\sum_{i=1}^{k} y_i p_i$ means "the sum of $y_i p_i$ for i running from 1 to k." The expected value of Y is also called the mean of Y or the expectation of Y and is denoted μ_Y.

Expected value of a Bernoulli random variable. An important special case of the general formula in Key Concept 2.1 is the mean of a Bernoulli random variable. Let G be the Bernoulli random variable with the probability distribution in Equation (2.1). The expected value of G is

$$E(G) = 1 \times p + 0 \times (1 - p) = p. \tag{2.4}$$

Thus the expected value of a Bernoulli random variable is p, the probability that it takes on the value "1."

Expected value of a continuous random variable. The expected value of a continuous random variable is also the probability-weighted average of the possible outcomes of the random variable. Because a continuous random variable can take on a continuum of possible values, the formal mathematical definition of its expectation involves calculus and its definition is given in Appendix 17.1.

The Standard Deviation and Variance

The variance and standard deviation measure the dispersion or the "spread" of a probability distribution. The **variance** of a random variable Y, denoted $\text{var}(Y)$, is the expected value of the square of the deviation of Y from its mean: $\text{var}(Y) = E[(Y - \mu_Y)^2]$.

Because the variance involves the square of Y, the units of the variance are the units of the square of Y, which makes the variance awkward to interpret. It is therefore common to measure the spread by the **standard deviation**, which is the square root of the variance and is denoted σ_Y. The standard deviation has the same units as Y. These definitions are summarized in Key Concept 2.2.

Variance and Standard Deviation

KEY CONCEPT 2.2

The variance of the discrete random variable Y, denoted σ_Y^2, is

$$\sigma_Y^2 = \text{var}(Y) = E[(Y - \mu_Y)^2] = \sum_{i=1}^{k} (y_i - \mu_Y)^2 p_i. \tag{2.5}$$

The standard deviation of Y is σ_Y, the square root of the variance. The units of the standard deviation are the same as the units of Y.

For example, the variance of the number of computer crashes M is the probability-weighted average of the squared difference between M and its mean, 0.35:

$$\begin{aligned}\text{var}(M) = &(0 - 0.35)^2 \times 0.80 + (1 - 0.35)^2 \times 0.10 + (2 - 0.35)^2 \times 0.06 \\ &+ (3 - 0.35)^2 \times 0.03 + (4 - 0.35)^2 \times 0.01 = 0.6475. \qquad (2.6)\end{aligned}$$

The standard deviation of M is the square root of the variance, so $\sigma_M = \sqrt{0.64750} \cong 0.80$.

Variance of a Bernoulli random variable. The mean of the Bernoulli random variable G with probability distribution in Equation (2.1) is $\mu_G = p$ [Equation (2.4)], so its variance is

$$\text{var}(G) = \sigma_G^2 = (0 - p)^2 \times (1 - p) + (1 - p)^2 \times p = p(1 - p). \qquad (2.7)$$

Thus the standard deviation of a Bernoulli random variable is $\sigma_G = \sqrt{p(1 - p)}$.

Mean and Variance of a Linear Function of a Random Variable

This section discusses random variables (say, X and Y) that are related by a linear function. For example, consider an income tax scheme under which a worker is taxed at a rate of 20% on his or her earnings and then given a (tax-free) grant of \$2000. Under this tax scheme, after-tax earnings Y are related to pre-tax earnings X by the equation

$$Y = 2000 + 0.8X. \qquad (2.8)$$

That is, after-tax earnings Y is 80% of pre-tax earnings X, plus \$2000.

Suppose an individual's pre-tax earnings next year are a random variable with mean μ_X and variance σ_X^2. Because pre-tax earnings are random, so are after-tax earnings. What are the mean and standard deviations of her after-tax earnings under this tax? After taxes, her earnings are 80% of the original pre-tax earnings, plus \$2000. Thus the expected value of her after-tax earnings is

$$E(Y) = \mu_Y = 2000 + 0.8\mu_X. \qquad (2.9)$$

The variance of after-tax earnings is the expected value of $(Y - \mu_Y)^2$. Because $Y = 2000 + 0.8X$, $Y - \mu_Y = 2000 + 0.8X - (2000 + 0.8\mu_X) = 0.8(X - \mu_X)$.

Thus $E[(Y - \mu_Y)^2] = E\{[0.8(X - \mu_X)]^2\} = 0.64E[(X - \mu_X)^2]$. It follows that $\text{var}(Y) = 0.64\text{var}(X)$, so, taking the square root of the variance, the standard deviation of Y is

$$\sigma_Y = 0.8\sigma_X. \tag{2.10}$$

That is, the standard deviation of the distribution of her after-tax earnings is 80% of the standard deviation of the distribution of pre-tax earnings.

This analysis can be generalized so that Y depends on X with an intercept a (instead of \$2000) and a slope b (instead of 0.8) so that

$$Y = a + bX. \tag{2.11}$$

Then the mean and variance of Y are

$$\mu_Y = a + b\mu_X \quad \text{and} \tag{2.12}$$

$$\sigma_Y^2 = b^2\sigma_X^2, \tag{2.13}$$

and the standard deviation of Y is $\sigma_Y = b\sigma_X$. The expressions in Equations (2.9) and (2.10) are applications of the more general formulas in Equations (2.12) and (2.13) with $a = 2000$ and $b = 0.8$.

Other Measures of the Shape of a Distribution

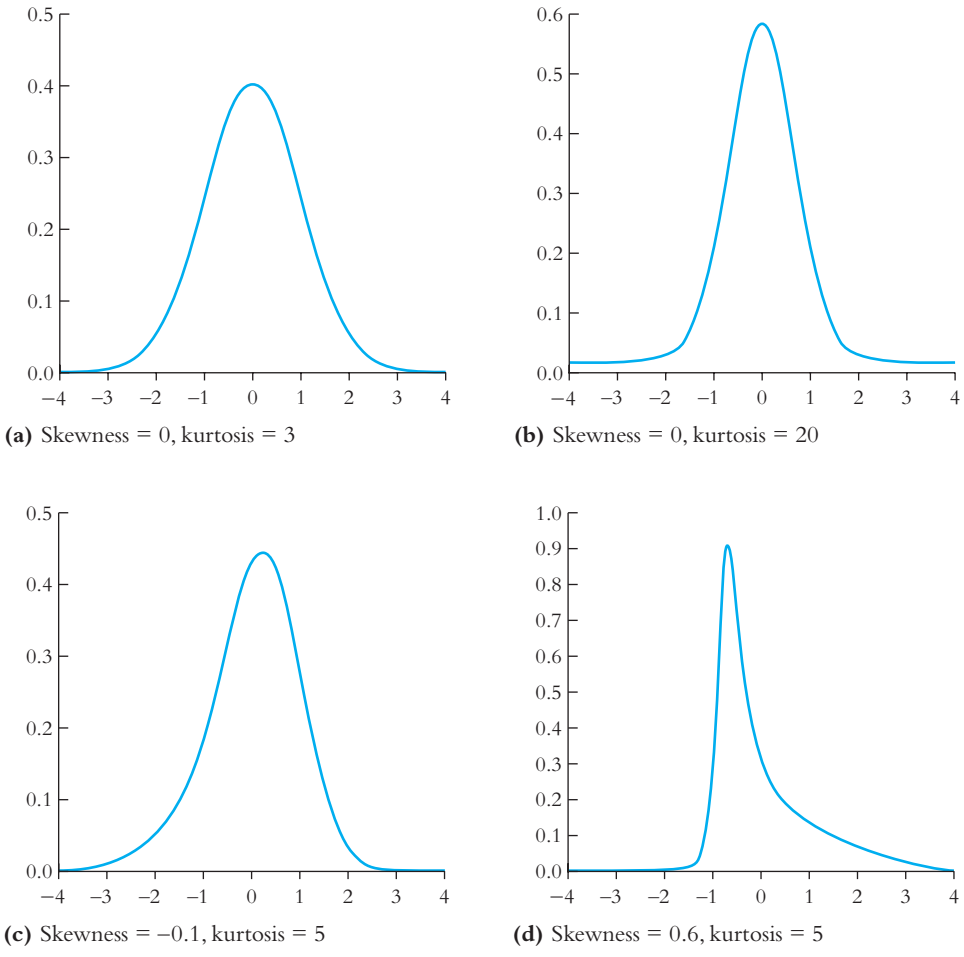
The mean and standard deviation measure two important features of a distribution: its center (the mean) and its spread (the standard deviation). This section discusses measures of two other features of a distribution: the skewness, which measures the lack of symmetry of a distribution, and the kurtosis, which measures how thick, or "heavy," are its tails. The mean, variance, skewness, and kurtosis are all based on what are called the **moments of a distribution**.

Skewness. Figure 2.3 plots four distributions, two which are symmetric (Figures 2.3a and 2.3b) and two which are not (Figures 2.3c and 2.3d). Visually, the distribution in Figure 2.3d appears to deviate more from symmetry than does the distribution in Figure 2.3c. The skewness of a distribution provides a mathematical way to describe how much a distribution deviates from symmetry.

The **skewness** of the distribution of a random variable Y is

$$\text{Skewness} = \frac{E[(Y - \mu_Y)^3]}{\sigma_Y^3}, \tag{2.14}$$

FIGURE 2.3 Four Distributions with Different Skewness and Kurtosis



(a) Skewness = 0, kurtosis = 3

(b) Skewness = 0, kurtosis = 20

(c) Skewness = −0.1, kurtosis = 5

(d) Skewness = 0.6, kurtosis = 5

All of these distributions have a mean of 0 and a variance of 1. The distributions with skewness of 0 (a and b) are symmetric; the distributions with nonzero skewness (c and d) are not symmetric. The distributions with kurtosis exceeding 3 (b–d) have heavy tails.

where σ_Y is the standard deviation of Y. For a symmetric distribution, a value of Y a given amount above its mean is just as likely as a value of Y the same amount below its mean. If so, then positive values of $(Y - \mu_Y)^3$ will be offset on average (in expectation) by equally likely negative values. Thus, for a symmetric distribution, $E[(Y - \mu_Y)^3] = 0$; the skewness of a symmetric distribution is zero. If a

distribution is not symmetric, then a positive value of $(Y - \mu_Y)^3$ generally is not offset on average by an equally likely negative value, so the skewness is nonzero for a distribution that is not symmetric. Dividing by σ_Y^3 in the denominator of Equation (2.14) cancels the units of Y^3 in the numerator, so the skewness is unit free; in other words, changing the units of Y does not change its skewness.

Below each of the four distributions in Figure 2.3 is its skewness. If a distribution has a long right tail, positive values of $(Y - \mu_Y)^3$ are not fully offset by negative values, and the skewness is positive. If a distribution has a long left tail, its skewness is negative.

Kurtosis. The **kurtosis** of a distribution is a measure of how much mass is in its tails and, therefore, is a measure of how much of the variance of Y arises from extreme values. An extreme value of Y is called an **outlier**. The greater the kurtosis of a distribution, the more likely are outliers.

The kurtosis of the distribution of Y is

$$\text{Kurtosis} = \frac{E[(Y - \mu_Y)^4]}{\sigma_Y^4}. \tag{2.15}$$

If a distribution has a large amount of mass in its tails, then some extreme departures of Y from its mean are likely, and these departures will lead to large values, on average (in expectation), of $(Y - \mu_Y)^4$. Thus, for a distribution with a large amount of mass in its tails, the kurtosis will be large. Because $(Y - \mu_Y)^4$ cannot be negative, the kurtosis cannot be negative.

The kurtosis of a normally distributed random variable is 3, so a random variable with kurtosis exceeding 3 has more mass in its tails than a normal random variable. A distribution with kurtosis exceeding 3 is called **leptokurtic** or, more simply, heavy-tailed. Like skewness, the kurtosis is unit free, so changing the units of Y does not change its kurtosis.

Below each of the four distributions in Figure 2.3 is its kurtosis. The distributions in Figures 2.3b–d are heavy-tailed.

Moments. The mean of Y, $E(Y)$, is also called the first moment of Y, and the expected value of the square of Y, $E(Y^2)$, is called the second moment of Y. In general, the expected value of Y^r is called the **r^{th} moment** of the random variable Y. That is, the r^{th} moment of Y is $E(Y^r)$. The skewness is a function of the first, second, and third moments of Y, and the kurtosis is a function of the first through fourth moments of Y.

2.3 Two Random Variables

Most of the interesting questions in economics involve two or more variables. Are college graduates more likely to have a job than nongraduates? How does the distribution of income for women compare to that for men? These questions concern the distribution of two random variables, considered together (education and employment status in the first example, income and gender in the second). Answering such questions requires an understanding of the concepts of joint, marginal, and conditional probability distributions.

Joint and Marginal Distributions

Joint distribution. The **joint probability distribution** of two discrete random variables, say X and Y, is the probability that the random variables simultaneously take on certain values, say x and y. The probabilities of all possible (x, y) combinations sum to 1. The joint probability distribution can be written as the function $\Pr(X = x, Y = y)$.

For example, weather conditions—whether or not it is raining—affect the commuting time of the student commuter in Section 2.1. Let Y be a binary random variable that equals 1 if the commute is short (less than 20 minutes) and equals 0 otherwise and let X be a binary random variable that equals 0 if it is raining and 1 if not. Between these two random variables, there are four possible outcomes: it rains and the commute is long ($X = 0, Y = 0$); rain and short commute ($X = 0, Y = 1$); no rain and long commute ($X = 1, Y = 0$); and no rain and short commute ($X = 1, Y = 1$). The joint probability distribution is the frequency with which each of these four outcomes occurs over many repeated commutes.

An example of a joint distribution of these two variables is given in Table 2.2. According to this distribution, over many commutes, 15% of the days have rain and a long commute ($X = 0, Y = 0$); that is, the probability of a long, rainy commute is 15%, or $\Pr(X = 0, Y = 0) = 0.15$. Also, $\Pr(X = 0, Y = 1) = 0.15$,

TABLE 2.2 Joint Distribution of Weather Conditions and Commuting Times

	Rain ($X = 0$)	No Rain ($X = 1$)	Total
Long commute ($Y = 0$)	0.15	0.07	0.22
Short commute ($Y = 1$)	0.15	0.63	0.78
Total	0.30	0.70	1.00

$\Pr(X = 1, Y = 0) = 0.07$, and $\Pr(X = 1, Y = 1) = 0.63$. These four possible outcomes are mutually exclusive and constitute the sample space so the four probabilities sum to 1.

Marginal probability distribution. The **marginal probability distribution** of a random variable Y is just another name for its probability distribution. This term is used to distinguish the distribution of Y alone (the marginal distribution) from the joint distribution of Y and another random variable.

The marginal distribution of Y can be computed from the joint distribution of X and Y by adding up the probabilities of all possible outcomes for which Y takes on a specified value. If X can take on l different values $x_1, \ldots, x_l$, then the marginal probability that Y takes on the value y is

$$\Pr(Y = y) = \sum_{i=1}^{l} \Pr(X = x_i, Y = y). \tag{2.16}$$

For example, in Table 2.2, the probability of a long rainy commute is 15% and the probability of a long commute with no rain is 7%, so the probability of a long commute (rainy or not) is 22%. The marginal distribution of commuting times is given in the final column of Table 2.2. Similarly, the marginal probability that it will rain is 30%, as shown in the final row of Table 2.2.

Conditional Distributions

Conditional distribution. The distribution of a random variable Y conditional on another random variable X taking on a specific value is called the **conditional distribution** of Y given X. The conditional probability that Y takes on the value y when X takes on the value x is written $\Pr(Y = y \mid X = x)$.

For example, what is the probability of a long commute ($Y = 0$) if you know it is raining ($X = 0$)? From Table 2.2, the joint probability of a rainy short commute is 15% and the joint probability of a rainy long commute is 15%, so if it is raining a long commute and a short commute are equally likely. Thus the probability of a long commute ($Y = 0$), conditional on it being rainy ($X = 0$), is 50%, or $\Pr(Y = 0 \mid X = 0) = 0.50$. Equivalently, the marginal probability of rain is 30%; that is, over many commutes it rains 30% of the time. Of this 30% of commutes, 50% of the time the commute is long (0.15/0.30).

In general, the conditional distribution of Y given $X = x$ is

$$\Pr(Y = y \mid X = x) = \frac{\Pr(X = x, Y = y)}{\Pr(X = x)}. \tag{2.17}$$

TABLE 2.3 Joint and Conditional Distributions of Computer Crashes (M) and Computer Age (A)

A. Joint Distribution

	$M = 0$	$M = 1$	$M = 2$	$M = 3$	$M = 4$	Total
Old computer ($A = 0$)	0.35	0.065	0.05	0.025	0.01	0.50
New computer ($A = 1$)	0.45	0.035	0.01	0.005	0.00	0.50
Total	0.80	0.10	0.06	0.03	0.01	1.00

B. Conditional Distributions of *M* given *A*

	$M = 0$	$M = 1$	$M = 2$	$M = 3$	$M = 2$	Total
$\Pr(M \mid A = 0)$	0.70	0.13	0.10	0.05	0.02	1.00
$\Pr(M \mid A = 1)$	0.90	0.07	0.02	0.01	0.00	1.00

For example, the conditional probability of a long commute given that it is rainy is $\Pr(Y = 0 \mid X = 0) = \Pr(X = 0, Y = 0)/\Pr(X = 0) = 0.15/0.30 = 0.50$.

As a second example, consider a modification of the crashing computer example. Suppose you use a computer in the library to type your term paper and the librarian randomly assigns you a computer from those available, half of which are new and half of which are old. Because you are randomly assigned to a computer, the age of the computer you use, A ($= 1$ if the computer is new, $= 0$ if it is old), is a random variable. Suppose the joint distribution of the random variables M and A is given in Part A of Table 2.3. Then the conditional distribution of computer crashes, given the age of the computer, is given in Part B of the table. For example, the joint probability $M = 0$ and $A = 0$ is 0.35; because half the computers are old, the conditional probability of no crashes, given that you are using an old computer, is $\Pr(M = 0 \mid A = 0) = \Pr(M = 0, A = 0)/\Pr(A = 0) = 0.35/0.50 = 0.70$, or 70%. In contrast, the conditional probability of no crashes given that you are assigned a new computer is 90%. According to the conditional distributions in Part B of Table 2.3, the newer computers are less likely to crash than the old ones; for example, the probability of three crashes is 5% with an old computer but 1% with a new computer.

Conditional expectation. The **conditional expectation** of Y given X, also called the **conditional mean** of Y given X, is the mean of the conditional distribution of Y given X. That is, the conditional expectation is the expected value of Y, computed

using the conditional distribution of Y given X. If Y takes on k values $y_1, \ldots, y_k$, then the conditional mean of Y given $X = x$ is

$$E(Y \mid X = x) = \sum_{i=1}^{k} y_i \Pr(Y = y_i \mid X = x). \tag{2.18}$$

For example, based on the conditional distributions in Table 2.3, the expected number of computer crashes, given that the computer is old, is $E(M \mid A = 0) = 0 \times 0.70 + 1 \times 0.13 + 2 \times 0.10 + 3 \times 0.05 + 4 \times 0.02 = 0.56$. The expected number of computer crashes, given that the computer is new, is $E(M \mid A = 1) = 0.14$, less than for the old computers.

The conditional expectation of Y given $X = x$ is just the mean value of Y when $X = x$. In the example of Table 2.3, the mean number of crashes is 0.56 for old computers, so the conditional expectation of Y given that the computer is old is 0.56. Similarly, among new computers, the mean number of crashes is 0.14, that is, the conditional expectation of Y given that the computer is new is 0.14.

The law of iterated expectations. The mean of Y is the weighted average of the conditional expectation of Y given X, weighted by the probability distribution of X. For example, the mean height of adults is the weighted average of the mean height of men and the mean height of women, weighted by the proportions of men and women. Stated mathematically, if X takes on the l values $x_1, \ldots, x_l$, then

$$E(Y) = \sum_{i=1}^{l} E(Y \mid X = x_i) \Pr(X = x_i). \tag{2.19}$$

Equation (2.19) follows from Equations (2.18) and (2.17) (see Exercise 2.19).

Stated differently, the expectation of Y is the expectation of the conditional expectation of Y given X,

$$E(Y) = E[E(Y \mid X)], \tag{2.20}$$

where the inner expectation on the right-hand side of Equation (2.20) is computed using the conditional distribution of Y given X and the outer expectation is computed using the marginal distribution of X. Equation (2.20) is known as the **law of iterated expectations**.

For example, the mean number of crashes M is the weighted average of the conditional expectation of M given that it is old and the conditional expectation of

M given that it is new, so $E(M) = E(M \mid A = 0) \times \Pr(A = 0) + E(M \mid A = 1) \times \Pr(A = 1) = 0.56 \times 0.50 + 0.14 \times 0.50 = 0.35$. This is the mean of the marginal distribution of M, as calculated in Equation (2.2).

The law of iterated expectations implies that if the conditional mean of Y given X is zero, then the mean of Y is zero. This is an immediate consequence of Equation (2.20): if $E(Y \mid X) = 0$, then $E(Y) = E[E(Y \mid X)] = E[0] = 0$. Said differently, if the mean of Y given X is zero, then it must be that the probability-weighted average of these conditional means is zero, that is, the mean of Y must be zero.

The law of iterated expectations also applies to expectations that are conditional on multiple random variables. For example, let X, Y, and Z be random variables that are jointly distributed. Then the law of iterated expectations says that $E(Y) = E[E(Y \mid X, Z)]$, where $E(Y \mid X, Z)$ is the conditional expectation of Y given both X and Z. For example, in the computer crash illustration of Table 2.3, let P denote the number of programs installed on the computer; then $E(M \mid A, P)$ is the expected number of crashes for a computer with age A that has P programs installed. The expected number of crashes overall, $E(M)$, is the weighted average of the expected number of crashes for a computer with age A and number of programs P, weighted by the proportion of computers with that value of both A and P.

Exercise 2.20 provides some additional properties of conditional expectations with multiple variables.

Conditional variance. The variance of Y conditional on X is the variance of the conditional distribution of Y given X. Stated mathematically, the **conditional variance** of Y given X is

$$\text{var}(Y \mid X = x) = \sum_{i=1}^{k} [y_i - E(Y \mid X = x)]^2 \Pr(Y = y_i \mid X = x). \quad (2.21)$$

For example, the conditional variance of the number of crashes given that the computer is old is $\text{var}(M \mid A = 0) = (0 - 0.56)^2 \times 0.70 + (1 - 0.56)^2 \times 0.13 + (2 - 0.56)^2 \times 0.10 + (3 - 0.56)^2 \times 0.05 + (4 - 0.56)^2 \times 0.02 \cong 0.99$. The standard deviation of the conditional distribution of M given that $A = 0$ is thus $\sqrt{0.99} = 0.99$. The conditional variance of M given that $A = 1$ is the variance of the distribution in the second row of Panel B of Table 2.3, which is 0.22, so the standard deviation of M for new computers is $\sqrt{0.22} = 0.47$. For the conditional distributions in Table 2.3, the expected number of crashes for new computers (0.14) is less than that for old computers (0.56), and the spread of the distribution of the number of crashes, as measured by the conditional standard deviation, is smaller for new computers (0.47) than for old (0.99).

Independence

Two random variables X and Y are **independently distributed**, or **independent**, if knowing the value of one of the variables provides no information about the other. Specifically, X and Y are independent if the conditional distribution of Y given X equals the marginal distribution of Y. That is, X and Y are independently distributed if, for all values of x and y,

$$\Pr(Y = y \mid X = x) = \Pr(Y = y) \quad \text{(independence of } X \text{ and } Y\text{)}. \tag{2.22}$$

Substituting Equation (2.22) into Equation (2.17) gives an alternative expression for independent random variables in terms of their joint distribution. If X and Y are independent, then

$$\Pr(X = x, Y = y) = \Pr(X = x)\Pr(Y = y). \tag{2.23}$$

That is, the joint distribution of two independent random variables is the product of their marginal distributions.

Covariance and Correlation

Covariance. One measure of the extent to which two random variables move together is their covariance. The **covariance** between X and Y is the expected value $E[(X - \mu_X)(Y - \mu_Y)]$, where μ_X, where μ_X is the mean of X and μ_Y is the mean of Y. The covariance is denoted $\text{cov}(X, Y)$ or σ_{XY}. If X can take on l values and Y can take on k values, then the covariance is given by the formula

$$\begin{aligned}\text{cov}(X, Y) = \sigma_{XY} &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= \sum_{i=1}^{k}\sum_{j=1}^{l}(x_j - \mu_X)(y_i - \mu_Y)\Pr(X = x_j, Y = y_i).\end{aligned} \tag{2.24}$$

To interpret this formula, suppose that when X is greater than its mean (so that $X - \mu_X$ is positive), then Y tends be greater than its mean (so that $Y - \mu_Y$ is positive), and when X is less than its mean (so that $X - \mu_X < 0$), then Y tends to be less than its mean (so that $Y - \mu_Y < 0$). In both cases, the product $(X - \mu_X) \times (Y - \mu_Y)$ tends to be positive, so the covariance is positive. In contrast, if X and Y tend to move in opposite directions (so that X is large when Y is small, and vice versa), then the covariance is negative. Finally, if X and Y are independent, then the covariance is zero (see Exercise 2.19).

Correlation. Because the covariance is the product of X and Y, deviated from their means, its units are, awkwardly, the units of X multiplied by the units of Y. This "units" problem can make numerical values of the covariance difficult to interpret.

The correlation is an alternative measure of dependence between X and Y that solves the "units" problem of the covariance. Specifically, the **correlation** between X and Y is the covariance between X and Y divided by their standard deviations:

$$\text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\,\text{var}(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}. \tag{2.25}$$

Because the units of the numerator in Equation (2.25) are the same as those of the denominator, the units cancel and the correlation is unitless. The random variables X and Y are said to be **uncorrelated** if $\text{corr}(X, Y) = 0$.

The correlation always is between –1 and 1; that is, as proven in Appendix 2.1,

$$-1 \le \text{corr}(X, Y) \le 1 \quad \text{(correlation inequality)}. \tag{2.26}$$

Correlation and conditional mean. If the conditional mean of Y does not depend on X, then Y and X are uncorrelated. That is,

$$\text{if } E(Y \mid X) = \mu_Y, \text{ then } \text{cov}(Y, X) = 0 \text{ and } \text{corr}(Y, X) = 0. \tag{2.27}$$

We now show this result. First suppose that Y and X have mean zero so that $\text{cov}(Y, X) = E[(Y - \mu_Y)(X - \mu_X)] = E(YX)$. By the law of iterated expectations [Equation (2.20)], $E(YX) = E[E(YX \mid X)] = E[E(Y \mid X)X] = 0$ because $E(Y \mid X) = 0$, so $\text{cov}(Y, X) = 0$. Equation (2.27) follows by substituting $\text{cov}(Y, X) = 0$ into the definition of correlation in Equation (2.25). If Y and X do not have mean zero, first subtract off their means, then the preceding proof applies.

It is *not* necessarily true, however, that if X and Y are uncorrelated, then the conditional mean of Y given X does not depend on X. Said differently, it is possible for the conditional mean of Y to be a function of X but for Y and X nonetheless to be uncorrelated. An example is given in Exercise 2.23.

The Mean and Variance of Sums of Random Variables

The mean of the sum of two random variables, X and Y, is the sum of their means:

$$E(X + Y) = E(X) + E(Y) = \mu_X + \mu_Y. \tag{2.28}$$

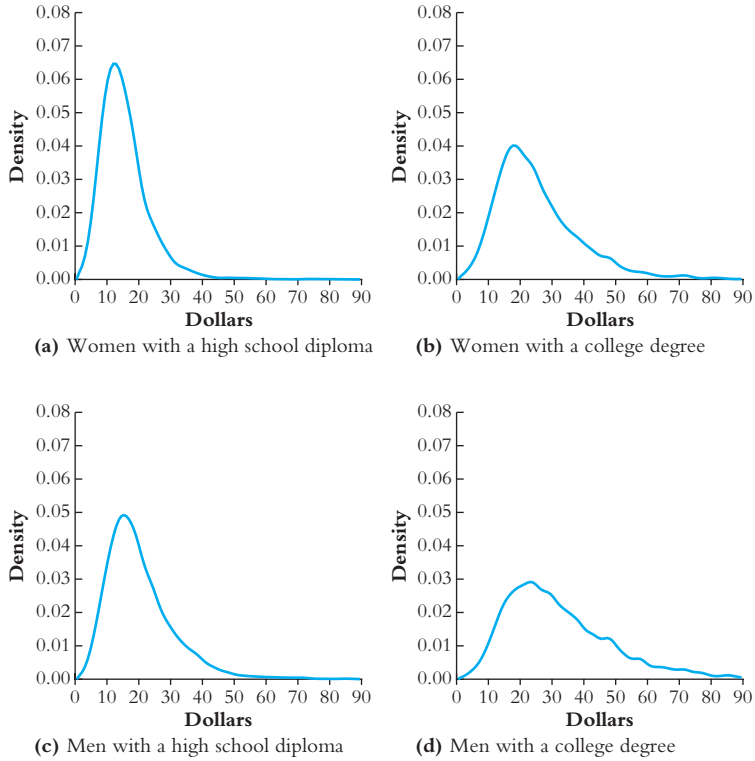
The Distribution of Earnings in the United States in 2012

Some parents tell their children that they will be able to get a better, higher-paying job if they get a college degree than if they skip higher education. Are these parents right? Does the distribution of earnings differ between workers who are college graduates and workers who have only a high school diploma, and, if so, how? Among workers with a similar education, does the distribution of earnings for men and women differ? For example, do the best-paid college-educated women earn as much as the best-paid college-educated men?

One way to answer these questions is to examine the distribution of earnings of full-time workers, conditional on the highest educational degree achieved (high school diploma or bachelor's degree) and on gender. These four conditional distributions are shown in Figure 2.4, and the mean, standard deviation, and

FIGURE 2.4 Conditional Distribution of Average Hourly Earnings of U.S. Full-Time Workers in 2012, Given Education Level and Gender

The four distributions of earnings are for women and men, for those with only a high school diploma (a and c) and those whose highest degree is from a four-year college (b and d).



(a) Women with a high school diploma

(b) Women with a college degree

(c) Men with a high school diploma

(d) Men with a college degree

continued on next page

TABLE 2.4 Summaries of the Conditional Distribution of Average Hourly Earnings of U.S. Full-Time Workers in 2012 Given Education Level and Gender

			Percentile			
	Mean	Standard Deviation	25%	50% (median)	75%	90%
(a) Women with high school diploma	$15.49	$8.42	$10.10	$14.03	$18.75	$24.52
(b) Women with four-year college degree	25.42	13.81	16.15	22.44	31.34	43.27
(c) Men with high school diploma	20.25	11.00	12.92	17.86	24.83	33.78
(d) Men with four-year college degree	32.73	18.11	19.61	28.85	41.68	57.30

Average hourly earnings are the sum of annual pretax wages, salaries, tips, and bonuses divided by the number of hours worked annually.

some percentiles of the conditional distributions are presented in Table 2.4.[1] For example, the conditional mean of earnings for women whose highest degree is a high school diploma—that is, $E(Earnings|Highest\ degree = high\ school\ diploma,\ Gender = female)$—is $15.49 per hour.

The distribution of average hourly earnings for female college graduates (Figure 2.4b) is shifted to the right of the distribution for women with only a high school degree (Figure 2.4a); the same shift can be seen for the two groups of men (Figure 2.4d and Figure 2.4c). For both men and women, mean earnings are higher for those with a college degree (Table 2.4, first numeric column). Interestingly, the spread of the distribution of earnings, as measured by the standard deviation, is greater for those with a college degree than for those with a high school diploma. In addition, for both men and women, the 90th percentile of earnings is much higher for workers with a college degree than for workers with only a high school diploma. This final comparison is consistent with the parental admonition that a college degree opens doors that remain closed to individuals with only a high school diploma.

Another feature of these distributions is that the distribution of earnings for men is shifted to the right of the distribution of earnings for women. This "gender gap" in earnings is an important—and, to many, troubling—aspect of the distribution of earnings. We return to this topic in later chapters.

[1]The distributions were estimated using data from the March 2013 Current Population Survey, which is discussed in more detail in Appendix 3.1.

Means, Variances, and Covariances of Sums of Random Variables

KEY CONCEPT 2.3

Let X, Y, and V be random variables, let μ_X and σ_X^2 be the mean and variance of X, let σ_{XY} be the covariance between X and Y (and so forth for the other variables), and let a, b, and c be constants. Equations (2.29) through (2.35) follow from the definitions of the mean, variance, and covariance:

$$E(a + bX + cY) = a + b\mu_X + c\mu_Y, \tag{2.29}$$

$$\text{var}(a + bY) = b^2\sigma_Y^2, \tag{2.30}$$

$$\text{var}(aX + bY) = a^2\sigma_X^2 + 2ab\sigma_{XY} + b^2\sigma_Y^2, \tag{2.31}$$

$$E(Y^2) = \sigma_Y^2 + \mu_Y^2, \tag{2.32}$$

$$\text{cov}(a + bX + cV, Y) = b\sigma_{XY} + c\sigma_{VY}, \tag{2.33}$$

$$E(XY) = \sigma_{XY} + \mu_X\mu_Y, \tag{2.34}$$

$$|\text{corr}(X, Y)| \leq 1 \text{ and } |\sigma_{XY}| \leq \sqrt{\sigma_X^2\sigma_Y^2} \text{ (correlation inequality)}. \tag{2.35}$$

The variance of the sum of X and Y is the sum of their variances plus two times their covariance:

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X, Y) = \sigma_X^2 + \sigma_Y^2 + 2\sigma_{XY}. \tag{2.36}$$

If X and Y are independent, then the covariance is zero and the variance of their sum is the sum of their variances:

$$\text{var}(X + Y) = \text{var}(X) + \text{var}(Y) = \sigma_X^2 + \sigma_Y^2 \quad \text{(if } X \text{ and } Y \text{ are independent)}. \tag{2.37}$$

Useful expressions for means, variances, and covariances involving weighted sums of random variables are collected in Key Concept 2.3. The results in Key Concept 2.3 are derived in Appendix 2.1.

2.4 The Normal, Chi-Squared, Student *t*, and *F* Distributions

The probability distributions most often encountered in econometrics are the normal, chi-squared, Student t, and F distributions.

The Normal Distribution

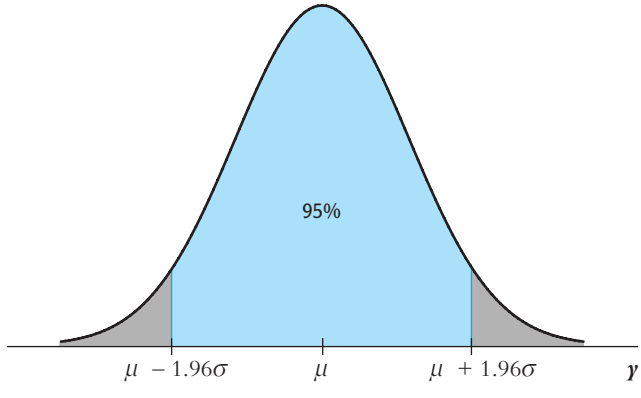
A continuous random variable with a **normal distribution** has the familiar bell-shaped probability density shown in Figure 2.5. The function defining the normal probability density is given in Appendix 17.1. As Figure 2.5 shows, the normal density with mean μ and variance σ^2 is symmetric around its mean and has 95% of its probability between $\mu - 1.96\sigma$ and $\mu + 1.96\sigma$.

Some special notation and terminology have been developed for the normal distribution. The normal distribution with mean μ and variance σ^2 is expressed concisely as "$N(\mu, \sigma^2)$." The **standard normal distribution** is the normal distribution with mean $\mu = 0$ and variance $\sigma^2 = 1$ and is denoted $N(0, 1)$. Random variables that have a $N(0, 1)$ distribution are often denoted Z, and the standard normal cumulative distribution function is denoted by the Greek letter Φ; accordingly, $\Pr(Z \leq c) = \Phi(c)$, where c is a constant. Values of the standard normal cumulative distribution function are tabulated in Appendix Table 1.

To look up probabilities for a normal variable with a general mean and variance, we must **standardize the variable** by first subtracting the mean, then by dividing

FIGURE 2.5 The Normal Probability Density

The normal probability density function with mean μ and variance σ^2 is a bell-shaped curve, centered at μ. The area under the normal p.d.f. between $\mu - 1.96\sigma$ and $\mu + 1.96\sigma$ is 0.95. The normal distribution is denoted $N(\mu, \sigma^2)$.



Computing Probabilities Involving Normal Random Variables

KEY CONCEPT 2.4

Suppose Y is normally distributed with mean μ and variance σ^2; in other words, Y is distributed $N(\mu, \sigma^2)$. Then Y is standardized by subtracting its mean and dividing by its standard deviation, that is, by computing $Z = (Y - \mu)/\sigma$.

Let c_1 and c_2 denote two numbers with $c_1 < c_2$ and let $d_1 = (c_1 - \mu)/\sigma$ and $d_2 = (c_2 - \mu)/\sigma$. Then

$$\Pr(Y \leq c_2) = \Pr(Z \leq d_2) = \Phi(d_2), \tag{2.38}$$

$$\Pr(Y \geq c_1) = \Pr(Z \geq d_1) = 1 - \Phi(d_1), \tag{2.39}$$

$$\Pr(c_1 \leq Y \leq c_2) = \Pr(d_1 \leq Z \leq d_2) = \Phi(d_2) - \Phi(d_1). \tag{2.40}$$

The normal cumulative distribution function Φ is tabulated in Appendix Table 1.

the result by the standard deviation. For example, suppose Y is distributed $N(1, 4)$—that is, Y is normally distributed with a mean of 1 and a variance of 4. What is the probability that $Y \leq 2$—that is, what is the shaded area in Figure 2.6a? The standardized version of Y is Y minus its mean, divided by its standard deviation, that is, $(Y - 1)/\sqrt{4} = \frac{1}{2}(Y - 1)$. Accordingly, the random variable $\frac{1}{2}(Y - 1)$ is normally distributed with mean zero and variance one (see Exercise 2.8); it has the standard normal distribution shown in Figure 2.6b. Now $Y \leq 2$ is equivalent to $\frac{1}{2}(Y - 1) \leq \frac{1}{2}(2 - 1)$—that is, $\frac{1}{2}(Y - 1) \leq \frac{1}{2}$. Thus,

$$\Pr(Y \leq 2) = \Pr[\tfrac{1}{2}(Y - 1) \leq \tfrac{1}{2}] = \Pr(Z \leq \tfrac{1}{2}) = \Phi(0.5) = 0.691, \tag{2.41}$$

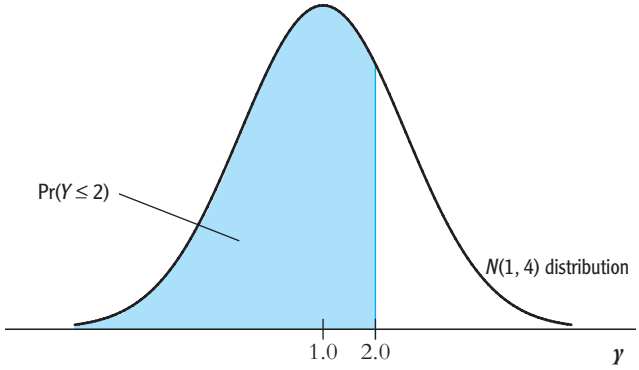
where the value 0.691 is taken from Appendix Table 1.

The same approach can be applied to compute the probability that a normally distributed random variable exceeds some value or that it falls in a certain range. These steps are summarized in Key Concept 2.4. The box "A Bad Day on Wall Street" presents an unusual application of the cumulative normal distribution.

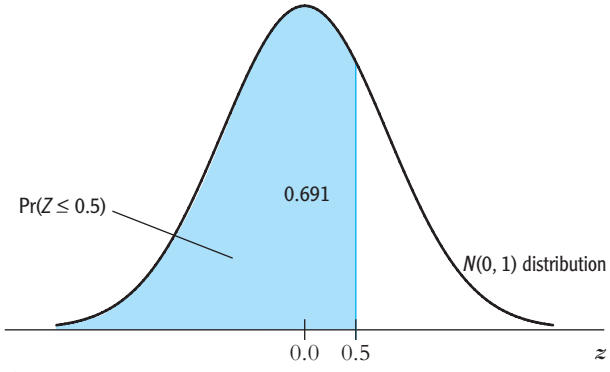
The normal distribution is symmetric, so its skewness is zero. The kurtosis of the normal distribution is 3.

FIGURE 2.6 Calculating the Probability That $Y \leq 2$ When Y Is Distributed $N(1, 4)$

To calculate $\Pr(Y \leq 2)$, standardize Y, then use the standard normal distribution table. Y is standardized by subtracting its mean ($\mu = 1$) and dividing by its standard deviation ($\sigma = 2$). The probability that $Y \leq 2$ is shown in Figure 2.6a, and the corresponding probability after standardizing Y is shown in Figure 2.6b. Because the standardized random variable, $(Y - 1)/2$, is a standard normal (Z) random variable, $\Pr(Y \leq 2) = \Pr\left(\frac{Y-1}{2} \leq \frac{2-1}{2}\right) = \Pr(Z \leq 0.5)$. From Appendix Table 1, $\Pr(Z \leq 0.5) = \Phi(0.5) = 0.691$.



(a) $N(1, 4)$



(b) $N(0, 1)$

The multivariate normal distribution. The normal distribution can be generalized to describe the joint distribution of a set of random variables. In this case, the distribution is called the **multivariate normal distribution**, or, if only two variables are being considered, the **bivariate normal distribution**. The formula for the bivariate normal p.d.f. is given in Appendix 17.1, and the formula for the general multivariate normal p.d.f. is given in Appendix 18.1.

The multivariate normal distribution has four important properties. If X and Y have a bivariate normal distribution with covariance σ_{XY} and if a and b are two constants, then $aX + bY$ has the normal distribution:

$$aX + bY \text{ is distributed } N(a\mu_X + b\mu_Y, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_{XY}) \quad (X, Y \text{ bivariate normal}). \tag{2.42}$$

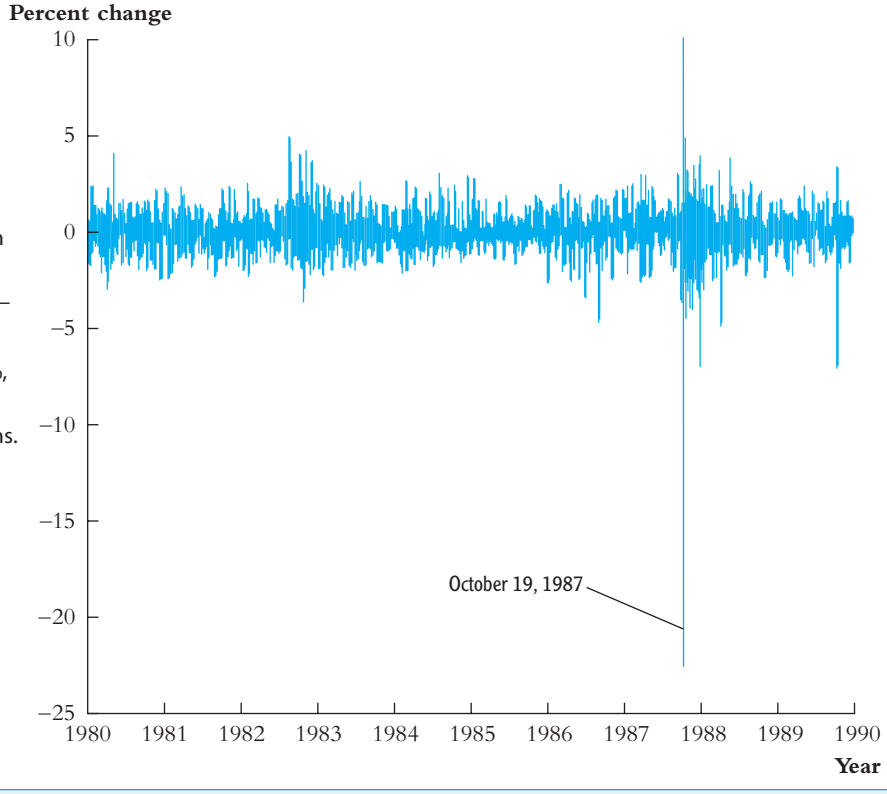
A Bad Day on Wall Street

On a typical day the overall value of stocks traded on the U.S. stock market can rise or fall by 1% or even more. This is a lot—but nothing compared to what happened on Monday, October 19, 1987. On "Black Monday," the Dow Jones Industrial Average (an average of 30 large industrial stocks) fell by 22.6%! From January 1, 1980, to December 31, 2012, the standard deviation of daily percentage price changes on the Dow was 1.12%, so the drop of 22.6% was a negative return of 20(= 22.6/1.12) standard deviations. The enormity of this drop can be seen in Figure 2.7, a plot of the daily returns on the Dow during the 1980s.

If daily percentage price changes are normally distributed, then the probability of a change of at least 20 standard deviations is $\Pr(|Z| \geq 20) = 2 \times \Phi(-20)$. You will not find this value in Appendix Table 1, but you can calculate it using a computer (try it!). This probability is 5.5×10^{-89}, that is, 0.000 . . . 00055, where there are a total of 88 zeros!

FIGURE 2.7 Daily Percentage Changes in the Dow Jones Industrial Average in the 1980s

From 1980 through 2012, the average percentage daily change of "the Dow" index was 0.04% and its standard deviation was 1.12%. On October 19, 1987—"Black Monday"—the Dow fell 22.6%, or more than 20 standard deviations.



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How small is 5.5×10^{-89}? Consider the following:

- The world population is about 7 billion, so the probability of winning a random lottery among all living people is about one in 7 billion, or 1.4×10^{-10}.
- The universe is believed to have existed for 14 billion years, or about 5×10^{17} seconds, so the probability of choosing a particular second at random from all the seconds since the beginning of time is 2×10^{-18}.
- There are approximately 10^{43} molecules of gas in the first kilometer above the earth's surface. The probability of choosing one at random is 10^{-43}.

Although Wall Street *did* have a bad day, the fact that it happened at all suggests its probability was more than 5.5×10^{-89}. In fact, there have been many days—good and bad—with stock price changes too large to be consistent with a normal distribution with a constant variance. Table 2.5 lists the ten largest daily percentage price changes in the Dow Jones Industrial Average in the 8325 trading days between January 1, 1980, and December 31, 2012, along with the standardized change using the mean and variance over this period. All ten changes exceed 6.4 standard deviations, an extremely rare event if stock prices are normally distributed.

Clearly, stock price percentage changes have a distribution with heavier tails than the normal distribution. For this reason, finance professionals use other models of stock price changes. One such model treats stock price changes as normally distributed with a variance that evolves over time, so periods like October 1987 and the financial crisis in the fall of 2008 have higher volatility than others (models with time-varying variances are discussed in Chapter 16). Other models abandon the normal distribution in favor of distributions with heavier tails, an idea popularized in Nassim Taleb's 2007 book, *The Black Swan*. These models are more consistent with the very bad—and very good—days we actually see on Wall Street.

TABLE 2.5 The Ten Largest Daily Percentage Changes in the Dow Jones Industrial Index, 1980–2012, and the Normal Probability of a Change at Least as Large

Date	Percentage Change (x)	Standardized Change $Z = (x - \mu)/\sigma$	Normal Probability of a Change at Least This Large $\Pr(\lvert Z\rvert \geq z) = 2\Phi(-z)$
October 19, 1987	−22.6	−20.2	5.5×10^{-89}
October 13, 2008	11.1	9.9	6.4×10^{-23}
October 28, 2008	10.9	9.7	3.8×10^{-22}
October 21, 1987	10.1	9.0	1.8×10^{-19}
October 26, 1987	−8.0	−7.2	5.6×10^{-13}
October 15, 2008	−7.9	−7.1	1.6×10^{-12}
December 01, 2008	−7.7	−6.9	4.9×10^{-12}
October 09, 2008	−7.3	−6.6	4.7×10^{-11}
October 27, 1997	−7.2	−6.4	1.2×10^{-10}
September 17, 2001	−7.1	−6.4	1.6×10^{-10}

More generally, if n random variables have a multivariate normal distribution, then any linear combination of these variables (such as their sum) is normally distributed.

Second, if a set of variables has a multivariate normal distribution, then the marginal distribution of each of the variables is normal [this follows from Equation (2.42) by setting $a = 1$ and $b = 0$].

Third, if variables with a multivariate normal distribution have covariances that equal zero, then the variables are independent. Thus, if X and Y have a bivariate normal distribution and $\sigma_{XY} = 0$, then X and Y are independent. In Section 2.3 it was shown that if X and Y are independent, then, regardless of their joint distribution, $\sigma_{XY} = 0$. If X and Y are jointly normally distributed, then the converse is also true. This result—that zero covariance implies independence—is a special property of the multivariate normal distribution that is not true in general.

Fourth, if X and Y have a bivariate normal distribution, then the conditional expectation of Y given X is linear in X; that is, $E(Y|X = x) = a + bx$, where a and b are constants (Exercise 17.11). Joint normality implies linearity of conditional expectations, but linearity of conditional expectations does not imply joint normality.

The Chi-Squared Distribution

The chi-squared distribution is used when testing certain types of hypotheses in statistics and econometrics.

The **chi-squared distribution** is the distribution of the sum of m squared independent standard normal random variables. This distribution depends on m, which is called the degrees of freedom of the chi-squared distribution. For example, let Z_1, Z_2, and Z_3 be independent standard normal random variables. Then $Z_1^2 + Z_2^2 + Z_3^2$ has a chi-squared distribution with 3 degrees of freedom. The name for this distribution derives from the Greek letter used to denote it: A chi-squared distribution with m degrees of freedom is denoted χ_m^2.

Selected percentiles of the χ_m^2 distribution are given in Appendix Table 3. For example, Appendix Table 3 shows that the 95th percentile of the χ_m^2 distribution is 7.81, so $\Pr(Z_1^2 + Z_2^2 + Z_3^3 \leq 7.81) = 0.95$.

The Student *t* Distribution

The **Student *t* distribution** with m degrees of freedom is defined to be the distribution of the ratio of a standard normal random variable, divided by the square root of an independently distributed chi-squared random variable with m degrees of freedom divided by m. That is, let Z be a standard normal random variable, let W be a random variable with a chi-squared distribution with m degrees of freedom,

and let Z and W be independently distributed. Then the random variable $Z/\sqrt{W/m}$ has a Student t distribution (also called the ***t* distribution**) with m degrees of freedom. This distribution is denoted t_m. Selected percentiles of the Student t distribution are given in Appendix Table 2.

The Student t distribution depends on the degrees of freedom m. Thus the 95th percentile of the t_m distribution depends on the degrees of freedom m. The Student t distribution has a bell shape similar to that of the normal distribution, but when m is small (20 or less), it has more mass in the tails—that is, it is a "fatter" bell shape than the normal. When m is 30 or more, the Student t distribution is well approximated by the standard normal distribution and the t_∞ distribution equals the standard normal distribution.

The *F* Distribution

The ***F* distribution** with m and n degrees of freedom, denoted $F_{m,n}$, is defined to be the distribution of the ratio of a chi-squared random variable with degrees of freedom m, divided by m, to an independently distributed chi-squared random variable with degrees of freedom n, divided by n. To state this mathematically, let W be a chi-squared random variable with m degrees of freedom and let V be a chi-squared random variable with n degrees of freedom, where W and V are independently distributed. Then $\frac{W/m}{V/n}$ has an $F_{m,n}$ distribution—that is, an F distribution with numerator degrees of freedom m and denominator degrees of freedom n.

In statistics and econometrics, an important special case of the F distribution arises when the denominator degrees of freedom is large enough that the $F_{m,n}$ distribution can be approximated by the $F_{m,\infty}$ distribution. In this limiting case, the denominator random variable V/n is the mean of infinitely many squared standard normal random variables, and that mean is 1 because the mean of a squared standard normal random variable is 1 (see Exercise 2.24). Thus the $F_{m,\infty}$ distribution is the distribution of a chi-squared random variable with m degrees of freedom, divided by m: W/m is distributed $F_{m,\infty}$. For example, from Appendix Table 4, the 95th percentile of the $F_{3,\infty}$ distribution is 2.60, which is the same as the 95th percentile of the χ^2_3 distribution, 7.81 (from Appendix Table 2), divided by the degrees of freedom, which is 3 ($7.81/3 = 2.60$).

The 90th, 95th, and 99th percentiles of the $F_{m,n}$ distribution are given in Appendix Table 5 for selected values of m and n. For example, the 95th percentile of the $F_{3,30}$ distribution is 2.92, and the 95th percentile of the $F_{3,90}$ distribution is 2.71. As the denominator degrees of freedom n increases, the 95th percentile of the $F_{3,n}$ distribution tends to the $F_{3,\infty}$ limit of 2.60.

2.5 Random Sampling and the Distribution of the Sample Average

Almost all the statistical and econometric procedures used in this book involve averages or weighted averages of a sample of data. Characterizing the distributions of sample averages therefore is an essential step toward understanding the performance of econometric procedures.

This section introduces some basic concepts about random sampling and the distributions of averages that are used throughout the book. We begin by discussing random sampling. The act of random sampling—that is, randomly drawing a sample from a larger population—has the effect of making the sample average itself a random variable. Because the sample average is a random variable, it has a probability distribution, which is called its sampling distribution. This section concludes with some properties of the sampling distribution of the sample average.

Random Sampling

Simple random sampling. Suppose our commuting student from Section 2.1 aspires to be a statistician and decides to record her commuting times on various days. She selects these days at random from the school year, and her daily commuting time has the cumulative distribution function in Figure 2.2a. Because these days were selected at random, knowing the value of the commuting time on one of these randomly selected days provides no information about the commuting time on another of the days; that is, because the days were selected at random, the values of the commuting time on each of the different days are independently distributed random variables.

The situation described in the previous paragraph is an example of the simplest sampling scheme used in statistics, called **simple random sampling**, in which n objects are selected at random from a **population** (the population of commuting days) and each member of the population (each day) is equally likely to be included in the sample.

The n observations in the sample are denoted $Y_1, \ldots, Y_n$, where Y_1 is the first observation, Y_2 is the second observation, and so forth. In the commuting example, Y_1 is the commuting time on the first of her n randomly selected days and Y_i is the commuting time on the i^{th} of her randomly selected days.

Because the members of the population included in the sample are selected at random, the values of the observations $Y_1, \ldots, Y_n$ are themselves random. If

KEY CONCEPT 2.5

Simple Random Sampling and i.i.d. Random Variables

In a simple random sample, n objects are drawn at random from a population and each object is equally likely to be drawn. The value of the random variable Y for the i^{th} randomly drawn object is denoted Y_i. Because each object is equally likely to be drawn and the distribution of Y_i is the same for all i, the random variables $Y_1, \ldots, Y_n$ are independently and identically distributed (i.i.d.); that is, the distribution of Y_i is the same for all $i = 1, \ldots, n$ and Y_1 is distributed independently of $Y_2, \ldots, Y_n$ and so forth.

different members of the population are chosen, their values of Y will differ. Thus the act of random sampling means that $Y_1, \ldots, Y_n$ can be treated as random variables. Before they are sampled, $Y_1, \ldots, Y_n$ can take on many possible values; after they are sampled, a specific value is recorded for each observation.

i.i.d. draws. Because $Y_1, \ldots, Y_n$ are randomly drawn from the same population, the marginal distribution of Y_i is the same for each $i = 1, \ldots, n$; this marginal distribution is the distribution of Y in the population being sampled. When Y_i has the same marginal distribution for $i = 1, \ldots, n$, then $Y_1, \ldots, Y_n$ are said to be **identically distributed**.

Under simple random sampling, knowing the value of Y_1 provides no information about Y_2, so the conditional distribution of Y_2 given Y_1 is the same as the marginal distribution of Y_2. In other words, under simple random sampling, Y_1 is distributed independently of $Y_2, \ldots, Y_n$.

When $Y_1, \ldots, Y_n$ are drawn from the same distribution and are independently distributed, they are said to be **independently and identically distributed** (or **i.i.d.**).

Simple random sampling and i.i.d. draws are summarized in Key Concept 2.5.

The Sampling Distribution of the Sample Average

The **sample average** or **sample mean**, $\overline{Y}$, of the n observations $Y_1, \ldots, Y_n$ is

$$\overline{Y} = \frac{1}{n}(Y_1 + Y_2 + \cdots + Y_n) = \frac{1}{n}\sum_{i=1}^{n} Y_i. \tag{2.43}$$

An essential concept is that the act of drawing a random sample has the effect of making the sample average $\overline{Y}$ a random variable. Because the sample was drawn

at random, the value of each Y_i is random. Because $Y_1, \ldots, Y_n$ are random, their average is random. Had a different sample been drawn, then the observations and their sample average would have been different: The value of $\overline{Y}$ differs from one randomly drawn sample to the next.

For example, suppose our student commuter selected five days at random to record her commute times, then computed the average of those five times. Had she chosen five different days, she would have recorded five different times—and thus would have computed a different value of the sample average.

Because $\overline{Y}$ is random, it has a probability distribution. The distribution of $\overline{Y}$ is called the **sampling distribution** of $\overline{Y}$ because it is the probability distribution associated with possible values of $\overline{Y}$ that could be computed for different possible samples $Y_1, \ldots, Y_n$.

The sampling distribution of averages and weighted averages plays a central role in statistics and econometrics. We start our discussion of the sampling distribution of $\overline{Y}$ by computing its mean and variance under general conditions on the population distribution of Y.

Mean and variance of $\overline{Y}$. Suppose that the observations $Y_1, \ldots, Y_n$ are i.i.d., and let μ_Y and σ_Y^2 denote the mean and variance of Y_i (because the observations are i.i.d. the mean and variance is the same for all $i = 1, \ldots, n$). When $n = 2$, the mean of the sum $Y_1 + Y_2$ is given by applying Equation (2.28): $E(Y_1 + Y_2) = \mu_Y + \mu_Y = 2\mu_Y$. Thus the mean of the sample average is $E[\frac{1}{2}(Y_1 + Y_2)] = \frac{1}{2} \times 2\mu_Y = \mu_Y$. In general,

$$E(\overline{Y}) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \mu_Y. \tag{2.44}$$

The variance of $\overline{Y}$ is found by applying Equation (2.37). For example, for $n = 2$, $\text{var}(Y_1 + Y_2) = 2\sigma_Y^2$, so [by applying Equation (2.31) with $a = b = \frac{1}{2}$ and $\text{cov}(Y_1, Y_2) = 0$], $\text{var}(\overline{Y}) = \frac{1}{2}\sigma_Y^2$. For general n, because $Y_1, \ldots, Y_n$ are i.i.d., Y_i and Y_j are independently distributed for $i \neq j$, so $\text{cov}(Y_i, Y_j) = 0$. Thus,

$$\begin{aligned}
\text{var}(\overline{Y}) &= \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(Y_i) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\text{cov}(Y_i, Y_j) \\
&= \frac{\sigma_Y^2}{n}.
\end{aligned} \tag{2.45}$$

The standard deviation of $\overline{Y}$ is the square root of the variance, $\sigma_Y/\sqrt{n}$.

Financial Diversification and Portfolios

The principle of diversification says that you can reduce your risk by holding small investments in multiple assets, compared to putting all your money into one asset. That is, you shouldn't put all your eggs in one basket.

The math of diversification follows from Equation (2.45). Suppose you divide \$1 equally among n assets. Let Y_i represent the payout in 1 year of \$1 invested in the i^{th} asset. Because you invested $1/n$ dollars in each asset, the actual payoff of your portfolio after 1 year is $(Y_1 + Y_2 + \cdots + Y_n)/n = \overline{Y}$. To keep things simple, suppose that each asset has the same expected payout, μ_Y, the same variance, σ^2, and the same positive correlation ρ across assets [so that $\text{cov}(Y_i, Y_j) = \rho\sigma^2$]. Then the expected payout is $E(\overline{Y}) = \mu_Y$, and, for large n, the variance of the portfolio payout is $\text{var}(\overline{Y}) = \rho\sigma^2$ (Exercise 2.26). Putting all your money into one asset or spreading it equally across all n assets has the same expected payout, but diversifying reduces the variance from σ^2 to $\rho\sigma^2$.

The math of diversification has led to financial products such as stock mutual funds, in which the fund holds many stocks and an individual owns a share of the fund, thereby owning a small amount of many stocks. But diversification has its limits: For many assets, payouts are positively correlated, so $\text{var}(\overline{Y})$ remains positive even if n is large. In the case of stocks, risk is reduced by holding a portfolio, but that portfolio remains subject to the unpredictable fluctuations of the overall stock market.

In summary, the mean, the variance, and the standard deviation of $\overline{Y}$ are

$$E(\overline{Y}) = \mu_Y. \tag{2.46}$$

$$\text{var}(\overline{Y}) = \sigma^2_{\overline{Y}} = \frac{\sigma^2_Y}{n}, \text{ and} \tag{2.47}$$

$$\text{std.dev}(\overline{Y}) = \sigma_{\overline{Y}} = \frac{\sigma_Y}{\sqrt{n}}. \tag{2.48}$$

These results hold whatever the distribution of Y_i is; that is, the distribution of Y_i does not need to take on a specific form, such as the normal distribution, for Equations (2.46) through (2.48) to hold.

The notation $\sigma^2_{\overline{Y}}$ denotes the variance of the sampling distribution of the sample average $\overline{Y}$. In contrast, σ^2_Y is the variance of each individual Y_i, that is, the variance of the population distribution from which the observation is drawn. Similarly, $\sigma_{\overline{Y}}$ denotes the standard deviation of the sampling distribution of $\overline{Y}$.

Sampling distribution of $\overline{Y}$ when Y is normally distributed. Suppose that $Y_1, \ldots, Y_n$ are i.i.d. draws from the $N(\mu_Y, \sigma^2_Y)$ distribution. As stated following Equation (2.42), the sum of n normally distributed random variables is itself

normally distributed. Because the mean of $\overline{Y}$ is μ_Y and the variance of $\overline{Y}$ is σ_Y^2/n, this means that, if $Y_1, \ldots, Y_n$ are i.i.d. draws from the $N(\mu_Y, \sigma_Y^2)$, then $\overline{Y}$ is distributed $N(\mu_Y, \sigma_Y^2/n)$.

2.6 Large-Sample Approximations to Sampling Distributions

Sampling distributions play a central role in the development of statistical and econometric procedures, so it is important to know, in a mathematical sense, what the sampling distribution of $\overline{Y}$ is. There are two approaches to characterizing sampling distributions: an "exact" approach and an "approximate" approach.

The "exact" approach entails deriving a formula for the sampling distribution that holds exactly for any value of n. The sampling distribution that exactly describes the distribution of $\overline{Y}$ for any n is called the **exact distribution** or **finite-sample distribution** of $\overline{Y}$. For example, if Y is normally distributed and $Y_1, \ldots, Y_n$ are i.i.d., then (as discussed in Section 2.5) the exact distribution of $\overline{Y}$ is normal with mean μ_Y and variance σ_Y^2/n. Unfortunately, if the distribution of Y is not normal, then in general the exact sampling distribution of $\overline{Y}$ is very complicated and depends on the distribution of Y.

The "approximate" approach uses approximations to the sampling distribution that rely on the sample size being large. The large-sample approximation to the sampling distribution is often called the **asymptotic distribution**—"asymptotic" because the approximations become exact in the limit that $n \to \infty$. As we see in this section, these approximations can be very accurate even if the sample size is only $n = 30$ observations. Because sample sizes used in practice in econometrics typically number in the hundreds or thousands, these asymptotic distributions can be counted on to provide very good approximations to the exact sampling distribution.

This section presents the two key tools used to approximate sampling distributions when the sample size is large: the law of large numbers and the central limit theorem. The law of large numbers says that, when the sample size is large, $\overline{Y}$ will be close to μ_Y with very high probability. The central limit theorem says that, when the sample size is large, the sampling distribution of the standardized sample average, $(\overline{Y} - \mu_Y)/\sigma_{\overline{Y}}$, is approximately normal.

Although exact sampling distributions are complicated and depend on the distribution of Y, the asymptotic distributions are simple. Moreover—remarkably—the asymptotic normal distribution of $(\overline{Y} - \mu_Y)/\sigma_{\overline{Y}}$ does *not* depend on the distribution of Y. This normal approximate distribution provides enormous simplifications and underlies the theory of regression used throughout this book.

KEY CONCEPT 2.6

Convergence in Probability, Consistency, and the Law of Large Numbers

The sample average $\overline{Y}$ converges in probability to μ_Y (or, equivalently, $\overline{Y}$ is consistent for μ_Y) if the probability that $\overline{Y}$ is in the range $(\mu_Y - c)$ to $(\mu_Y + c)$ becomes arbitrarily close to 1 as n increases for any constant $c > 0$. The convergence of $\overline{Y}$ to μ_Y in probability is written, $\overline{Y} \xrightarrow{p} \mu_Y$.

The law of large numbers says that if $Y_i, i = 1, \ldots, n$ are independently and identically distributed with $E(Y_i) = \mu_Y$ and if large outliers are unlikely (technically if $\text{var}(Y_i) = \sigma_Y^2 < \infty$), then $\overline{Y} \xrightarrow{p} \mu_Y$.

The Law of Large Numbers and Consistency

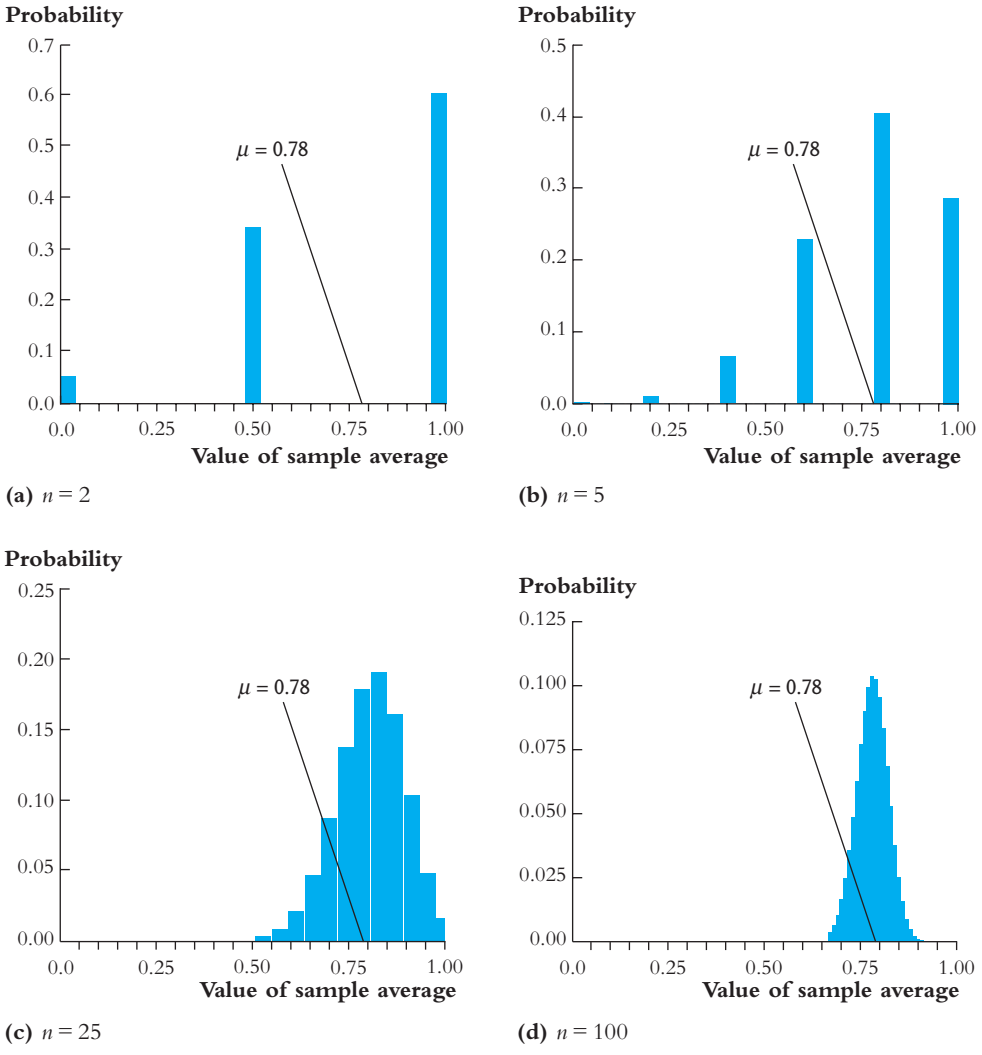
The **law of large numbers** states that, under general conditions, $\overline{Y}$ will be near μ_Y with very high probability when n is large. This is sometimes called the "law of averages." When a large number of random variables with the same mean are averaged together, the large values balance the small values and their sample average is close to their common mean.

For example, consider a simplified version of our student commuter's experiment in which she simply records whether her commute was short (less than 20 minutes) or long. Let $Y_i = 1$ if her commute was short on the i^{th} randomly selected day and $Y_i = 0$ if it was long. Because she used simple random sampling, $Y_1, \ldots, Y_n$ are i.i.d. Thus Y_i, $i = 1, \ldots, n$ are i.i.d. draws of a Bernoulli random variable, where (from Table 2.2) the probability that $Y_i = 1$ is 0.78. Because the expectation of a Bernoulli random variable is its success probability, $E(Y_i) = \mu_Y = 0.78$. The sample average $\overline{Y}$ is the fraction of days in her sample in which her commute was short.

Figure 2.8 shows the sampling distribution of $\overline{Y}$ for various sample sizes n. When $n = 2$ (Figure 2.8a), $\overline{Y}$ can take on only three values: 0, $\frac{1}{2}$, and 1 (neither commute was short, one was short, and both were short), none of which is particularly close to the true proportion in the population, 0.78. As n increases, however (Figures 2.8b–d), $\overline{Y}$ takes on more values and the sampling distribution becomes tightly centered on μ_Y.

The property that $\overline{Y}$ is near μ_Y with increasing probability as n increases is called **convergence in probability** or, more concisely, **consistency** (see Key Concept 2.6). The law of large numbers states that, under certain conditions, $\overline{Y}$ converges in probability to μ_Y or, equivalently, that $\overline{Y}$ is consistent for μ_Y.

FIGURE 2.8 Sampling Distribution of the Sample Average of *n* Bernoulli Random Variables



The distributions are the sampling distributions of $\overline{Y}$, the sample average of n independent Bernoulli random variables with $p = \Pr(Y_i = 1) = 0.78$ (the probability of a short commute is 78%). The variance of the sampling distribution of $\overline{Y}$ decreases as n gets larger, so the sampling distribution becomes more tightly concentrated around its mean $\mu = 0.78$ as the sample size n increases.

The conditions for the law of large numbers that we will use in this book are that $Y_i, i = 1, \ldots, n$ are i.i.d. and that the variance of Y_i, σ_Y^2, is finite. The mathematical role of these conditions is made clear in Section 17.2, where the law of large numbers is proven. If the data are collected by simple random sampling, then the i.i.d. assumption holds. The assumption that the variance is finite says that extremely large values of Y_i—that is, outliers—are unlikely and observed infrequently; otherwise, these large values could dominate $\overline{Y}$ and the sample average would be unreliable. This assumption is plausible for the applications in this book. For example, because there is an upper limit to our student's commuting time (she could park and walk if the traffic is dreadful), the variance of the distribution of commuting times is finite.

The Central Limit Theorem

The **central limit theorem** says that, under general conditions, the distribution of $\overline{Y}$ is well approximated by a normal distribution when n is large. Recall that the mean of $\overline{Y}$ is μ_Y and its variance is $\sigma_{\overline{Y}}^2 = \sigma_Y^2/n$. According to the central limit theorem, when n is large, the distribution of $\overline{Y}$ is approximately $N(\mu_Y, \sigma_{\overline{Y}}^2)$. As discussed at the end of Section 2.5, the distribution of $\overline{Y}$ is *exactly* $N(\mu_Y, \sigma_{\overline{Y}}^2)$ when the sample is drawn from a population with the normal distribution $N(\mu_Y, \sigma_Y^2)$. The central limit theorem says that this same result is *approximately* true when n is large even if $Y_1, \ldots, Y_n$ are not themselves normally distributed.

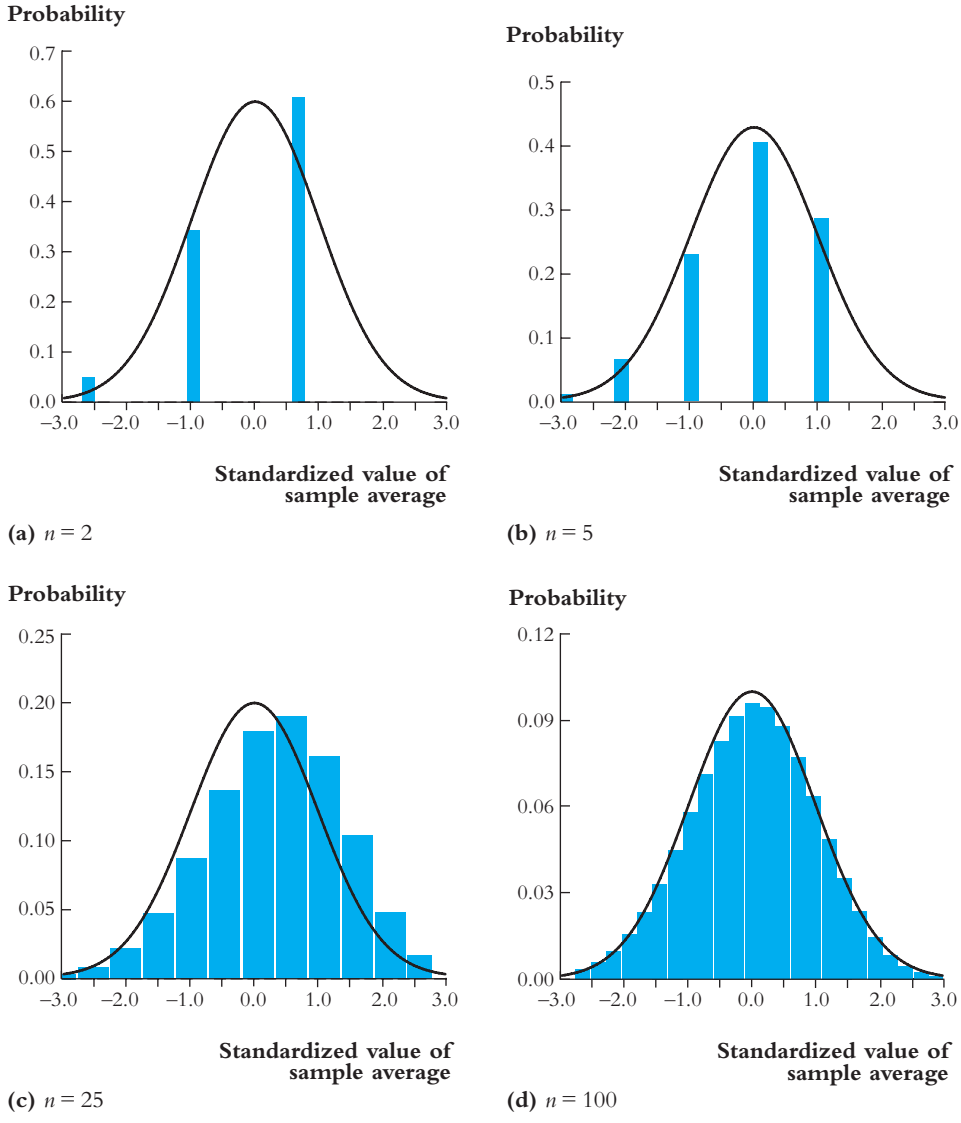
The convergence of the distribution of $\overline{Y}$ to the bell-shaped, normal approximation can be seen (a bit) in Figure 2.8. However, because the distribution gets quite tight for large n, this requires some squinting. It would be easier to see the shape of the distribution of $\overline{Y}$ if you used a magnifying glass or had some other way to zoom in or to expand the horizontal axis of the figure.

One way to do this is to standardize $\overline{Y}$ by subtracting its mean and dividing by its standard deviation so that it has a mean of 0 and a variance of 1. This process leads to examining the distribution of the standardized version of $\overline{Y}$, $(\overline{Y} - \mu_Y)/\sigma_{\overline{Y}}$. According to the central limit theorem, this distribution should be well approximated by a $N(0, 1)$ distribution when n is large.

The distribution of the standardized average $(\overline{Y} - \mu_Y)/\sigma_{\overline{Y}}$ is plotted in Figure 2.9 for the distributions in Figure 2.8; the distributions in Figure 2.9 are exactly the same as in Figure 2.8, except that the scale of the horizontal axis is changed so that the standardized variable has a mean of 0 and a variance of 1. After this change of scale, it is easy to see that, if n is large enough, the distribution of $\overline{Y}$ is well approximated by a normal distribution.

One might ask, how large is "large enough"? That is, how large must n be for the distribution of $\overline{Y}$ to be approximately normal? The answer is, "It depends." The

FIGURE 2.9 Distribution of the Standardized Sample Average of n Bernoulli Random Variables with $p = 0.78$



The sampling distribution of $\overline{Y}$ in Figure 2.8 is plotted here after standardizing $\overline{Y}$. This plot centers the distributions in Figure 2.8 and magnifies the scale on the horizontal axis by a factor of $\sqrt{n}$. When the sample size is large, the sampling distributions are increasingly well approximated by the normal distribution (the solid line), as predicted by the central limit theorem. The normal distribution is scaled so that the height of the distributions is approximately the same in all figures.

KEY CONCEPT 2.7

The Central Limit Theorem

Suppose that $Y_1, \ldots, Y_n$ are i.i.d. with $E(Y_i) = \mu_Y$ and $\text{var}(Y_i) = \sigma_Y^2$, where $0 < \sigma_Y^2 < \infty$. As $n \to \infty$, the distribution of $(\overline{Y} - \mu_Y)/\sigma_{\overline{Y}}$ (where $\sigma_{\overline{Y}}^2 = \sigma_Y^2/n$) becomes arbitrarily well approximated by the standard normal distribution.

quality of the normal approximation depends on the distribution of the underlying Y_i that make up the average. At one extreme, if the Y_i are themselves normally distributed, then $\overline{Y}$ is exactly normally distributed for all n. In contrast, when the underlying Y_i themselves have a distribution that is far from normal, then this approximation can require $n = 30$ or even more.

This point is illustrated in Figure 2.10 for a population distribution, shown in Figure 2.10a, that is quite different from the Bernoulli distribution. This distribution has a long right tail (it is "skewed" to the right). The sampling distribution of $\overline{Y}$, after centering and scaling, is shown in Figures 2.10b–d for $n = 5, 25$, and 100, respectively. Although the sampling distribution is approaching the bell shape for $n = 25$, the normal approximation still has noticeable imperfections. By $n = 100$, however, the normal approximation is quite good. In fact, for $n \geq 100$, the normal approximation to the distribution of $\overline{Y}$ typically is very good for a wide variety of population distributions.

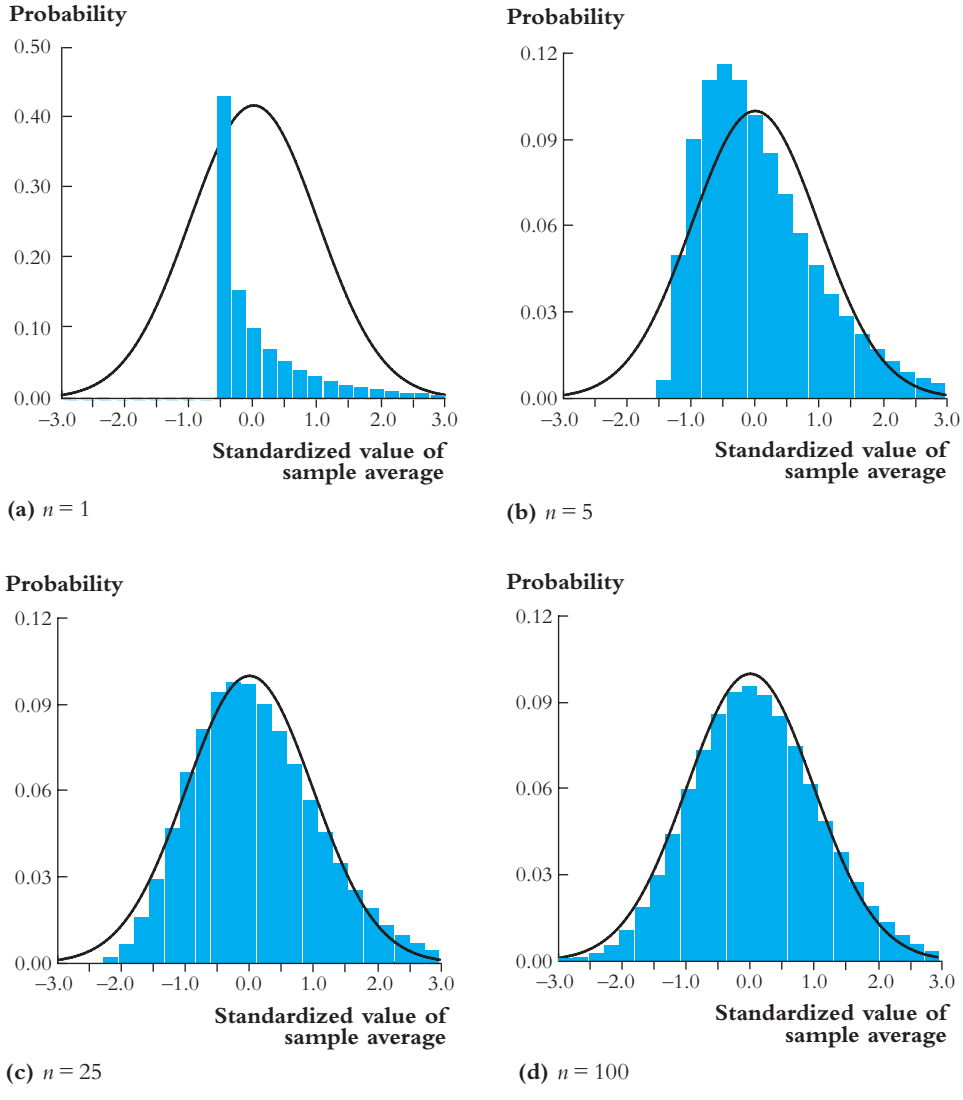
The central limit theorem is a remarkable result. While the "small n" distributions of $\overline{Y}$ in parts b and c of Figures 2.9 and 2.10 are complicated and quite different from each other, the "large n" distributions in Figures 2.9d and 2.10d are simple and, amazingly, have a similar shape. Because the distribution of $\overline{Y}$ approaches the normal as n grows large, $\overline{Y}$ is said to have an **asymptotic normal distribution**.

The convenience of the normal approximation, combined with its wide applicability because of the central limit theorem, makes it a key underpinning of modern applied econometrics. The central limit theorem is summarized in Key Concept 2.7.

Summary

1. The probabilities with which a random variable takes on different values are summarized by the cumulative distribution function, the probability distribution function (for discrete random variables), and the probability density function (for continuous random variables).

FIGURE 2.10 Distribution of the Standardized Sample Average of *n* Draws from a Skewed Distribution



The figures show the sampling distribution of the standardized sample average of *n* draws from the skewed (asymmetric) population distribution shown in Figure 2.10a. When *n* is small ($n = 5$), the sampling distribution, like the population distribution, is skewed. But when *n* is large ($n = 100$), the sampling distribution is well approximated by a standard normal distribution (solid line), as predicted by the central limit theorem. The normal distribution is scaled so that the height of the distributions is approximately the same in all figures.

2. The expected value of a random variable Y (also called its mean, μ_Y), denoted $E(Y)$, is its probability-weighted average value. The variance of Y is $\sigma_Y^2 = E[(Y - \mu_Y)^2]$, and the standard deviation of Y is the square root of its variance.
3. The joint probabilities for two random variables X and Y are summarized by their joint probability distribution. The conditional probability distribution of Y given $X = x$ is the probability distribution of Y, conditional on X taking on the value x.
4. A normally distributed random variable has the bell-shaped probability density in Figure 2.5. To calculate a probability associated with a normal random variable, first standardize the variable and then use the standard normal cumulative distribution tabulated in Appendix Table 1.
5. Simple random sampling produces n random observations $Y_1, \ldots, Y_n$ that are independently and identically distributed (i.i.d.).
6. The sample average, $\overline{Y}$, varies from one randomly chosen sample to the next and thus is a random variable with a sampling distribution. If $Y_1, \ldots, Y_n$ are i.i.d., then:
   a. the sampling distribution of $\overline{Y}$ has mean μ_Y and variance $\sigma_{\overline{Y}}^2 = \sigma_Y^2/n$;
   b. the law of large numbers says that $\overline{Y}$ converges in probability to μ_Y; and
   c. the central limit theorem says that the standardized version of $\overline{Y}$, $(\overline{Y} - \mu_Y)/\sigma_{\overline{Y}}$, has a standard normal distribution [$N(0, 1)$ distribution] when n is large.

Key Terms

outcomes (61)
probability (61)
sample space (61)
event (61)
discrete random variable (61)
continuous random variable (61)
probability distribution (62)
cumulative probability distribution (62)
cumulative distribution function (c.d.f.) (63)
Bernoulli random variable (63)
Bernoulli distribution (63)
probability density function (p.d.f.) (65)
density function (65)
density (65)
expected value (65)
expectation (65)
mean (65)
variance (67)
standard deviation (67)
moments of a distribution (69)
skewness (69)
kurtosis (71)
outlier (71)

leptokurtic (71)
r^{th} moment (71)
joint probability distribution (72)
marginal probability distribution (73)
conditional distribution (73)
conditional expectation (74)
conditional mean (74)
law of iterated expectations (75)
conditional variance (76)
independently distributed (77)
independent (77)
covariance (77)
correlation (78)
uncorrelated (78)
normal distribution (82)
standard normal distribution (82)
standardize a variable (82)
multivariate normal distribution (84)
bivariate normal distribution (84)
chi-squared distribution (87)
Student t distribution (87)
t distribution (88)
F distribution (88)
simple random sampling (89)
population (89)
identically distributed (90)
independently and identically distributed (i.i.d.) (90)
sample average (90)
sample mean (90)
sampling distribution (91)
exact (finite-sample) distribution (93)
asymptotic distribution (93)
law of large numbers (94)
convergence in probability (94)
consistency (94)
central limit theorem (96)
asymptotic normal distribution (98)

Review the Concepts

2.1. Examples of random variables used in this chapter included (a) the gender of the next person you meet, (b) the number of times a computer crashes, (c) the time it takes to commute to school, (d) whether the computer you are assigned in the library is new or old, and (e) whether it is raining or not. Explain why each can be thought of as random.

2.2. Suppose that the random variables X and Y are independent and you know their distributions. Explain why knowing the value of X tells you nothing about the value of Y.

2.3. Suppose that X denotes the amount of rainfall in your hometown during a randomly selected month and Y denotes the number of children born in Los Angeles during the same month. Are X and Y independent? Explain.

2.4. An econometrics class has 80 students, and the mean student weight is 145 lb. A random sample of 4 students is selected from the class, and their average weight is calculated. Will the average weight of the students in the sample equal 145 lb? Why or why not? Use this example to explain why the sample average, $\overline{Y}$, is a random variable.

2.5. Suppose that $Y_1, \ldots, Y_n$ are i.i.d. random variables with a $N(1, 4)$ distribution. Sketch the probability density of $\overline{Y}$ when $n = 2$. Repeat this for $n = 10$ and $n = 100$. In words, describe how the densities differ. What is the relationship between your answer and the law of large numbers?

2.6. Suppose that $Y_1, \ldots, Y_n$ are i.i.d. random variables with the probability distribution given in Figure 2.10a. You want to calculate $\Pr(\overline{Y} \leq 0.1)$. Would it be reasonable to use the normal approximation if $n = 5$? What about $n = 25$ or $n = 100$? Explain.

2.7. Y is a random variable with $\mu_Y = 0, \sigma_Y = 1$, skewness $= 0$, and kurtosis $= 100$. Sketch a hypothetical probability distribution of Y. Explain why n random variables drawn from this distribution might have some large outliers.

Exercises

2.1 Let Y denote the number of "heads" that occur when two loaded coins are tossed. Assume the probability of getting "heads" is 0.4 on either coin.

a. Derive the probability distribution of Y.

b. Derive the mean and variance of Y.

2.2 Use the probability distribution given in Table 2.2 to compute (a) $E(Y)$ and $E(X)$; (b) σ_X^2 and σ_Y^2; and (c) σ_{XY} and $\text{corr}(X, Y)$.

2.3 Using the random variables X and Y from Table 2.2, consider two new random variables $W = 4 + 8X$ and $V = 11 - 2Y$. Compute (a) $E(W)$ and $E(V)$; (b) J_W^2 and J_V^2; and (c) J_{WV} and $\text{corr}(W, V)$.

2.4 Suppose X is a Bernoulli random variable with $P(X = 1) = p$.

a. Show $E(X^3) = p$.

b. Show $E(X^k) = p$ for $k > 0$.

c. Suppose that $p = 0.3$. Compute the mean, variance, skewness, and kurtosis of X. (*Hint:* You might find it helpful to use the formulas given in Exercise 2.21.)

2.5 In July, Fairtown's daily maximum temperature has a mean of 65°F and a standard deviation of 5°F. What are the mean, standard deviation, and variance in °C?

2.6 The following table gives the joint probability distribution between employment status and college graduation among those either employed or looking for work (unemployed) in the working-age population of country A.

	Unemployed ($Y = 0$)	Employed ($Y = 1$)	Total
Non–college grads ($X = 0$)	0.078	0.673	0.751
College grads ($X = 1$)	0.042	0.207	0.249
Total	0.12	0.88	1.000

a. Compute $E(Y)$.

b. The unemployment rate is a fraction of the labor force that is unemployed. Show that the unemployment rate is given by $1 - E(Y)$.

c. Calculate $E(Y|X = 1)$ and $E(Y|X = 0)$.

d. Calculate the unemployment rate for (i) college graduates and (ii) non–graduates.

e. A randomly selected member of this population reports being unemployed. What is the probability that this worker is a (i) college graduate, (ii) non–graduate?

f. Are educational achievement and employment status independent? Explain.

2.7 In a given population of two-earning male-female couples, male earnings have a mean of \$50,000 per year and a standard deviation of \$15,000. Female earnings have a mean of \$48,000 per year and a standard deviation of \$13,000. The correlation between male and female earnings for a couple is 0.90. Let C denote the combined earnings for a randomly selected couple.

a. What is the mean of C?

b. What is the covariance between male and female earnings?

c. What is the standard deviation of C?

d. Convert the answers to (a) through (c) from U.S. dollars ($) to euros (€).

2.8 The random variable Y has a mean of 4 and a variance of 1/9. Let $Z = 3(Y - 4)$. Find the mean and the variance of Z.

2.9 X and Y are discrete random variables with the following joint distribution:

		Value of Y				
		2	**4**	**6**	**8**	**10**
Value of X	**3**	0.04	0.09	0.03	0.12	0.01
	6	0.10	0.06	0.15	0.03	0.02
	9	0.13	0.11	0.04	0.06	0.01

That is, $\Pr(X = 3, Y = 2) = 0.04$, and so forth.

a. Calculate the probability distribution, mean, and variance of Y.

b. Calculate the probability distribution, mean, and variance of Y given $X = 6$.

c. Calculate the covariance and correlation between X and Y.

2.10 Compute the following probabilities:

a. If Y is distributed $N(4, 9)$, find $\Pr(Y \leq 5)$.

b. If Y is distributed $N(5, 16)$, find $\Pr(Y > 2)$.

c. If Y is distributed $N(1, 4)$, find $\Pr(2 \leq Y \leq 5)$.

d. If Y is distributed $N(2, 1)$, find $\Pr(1 \leq Y \leq 4)$.

2.11 Compute the following probabilities:

a. If Y is distributed χ^2_3, find $\Pr(Y \leq 6.25)$.

b. If Y is distributed χ^2_8, find $\Pr(Y \leq 15.51)$.

c. If Y is distributed $F_{8,\infty}$, find $\Pr(Y \leq 1.94)$.

d. Why are the answers to (b) and (c) the same?

e. If Y is distributed χ^2_1, find $\Pr(Y \leq 0.5)$. (*Hint:* Use the definition of the χ^2_1 distribution.)

2.12 Compute the following probabilities:

a. If Y is distributed t_{12}, find $\Pr(Y \leq 1.36)$.

b. If Y is distributed t_{30}, find $\Pr(-1.70 \leq Y \leq 1.70)$.

c. If Y is distributed $N(0, 1)$, find $\Pr(-1.70 \leq Y \leq 1.70)$.

d. When do the critical values of the normal and the t distribution coincide?

e. If Y is distributed $F_{4,11}$, find $\Pr(Y > 3.36)$.

f. If Y is distributed $F_{3,21}$, find $\Pr(Y > 4.87)$.

2.13 X is a Bernoulli random variable with $\Pr(X = 1) = 0.90$, Y is distributed $N(0, 4)$, W is distributed $N(0, 16)$, and X, Y, and W are independent. Let $S = XY + (1 - X)W$. (That is, $S = Y$ when $X = 1$, and $S = W$ when $X = 0$.)

a. Show that $E(Y^2) = 4$ and $E(W^2) = 16$.

b. Show that $E(Y^3) = 0$ and $E(W^3) = 0$. (*Hint:* What is the skewness for a symmetric distribution?)

c. Show that $E(Y^4) = 3 \times 4^2$ and $E(W^4) = 3 \times 16^2$. (*Hint:* Use the fact that the kurtosis is 3 for a normal distribution.)

d. Derive $E(S)$, $E(S^2)$, $E(S^3)$ and $E(S^4)$. (*Hint:* Use the law of iterated expectations conditioning on $X = 0$ and $X = 1$.)

e. Derive the skewness and kurtosis for S.

2.14 In a population $\mu_Y = 50$ and $J_Y^2 = 21$. Use the central limit theorem to answer the following questions:

a. In a random sample of size $n = 50$, find $\Pr(\overline{Y} \leq 51)$.

b. In a random sample of size $n = 150$, find $\Pr(\overline{Y} > 49)$.

c. In a random sample of size $n = 45$, find $\Pr(50.5 \leq \overline{Y} \leq 51)$.

2.15 Suppose $Y_i, i = 1, 2, \ldots, n$, are i.i.d. random variables, each distributed $N(10, 4)$.

a. Compute $\Pr(9.6 \leq \overline{Y} \leq 10.4)$ when (i) $n = 20$, (ii) $n = 100$, and (iii) $n = 1000$.

b. Suppose c is a positive number. Show that $\Pr(10 - c \leq \overline{Y} \leq 10 + c)$ becomes close to 1.0 as n grows large.

c. Use your answer in (b) to argue that $\overline{Y}$ converges in probability to 10.

2.16 Y is distributed $N(5, 100)$ and you want to calculate $\Pr(Y < 3.6)$. Unfortunately, you do not have your textbook, and do not have access to a normal probability table like Appendix Table 1. However, you do have your

computer and a computer program that can generate i.i.d. draws from the $N(5, 100)$ distribution. Explain how you can use your computer to compute an accurate approximation for $\Pr(Y < 3.6)$.

2.17 $Y_i, i = 1, \text{c}, n$, are i.i.d. Bernoulli random variables with $p = 0.6$. Let Y denote the sample mean.

a. Use the central limit to compute approximations for

i. $\Pr(\overline{Y} > 0.64)$ when $n = 50$.

ii. $\Pr(\overline{Y} < 0.56)$ when $n = 200$.

b. How large would n need to be to ensure that $\Pr(0.65 > \overline{Y} > 0.55) = 0.95$? (*Hint:* Use the central limit theorem to compute an approximate answer.)

2.18 In any year, the weather may cause damages to a home. On a year-to-year basis, the damage is random. Let Y denote the dollar value of damages in any given year. Suppose that during 95% of the year $Y = \$0$, but during the other 5% $Y = \$30{,}000$.

a. What are the mean and standard deviation of damages caused in a year?

b. Consider an "insurance pool" of 120 people whose homes are sufficiently dispersed so that, in any year, the damage to different homes can be viewed as independently distributed random variables. Let $\overline{Y}$ denote the average damage caused to these 120 homes in one year. (i) What is the expected value of the average damage $\overline{Y}$? (ii) What is the probability that $\overline{Y}$ exceeds \$3,000?

2.19 Consider two random variables X and Y. Suppose that Y takes on k values $y_1, \ldots, y_k$ and that X takes on l values $x_1, \ldots, x_l$.

a. Show that $\Pr(Y = y_j) = \sum_{i=1}^{l} \Pr(Y = y_j \mid X = x_i) \Pr(X = x_i)$. [*Hint:* Use the definition of $\Pr(Y = y_j \mid X = x_i)$.]

b. Use your answer to (a) to verify Equation (2.19).

c. Suppose that X and Y are independent. Show that $\sigma_{XY} = 0$ and $\text{corr}(X, Y) = 0$.

2.20 Consider three random variables X, Y, and Z. Suppose that Y takes on k values $y_1, \ldots, y_k$, that X takes on l values $x_1, \ldots, x_l$, and that Z takes on m values $z_1, \ldots, z_m$. The joint probability distribution of X, Y, Z is $\Pr(X = x, Y = y, Z = z)$, and the conditional probability distribution of Y given X and Z is $\Pr(Y = y \mid X = x, Z = z) = \frac{\Pr(Y = y, X = x, Z = z)}{\Pr(X = x, Z = z)}$.

a. Explain how the marginal probability that $Y = y$ can be calculated from the joint probability distribution. [*Hint:* This is a generalization of Equation (2.16).]

b. Show that $E(Y) = E[E(Y|X, Z)]$. [*Hint:* This is a generalization of Equations (2.19) and (2.20).]

2.21 X is a random variable with moments $E(X)$, $E(X^2)$, $E(X^3)$, and so forth.

a. Show $E(X - \mu)^3 = E(X^3) - 3[E(X^2)][E(X)] + 2[E(X)]^3$.

b. Show $E(X - \mu)^4 = E(X^4) - 4[E(X)][E(X^3)] + 6[E(X)]^2[E(X^2)] - 3[E(X)]^4$.

2.22 Suppose you have some money to invest—for simplicity, \$1—and you are planning to put a fraction w into a stock market mutual fund and the rest, $1 - w$, into a bond mutual fund. Suppose that \$1 invested in a stock fund yields R_s after 1 year and that \$1 invested in a bond fund yields R_b, suppose that R_s is random with mean 0.08 (8%) and standard deviation 0.07, and suppose that R_b is random with mean 0.05 (5%) and standard deviation 0.04. The correlation between R_s and R_b is 0.25. If you place a fraction w of your money in the stock fund and the rest, $1 - w$, in the bond fund, then the return on your investment is $R = wR_s + (1 - w)R_b$.

a. Suppose that $w = 0.5$. Compute the mean and standard deviation of R.

b. Suppose that $w = 0.75$. Compute the mean and standard deviation of R.

c. What value of w makes the mean of R as large as possible? What is the standard deviation of R for this value of w?

d. (Harder) What is the value of w that minimizes the standard deviation of R? (Show using a graph, algebra, or calculus.)

2.23 This exercise provides an example of a pair of random variables X and Y for which the conditional mean of Y given X depends on X but $\text{corr}(X, Y) = 0$. Let X and Z be two independently distributed standard normal random variables, and let $Y = X^2 + Z$.

a. Show that $E(Y|X) = X^2$.

b. Show that $\mu_Y = 1$.

c. Show that $E(XY) = 0$. (*Hint:* Use the fact that the odd moments of a standard normal random variable are all zero.)

d. Show that $\text{cov}(X, Y) = 0$ and thus $\text{corr}(X, Y) = 0$.

2.24 Suppose Y_i is distributed i.i.d. $N(0, \sigma^2)$ for $i = 1, 2, \ldots, n$.

a. Show that $E(Y_i^2/\sigma^2) = 1$.

b. Show that $W = (1/\sigma^2)\sum_{i=1}^{n} Y_i^2$ is distributed χ_n^2.

c. Show that $E(W) = n$. [*Hint:* Use your answer to (a).]

d. Show that $V = Y_1 \Big/ \sqrt{\dfrac{\sum_{i=2}^{n} Y_i^2}{n-1}}$ is distributed t_{n-1}.

2.25 (Review of summation notation) Let $x_1, \ldots, x_n$ denote a sequence of numbers, $y_1, \ldots, y_n$ denote another sequence of numbers, and a, b, and c denote three constants. Show that

a. $\displaystyle\sum_{i=1}^{n} a x_i = a \sum_{i=1}^{n} x_i$

b. $\displaystyle\sum_{i=1}^{n} (x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$

c. $\displaystyle\sum_{i=1}^{n} a = na$

d. $\displaystyle\sum_{i=1}^{n} (a + bx_i + cy_i)^2 = na^2 + b^2 \sum_{i=1}^{n} x_i^2 + c^2 \sum_{i=1}^{n} y_i^2 + 2ab \sum_{i=1}^{n} x_i + 2ac \sum_{i=1}^{n} y_i + 2bc \sum_{i=1}^{n} x_i y_i$

2.26 Suppose that $Y_1, Y_2, \ldots, Y_n$ are random variables with a common mean μ_Y, a common variance σ_Y^2, and the same correlation ρ (so that the correlation between Y_i and Y_j is equal to ρ for all pairs i and j, where $i \neq j$).

a. Show that $\text{cov}(Y_i, Y_j) = \rho\sigma_Y^2$ for $i \neq j$.

b. Suppose that $n = 2$. Show that $E(\overline{Y}) = \mu_Y$ and $\text{var}(\overline{Y}) = \frac{1}{2}\sigma_Y^2 + \frac{1}{2}\rho\sigma_Y^2$.

c. For $n \geq 2$, show that $E(\overline{Y}) = \mu_Y$ and $\text{var}(\overline{Y}) = \sigma_Y^2/n + [(n-1)/n]\rho\sigma_Y^2$.

d. When n is very large, show that $\text{var}(\overline{Y}) \approx \rho\sigma_Y^2$.

2.27 X and Z are two jointly distributed random variables. Suppose you know the value of Z, but not the value of X. Let $\widetilde{X} = E(X \mid Z)$ denote a guess of the value of X using the information on Z, and let $W = X - \widetilde{X}$ denote the error associated with this guess.

a. Show that $E(W) = 0$. (*Hint:* Use the law of iterated expectations.)

b. Show that $E(WZ) = 0$.

c. Let $\hat{X} = g(Z)$ denote another guess of X using Z, and $V = X - \hat{X}$ denote its error. Show that $E(V^2) \geq E(W^2)$. [*Hint:* Let $h(Z) = g(Z) - E(X \mid Z)$, so that $V = [X - E(X \mid Z)] - h(Z)$. Derive $E(V^2)$.]

Empirical Exercise

E2.1 On the text website, **http://www.pearsonglobaleditions.com/Stock_Watson**, you will find the spreadsheet **Age_HourlyEarnings**, which contains the joint distribution of age (*Age*) and average hourly earnings (*AHE*) for 25- to 34-year-old full-time workers in 2012 with an education level that exceeds a high school diploma. Use this joint distribution to carry out the following exercises. (*Note:* For these exercises, you need to be able to carry out calculations and construct charts using a spreadsheet.)

a. Compute the marginal distribution of *Age*.

b. Compute the mean of *AHE* for each value of *Age*; that is, compute, $E(AHE|Age = 25)$, and so forth.

c. Compute and plot the mean of *AHE* versus *Age*. Are average hourly earnings and age related? Explain.

d. Use the law of iterated expectations to compute the mean of *AHE*; that is, compute $E(AHE)$.

e. Compute the variance of *AHE*.

f. Compute the covariance between *AHE* and *Age*.

g. Compute the correlation between *AHE* and *Age*.

h. Relate your answers in parts (f) and (g) to the plot you constructed in (c).

APPENDIX

2.1 Derivation of Results in Key Concept 2.3

This appendix derives the equations in Key Concept 2.3.

Equation (2.29) follows from the definition of the expectation.

To derive Equation (2.30), use the definition of the variance to write $\text{var}(a + bY) = E\{[a + bY - E(a + bY)]^2\} = E\{[b(Y - \mu_Y)]^2\} = b^2E[(Y - \mu_Y)^2] = b^2\sigma_Y^2$.

To derive Equation (2.31), use the definition of the variance to write

$$\begin{aligned}\text{var}(aX + bY) &= E\{[(aX + bY) - (a\mu_X + b\mu_Y)]^2\} \\ &= E\{[a(X - \mu_X) + b(Y - \mu_Y)]^2\} \\ &= E[a^2(X - \mu_X)^2] + 2E[ab(X - \mu_X)(Y - \mu_Y)] \\ &\quad + E[b^2(Y - \mu_Y)^2] \\ &= a^2\text{var}(X) + 2ab\,\text{cov}(X, Y) + b^2\,\text{var}(Y) \\ &= a^2\sigma_X^2 + 2ab\sigma_{XY} + b^2\sigma_Y^2, \end{aligned} \tag{2.49}$$

where the second equality follows by collecting terms, the third equality follows by expanding the quadratic, and the fourth equality follows by the definition of the variance and covariance.

To derive Equation (2.32), write $E(Y^2) = E\{[(Y - \mu_Y) + \mu_Y]^2\} = E[(Y - \mu_Y)^2] + 2\,\mu_Y E(Y - \mu_Y) + \mu_Y^2 = \sigma_Y^2 + \mu_Y^2$ because $E(Y - \mu_Y) = 0$.

To derive Equation (2.33), use the definition of the covariance to write

$$\begin{aligned}\text{cov}(a + bX + cV, Y) &= E\{[a + bX + cV - E(a + bX + cV)][Y - \mu_Y]\} \\ &= E\{[b(X - \mu_X) + c(V - \mu_V)][Y - \mu_Y]\} \\ &= E\{[b(X - \mu_X)][Y - \mu_Y]\} + E\{[c(V - \mu_V)][Y - \mu_Y]\} \\ &= b\sigma_{XY} + c\sigma_{VY}, \end{aligned} \tag{2.50}$$

which is Equation (2.33).

To derive Equation (2.34), write $E(XY) = E\{[(X - \mu_X) + \mu_X][(Y - \mu_Y) + \mu_Y]\} = E[(X - \mu_X)(Y - \mu_Y)] + \mu_X E(Y - \mu_Y) + \mu_Y E(X - \mu_X) + \mu_X\mu_Y = \sigma_{XY} + \mu_X\mu_Y$.

We now prove the correlation inequality in Equation (2.35); that is, $|\text{corr}\,(X, Y)| \le 1$. Let $a = -\sigma_{XY}/\sigma_X^2$ and $b = 1$. Applying Equation (2.31), we have that

$$\begin{aligned}\text{var}(aX + Y) &= a^2\sigma_X^2 + \sigma_Y^2 + 2a\sigma_{XY} \\ &= (-\sigma_{XY}/\sigma_X^2)^2\,\sigma_X^2 + \sigma_Y^2 + 2(-\sigma_{XY}/\sigma_X^2)\sigma_{XY} \\ &= \sigma_Y^2 - \sigma_{XY}^2/\sigma_X^2. \end{aligned} \tag{2.51}$$

Because $\text{var}(aX + Y)$ is a variance, it cannot be negative, so from the final line of Equation (2.51), it must be that $\sigma_Y^2 - \sigma_{XY}^2/\sigma_X^2 \ge 0$. Rearranging this inequality yields

$$\sigma_{XY}^2 \le \sigma_X^2\sigma_Y^2 \quad \text{(covariance inequality)}. \tag{2.52}$$

The covariance inequality implies that $\sigma_{XY}^2/(\sigma_X^2\sigma_Y^2) \le 1$ or, equivalently, $|\sigma_{XY}/(\sigma_X\sigma_Y)| \le 1$, which (using the definition of the correlation) proves the correlation inequality, $|\text{corr}\,(X\,Y)| \le 1$.

CHAPTER

3 Review of Statistics

Statistics is the science of using data to learn about the world around us. Statistical tools help us answer questions about unknown characteristics of distributions in populations of interest. For example, what is the mean of the distribution of earnings of recent college graduates? Do mean earnings differ for men and women, and, if so, by how much?

These questions relate to the distribution of earnings in the population of workers. One way to answer these questions would be to perform an exhaustive survey of the population of workers, measuring the earnings of each worker and thus finding the population distribution of earnings. In practice, however, such a comprehensive survey would be extremely expensive. The only comprehensive survey of the U.S. population is the decennial census, which cost $13 billion to carry out in 2010. The process of designing the census forms, managing and conducting the surveys, and compiling and analyzing the data takes ten years. Despite this extraordinary commitment, many members of the population slip through the cracks and are not surveyed. Thus a different, more practical approach is needed.

The key insight of statistics is that one can learn about a population distribution by selecting a random sample from that population. Rather than survey the entire U.S. population, we might survey, say, 1000 members of the population, selected at random by simple random sampling. Using statistical methods, we can use this sample to reach tentative conclusions—to draw statistical inferences—about characteristics of the full population.

Three types of statistical methods are used throughout econometrics: estimation, hypothesis testing, and confidence intervals. Estimation entails computing a "best guess" numerical value for an unknown characteristic of a population distribution, such as its mean, from a sample of data. Hypothesis testing entails formulating a specific hypothesis about the population, then using sample evidence to decide whether it is true. Confidence intervals use a set of data to estimate an interval or range for an unknown population characteristic. Sections 3.1, 3.2, and 3.3 review estimation, hypothesis testing, and confidence intervals in the context of statistical inference about an unknown population mean.

Most of the interesting questions in economics involve relationships between two or more variables or comparisons between different populations. For example,

is there a gap between the mean earnings for male and female recent college graduates? In Section 3.4, the methods for learning about the mean of a single population in Sections 3.1 through 3.3 are extended to compare means in two different populations. Section 3.5 discusses how the methods for comparing the means of two populations can be used to estimate causal effects in experiments. Sections 3.2 through 3.5 focus on the use of the normal distribution for performing hypothesis tests and for constructing confidence intervals when the sample size is large. In some special circumstances, hypothesis tests and confidence intervals can be based on the Student *t* distribution instead of the normal distribution; these special circumstances are discussed in Section 3.6. The chapter concludes with a discussion of the sample correlation and scatterplots in Section 3.7.

3.1 Estimation of the Population Mean

Suppose you want to know the mean value of Y (that is, μ_Y) in a population, such as the mean earnings of women recently graduated from college. A natural way to estimate this mean is to compute the sample average $\overline{Y}$ from a sample of n independently and identically distributed (i.i.d.) observations, $Y_1, \ldots, Y_n$ (recall that $Y_1, \ldots, Y_n$ are i.i.d. if they are collected by simple random sampling). This section discusses estimation of μ_Y and the properties of $\overline{Y}$ as an estimator of μ_Y.

Estimators and Their Properties

Estimators. The sample average $\overline{Y}$ is a natural way to estimate μ_Y, but it is not the only way. For example, another way to estimate μ_Y is simply to use the first observation, Y_1. Both $\overline{Y}$ and Y_1 are functions of the data that are designed to estimate μ_Y; using the terminology in Key Concept 3.1, both are estimators of μ_Y. When evaluated in repeated samples, $\overline{Y}$ and Y_1 take on different values (they produce different estimates) from one sample to the next. Thus the estimators $\overline{Y}$ and Y_1 both have sampling distributions. There are, in fact, many estimators of μ_Y, of which $\overline{Y}$ and Y_1 are two examples.

There are many possible estimators, so what makes one estimator "better" than another? Because estimators are random variables, this question can be phrased more precisely: What are desirable characteristics of the sampling distribution of an estimator? In general, we would like an estimator that gets as close as possible to the unknown true value, at least in some average sense; in other words, we would like the sampling distribution of an estimator to be as tightly

Estimators and Estimates

KEY CONCEPT 3.1

An **estimator** is a function of a sample of data to be drawn randomly from a population. An **estimate** is the numerical value of the estimator when it is actually computed using data from a specific sample. An estimator is a random variable because of randomness in selecting the sample, while an estimate is a nonrandom number.

centered on the unknown value as possible. This observation leads to three specific desirable characteristics of an estimator: unbiasedness (a lack of bias), consistency, and efficiency.

Unbiasedness. Suppose you evaluate an estimator many times over repeated randomly drawn samples. It is reasonable to hope that, on average, you would get the right answer. Thus a desirable property of an estimator is that the mean of its sampling distribution equals μ_Y; if so, the estimator is said to be unbiased.

To state this concept mathematically, let $\hat{\mu}_Y$ denote some estimator of μ_Y, such as $\overline{Y}$ or Y_1. The estimator $\hat{\mu}_Y$ is unbiased if $E(\hat{\mu}_Y) = \mu_Y$, where $E(\hat{\mu}_Y)$ is the mean of the sampling distribution of $\hat{\mu}_Y$; otherwise, $\hat{\mu}_Y$ is biased.

Consistency. Another desirable property of an estimator μ_Y is that, when the sample size is large, the uncertainty about the value of μ_Y arising from random variations in the sample is very small. Stated more precisely, a desirable property of $\hat{\mu}_Y$ is that the probability that it is within a small interval of the true value μ_Y approaches 1 as the sample size increases, that is, $\hat{\mu}_Y$ is consistent for μ_Y (Key Concept 2.6).

Variance and efficiency. Suppose you have two candidate estimators, $\hat{\mu}_Y$ and $\tilde{\mu}_Y$, both of which are unbiased. How might you choose between them? One way to do so is to choose the estimator with the tightest sampling distribution. This suggests choosing between $\hat{\mu}_Y$ and $\tilde{\mu}_Y$ by picking the estimator with the smallest variance. If $\hat{\mu}_Y$ has a smaller variance than $\tilde{\mu}_Y$, then $\hat{\mu}_Y$ is said to be more efficient than $\tilde{\mu}_Y$. The terminology "efficiency" stems from the notion that if $\hat{\mu}_Y$ has a smaller variance than $\tilde{\mu}_Y$, then it uses the information in the data more efficiently than does $\tilde{\mu}_Y$.

KEY CONCEPT 3.2

Bias, Consistency, and Efficiency

Let $\hat{\mu}_Y$ be an estimator of μ_Y. Then:

- The *bias* of $\hat{\mu}_Y$ is $E(\hat{\mu}_Y) - \mu_Y$.
- $\hat{\mu}_Y$ is an *unbiased estimator* of μ_Y if $E(\hat{\mu}_Y) = \mu_Y$.
- $\hat{\mu}_Y$ is a *consistent estimator* of μ_Y if $\hat{\mu}_Y \xrightarrow{p} \mu_Y$.
- Let $\tilde{\mu}_Y$ be another estimator of μ_Y and suppose that both $\hat{\mu}_Y$ and $\tilde{\mu}_Y$ are unbiased. Then $\hat{\mu}_Y$ is said to be more *efficient* than $\hat{\mu}_Y$ if $\text{var}(\hat{\mu}_Y) < \text{var}(\tilde{\mu}_Y)$.

Bias, **consistency**, and **efficiency** are summarized in Key Concept 3.2.

Properties of $\overline{Y}$

How does $\overline{Y}$ fare as an estimator of μ_Y when judged by the three criteria of bias, consistency, and efficiency?

Bias and consistency. The sampling distribution of $\overline{Y}$ has already been examined in Sections 2.5 and 2.6. As shown in Section 2.5, $E(\overline{Y}) = \mu_Y$, so $\overline{Y}$ is an unbiased estimator of μ_Y. Similarly, the law of large numbers (Key Concept 2.6) states that $\overline{Y} \xrightarrow{p} \mu_Y$; that is, $\overline{Y}$ is consistent.

Efficiency. What can be said about the efficiency of $\overline{Y}$? Because efficiency entails a comparison of estimators, we need to specify the estimator or estimators to which $\overline{Y}$ is to be compared.

We start by comparing the efficiency of $\overline{Y}$ to the estimator Y_1. Because $Y_1, \ldots, Y_n$ are i.i.d., the mean of the sampling distribution of Y_1 is $E(Y_1) = \mu_Y$; thus Y_1 is an unbiased estimator of μ_Y. Its variance is $\text{var}(Y_1) = \sigma_Y^2$. From Section 2.5, the variance of $\overline{Y}$ is σ_Y^2/n. Thus, for $n \geq 2$, the variance of $\overline{Y}$ is less than the variance of Y_1; that is, $\overline{Y}$ is a more efficient estimator than Y_1, so, according to the criterion of efficiency, $\overline{Y}$ should be used instead of Y_1. The estimator Y_1 might strike you as an obviously poor estimator—why would you go to the trouble of collecting a sample of n observations only to throw away all but the first?—and the concept of efficiency provides a formal way to show that $\overline{Y}$ is a more desirable estimator than Y_1.

Efficiency of $\overline{Y}$: $\overline{Y}$ Is BLUE

KEY CONCEPT
3.3

Let $\hat{\mu}_Y$ be an estimator of μ_Y that is a weighted average of $Y_1, \ldots, Y_n$, that is, $\hat{\mu}_Y = (1/n)\sum_{i=1}^{n} a_i Y_i$, where $a_1, \ldots, a_n$ are nonrandom constants. If $\hat{\mu}_Y$ is unbiased, then $\text{var}(\overline{Y}) < \text{var}(\hat{\mu}_Y)$ unless $\hat{\mu}_Y = \overline{Y}$. Thus $\overline{Y}$ is the Best Linear Unbiased Estimator (BLUE); that is, $\overline{Y}$ is the most efficient estimator of μ_Y among all unbiased estimators that are weighted averages of $Y_1, \ldots, Y_n$.

What about a less obviously poor estimator? Consider the weighted average in which the observations are alternately weighted by $\frac{1}{2}$ and $\frac{3}{2}$:

$$\widetilde{Y} = \frac{1}{n}\left(\frac{1}{2}Y_1 + \frac{3}{2}Y_2 + \frac{1}{2}Y_3 + \frac{3}{2}Y_4 + \cdots + \frac{1}{2}Y_{n-1} + \frac{3}{2}Y_n\right), \qquad (3.1)$$

where the number of observations n is assumed to be even for convenience. The mean of $\widetilde{Y}$ is μ_Y and its variance is $\text{var}(\widetilde{Y}) = 1.25\sigma_Y^2/n$ (Exercise 3.11). Thus $\widetilde{Y}$ is unbiased and, because $\text{var}(\widetilde{Y}) \to 0$ as $n \to \infty$, $\widetilde{Y}$ is consistent. However, $\widetilde{Y}$ has a larger variance than $\overline{Y}$. Thus $\overline{Y}$ is more efficient than $\widetilde{Y}$.

The estimators $\overline{Y}$, Y_1, and $\widetilde{Y}$ have a common mathematical structure: They are weighted averages of $Y_1, \ldots, Y_n$. The comparisons in the previous two paragraphs show that the weighted averages Y_1 and $\widetilde{Y}$ have larger variances than $\overline{Y}$. In fact, these conclusions reflect a more general result: $\overline{Y}$ is the most efficient estimator of *all* unbiased estimators that are weighted averages of $Y_1, \ldots, Y_n$. Said differently, $\overline{Y}$ is the **B**est **L**inear **U**nbiased **E**stimator (**BLUE**); that is, it is the most efficient (best) estimator among all estimators that are unbiased and are linear functions of $Y_1, \ldots, Y_n$. This result is stated in Key Concept 3.3 and is proved in Chapter 5.

$\overline{Y}$ is the least squares estimator of μ_Y. The sample average $\overline{Y}$ provides the best fit to the data in the sense that the average squared differences between the observations and $\overline{Y}$ are the smallest of all possible estimators.

Consider the problem of finding the estimator m that minimizes

$$\sum_{i=1}^{n}(Y_i - m)^2, \qquad (3.2)$$

which is a measure of the total squared gap or distance between the estimator m and the sample points. Because m is an estimator of $E(Y)$, you can think of it as a

Off the Mark!

In 2009, just before India's National Elections, pollsters predicted a close fight between the coalition parties, UPA and NDA. Psephologists envisaged that while UPA might have had the upper hand, NDA could not be written off. They predicted UPA would get between 201 and 235 seats, and NDA between 165 and 186. The actual results were surprising: UPA got 262 seats, while NDA could only manage to get 157 seats.

What could be the possible reasons for opinion polls being wide off the mark?

In countries that do not have a homogenous population, such as India, caste, religion, and region influence electoral outcomes greatly. Vulnerable sections of the population may be afraid to disclose their actual preference. If opinion polls do not randomly select samples across various locations and sections of people they may not hit the mark.

Source: http://timesofindia.indiatimes.com/news/Why-opinion-polls-are-often-wide-of-the-mark/articleshow/33678070.cms?

prediction of the value of Y_i, so the gap $Y_i - m$ can be thought of as a prediction mistake. The sum of squared gaps in Expression (3.2) can be thought of as the sum of squared prediction mistakes.

The estimator m that minimizes the sum of squared gaps $Y_i - m$ in Expression (3.2) is called the **least squares estimator**. One can imagine using trial and error to solve the least squares problem: Try many values of m until you are satisfied that you have the value that makes Expression (3.2) as small as possible. Alternatively, as is done in Appendix 3.2, you can use algebra or calculus to show that choosing $m = \overline{Y}$ minimizes the sum of squared gaps in Expression (3.2) so that $\overline{Y}$ is the least squares estimator of μ_Y.

The Importance of Random Sampling

We have assumed that $Y_1, \ldots, Y_n$ are i.i.d. draws, such as those that would be obtained from simple random sampling. This assumption is important because nonrandom sampling can result in $\overline{Y}$ being biased. Suppose that, to estimate the monthly national unemployment rate, a statistical agency adopts a sampling scheme in which interviewers survey working-age adults sitting in city parks at 10 A.M. on the second Wednesday of the month. Because most employed people are at work at that hour (not sitting in the park!), the unemployed are overly represented in the sample, and an estimate of the unemployment rate based on this sampling plan would be biased. This bias arises because this sampling scheme overrepresents, or oversamples, the unemployed members of the population. This example is fictitious, but the "Off the Mark!" box gives a real-world example of biases introduced by sampling that is not entirely random.

It is important to design sample selection schemes in a way that minimizes bias. Appendix 3.1 includes a discussion of what the Bureau of Labor Statistics actually does when it conducts the U.S. Current Population Survey (CPS), the survey it uses to estimate the monthly U.S. unemployment rate.

3.2 Hypothesis Tests Concerning the Population Mean

Many hypotheses about the world around us can be phrased as yes/no questions. Do the mean hourly earnings of recent U.S. college graduates equal $20 per hour? Are mean earnings the same for male and female college graduates? Both these questions embody specific hypotheses about the population distribution of earnings. The statistical challenge is to answer these questions based on a sample of evidence. This section describes **hypothesis tests** concerning the population mean (Does the population mean of hourly earnings equal $20?). Hypothesis tests involving two populations (Are mean earnings the same for men and women?) are taken up in Section 3.4.

Null and Alternative Hypotheses

The starting point of statistical hypotheses testing is specifying the hypothesis to be tested, called the **null hypothesis**. Hypothesis testing entails using data to compare the null hypothesis to a second hypothesis, called the **alternative hypothesis**, that holds if the null does not.

The null hypothesis is that the population mean, $E(Y)$, takes on a specific value, denoted $\mu_{Y,0}$. The null hypothesis is denoted H_0 and thus is

$$H_0: E(Y) = \mu_{Y,0}. \tag{3.3}$$

For example, the conjecture that, on average in the population, college graduates earn $20 per hour constitutes a null hypothesis about the population distribution of hourly earnings. Stated mathematically, if Y is the hourly earning of a randomly selected recent college graduate, then the null hypothesis is that $E(Y) = 20$; that is, $\mu_{Y,0} = 20$ in Equation (3.3).

The alternative hypothesis specifies what is true if the null hypothesis is not. The most general alternative hypothesis is that $E(Y) \neq \mu_{Y,0}$, which is called a **two-sided alternative hypothesis** because it allows $E(Y)$ to be either less than or greater than $\mu_{Y,0}$. The two-sided alternative is written as

$$H_1: E(Y) \neq \mu_{Y,0} \quad \text{(two-sided alternative)}. \tag{3.4}$$

One-sided alternatives are also possible, and these are discussed later in this section.

The problem facing the statistician is to use the evidence in a randomly selected sample of data to decide whether to accept the null hypothesis H_0 or to reject it in favor of the alternative hypothesis H_1. If the null hypothesis is "accepted," this does not mean that the statistician declares it to be true; rather, it is accepted tentatively with the recognition that it might be rejected later based on additional evidence. For this reason, statistical hypothesis testing can be posed as either rejecting the null hypothesis or failing to do so.

The *p*-Value

In any given sample, the sample average $\overline{Y}$ will rarely be exactly equal to the hypothesized value $\mu_{Y,0}$. Differences between $\overline{Y}$ and $\mu_{Y,0}$ can arise because the true mean in fact does not equal $\mu_{Y,0}$ (the null hypothesis is false) or because the true mean equals $\mu_{Y,0}$ (the null hypothesis is true) but $\overline{Y}$ differs from $\mu_{Y,0}$ because of random sampling. It is impossible to distinguish between these two possibilities with certainty. Although a sample of data cannot provide conclusive evidence about the null hypothesis, it is possible to do a probabilistic calculation that permits testing the null hypothesis in a way that accounts for sampling uncertainty. This calculation involves using the data to compute the *p*-value of the null hypothesis.

The ***p*-value**, also called the **significance probability**, is the probability of drawing a statistic at least as adverse to the null hypothesis as the one you actually computed in your sample, assuming the null hypothesis is correct. In the case at hand, the *p*-value is the probability of drawing $\overline{Y}$ at least as far in the tails of its distribution under the null hypothesis as the sample average you actually computed.

For example, suppose that, in your sample of recent college graduates, the average wage is \$22.64. The *p*-value is the probability of observing a value of $\overline{Y}$ at least as different from \$20 (the population mean under the null) as the observed value of \$22.64 by pure random sampling variation, assuming that the null hypothesis is true. If this *p*-value is small, say 0.5%, then it is very unlikely that this sample would have been drawn if the null hypothesis is true; thus it is reasonable to conclude that the null hypothesis is not true. By contrast, if this *p*-value is large, say 40%, then it is quite likely that the observed sample average of \$22.64 could have arisen just by random sampling variation if the null hypothesis is true; accordingly, the evidence against the null hypothesis is weak in this probabilistic sense, and it is reasonable not to reject the null hypothesis.

To state the definition of the *p*-value mathematically, let $\overline{Y}^{act}$ denote the value of the sample average actually computed in the data set at hand and let $\Pr_{H_0}$

denote the probability computed under the null hypothesis (that is, computed assuming that $E(Y_i) = \mu_{Y,0}$). The p-value is

$$p\text{-value} = \Pr_{H_0}\left[\left|\overline{Y} - \mu_{Y,0}\right| > \left|\overline{Y}^{act} - \mu_{Y,0}\right|\right]. \tag{3.5}$$

That is, the p-value is the area in the tails of the distribution of $\overline{Y}$ under the null hypothesis beyond $\mu_{Y,0} \pm \left|\overline{Y}^{act} - \mu_{Y,0}\right|$. If the p-value is large, then the observed value $\overline{Y}^{act}$ is consistent with the null hypothesis, but if the p-value is small, it is not.

To compute the p-value, it is necessary to know the sampling distribution of $\overline{Y}$ under the null hypothesis. As discussed in Section 2.6, when the sample size is small this distribution is complicated. However, according to the central limit theorem, when the sample size is large, the sampling distribution of $\overline{Y}$ is well approximated by a normal distribution. Under the null hypothesis the mean of this normal distribution is $\mu_{Y,0}$, so under the null hypothesis $\overline{Y}$ is distributed $N(\mu_{Y,0}, \sigma_{\overline{Y}}^2)$, where $\sigma_{\overline{Y}}^2 = \sigma_Y^2/n$. This large-sample normal approximation makes it possible to compute the p-value without needing to know the population distribution of Y, as long as the sample size is large. The details of the calculation, however, depend on whether σ_Y^2 is known.

Calculating the p-Value When σ_Y Is Known

The calculation of the p-value when σ_Y is known is summarized in Figure 3.1. If the sample size is large, then under the null hypothesis the sampling distribution of $\overline{Y}$ is $N(\mu_{Y,0}, \sigma_{\overline{Y}}^2)$, where $\sigma_{\overline{Y}}^2 = \sigma_Y^2/n$. Thus, under the null hypothesis, the standardized version of $\overline{Y}$, $(\overline{Y} - \mu_{Y,0})/\sigma_{\overline{Y}}$, has a standard normal distribution. The p-value is the probability of obtaining a value of $\overline{Y}$ farther from $\mu_{Y,0}$ than $\overline{Y}^{act}$ under the null hypothesis or, equivalently, is the probability of obtaining $(\overline{Y} - \mu_{Y,0})/\sigma_{\overline{Y}}$ greater than $(\overline{Y}^{act} - \mu_{Y,0})/\sigma_{\overline{Y}}$ in absolute value. This probability is the shaded area shown in Figure 3.1. Written mathematically, the shaded tail probability in Figure 3.1 (that is, the p-value) is

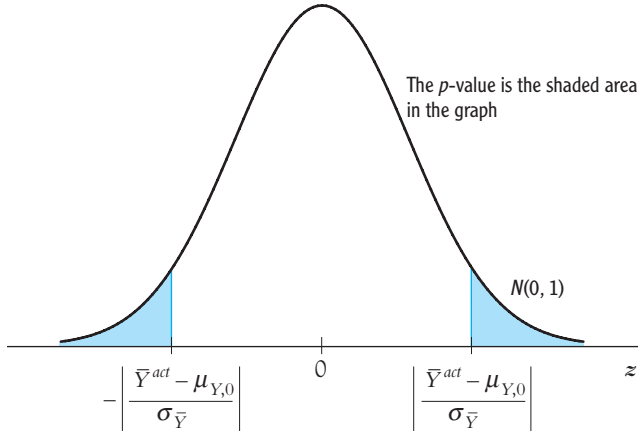
$$p\text{-value} = \Pr_{H_0}\left(\left|\frac{\overline{Y} - \mu_{Y,0}}{\sigma_{\overline{Y}}}\right| > \left|\frac{\overline{Y}^{act} - \mu_{Y,0}}{\sigma_{\overline{Y}}}\right|\right) = 2\Phi\left(-\left|\frac{\overline{Y}^{act} - \mu_{Y,0}}{\sigma_{\overline{Y}}}\right|\right), \tag{3.6}$$

where Φ is the standard normal cumulative distribution function. That is, the p-value is the area in the tails of a standard normal distribution outside $\pm\left|\overline{Y}^{act} - \mu_{Y,0}\right|/\sigma_{\overline{Y}}$.

The formula for the p-value in Equation (3.6) depends on the variance of the population distribution, σ_Y^2. In practice, this variance is typically unknown. [An exception is when Y_i is binary so that its distribution is Bernoulli, in which case

FIGURE 3.1 Calculating a *p*-value

The *p*-value is the probability of drawing a value of $\overline{Y}$ that differs from $\mu_{Y,0}$ by at least as much as $\overline{Y}^{act}$. In large samples, $\overline{Y}$ is distributed $N(\mu_{Y,0}, \sigma^2_{\overline{Y}})$, under the null hypothesis, so $(\overline{Y} - \mu_{Y,0})/\sigma_{\overline{Y}}$ is distributed $N(0, 1)$. Thus the *p*-value is the shaded standard normal tail probability outside $\pm|(\overline{Y}^{act} - \mu_{Y,0})/\sigma_{\overline{Y}}|$.



the variance is determined by the null hypothesis; see Equation (2.7) and Exercise 3.2.] Because in general $\sigma^2_{\overline{Y}}$ must be estimated before the *p*-value can be computed, we now turn to the problem of estimating $\sigma^2_{\overline{Y}}$.

The Sample Variance, Sample Standard Deviation, and Standard Error

The sample variance $s^2_{\overline{Y}}$ is an estimator of the population variance $\sigma^2_{\overline{Y}}$, the sample standard deviation s_Y is an estimator of the population standard deviation σ_Y, and the standard error of the sample average $\overline{Y}$ is an estimator of the standard deviation of the sampling distribution of $\overline{Y}$.

The sample variance and standard deviation. The **sample variance**, $s^2_{\overline{Y}}$, is

$$s^2_{\overline{Y}} = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2. \tag{3.7}$$

The **sample standard deviation**, s_Y, is the square root of the sample variance.

The formula for the sample variance is much like the formula for the population variance. The population variance, $E(Y - \mu_Y)^2$, is the average value of $(Y - \mu_Y)^2$ in the population distribution. Similarly, the sample variance is the sample average of $(Y_i - \mu_Y)^2, i = 1, \ldots, n$, with two modifications: First, μ_Y is replaced by $\overline{Y}$, and second, the average uses the divisor $n - 1$ instead of n.

The Standard Error of $\overline{Y}$

KEY CONCEPT 3.4

The standard error of $\overline{Y}$ is an estimator of the standard deviation of $\overline{Y}$. The standard error of $\overline{Y}$ is denoted $SE(\overline{Y})$ or $\hat{\sigma}_{\overline{Y}}$. When $Y_1, \ldots, Y_n$ are i.i.d.,

$$SE(\overline{Y}) = \hat{\sigma}_{\overline{Y}} = s_Y/\sqrt{n}. \quad (3.8)$$

The reason for the first modification—replacing μ_Y by $\overline{Y}$—is that μ_Y is unknown and thus must be estimated; the natural estimator of μ_Y is $\overline{Y}$. The reason for the second modification—dividing by $n - 1$ instead of by n—is that estimating μ_Y by $\overline{Y}$ introduces a small downward bias in $(Y_i - \overline{Y})^2$. Specifically, as is shown in Exercise 3.18, $E[(Y_i - \overline{Y})^2] = [(n-1)/n]\sigma_Y^2$. Thus $E\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = nE[(Y_i - \overline{Y})^2] = (n-1)\sigma_Y^2$. Dividing by $n - 1$ in Equation (3.7) instead of n corrects for this small downward bias, and as a result s_Y^2 is unbiased.

Dividing by $n - 1$ in Equation (3.7) instead of n is called a **degrees of freedom** correction: Estimating the mean uses up some of the information—that is, uses up 1 "degree of freedom"—in the data, so that only $n - 1$ degrees of freedom remain.

Consistency of the sample variance. The sample variance is a consistent estimator of the population variance:

$$s_Y^2 \longrightarrow \sigma_Y^2. \quad (3.9)$$

In other words, the sample variance is close to the population variance with high probability when n is large.

The result in Equation (3.9) is proven in Appendix 3.3 under the assumptions that $Y_1, \ldots, Y_n$ are i.i.d. and Y_i has a finite fourth moment; that is, $E(Y_i^4) < \infty$. Intuitively, the reason that s_Y^2 is consistent is that it is a sample average, so s_Y^2 obeys the law of large numbers. But for s_Y^2 to obey the law of large numbers in Key Concept 2.6, $(Y_i - \mu_Y)^2$ must have finite variance, which in turn means that $E(Y_i^4)$ must be finite; in other words, Y_i must have a finite fourth moment.

The standard error of $\overline{Y}$. Because the standard deviation of the sampling distribution of $\overline{Y}$ is $\sigma_{\overline{Y}} = \sigma_Y/\sqrt{n}$, Equation (3.9) justifies using $s_Y/\sqrt{n}$ as an estimator of $\sigma_{\overline{Y}}$. The estimator of $\sigma_{\overline{Y}}$, $s_Y/\sqrt{n}$, is called the **standard error of $\overline{Y}$** and is denoted $SE(\overline{Y})$ or $\hat{\sigma}_{\overline{Y}}$ (the caret "^" over the symbol means that it is an estimator of $\sigma_{\overline{Y}}$). The standard error of $\overline{Y}$ is summarized as in Key Concept 3.4.

When $Y_1, \ldots, Y_n$ are i.i.d. draws from a Bernoulli distribution with success probability p, the formula for the variance of $\overline{Y}$ simplifies to $p(1-p)/n$ (see Exercise 3.2). The formula for the standard error also takes on a simple form that depends only on $\overline{Y}$ and n: $SE(\overline{Y}) = \sqrt{\overline{Y}(1-\overline{Y})/n}$.

Calculating the *p*-Value When σ_Y Is Unknown

Because s_Y^2 is a consistent estimator of σ_Y^2, the p-value can be computed by replacing $\sigma_{\overline{Y}}$ in Equation (3.6) by the standard error, $SE(\overline{Y}) = \hat{\sigma}_{\overline{Y}}$. That is, when σ_Y is unknown and $Y_1, \ldots, Y_n$ are i.i.d., the p-value is calculated using the formula

$$p\text{-value} = 2\Phi\left(-\left|\frac{\overline{Y}^{act} - \mu_{Y,0}}{SE(\overline{Y})}\right|\right). \tag{3.10}$$

The *t*-Statistic

The standardized sample average $(\overline{Y} - \mu_{Y,0})/SE(\overline{Y})$ plays a central role in testing statistical hypotheses and has a special name, the ***t*-statistic** or ***t*-ratio**:

$$t = \frac{\overline{Y} - \mu_{Y,0}}{SE(\overline{Y})}. \tag{3.11}$$

In general, a **test statistic** is a statistic used to perform a hypothesis test. The t-statistic is an important example of a test statistic.

Large-sample distribution of the t-statistic. When n is large, s_Y^2 is close to σ_Y^2 with high probability. Thus the distribution of the t-statistic is approximately the same as the distribution of $(\overline{Y} - \mu_{Y,0})/\sigma_{\overline{Y}}$, which in turn is well approximated by the standard normal distribution when n is large because of the central limit theorem (Key Concept 2.7). Accordingly, under the null hypothesis,

$$t \text{ is approximately distributed } N(0,1) \text{ for large } n. \tag{3.12}$$

The formula for the p-value in Equation (3.10) can be rewritten in terms of the t-statistic. Let t^{act} denote the value of the t-statistic actually computed:

$$t^{act} = \frac{\overline{Y}^{act} - \mu_{Y,0}}{SE(\overline{Y})}. \tag{3.13}$$

Accordingly, when n is large, the p-value can be calculated using

$$p\text{-value} = 2\Phi(-|t^{act}|). \tag{3.14}$$

As a hypothetical example, suppose that a sample of $n = 200$ recent college graduates is used to test the null hypothesis that the mean wage, $E(Y)$, is \$20 per hour. The sample average wage is $\overline{Y}^{act} = \$22.64$, and the sample standard deviation is $s_Y = \$18.14$. Then the standard error of $\overline{Y}$ is $s_Y/\sqrt{n} = 18.14/\sqrt{200} = 1.28$. The value of the t-statistic is $t^{act} = (22.64 - 20)/1.28 = 2.06$. From Appendix Table 1, the p-value is $2\Phi(-2.06) = 0.039$, or 3.9%. That is, assuming the null hypothesis to be true, the probability of obtaining a sample average at least as different from the null as the one actually computed is 3.9%.

Hypothesis Testing with a Prespecified Significance Level

When you undertake a statistical hypothesis test, you can make two types of mistakes: You can incorrectly reject the null hypothesis when it is true, or you can fail to reject the null hypothesis when it is false. Hypothesis tests can be performed without computing the p-value if you are willing to specify in advance the probability you are willing to tolerate of making the first kind of mistake—that is, of incorrectly rejecting the null hypothesis when it is true. If you choose a prespecified probability of rejecting the null hypothesis when it is true (for example, 5%), then you will reject the null hypothesis if and only if the p-value is less than 0.05. This approach gives preferential treatment to the null hypothesis, but in many practical situations this preferential treatment is appropriate.

Hypothesis tests using a fixed significance level. Suppose it has been decided that the hypothesis will be rejected if the p-value is less than 5%. Because the area under the tails of the standard normal distribution outside ± 1.96 is 5%, this gives a simple rule:

$$\text{Reject } H_0 \text{ if } |t^{act}| > 1.96. \tag{3.15}$$

That is, reject if the absolute value of the t-statistic computed from the sample is greater than 1.96. If n is large enough, then under the null hypothesis the t-statistic has a $N(0, 1)$ distribution. Thus the probability of erroneously rejecting the null hypothesis (rejecting the null hypothesis when it is in fact true) is 5%.

KEY CONCEPT 3.5

The Terminology of Hypothesis Testing

A statistical hypothesis test can make two types of mistakes: a **type I error**, in which the null hypothesis is rejected when in fact it is true, and a **type II error**, in which the null hypothesis is not rejected when in fact it is false. The prespecified rejection probability of a statistical hypothesis test when the null hypothesis is true—that is, the prespecified probability of a type I error—is the **significance level** of the test. The **critical value** of the test statistic is the value of the statistic for which the test just rejects the null hypothesis at the given significance level. The set of values of the test statistic for which the test rejects the null hypothesis is the **rejection region**, and the values of the test statistic for which it does not reject the null hypothesis is the **acceptance region**. The probability that the test actually incorrectly rejects the null hypothesis when it is true is the **size of the test**, and the probability that the test correctly rejects the null hypothesis when the alternative is true is the **power of the test**.

The p-value is the probability of obtaining a test statistic, by random sampling variation, at least as adverse to the null hypothesis value as is the statistic actually observed, assuming that the null hypothesis is correct. Equivalently, the p-value is the smallest significance level at which you can reject the null hypothesis.

This framework for testing statistical hypotheses has some specialized terminology, summarized in Key Concept 3.5. The significance level of the test in Equation (3.15) is 5%, the critical value of this two-sided test is 1.96, and the rejection region is the values of the t-statistic outside ± 1.96. If the test rejects at the 5% significance level, the population mean μ_Y is said to be statistically significantly different from $\mu_{Y,0}$ at the 5% significance level.

Testing hypotheses using a prespecified significance level does not require computing p-values. In the previous example of testing the hypothesis that the mean earnings of recent college graduates is \$20 per hour, the t-statistic was 2.06. This value exceeds 1.96, so the hypothesis is rejected at the 5% level. Although performing the test with a 5% significance level is easy, reporting only whether the null hypothesis is rejected at a prespecified significance level conveys less information than reporting the p-value.

What significance level should you use in practice? In many cases, statisticians and econometricians use a 5% significance level. If you were to test many statistical

Testing the Hypothesis $E(Y) = \mu_{Y,0}$ Against the Alternative $E(Y) \neq \mu_{Y,0}$

KEY CONCEPT 3.6

1. Compute the standard error of $\overline{Y}$, $SE(\overline{Y})$ [Equation (3.8)].
2. Compute the t-statistic [Equation (3.13)].
3. Compute the p-value [Equation (3.14)]. Reject the hypothesis at the 5% significance level if the p-value is less than 0.05 (equivalently, if $|t^{act}| > 1.96$).

hypotheses at the 5% level, you would incorrectly reject the null on average once in 20 cases. Sometimes a more conservative significance level might be in order. For example, legal cases sometimes involve statistical evidence, and the null hypothesis could be that the defendant is not guilty; then one would want to be quite sure that a rejection of the null (conclusion of guilt) is not just a result of random sample variation. In some legal settings, the significance level used is 1%, or even 0.1%, to avoid this sort of mistake. Similarly, if a government agency is considering permitting the sale of a new drug, a very conservative standard might be in order so that consumers can be sure that the drugs available in the market actually work.

Being conservative, in the sense of using a very low significance level, has a cost: The smaller the significance level, the larger the critical value and the more difficult it becomes to reject the null when the null is false. In fact, the most conservative thing to do is never to reject the null hypothesis—but if that is your view, then you never need to look at any statistical evidence for you will never change your mind! The lower the significance level, the lower the power of the test. Many economic and policy applications can call for less conservatism than a legal case, so a 5% significance level is often considered to be a reasonable compromise.

Key Concept 3.6 summarizes hypothesis tests for the population mean against the two-sided alternative.

One-Sided Alternatives

In some circumstances, the alternative hypothesis might be that the mean exceeds $\mu_{Y,0}$. For example, one hopes that education helps in the labor market, so the relevant alternative to the null hypothesis that earnings are the same for college graduates and non–college graduates is not just that their earnings differ, but

rather that graduates earn more than nongraduates. This is called a **one-sided alternative hypothesis** and can be written

$$H_1: E(Y) > \mu_{Y,0} \quad \text{(one-sided alternative)}. \tag{3.16}$$

The general approach to computing p-values and to hypothesis testing is the same for one-sided alternatives as it is for two-sided alternatives, with the modification that only large positive values of the t-statistic reject the null hypothesis rather than values that are large in absolute value. Specifically, to test the one-sided hypothesis in Equation (3.16), construct the t-statistic in Equation (3.13). The p-value is the area under the standard normal distribution to the right of the calculated t-statistic. That is, the p-value, based on the $N(0, 1)$ approximation to the distribution of the t-statistic, is

$$p\text{-value} = \Pr_{H_0}(Z > t^{act}) = 1 - \Phi(t^{act}). \tag{3.17}$$

The $N(0, 1)$ critical value for a one-sided test with a 5% significance level is 1.64. The rejection region for this test is all values of the t-statistic exceeding 1.64.

The one-sided hypothesis in Equation (3.16) concerns values of μ_Y exceeding $\mu_{Y,0}$. If instead the alternative hypothesis is that $E(Y) < \mu_{Y,0}$, then the discussion of the previous paragraph applies except that the signs are switched; for example, the 5% rejection region consists of values of the t-statistic less than –1.64.

3.3 Confidence Intervals for the Population Mean

Because of random sampling error, it is impossible to learn the exact value of the population mean of Y using only the information in a sample. However, it is possible to use data from a random sample to construct a set of values that contains the true population mean μ_Y with a certain prespecified probability. Such a set is called a **confidence set**, and the prespecified probability that μ_Y is contained in this set is called the **confidence level**. The confidence set for μ_Y turns out to be all the possible values of the mean between a lower and an upper limit, so that the confidence set is an interval, called a **confidence interval**.

Here is one way to construct a 95% confidence set for the population mean. Begin by picking some arbitrary value for the mean; call it $\mu_{Y,0}$. Test the null hypothesis that $\mu_Y = \mu_{Y,0}$ against the alternative that $\mu_Y \neq \mu_{Y,0}$ by computing the t-statistic; if its absolute value is less than 1.96, this hypothesized value $\mu_{Y,0}$ is not rejected at the 5% level, and write down this nonrejected value $\mu_{Y,0}$. Now pick another arbitrary value of $\mu_{Y,0}$ and test it; if you cannot reject it, write down this value on your list.

Confidence Intervals for the Population Mean

KEY CONCEPT 3.7

A 95% two-sided confidence interval for μ_Y is an interval constructed so that it contains the true value of μ_Y in 95% of all possible random samples. When the sample size n is large, 95%, 90%, and 99% confidence intervals for μ_Y are

$$95\% \text{ confidence interval for } \mu_Y = \{\overline{Y} \pm 1.96SE(\overline{Y})\}.$$
$$90\% \text{ confidence interval for } \mu_Y = \{\overline{Y} \pm 1.64SE(\overline{Y})\}.$$
$$99\% \text{ confidence interval for } \mu_Y = \{\overline{Y} \pm 2.58SE(\overline{Y})\}.$$

Do this again and again; indeed, do so for all possible values of the population mean. Continuing this process yields the set of all values of the population mean that cannot be rejected at the 5% level by a two-sided hypothesis test.

This list is useful because it summarizes the set of hypotheses you can and cannot reject (at the 5% level) based on your data: If someone walks up to you with a specific number in mind, you can tell him whether his hypothesis is rejected or not simply by looking up his number on your handy list. A bit of clever reasoning shows that this set of values has a remarkable property: The probability that it contains the true value of the population mean is 95%.

The clever reasoning goes like this: Suppose the true value of μ_Y is 21.5 (although we do not know this). Then $\overline{Y}$ has a normal distribution centered on 21.5, and the t-statistic testing the null hypothesis $\mu_Y = 21.5$ has a $N(0, 1)$ distribution. Thus, if n is large, the probability of rejecting the null hypothesis $\mu_Y = 21.5$ at the 5% level is 5%. But because you tested all possible values of the population mean in constructing your set, in particular you tested the true value, $\mu_Y = 21.5$. In 95% of all samples, you will correctly accept 21.5; this means that in 95% of all samples, your list will contain the true value of μ_Y. Thus the values on your list constitute a 95% confidence set for μ_Y.

This method of constructing a confidence set is impractical, for it requires you to test all possible values of μ_Y as null hypotheses. Fortunately, there is a much easier approach. According to the formula for the t-statistic in Equation (3.13), a trial value of $\mu_{Y,0}$ is rejected at the 5% level if it is more than 1.96 standard errors away from $\overline{Y}$. Thus the set of values of μ_Y that are not rejected at the 5% level consists of those values within $\pm 1.96SE(\overline{Y})$ of $\overline{Y}$; that is, a 95% confidence interval for μ_Y is $\overline{Y} - 1.96SE(\overline{Y}) \leq \mu_Y \leq \overline{Y} + 1.96SE(\overline{Y})$. Key Concept 3.7 summarizes this approach.

As an example, consider the problem of constructing a 95% confidence interval for the mean hourly earnings of recent college graduates using a hypothetical random sample of 200 recent college graduates where $\overline{Y} = \$22.64$ and $SE(\overline{Y}) = 1.28$. The 95% confidence interval for mean hourly earnings is $22.64 \pm 1.96 \times 1.28 = 22.64 \pm 2.51 = [\$20.13, \$25.15]$.

This discussion so far has focused on two-sided confidence intervals. One could instead construct a one-sided confidence interval as the set of values of μ_Y that cannot be rejected by a one-sided hypothesis test. Although one-sided confidence intervals have applications in some branches of statistics, they are uncommon in applied econometric analysis.

Coverage probabilities. The **coverage probability** of a confidence interval for the population mean is the probability, computed over all possible random samples, that it contains the true population mean.

3.4 Comparing Means from Different Populations

Do recent male and female college graduates earn the same amount on average? This question involves comparing the means of two different population distributions. This section summarizes how to test hypotheses and how to construct confidence intervals for the difference in the means from two different populations.

Hypothesis Tests for the Difference Between Two Means

To illustrate a **test for the difference between two means**, let μ_w be the mean hourly earning in the population of women recently graduated from college and let μ_m be the population mean for recently graduated men. Consider the null hypothesis that mean earnings for these two populations differ by a certain amount, say d_0. Then the null hypothesis and the two-sided alternative hypothesis are

$$H_0: \mu_m - \mu_w = d_0 \text{ vs. } H_1: \mu_m - \mu_w \neq d_0. \tag{3.18}$$

The null hypothesis that men and women in these populations have the same mean earnings corresponds to H_0 in Equation (3.18) with $d_0 = 0$.

Because these population means are unknown, they must be estimated from samples of men and women. Suppose we have samples of n_m men and n_w women drawn at random from their populations. Let the sample average annual earnings be $\overline{Y}_m$ for men and $\overline{Y}_w$ for women. Then an estimator of $\mu_m - \mu_w$ is $\overline{Y}_m - \overline{Y}_w$.

To test the null hypothesis that $\mu_m - \mu_w = d_0$ using $\overline{Y}_m - \overline{Y}_w$, we need to know the distribution of $\overline{Y}_m - \overline{Y}_w$. Recall that $\overline{Y}_m$ is, according to the central limit theorem, approximately distributed $N(\mu_m, \sigma_m^2/n_m)$, where σ_m^2 is the population variance of earnings for men. Similarly, $\overline{Y}_w$ is approximately distributed $N(\mu_w, \sigma_w^2/n_w)$ where σ_w^2 is the population variance of earnings for women. Also, recall from Section 2.4 that a weighted average of two normal random variables is itself normally distributed. Because $\overline{Y}_m$ and $\overline{Y}_w$ are constructed from different randomly selected samples, they are independent random variables. Thus $\overline{Y}_m - \overline{Y}_w$ is distributed $N[\mu_m - \mu_w, (\sigma_m^2/n_m) + (\sigma_w^2/n_w)]$.

If σ_m^2 and σ_w^2 are known, then this approximate normal distribution can be used to compute p-values for the test of the null hypothesis that $\mu_m - \mu_w = d_0$. In practice, however, these population variances are typically unknown so they must be estimated. As before, they can be estimated using the sample variances, s_m^2 and s_w^2 where s_m^2 is defined as in Equation (3.7), except that the statistic is computed only for the men in the sample, and s_w^2 is defined similarly for the women. Thus the standard error of $\overline{Y}_m - \overline{Y}_w$ is

$$SE(\overline{Y}_m - \overline{Y}_w) = \sqrt{\frac{s_m^2}{n_m} + \frac{s_w^2}{n_w}}. \tag{3.19}$$

For a simplified version of Equation (3.19) when Y is a Bernoulli random variable, see Exercise 3.15.

The t-statistic for testing the null hypothesis is constructed analogously to the t-statistic for testing a hypothesis about a single population mean, by subtracting the null hypothesized value of $\mu_m - \mu_w$ from the estimator $\overline{Y}_m - \overline{Y}_w$ and dividing the result by the standard error of $\overline{Y}_m - \overline{Y}_w$:

$$t = \frac{(\overline{Y}_m - \overline{Y}_w) - d_0}{SE(\overline{Y}_m - \overline{Y}_w)} \quad (t\text{-statistic for comparing two means}). \tag{3.20}$$

If both n_m and n_w are large, then this t-statistic has a standard normal distribution when the null hypothesis is true.

Because the t-statistic in Equation (3.20) has a standard normal distribution under the null hypothesis when n_m and n_w are large, the p-value of the two-sided

test is computed exactly as it was in the case of a single population. That is, the p-value is computed using Equation (3.14).

To conduct a test with a prespecified significance level, simply calculate the t-statistic in Equation (3.20) and compare it to the appropriate critical value. For example, the null hypothesis is rejected at the 5% significance level if the absolute value of the t-statistic exceeds 1.96.

If the alternative is one-sided rather than two-sided (that is, if the alternative is that $\mu_m - \mu_w > d_0$), then the test is modified as outlined in Section 3.2. The p-value is computed using Equation (3.17), and a test with a 5% significance level rejects when $t > 1.64$.

Confidence Intervals for the Difference Between Two Population Means

The method for constructing confidence intervals summarized in Section 3.3 extends to constructing a confidence interval for the difference between the means, $d = \mu_m - \mu_w$. Because the hypothesized value d_0 is rejected at the 5% level if $|t| > 1.96$, d_0 will be in the confidence set if $|t| \leq 1.96$. But $|t| \leq 1.96$ means that the estimated difference, $\overline{Y}_m - \overline{Y}_w$, is less than 1.96 standard errors away from d_0. Thus the 95% two-sided confidence interval for d consists of those values of d within $\pm$ 1.96 standard errors of $\overline{Y}_m - \overline{Y}_w$:

$$\text{95\% confidence interval for } d = \mu_m - \mu_w \text{ is}$$
$$(\overline{Y}_m - \overline{Y}_w) \pm 1.96SE(\overline{Y}_m - \overline{Y}_w). \tag{3.21}$$

With these formulas in hand, the box "The Gender Gap of Earnings of College Graduates in the United States" contains an empirical investigation of gender differences in earnings of U.S. college graduates.

3.5 Differences-of-Means Estimation of Causal Effects Using Experimental Data

Recall from Section 1.2 that a randomized controlled experiment randomly selects subjects (individuals or, more generally, entities) from a population of interest, then randomly assigns them either to a treatment group, which receives the experimental treatment, or to a control group, which does not receive the treatment. The difference between the sample means of the treatment and control groups is an estimator of the causal effect of the treatment.

The Causal Effect as a Difference of Conditional Expectations

The causal effect of a treatment is the expected effect on the outcome of interest of the treatment as measured in an ideal randomized controlled experiment. This effect can be expressed as the difference of two conditional expectations. Specifically, the **causal effect** on Y of treatment level x is the difference in the conditional expectations, $E(Y \mid X = x) - E(Y \mid X = 0)$, where $E(Y \mid X = x)$ is the expected value of Y for the treatment group (which receives treatment level $X = x$) in an ideal randomized controlled experiment and $E(Y \mid X = 0)$ is the expected value of Y for the control group (which receives treatment level $X = 0$). In the context of experiments, the causal effect is also called the **treatment effect**. If there are only two treatment levels (that is, if the treatment is binary), then we can let $X = 0$ denote the control group and $X = 1$ denote the treatment group. If the treatment is binary treatment, then the causal effect (that is, the treatment effect) is $E(Y \mid X = 1) - E(Y \mid X = 0)$ in an ideal randomized controlled experiment.

Estimation of the Causal Effect Using Differences of Means

If the treatment in a randomized controlled experiment is binary, then the causal effect can be estimated by the difference in the sample average outcomes between the treatment and control groups. The hypothesis that the treatment is ineffective is equivalent to the hypothesis that the two means are the same, which can be tested using the t-statistic for comparing two means, given in Equation (3.20). A 95% confidence interval for the difference in the means of the two groups is a 95% confidence interval for the causal effect, so a 95% confidence interval for the causal effect can be constructed using Equation (3.21).

A well-designed, well-run experiment can provide a compelling estimate of a causal effect. For this reason, randomized controlled experiments are commonly conducted in some fields, such as medicine. The box "A Way to Increase Voter Turnout" provides an example of such a randomized experiment that yielded such casual effects. In economics, however, experiments tend to be expensive, difficult to administer, and, in some cases, ethically questionable, so they are used less often. For this reason, econometricians sometimes study "natural experiments," also called quasi-experiments, in which some event unrelated to the treatment or subject characteristics has the effect of assigning different treatments to different subjects *as if* they had been part of a randomized controlled experiment.

The Gender Gap of Earnings of College Graduates in the United States

The box in Chapter 2 "The Distribution of Earnings in the United States in 2012" shows that, on average, male college graduates earn more than female college graduates. What are the recent trends in this "gender gap" in earnings? Social norms and laws governing gender discrimination in the workplace have changed substantially in the United States. Is the gender gap in earnings of college graduates stable, or has it diminished over time?

Table 3.1 gives estimates of hourly earnings for college-educated full-time workers ages 25–34 in the United States in 1992, 1996, 2000, 2004, 2008, and 2012, using data collected by the Current Population Survey. Earnings for 1992, 1996, 2000, 2004, and 2008 were adjusted for inflation by putting them in 2012 dollars using the Consumer Price Index (CPI).[1] In 2012, the average hourly earnings of the 2004 men surveyed was \$25.30, and the standard deviation of earnings for men was \$12.09. The average hourly earnings in 2012 of the 1951 women surveyed was \$21.50, and the standard deviation of earnings was \$9.99. Thus the estimate of the gender gap in earnings for 2012 is \$3.80 (= \$25.30 − \$21.50), with a standard error of \$0.35 ($= \sqrt{12.09^2/2004 + 9.99^2/1951}$). The 95% confidence interval for the gender gap in earnings in 2012 is $3.80 \pm 1.96 \times 0.35 = (\$3.11, \$4.49)$.

The results in Table 3.1 suggest four conclusions. First, the gender gap is large. An hourly gap of \$3.80 might not sound like much, but over a year it adds up to \$7600, assuming a 40-hour workweek and 50 paid weeks per year. Second, from 1992 to 2012, the estimated gender gap increased by \$0.36 per hour in real terms, from \$3.44 per hour to \$3.80 per hour;

TABLE 3.1 Trends in Hourly Earnings in the United States of Working College Graduates, Ages 25–34, 1992 to 2012, in 2012 Dollars

	Men			Women			Difference, Men vs. Women		
Year	$\bar{Y}_m$	s_m	n_m	$\bar{Y}_w$	s_w	n_w	$\bar{Y}_m - \bar{Y}_w$	$SE(\bar{Y}_m - \bar{Y}_w)$	95% Confidence Interval for d
1992	24.83	10.85	1594	21.39	8.39	1368	3.44**	0.35	2.75–4.14
1996	23.97	10.79	1380	20.26	8.48	1230	3.71**	0.38	2.97–4.46
2000	26.55	12.38	1303	22.13	9.98	1181	4.42**	0.45	3.54–5.30
2004	26.80	12.81	1894	22.43	9.99	1735	4.37**	0.38	3.63–5.12
2008	26.63	12.57	1839	22.26	10.30	1871	4.36**	0.38	3.62–5.10
2012	25.30	12.09	2004	21.50	9.99	1951	3.80**	0.35	3.11–4.49

These estimates are computed using data on all full-time workers ages 25–34 surveyed in the Current Population Survey conducted in March of the next year (for example, the data for 2012 were collected in March 2013). The difference is significantly different from zero at the **1% significance level.

(*continued*)

however, this increase is not statistically significant at the 5% significance level (Exercise 3.17). Third, the gap is large if it is measured instead in percentage terms: According to the estimates in Table 3.1, in 2012 women earned 15% less per hour than men did ($3.80/$25.30), slightly more than the gap of 14% seen in 1992 ($3.44/$24.83). Fourth, the gender gap is smaller for young college graduates (the group analyzed in Table 3.1) than it is for all college graduates (analyzed in Table 2.4): As reported in Table 2.4, the mean earnings for all college-educated women working full-time in 2012 was $25.42, while for men this mean was $32.73, which corresponds to a gender gap of 22% $[= (32.73 - 25.42)/32.73]$ among all full-time college-educated workers.

This empirical analysis documents that the "gender gap" in hourly earnings is large and has been fairly stable (or perhaps increased slightly) over the recent past. The analysis does not, however, tell us *why* this gap exists. Does it arise from gender discrimination in the labor market? Does it reflect differences in skills, experience, or education between men and women? Does it reflect differences in choice of jobs? Or is there some other cause? We return to these questions once we have in hand the tools of multiple regression analysis, the topic of Part II.

[1]Because of inflation, a dollar in 1992 was worth more than a dollar in 2012, in the sense that a dollar in 1992 could buy more goods and services than a dollar in 2012 could. Thus earnings in 1992 cannot be directly compared to earnings in 2012 without adjusting for inflation. One way to make this adjustment is to use the CPI, a measure of the price of a "market basket" of consumer goods and services constructed by the Bureau of Labor Statistics. Over the 20 years from 1992 to 2012, the price of the CPI market basket rose by 63.6%; in other words, the CPI basket of goods and services that cost $100 in 1992 cost $163.64 in 2012. To make earnings in 1992 and 2012 comparable in Table 3.1, 1992 earnings are inflated by the amount of overall CPI price inflation, that is, by multiplying 1992 earnings by 1.636 to put them into "2012 dollars."

3.6 Using the *t*-Statistic When the Sample Size Is Small

In Sections 3.2 through 3.5, the *t*-statistic is used in conjunction with critical values from the standard normal distribution for hypothesis testing and for the construction of confidence intervals. The use of the standard normal distribution is justified by the central limit theorem, which applies when the sample size is large. When the sample size is small, the standard normal distribution can provide a poor approximation to the distribution of the *t*-statistic. If, however, the population distribution is itself normally distributed, then the exact distribution (that is, the finite-sample distribution; see Section 2.6) of the *t*-statistic testing the mean of a single population is the Student *t* distribution with $n - 1$ degrees of freedom, and critical values can be taken from the Student *t* distribution.

The *t*-Statistic and the Student *t* Distribution

The t-statistic testing the mean. Consider the *t*-statistic used to test the hypothesis that the mean of Y is $\mu_{Y,0}$, using data $Y_1, \ldots, Y_n$. The formula for this statistic is

given by Equation (3.10), where the standard error of $\overline{Y}$ is given by Equation (3.8). Substitution of the latter expression into the former yields the formula for the *t*-statistic:

$$t = \frac{\overline{Y} - \mu_{Y,0}}{\sqrt{s_Y^2/n}}, \tag{3.22}$$

where s_Y^2 is given in Equation (3.7).

As discussed in Section 3.2, under general conditions the *t*-statistic has a standard normal distribution if the sample size is large and the null hypothesis is true [see Equation (3.12)]. Although the standard normal approximation to the *t*-statistic is reliable for a wide range of distributions of Y if n is large, it can be unreliable if n is small. The exact distribution of the *t*-statistic depends on the distribution of Y, and it can be very complicated. There is, however, one special case in which the exact distribution of the *t*-statistic is relatively simple: If Y is normally distributed, then the *t*-statistic in Equation (3.22) has a Student *t* distribution with $n - 1$ degrees of freedom. (The mathematics behind this result is provided in Sections 17.4 and 18.4.)

If the population distribution is normally distributed, then critical values from the Student *t* distribution can be used to perform hypothesis tests and to construct confidence intervals. As an example, consider a hypothetical problem in which $t^{act} = 2.15$ and $n = 20$ so that the degrees of freedom is $n - 1 = 19$. From Appendix Table 2, the 5% two-sided critical value for the t_{19} distribution is 2.09. Because the *t*-statistic is larger in absolute value than the critical value ($2.15 > 2.09$), the null hypothesis would be rejected at the 5% significance level against the two-sided alternative. The 95% confidence interval for μ_Y, constructed using the t_{19} distribution, would be $\overline{Y} \pm 2.09\, SE(\overline{Y})$. This confidence interval is somewhat wider than the confidence interval constructed using the standard normal critical value of 1.96.

The t-statistic testing differences of means. The *t*-statistic testing the difference of two means, given in Equation (3.20), does not have a Student *t* distribution, even if the population distribution of Y is normal. (The Student *t* distribution does not apply here because the variance estimator used to compute the standard error in Equation (3.19) does not produce a denominator in the *t*-statistic with a chi-squared distribution.)

A modified version of the differences-of-means *t*-statistic, based on a different standard error formula—the "pooled" standard error formula—has an exact Student *t* distribution when Y is normally distributed; however, the pooled

standard error formula applies only in the special case that the two groups have the same variance or that each group has the same number of observations (Exercise 3.21). Adopt the notation of Equation (3.19) so that the two groups are denoted as m and w. The pooled variance estimator is

$$s^2_{pooled} = \frac{1}{n_m + n_w - 2}\left[\sum_{\substack{i=1 \\ \text{group } m}}^{n_m} (Y_i - \overline{Y}_m)^2 + \sum_{\substack{i=1 \\ \text{group } w}}^{n_w} (Y_i - \overline{Y}_m)^2\right], \quad (3.23)$$

where the first summation is for the observations in group m and the second summation is for the observations in group w. The pooled standard error of the difference in means is $SE_{pooled}(\overline{Y}_m - \overline{Y}_w) = s_{pooled} \times \sqrt{1/n_m + 1/n_w}$, and the pooled t-statistic is computed using Equation (3.20), where the standard error is the pooled standard error, $SE_{pooled}(\overline{Y}_m - \overline{Y}_w)$.

If the population distribution of Y in group m is $N(\mu_m, \sigma^2_m)$, if the population distribution of Y in group w is $N(\mu_w, \sigma^2_w)$, *and* if the two group variances are the same (that is, $\sigma^2_m = \sigma^2_w$), then under the null hypothesis the t-statistic computed using the pooled standard error has a Student t distribution with $n_m + n_w - 2$ degrees of freedom.

The drawback of using the pooled variance estimator s^2_{pooled} is that it applies only if the two population variances are the same (assuming $n_m \neq n_w$). If the population variances are different, the pooled variance estimator is biased and inconsistent. If the population variances are different but the pooled variance formula is used, the null distribution of the pooled t-statistic is not a Student t distribution, even if the data are normally distributed; in fact, it does not even have a standard normal distribution in large samples. Therefore, the pooled standard error and the pooled t-statistic should not be used unless you have a good reason to believe that the population variances are the same.

Use of the Student *t* Distribution in Practice

For the problem of testing the mean of Y, the Student t distribution is applicable if the underlying population distribution of Y is normal. For economic variables, however, normal distributions are the exception (for example, see the boxes in Chapter 2 "The Distribution of Earnings in the United States in 2012" and "A Bad Day on Wall Street"). Even if the underlying data are not normally distributed, the normal approximation to the distribution of the t-statistic is valid if the sample size is large. Therefore, inferences—hypothesis tests and confidence intervals—about the mean of a distribution should be based on the large-sample normal approximation.

A Way to Increase Voter Turnout

Apathy among citizens toward political participation, especially in voting, has been noted in the United Kingdom and other democratic countries. This kind of behavior is generally seen in economies where people have greater mobility, maintain an intensive work culture, and work for private corporate entities. Apart from these there could be other dominant factors like politicians failing to keep their promises, inappropriately using public funds that have had a negative impact on the citizens' willingness to participate in elections.

A study was conducted in a Manchester constituency in the United Kingdom, in 2005, during the campaign period before the general election. The constituency's voter turnout rate in the 2001 general election had been 48.6%, while the national average had been 59.4%. Thus, voter participation in this constituency was far below the national average. For the experiment, three groups (two treatment groups and one control group) were randomly selected out of the registered voters from whom landline numbers could be obtained. One of the treatment groups was exposed to strong canvassing in the form of telephone calls, and the other treatment group was exposed to strong canvassing in the form of door-to-door visits. No contacts were made with the control group. The callers and the door-to-door canvassers were given instructions to ask respondents three questions, namely, whether the respondents thought voting is important, whether the respondents intended to vote, and whether they would vote by post. The conversations were informal and the main objective of this exercise was to persuade citizens to vote, by focusing on the importance of voting. The callers and canvassers were also advised to respond to any concerns of the voters regarding the voting process.

The researchers got interesting results from the elections. The participation rate was 55.1% in the group which was exposed to canvassing. The participation rate for the treatment group which was treated with telephone calls was 55%. Both these rates had a difference with the control group, which was not exposed to any experiment. Further calculations using suitable methodologies gave estimates of the effects of canvassing and telephone calls. 6.7% and 7.3% were the estimates of the two. The overall experiment was a success as the two interventions done on the two treatments groups by a non-partisan source had impacts that were statistically significant.

This exercise showed that citizens can be nudged to participate in elections by creating awareness through personal contacts. In yet another democracy, India, the 2014 general election saw a record voter turnout. A top Election Commission official has said that the Election Commission's efforts to increase voters' awareness and their registration has helped the process.

Sources: 1. http://www.bloomsburycollections.com/book/nudge-nudge-think-think-experimenting-with-ways-to-change-civic-behaviour/
2. Lok Sabha polls 2014: Country records highest voter turnout since independence, The Economic Times

When comparing two means, any economic reason for two groups having different means typically implies that the two groups also could have different variances. Accordingly, the pooled standard error formula is inappropriate, and the correct standard error formula, which allows for different group variances, is as given in Equation (3.19). Even if the population distributions are normal, the t-statistic computed using the standard error formula in Equation (3.19) does not have a Student t distribution. In practice, therefore, inferences about differences in means should be based on Equation (3.19), used in conjunction with the large-sample standard normal approximation.

Even though the Student t distribution is rarely applicable in economics, some software uses the Student t distribution to compute p-values and confidence intervals. In practice, this does not pose a problem because the difference between the Student t distribution and the standard normal distribution is negligible if the sample size is large. For $n > 15$, the difference in the p-values computed using the Student t and standard normal distributions never exceeds 0.01; for $n > 80$, the difference never exceeds 0.002. In most modern applications, and in all applications in this textbook, the sample sizes are in the hundreds or thousands, large enough for the difference between the Student t distribution and the standard normal distribution to be negligible.

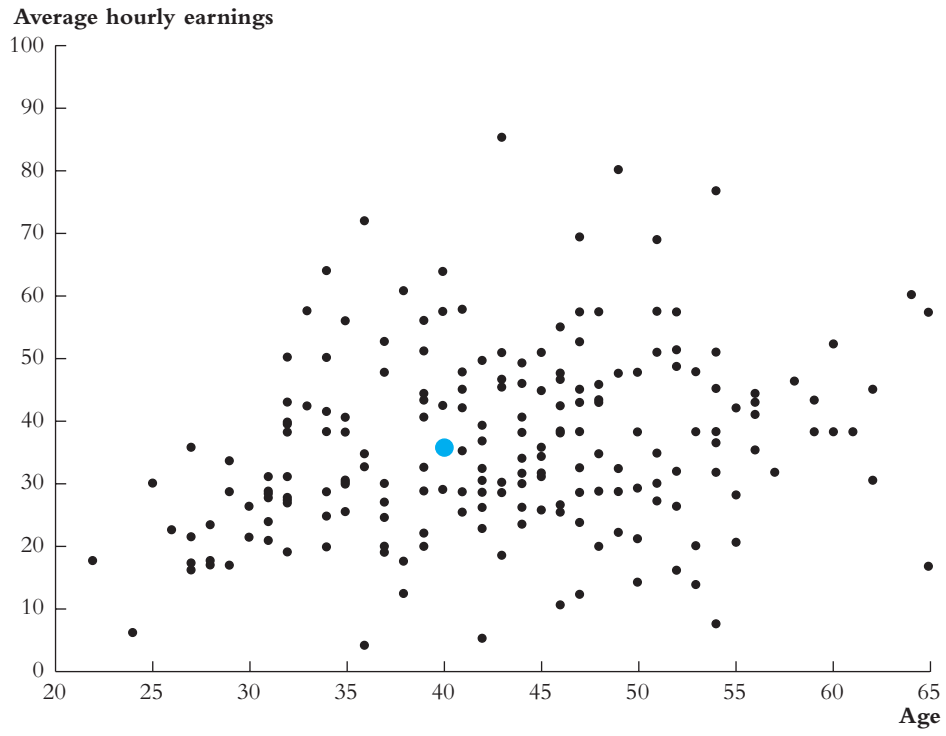
3.7 Scatterplots, the Sample Covariance, and the Sample Correlation

What is the relationship between age and earnings? This question, like many others, relates one variable, X (age), to another, Y (earnings). This section reviews three ways to summarize the relationship between variables: the scatterplot, the sample covariance, and the sample correlation coefficient.

Scatterplots

A **scatterplot** is a plot of n observations on X_i and Y_i, in which each observation is represented by the point (X_i, Y_i). For example, Figure 3.2 is a scatterplot of age (X) and hourly earnings (Y) for a sample of 200 managers in the information industry from the March 2009 CPS. Each dot in Figure 3.2 corresponds to an (X, Y) pair for one of the observations. For example, one of the workers in this sample is 40 years old and earns $35.78 per hour; this worker's age and earnings are indicated by the highlighted dot in Figure 3.2. The scatterplot shows a positive

FIGURE 3.2 Scatterplot of Average Hourly Earnings vs. Age



Each point in the plot represents the age and average earnings of one of the 200 workers in the sample. The highlighted dot corresponds to a 40-year-old worker who earns $35.78 per hour. The data are for computer and information systems managers from the March 2009 CPS.

relationship between age and earnings in this sample: Older workers tend to earn more than younger workers. This relationship is not exact, however, and earnings could not be predicted perfectly using only a person's age.

Sample Covariance and Correlation

The covariance and correlation were introduced in Section 2.3 as two properties of the joint probability distribution of the random variables X and Y. Because the population distribution is unknown, in practice we do not know the population covariance or correlation. The population covariance and correlation can, however, be estimated by taking a random sample of n members of the population and collecting the data (X_i, Y_i), $i = 1, \ldots, n$.

The sample covariance and correlation are estimators of the population covariance and correlation. Like the estimators discussed previously in this chapter, they are computed by replacing a population mean (the expectation) with a sample mean. The **sample covariance**, denoted s_{XY}, is

$$s_{XY} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y}). \tag{3.24}$$

Like the sample variance, the average in Equation (3.24) is computed by dividing by $n - 1$ instead of n; here, too, this difference stems from using $\overline{X}$ and $\overline{Y}$ to estimate the respective population means. When n is large, it makes little difference whether division is by n or $n - 1$.

The **sample correlation coefficient**, or **sample correlation**, is denoted r_{XY} and is the ratio of the sample covariance to the sample standard deviations:

$$r_{XY} = \frac{s_{XY}}{s_X s_Y}. \tag{3.25}$$

The sample correlation measures the strength of the linear association between X and Y in a sample of n observations. Like the population correlation, the sample correlation is unitless and lies between –1 and 1: $|r_{XY}| \leq 1$.

The sample correlation equals 1 if $X_i = Y_i$ for all i and equals –1 if $X_i = -Y_i$ for all i. More generally, the correlation is ±1 if the scatterplot is a straight line. If the line slopes upward, then there is a positive relationship between X and Y and the correlation is 1. If the line slopes down, then there is a negative relationship and the correlation is –1. The closer the scatterplot is to a straight line, the closer is the correlation to ±1. A high correlation coefficient does not necessarily mean that the line has a steep slope; rather, it means that the points in the scatterplot fall very close to a straight line.

Consistency of the sample covariance and correlation. Like the sample variance, the sample covariance is consistent. That is,

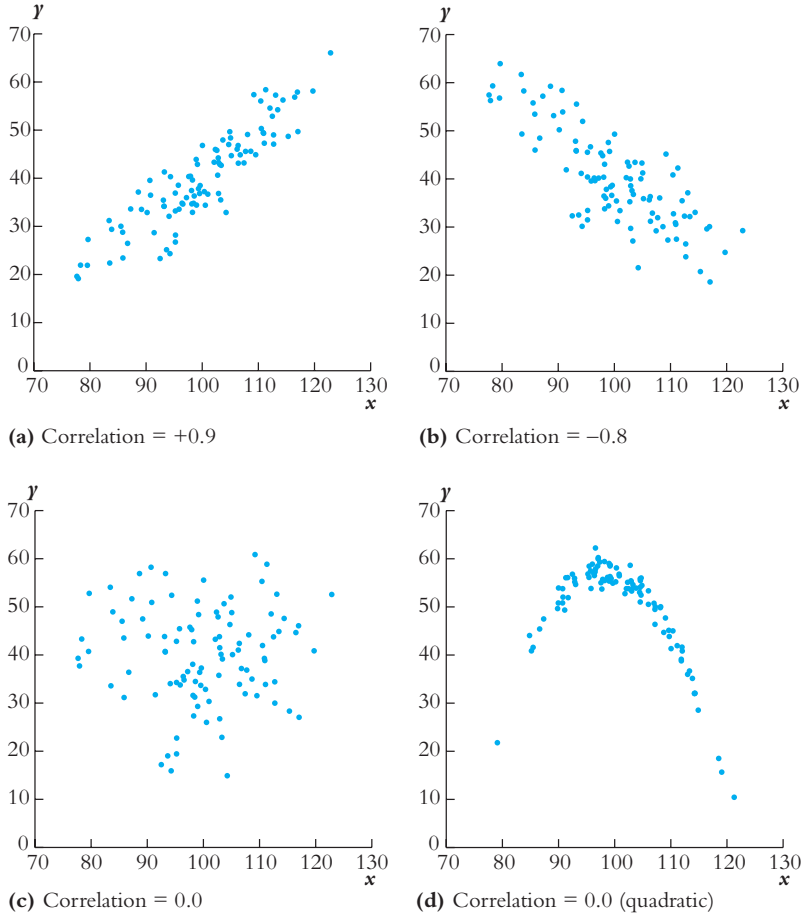
$$s_{XY} \xrightarrow{p} \sigma_{XY}. \tag{3.26}$$

In other words, in large samples the sample covariance is close to the population covariance with high probability.

The proof of the result in Equation (3.26) under the assumption that (X_i, Y_i) are i.i.d. and that X_i and Y_i have finite fourth moments is similar to the proof in Appendix 3.3 that the sample covariance is consistent and is left as an exercise (Exercise 3.20).

FIGURE 3.3 Scatterplots for Four Hypothetical Data Sets

The scatterplots in Figures 3.3a and 3.3b show strong linear relationships between *X* and *Y*. In Figure 3.3c, *X* is independent of *Y* and the two variables are uncorrelated. In Figure 3.3d, the two variables also are uncorrelated even though they are related nonlinearly.



(a) Correlation = +0.9

(b) Correlation = −0.8

(c) Correlation = 0.0

(d) Correlation = 0.0 (quadratic)

Because the sample variance and sample covariance are consistent, the sample correlation coefficient is consistent, that is, $r_{XY} \xrightarrow{p} \text{corr}(X_i, Y_i)$.

Example. As an example, consider the data on age and earnings in Figure 3.2. For these 200 workers, the sample standard deviation of age is $s_A = 9.07$ years and the sample standard deviation of earnings is $s_E = \$14.37$ per hour. The sample covariance between age and earnings is $s_{AE} = 33.16$ (the units are years $\times$ dollars per hour, not readily interpretable). Thus the sample correlation coefficient is $r_{AE} = 33.16/(9.07 \times 14.37) = 0.25$ or 25%. The correlation of 0.25 means that there

is a positive relationship between age and earnings, but as is evident in the scatterplot, this relationship is far from perfect.

To verify that the correlation does not depend on the units of measurement, suppose that earnings had been reported in cents, in which case the sample standard deviations of earnings is 1437¢ per hour and the covariance between age and earnings is 3316 (units are years × cents per hour); then the correlation is $3316/(9.07 \times 1437) = 0.25$ or 25%.

Figure 3.3 gives additional examples of scatterplots and correlation. Figure 3.3a shows a strong positive linear relationship between these variables, and the sample correlation is 0.9.

Figure 3.3b shows a strong negative relationship with a sample correlation of –0.8. Figure 3.3c shows a scatterplot with no evident relationship, and the sample correlation is zero. Figure 3.3d shows a clear relationship: As X increases, Y initially increases, but then decreases. Despite this discernable relationship between X and Y, the sample correlation is zero; the reason is that, for these data, small values of Y are associated with *both* large and small values of X.

This final example emphasizes an important point: The correlation coefficient is a measure of *linear* association. There is a relationship in Figure 3.3d, but it is not linear.

Summary

1. The sample average, $\overline{Y}$, is an estimator of the population mean, μ_Y. When $Y_1, \ldots, Y_n$ are i.i.d.,
   a. the sampling distribution of $\overline{Y}$ has mean μ_Y and variance $\sigma^2_{\overline{Y}} = \sigma^2_Y/n$;
   b. $\overline{Y}$ is unbiased;
   c. by the law of large numbers, $\overline{Y}$ is consistent; and
   d. by the central limit theorem, $\overline{Y}$ has an approximately normal sampling distribution when the sample size is large.
2. The t-statistic is used to test the null hypothesis that the population mean takes on a particular value. If n is large, the t-statistic has a standard normal sampling distribution when the null hypothesis is true.
3. The t-statistic can be used to calculate the p-value associated with the null hypothesis. A small p-value is evidence that the null hypothesis is false.
4. A 95% confidence interval for μ_Y is an interval constructed so that it contains the true value of μ_Y in 95% of all possible samples.
5. Hypothesis tests and confidence intervals for the difference in the means of two populations are conceptually similar to tests and intervals for the mean of a single population.

6. The sample correlation coefficient is an estimator of the population correlation coefficient and measures the linear relationship between two variables—that is, how well their scatterplot is approximated by a straight line.

Key Terms

estimator (113)
estimate (113)
bias, consistency, and efficiency (114)
BLUE (Best Linear Unbiased Estimator) (115)
least squares estimator (116)
hypothesis tests (117)
null hypothesis (117)
alternative hypothesis (117)
two-sided alternative hypothesis (117)
p-value (significance probability) (118)
sample variance (120)
sample standard deviation (120)
degrees of freedom (121)
standard error of $\overline{Y}$ (121)
t-statistic (*t*-ratio) (122)
test statistic (122)
type I error (124)
type II error (124)
significance level (124)
critical value (124)
rejection region (124)
acceptance region (124)
size of a test (124)
power of a test (124)
one-sided alternative hypothesis (126)
confidence set (126)
confidence level (126)
confidence interval (126)
coverage probability (128)
test for the difference between two means (128)
causal effect (131)
treatment effect (131)
scatterplot (137)
sample covariance (139)
sample correlation coefficient (sample correlation) (139)

Review the Concepts

3.1 What is the difference between an unbiased estimator and a consistent estimator?

3.2 What is meant by the efficiency of an estimator? Which estimator is known as BLUE?

3.3 A population distribution has a mean of 15 and a variance of 10. Determine the mean and variance of $\overline{Y}$ from an i.i.d. sample from this population for (a) $n = 50$; (b) $n = 500$; and (c) $n = 5000$. Relate your answers to the law of large numbers.

3.4 Differentiate between standard error and standard deviation. How is the standard error of a sample mean calculated?

3.5 What is the difference between a null hypothesis and an alternative hypothesis? Among size, significance level, and power? Between a one-sided alternative hypothesis and a two-sided alternative hypothesis?

3.6 Why does a confidence interval contain more information than the result of a single hypothesis test?

3.7 What is a scatterplot? What statistical features of a dataset can be represented using a scatterplot diagram?

3.8 Sketch a hypothetical scatterplot for a sample of size 10 for two random variables with a population correlation of (a) 1.0; (b) –1.0; (c) 0.9; (d) –0.5; (e) 0.0.

Exercises

3.1 In a population, $\mu_Y = 75$ and $\sigma_Y^2 = 45$. Use the central limit theorem to answer the following questions:

- **a.** In a random sample of size $n = 50$, find $\Pr(\overline{Y} < 73)$.
- **b.** In a random sample of size $n = 90$, find $\Pr(76 < \overline{Y} < 78)$.
- **c.** In a random sample of size $n = 120$, find $\Pr(\overline{Y} > 68)$.

3.2 Let Y be a Bernoulli random variable with success probability $\Pr(Y = 1) = p$, and let $Y_1, \ldots, Y_n$ be i.i.d. draws from this distribution. Let $\hat{p}$ be the fraction of successes (1s) in this sample.

a. Show that $\hat{p} = \overline{Y}$.

b. Show that $\hat{p}$ is an unbiased estimator of p.

c. Show that $\text{var}(\hat{p}) = p(1 - p)/n$.

3.3 In a survey of 400 likely voters, 215 responded that they would vote for the incumbent, and 185 responded that they would vote for the challenger. Let p denote the fraction of all likely voters who preferred the incumbent at the time of the survey, and let $\hat{p}$ be the fraction of survey respondents who preferred the incumbent.

a. Use the survey results to estimate p.

b. Use the estimator of the variance of $\hat{p}$, $\hat{p}(1 - \hat{p})/n$, to calculate the standard error of your estimator.

c. What is the p-value for the test $H_0: p = 0.5$ vs. $H_1: p \neq 0.5$?

d. What is the p-value for the test $H_0: p = 0.5$ vs. $H_1: p > 0.5$?

e. Why do the results from (c) and (d) differ?

f. Did the survey contain statistically significant evidence that the incumbent was ahead of the challenger at the time of the survey? Explain.

3.4 Using the data in Exercise 3.3:

a. Construct a 95% confidence interval for p.

b. Construct a 99% confidence interval for p.

c. Why is the interval in (b) wider than the interval in (a)?

a. Without doing any additional calculations, test the hypothesis $H_0: p = 0.50$ vs. $H_1: p \neq 0.50$ at the 5% significance level.

3.5 A survey is conducted using 1000 registered voters, who are asked to choose between candidate A and candidate B. Let p denote the fraction of voters in the population who prefer candidate A, and let $\hat{p}$ denote the fraction of voters who prefer Candidate B.

a. You are interested in the competing hypotheses $H_0: p = 0.4$ vs. $H_1: p \neq 0.4$. Suppose that you decide to reject H_0 if $|\hat{p} - 0.4| > 0.01$. Then

i. What is the size of this test?

ii. Compute the power of this test if $p = 0.45$.

b. In the survey, $\hat{p} = 0.44$.

i. Test $H_0: p = 0.4$ vs. $H_1: p \neq 0.4$ using a 10% significance level.

ii. Test $H_0: p = 0.4$ vs. $H_1: p < 0.4$ using a 10% significance level.

iii. Construct a 90% confidence interval for p.

iv. Construct a 99% confidence interval for p.

v. Construct a 60% confidence interval for p.

c. Suppose the survey is carried out 30 times, using independently selected voters in each survey. For each of these 30 surveys, a 90% confidence interval for p is constructed.

i. What is the probability that the true value of p is contained in all 30 of these confidence intervals?

ii. How many of these confidence intervals do you expect to contain the true value of p?

d. In survey jargon, the "margin of error" is $1.96 \times SE(\hat{p})$; that means it is half the length of 95% confidence interval. Suppose you want to design a survey that has a margin of error of at most 0.5%, i.e., $Pr(|\hat{p} - p| > 0.005) \leq 0.05$. How large should n be if the survey uses simple random sampling?

3.6 Let $Y_1, \ldots, Y_n$ be i.i.d. draws from a distribution with mean μ. A test of $H_0: \mu = 10$ vs. $H_1: \mu \neq 10$ using the usual t-statistic yields a p-value of 0.07.

a. Does the 90% confidence interval contain $\mu = 10$? Explain.

b. Can you determine if $\mu = 8$ is contained in the 95% confidence interval? Briefly explain your answers.

3.7 In a given population, 50% of the likely voters are women. A survey using a simple random sample of 1000 landline telephone numbers finds the percentage of female voters to be 55%. Is there evidence that this survey is biased? Explain.

3.8 A new version of the SAT is given to 1500 randomly selected high school seniors. The sample mean test score is 1230, and the sample standard deviation is 145. Construct a 95% confidence interval for the population mean test score for high school seniors.

3.9 Suppose a light bulb manufacturing plant produces bulbs with a mean life of 1000 hours, and a standard deviation of 100 hours. An inventor claims to have developed an improved process that produces bulbs with a longer mean life and the same standard deviation. The plant manager randomly

selects 50 bulbs produced by the process. She says that she will believe the inventor's claim if the sample mean life of the bulbs is greater than 1100 hours; otherwise, she will conclude that the new process is no better than the old one. Let μ denote the mean of the new process. Consider the null and alternative hypotheses H_0: $\mu = 1000$ vs. H_1: $\mu > 1000$.

a. What is the size of the plant manager's testing procedure?

b. Suppose the new process is, in fact, better and has a mean bulb life of 1150 hours. What is the power of the plant manager's testing procedure?

c. What testing procedure should the plant manager use if she wants the size of her test to be 1%?

3.10 Suppose a new standardized test is given to 150 randomly selected third-grade students in New Jersey. The sample average score Y on the test is 42 points, and the sample standard deviation, s_Y, is 6 points.

a. The authors plan to administer the test to all third-grade students in New Jersey. Construct a 99% confidence interval for the mean score of all New Jersey third graders.

b. Suppose the same test is given to 300 randomly selected third graders from Iowa, producing a sample average of 48 points and sample standard deviation of 10 points. Construct a 95% confidence interval for the difference in mean scores between Iowa and New Jersey.

c. Can you conclude with a high degree of confidence that the population means for Iowa and New Jersey students are different? (What is the standard error of the difference in the two sample means? What is the p-value of the test of no difference in means versus some difference?)

3.11 Consider the estimator $\widetilde{Y}$, defined in Equation (3.1). Show that (a) $E(\widetilde{Y}) = \mu_Y$ and (b) $\text{var}(\widetilde{Y}) = 1.25\sigma_Y^2/n$.

3.12 To investigate possible gender discrimination in a firm, a sample of 120 men and 150 women with similar job descriptions are selected at random. A summary of the resulting monthly salaries follows:

	Average Salary	Standard Deviation	n
Men	\$8,200	\$450	120
Women	\$7,900	\$520	150

a. What do the data suggest about wage differences in the firm? Do they represent statistically significant evidence that average wages of men and women are different? (To answer this question, first state the null and alternative hypotheses; second, compute the relevant t-statistic; third, compute the p-value associated with the t-statistic; and finally, use the p-value to answer the question.)

b. Does the data suggest that the firm is guilty of gender discrimination in its compensation policies? Explain.

3.13 Data on fifth-grade test scores (reading and mathematics) for 400 school districts in California yield $\overline{Y} = 712.1$ and standard deviation $s_Y = 23.2$.

a. Construct a 90% confidence interval for the mean test score in the population.

b. When the districts were divided into districts with small classes (< 20 students per teacher) and large classes (≥ 20 students per teacher), the following results were found:

Class Size	Average Score	Standard Deviation	n
Small	721.8	24.4	150
Large	710.9	20.6	250

Is there statistically significant evidence that the districts with smaller classes have higher average test scores? Explain.

3.14 Values of height in inches (X) and weight in pounds (Y) are recorded from a sample of 200 male college students. The resulting summary statistics are $\overline{X} = 71.2$ in, $\overline{Y} = 164$ lb., $s_X = 1.9$ in, $s_Y = 16.4$ lb., $s_{XY} = 22.54$ in. $\times$ lb., and $r_{XY} = 0.8$. Convert these statistics to the metric system (meters and kilograms).

3.15 Let Y_a and Y_b denote Bernoulli random variables from two different populations, denoted a and b. Suppose that $E(Y_a) = p_a$ and $E(Y_b) = p_b$. A random sample of size n_a is chosen from population a, with sample average denoted $\hat{p}_a$, and a random sample of size n_b is chosen from population b, with sample average denoted $\hat{p}_b$. Suppose the sample from population a is independent of the sample from population b.

a. Show that $E(\hat{p}_a) = p_a$ and $\text{var}(\hat{p}_a) = p_a(1 - p_a)/n_a$. Show that $E(\hat{p}_b) = p_b$ and $\text{var}(\hat{p}_b) = p_b(1 - p_b)/n_b$.

b. Show that $\text{var}(\hat{p}_a - \hat{p}_b) = \frac{p_a(1 - p_a)}{n_a} + \frac{p_b(1 - p_b)}{n_b}$. (*Hint:* Remember that the samples are independent.)

c. Suppose that n_a and n_b are large. Show that a 95% confidence interval for $p_a - p_b$ is given by $(\hat{p}_a - \hat{p}_b) \pm 1.96\sqrt{\frac{\hat{p}_a(1 - \hat{p}_a)}{n_a} + \frac{\hat{p}_b(1 - \hat{p}_b)}{n_b}}$. How would you construct a 90% confidence interval for $p_a - p_b$?

3.16 Grades on a standardized test are known to have a mean of 500 for students in the United States. The test is administered to 600 randomly selected students in Florida; in this sample, the mean is 508, and the standard deviation (s) is 75.

a. Construct a 95% confidence interval for the average test score for students in Florida.

b. Is there statistically significant evidence that students in Florida perform differently from other students in the United States?

c. Another 500 students are selected at random from Florida. They are given a 3-hour preparation course before the test is administered. Their average test score is 514, with a standard deviation of 65.

i. Construct a 95% confidence interval for the change in average test score associated with the prep course.

ii. Is there statistically significant evidence that the prep course helped?

d. The original 600 students are given the prep course and are then asked to take the test a second time. The average change in their test scores is 7 points, and the standard deviation of the change is 40 points.

i. Construct a 95% confidence interval for the change in average test scores.

ii. Is there statistically significant evidence that students will perform better on their second attempt, after taking the prep course?

iii. Students may have performed better in their second attempt because of the prep course or because they gained test-taking experience in their first attempt. Describe an experiment that would quantify these two effects.

3.17 Read the box "The Gender Gap of Earnings of College Graduates in the United States" in Section 3.5.

a. Construct a 95% confidence interval for the change in men's average hourly earnings between 1992 and 2012.

b. Construct a 95% confidence interval for the change in women's average hourly earnings between 1992 and 2012.

c. Construct a 95% confidence interval for the change in the gender gap in average hourly earnings between 1992 and 2012. (*Hint:* $\overline{Y}_{m,1992} - \overline{Y}_{w,1992}$ is independent of $\overline{Y}_{m,2012} - \overline{Y}_{w,2012}$.)

3.18 This exercise shows that the sample variance is an unbiased estimator of the population variance when $Y_1, \ldots, Y_n$ are i.i.d. with mean μ_Y and variance σ_Y^2.

a. Use Equation (2.31) to show that $E[(Y_i - \overline{Y})^2] = \text{var}(Y_i) - 2\text{cov}(Y_i, \overline{Y}) + \text{var}(\overline{Y})$.

b. Use Equation (2.33) to show that $\text{cov}(\overline{Y}, Y_i) = \sigma_Y^2/n$.

c. Use the results in (a) and (b) to show that $E(s_Y^2) = \sigma_Y^2$.

3.19 **a.** $\overline{Y}$ is an unbiased estimator of μ_Y. Is $\overline{Y}^2$ an unbiased estimator of μ_Y^2?

b. $\overline{Y}$ is a consistent estimator of μ_Y. Is $\overline{Y}^2$ a consistent estimator of μ_Y^2?

3.20 Suppose that (X_i, Y_i) are i.i.d. with finite fourth moments. Prove that the sample covariance is a consistent estimator of the population covariance, that is, $s_{XY} \xrightarrow{p} \sigma_{XY}$, where s_{XY} is defined in Equation (3.24). (*Hint:* Use the strategy of Appendix 3.3.)

3.21 Show that the pooled standard error $[SE_{pooled}(\overline{Y}_m - \overline{Y}_w)]$ given following Equation (3.23) equals the usual standard error for the difference in means in Equation (3.19) when the two group sizes are the same ($n_m = n_w$).

Empirical Exercises

E3.1 On the text website, **www.pearsonglobaleditions.com/Stock_Watson**, you will find the data file **CPS92_12**, which contains an extended version of the data set used in Table 3.1 of the text for the years 1992 and 2012. It contains data on full-time workers, ages 25–34, with a high school diploma or B.A./B.S. as their highest degree. A detailed description is given in

CPS92_12_Description, available on the website. Use these data to answer the following questions.

a. i. Compute the sample mean for average hourly earnings (*AHE*) in 1992 and 2012.

ii. Compute the sample standard deviation for *AHE* in 1992 and 2012.

iii. Construct a 95% confidence interval for the population means of *AHE* in 1992 and 2012.

iv. Construct a 95% confidence interval for the change in the population mean of *AHE* between 1992 and 2012.

b. In 2012, the value of the Consumer Price Index (CPI) was 229.6. In 1992, the value of the CPI was 140.3. Repeat (a) but use *AHE* measured in real 2012 dollars ($2012); that is, adjust the 1992 data for the price inflation that occurred between 1992 and 2012.

c. If you were interested in the change in workers' purchasing power from 1992 to 2012, would you use the results from (a) or (b)? Explain.

d. Using the data for 2012:

i. Construct a 95% confidence interval for the mean of *AHE* for high school graduates.

ii. Construct a 95% confidence interval for the mean of *AHE* for workers with a college degree.

iii. Construct a 95% confidence interval for the difference between the two means.

e. Repeat (d) using the 1992 data expressed in $2012.

f. Using appropriate estimates, confidence intervals, and test statistics, answer the following questions:

i. Did real (inflation-adjusted) wages of high school graduates increase from 1992 to 2012?

ii. Did real wages of college graduates increase?

iii. Did the gap between earnings of college and high school graduates increase? Explain.

g. Table 3.1 presents information on the gender gap for college graduates. Prepare a similar table for high school graduates, using the 1992 and 2012 data. Are there any notable differences between the results for high school and college graduates?

E3.2 A consumer is given the chance to buy a baseball card for \$1, but he declines the trade. If the consumer is now given the baseball card, will he be willing to sell it for \$1? Standard consumer theory suggests yes, but behavioral economists have found that "ownership" tends to increase the value of goods to consumers. That is, the consumer may hold out for some amount more than \$1 (for example, \$1.20) when selling the card, even though he was willing to pay only some amount less than \$1 (for example, \$0.88) when buying it. Behavioral economists call this phenomenon the "endowment effect." John List investigated the endowment effect in a randomized experiment involving sports memorabilia traders at a sports-card show. Traders were randomly given one of two sports collectibles, say good A or good B, that had approximately equal market value.[1] Those receiving good A were then given the option of trading good A for good B with the experimenter; those receiving good B were given the option of trading good B for good A with the experimenter. Data from the experiment and a detailed description can be found on the textbook website, **www.pearsonglobaleditions.com/Stock_Watson**, in the files **Sportscards** and **Sportscards_Description**.[2]

a. i. Suppose that, absent any endowment effect, all the subjects prefer good A to good B. What fraction of the experiment's subjects would you expect to trade the good that they were given for the other good? (*Hint:* Because of random assignment of the two treatments, approximately 50% of the subjects received good A and 50% received good B.)

ii. Suppose that, absent any endowment effect, 50% of the subjects prefer good A to good B, and the other 50% prefer good B to good A. What fraction of the subjects would you expect to trade the good that they were given for the other good?

iii. Suppose that, absent any endowment effect, X% of the subjects prefer good A to good B, and the other $(100 - X)$% prefer good B to good A. Show that you would expect 50% of the subjects to trade the good that they were given for the other good.

---

[1]Good A was a ticket stub from the game in which Cal Ripken, Jr., set the record for consecutive games played, and good B was a souvenir from the game in which Nolan Ryan won his 300th game.

[2]These data were provided by Professor John List of the University of Chicago and were used in his paper "Does Market Experience Eliminate Market Anomalies," *Quarterly Journal of Economics*, 2003, 118(1): 41–71.

b. Using the sports-card data, what fraction of the subjects traded the good they were given? Is the fraction significantly different from 50%? Is there evidence of an endowment effect? (*Hint:* Review Exercises 3.2 and 3.3)

c. Some have argued that the endowment effect may be present, but that it is likely to disappear as traders gain more trading experience. Half of the experimental subjects were dealers, and the other half were nondealers. Dealers have more experience than nondealers. Repeat (b) for dealers and nondealers. Is there a significant difference in their behavior? Is the evidence consistent with the hypothesis that the endowment effect disappears as traders gain more experience? (*Hint:* Review Exercise 3.15).

APPENDIX

3.1 The U.S. Current Population Survey

Each month, the U.S. Census Bureau and the U.S. Bureau of Labor Statistics conduct the Current Population Survey (CPS), which provides data on labor force characteristics of the population, including the levels of employment, unemployment, and earnings. Approximately 60,000 U.S. households are surveyed each month. The sample is chosen by randomly selecting addresses from a database of addresses from the most recent decennial census augmented with data on new housing units constructed after the last census. The exact random sampling scheme is rather complicated (first, small geographical areas are randomly selected, then housing units within these areas are randomly selected); details can be found in the *Handbook of Labor Statistics* and on the Bureau of Labor Statistics website (www.bls.gov).

The survey conducted each March is more detailed than in other months and asks questions about earnings during the previous year. The statistics in Tables 2.4 and 3.1 were computed using the March surveys. The CPS earnings data are for full-time workers, defined to be somebody employed more than 35 hours per week for at least 48 weeks in the previous year.

APPENDIX

3.2 Two Proofs That $\overline{Y}$ Is the Least Squares Estimator of μ_Y

This appendix provides two proofs, one using calculus and one not, that $\overline{Y}$ minimizes the sum of squared prediction mistakes in Equation (3.2)—that is, that $\overline{Y}$ is the least squares estimator of $E(Y)$.

Calculus Proof

To minimize the sum of squared prediction mistakes, take its derivative and set it to zero:

$$\frac{d}{dm}\sum_{i=1}^{n}(Y_i - m)^2 = -2\sum_{i=1}^{n}(Y_i - m) = -2\sum_{i=1}^{n}Y_i + 2nm = 0. \qquad (3.27)$$

Solving for the final equation for m shows that $\sum_{i=1}^{n}(Y_i - m)^2$ is minimized when $m = \overline{Y}$.

Noncalculus Proof

The strategy is to show that the difference between the least squares estimator and $\overline{Y}$ must be zero, from which it follows that $\overline{Y}$ is the least squares estimator. Let $d = \overline{Y} - m$, so that $m = \overline{Y} - d$. Then $(Y_i - m)^2 = (Y_i - [\overline{Y} - d])^2 = ([Y_i - \overline{Y}] + d)^2 = (Y_i - \overline{Y})^2 + 2d(Y_i - \overline{Y}) + d^2$. Thus the sum of squared prediction mistakes [Equation (3.2)] is

$$\sum_{i=1}^{n}(Y_i - m)^2 = \sum_{i=1}^{n}(Y_i - \overline{Y})^2 + 2d\sum_{i=1}^{n}(Y_i - \overline{Y}) + nd^2 = \sum_{i=1}^{n}(Y_i - \overline{Y})^2 + nd^2, \qquad (3.28)$$

where the second equality uses the fact that $\sum_{i=1}^{n}(Y_i - \overline{Y}) = 0$. Because both terms in the final line of Equation (3.28) are nonnegative and because the first term does not depend on d, $\sum_{i=1}^{n}(Y_i - m)^2$ is minimized by choosing d to make the second term, nd^2, as small as possible. This is done by setting $d = 0$—that is, by setting $m = \overline{Y}$—so that $\overline{Y}$ is the least squares estimator of $E(Y)$.

APPENDIX

3.3 A Proof That the Sample Variance Is Consistent

This appendix uses the law of large numbers to prove that the sample variance s_Y^2 is a consistent estimator of the population variance σ_Y^2, as stated in Equation (3.9), when $Y_1, \ldots, Y_n$ are i.i.d. and $E(Y_i^4) < \infty$.

First, consider a version of the sample variance that uses n instead of $n - 1$ as a divisor:

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 &= \frac{1}{n}\sum_{i=1}^{n}Y_i^2 - 2\overline{Y}\frac{1}{n}\sum_{i=1}^{n}Y_i + \overline{Y}^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}Y_i^2 - \overline{Y}^2 \\ &\xrightarrow{p} (\sigma_Y^2 + \mu_Y^2) - \mu_Y^2 \\ &= \sigma_Y^2, \end{aligned} \tag{3.29}$$

where the first equality uses $(Y_i - \overline{Y})^2 = Y_i^2 - 2\overline{Y}Y_i + \overline{Y}^2$, and the second uses $\frac{1}{n}\sum_{i=1}^{n}Y_i = \overline{Y}$. The convergence in the third line follows from (i) applying the law of large numbers to $\frac{1}{n}\sum_{i=1}^{n}Y_i^2 \xrightarrow{p} E(Y^2)$ (which follows because Y_i^2 are i.i.d. and have finite variance because $E(Y_i^4)$ is finite), (ii) recognizing that $E(Y_i^2) = \sigma_Y^2 + \mu_Y^2$ (Key Concept 2.3), and (iii) noting $\overline{Y} \xrightarrow{p} \mu_Y$ so that $\overline{Y}^2 \xrightarrow{p} \mu_Y^2$. Finally, $s_Y^2 = (\frac{n}{n-1})(\frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2) \xrightarrow{p} \sigma_Y^2$ follows from Equation (3.29) and $(\frac{n}{n-1}) \rightarrow 1$.

CHAPTER 4

Linear Regression with One Regressor

A state implements tough new penalties on drunk drivers: What is the effect on highway fatalities? A school district cuts the size of its elementary school classes: What is the effect on its students' standardized test scores? You successfully complete one more year of college classes: What is the effect on your future earnings?

All three of these questions are about the unknown effect of changing one variable, X (X being penalties for drunk driving, class size, or years of schooling), on another variable, Y (Y being highway deaths, student test scores, or earnings).

This chapter introduces the linear regression model relating one variable, X, to another, Y. This model postulates a linear relationship between X and Y; the slope of the line relating X and Y is the effect of a one-unit change in X on Y. Just as the mean of Y is an unknown characteristic of the population distribution of Y, the slope of the line relating X and Y is an unknown characteristic of the population joint distribution of X and Y. The econometric problem is to estimate this slope—that is, to estimate the effect on Y of a unit change in X—using a sample of data on these two variables.

This chapter describes methods for estimating this slope using a random sample of data on X and Y. For instance, using data on class sizes and test scores from different school districts, we show how to estimate the expected effect on test scores of reducing class sizes by, say, one student per class. The slope and the intercept of the line relating X and Y can be estimated by a method called ordinary least squares (OLS).

4.1 The Linear Regression Model

The superintendent of an elementary school district must decide whether to hire additional teachers and she wants your advice. If she hires the teachers, she will reduce the number of students per teacher (the student–teacher ratio) by two. She faces a trade-off. Parents want smaller classes so that their children can receive more individualized attention. But hiring more teachers means spending more money, which is not to the liking of those paying the bill! So she asks you: If she cuts class sizes, what will the effect be on student performance?

In many school districts, student performance is measured by standardized tests, and the job status or pay of some administrators can depend in part on how well their students do on these tests. We therefore sharpen the superintendent's question: If she reduces the average class size by two students, what will the effect be on standardized test scores in her district?

A precise answer to this question requires a quantitative statement about changes. If the superintendent *changes* the class size by a certain amount, what would she expect the *change* in standardized test scores to be? We can write this as a mathematical relationship using the Greek letter beta, $\beta_{ClassSize}$, where the subscript *ClassSize* distinguishes the effect of changing the class size from other effects. Thus,

$$\beta_{ClassSize} = \frac{change\ in\ TestScore}{change\ in\ ClassSize} = \frac{\Delta TestScore}{\Delta ClassSize}, \tag{4.1}$$

where the Greek letter Δ (delta) stands for "change in." That is, $\beta_{ClassSize}$ is the change in the test score that results from changing the class size divided by the change in the class size.

If you were lucky enough to know $\beta_{ClassSize}$, you would be able to tell the superintendent that decreasing class size by one student would change district-wide test scores by $\beta_{ClassSize}$. You could also answer the superintendent's actual question, which concerned changing class size by two students per class. To do so, rearrange Equation (4.1) so that

$$\Delta TestScore = \beta_{ClassSize} \times \Delta ClassSize. \tag{4.2}$$

Suppose that $\beta_{ClassSize} = -0.6$. Then a reduction in class size of two students per class would yield a predicted change in test scores of $(-0.6) \times (-2) = 1.2$; that is, you would predict that test scores would *rise* by 1.2 points as a result of the *reduction* in class sizes by two students per class.

Equation (4.1) is the definition of the slope of a straight line relating test scores and class size. This straight line can be written

$$TestScore = \beta_0 + \beta_{ClassSize} \times ClassSize, \tag{4.3}$$

where β_0 is the intercept of this straight line and, as before, $\beta_{ClassSize}$ is the slope. According to Equation (4.3), if you knew β_0 and $\beta_{ClassSize}$, not only would you be able to determine the *change* in test scores at a district associated with a *change* in class size, but you also would be able to predict the average test score itself for a given class size.

When you propose Equation (4.3) to the superintendent, she tells you that something is wrong with this formulation. She points out that class size is just one of many facets of elementary education and that two districts with the same class sizes will have different test scores for many reasons. One district might have better teachers or it might use better textbooks. Two districts with comparable class sizes, teachers, and textbooks still might have very different student populations; perhaps one district has more immigrants (and thus fewer native English speakers) or wealthier families. Finally, she points out that even if two districts are the same in all these ways they might have different test scores for essentially random reasons having to do with the performance of the individual students on the day of the test. She is right, of course; for all these reasons, Equation (4.3) will not hold exactly for all districts. Instead, it should be viewed as a statement about a relationship that holds *on average* across the population of districts.

A version of this linear relationship that holds for *each* district must incorporate these other factors influencing test scores, including each district's unique characteristics (for example, quality of their teachers, background of their students, how lucky the students were on test day). One approach would be to list the most important factors and to introduce them explicitly into Equation (4.3) (an idea we return to in Chapter 6). For now, however, we simply lump all these "other factors" together and write the relationship for a given district as

$$TestScore = \beta_0 + \beta_{ClassSize} \times ClassSize + \text{other factors.} \tag{4.4}$$

Thus the test score for the district is written in terms of one component, $\beta_0 + \beta_{ClassSize} \times ClassSize$, that represents the average effect of class size on scores in the population of school districts and a second component that represents all other factors.

Although this discussion has focused on test scores and class size, the idea expressed in Equation (4.4) is much more general, so it is useful to introduce more general notation. Suppose you have a sample of n districts. Let Y_i be the average test score in the i^{th} district, let X_i be the average class size in the i^{th} district, and let u_i denote the other factors influencing the test score in the i^{th} district. Then Equation (4.4) can be written more generally as

$$Y_i = \beta_0 + \beta_1 X_i + u_i, \tag{4.5}$$

for each district (that is, $i = 1, \ldots, n$), where β_0 is the intercept of this line and β_1 is the slope. [The general notation β_1 is used for the slope in Equation (4.5) instead of $\beta_{ClassSize}$ because this equation is written in terms of a general variable X_i.]

Equation (4.5) is the **linear regression model with a single regressor**, in which Y is the **dependent variable** and X is the **independent variable** or the **regressor**.

The first part of Equation (4.5), $\beta_0 + \beta_1 X_i$, is the **population regression line** or the **population regression function**. This is the relationship that holds between Y and X on average over the population. Thus, if you knew the value of X, according to this population regression line you would predict that the value of the dependent variable, Y, is $\beta_0 + \beta_1 X$.

The **intercept** β_0 and the **slope** β_1 are the **coefficients** of the population regression line, also known as the **parameters** of the population regression line. The slope β_1 is the change in Y associated with a unit change in X. The intercept is the value of the population regression line when $X = 0$; it is the point at which the population regression line intersects the Y axis. In some econometric applications, the intercept has a meaningful economic interpretation. In other applications, the intercept has no real-world meaning; for example, when X is the class size, strictly speaking the intercept is the predicted value of test scores when there are no students in the class! When the real-world meaning of the intercept is nonsensical, it is best to think of it mathematically as the coefficient that determines the level of the regression line.

The term u_i in Equation (4.5) is the **error term**. The error term incorporates all of the factors responsible for the difference between the i^{th} district's average test score and the value predicted by the population regression line. This error term contains all the other factors besides X that determine the value of the dependent variable, Y, for a specific observation, i. In the class size example, these other factors include all the unique features of the i^{th} district that affect the performance of its students on the test, including teacher quality, student economic background, luck, and even any mistakes in grading the test.

The linear regression model and its terminology are summarized in Key Concept 4.1.

Figure 4.1 summarizes the linear regression model with a single regressor for seven hypothetical observations on test scores (Y) and class size (X). The population regression line is the straight line $\beta_0 + \beta_1 X$. The population regression line slopes down ($\beta_1 < 0$), which means that districts with lower student–teacher ratios (smaller classes) tend to have higher test scores. The intercept β_0 has a mathematical meaning as the value of the Y axis intersected by the population regression line, but, as mentioned earlier, it has no real-world meaning in this example.

Because of the other factors that determine test performance, the hypothetical observations in Figure 4.1 do not fall exactly on the population regression line. For example, the value of Y for district #1, Y_1, is above the population regression line. This means that test scores in district #1 were better than predicted by the

Terminology for the Linear Regression Model with a Single Regressor

KEY CONCEPT 4.1

The linear regression model is

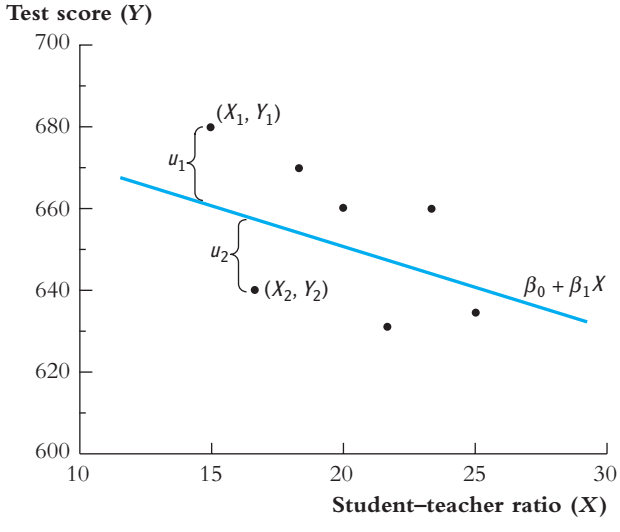
$$Y_i = \beta_0 + \beta_1 X_i + u_i,$$

where

the subscript i runs over observations, $i = 1, \ldots, n$;
Y_i is the *dependent variable*, the *regressand*, or simply the *left-hand variable*;
X_i is the *independent variable*, the *regressor*, or simply the *right-hand variable*;
$\beta_0 + \beta_1 X$ is the *population regression line* or the *population regression function*;
β_0 is the *intercept* of the population regression line;
β_1 is the *slope* of the population regression line; and
u_i is the *error term*.

FIGURE 4.1 Scatterplot of Test Score vs. Student–Teacher Ratio (Hypothetical Data)

The scatterplot shows hypothetical observations for seven school districts. The population regression line is $\beta_0 + \beta_1 X$. The vertical distance from the i^{th} point to the population regression line is $Y_i - (\beta_0 + \beta_1 X_i)$, which is the population error term u_i for the i^{th} observation.



population regression line, so the error term for that district, u_1, is positive. In contrast, Y_2 is below the population regression line, so test scores for that district were worse than predicted, and $u_2 < 0$.

Now return to your problem as advisor to the superintendent: What is the expected effect on test scores of reducing the student–teacher ratio by two students per teacher? The answer is easy: The expected change is $(-2) \times \beta_{ClassSize}$. But what is the value of $\beta_{ClassSize}$?

4.2 Estimating the Coefficients of the Linear Regression Model

In a practical situation such as the application to class size and test scores, the intercept β_0 and slope β_1 of the population regression line are unknown. Therefore, we must use data to estimate the unknown slope and intercept of the population regression line.

This estimation problem is similar to others you have faced in statistics. For example, suppose you want to compare the mean earnings of men and women who recently graduated from college. Although the population mean earnings are unknown, we can estimate the population means using a random sample of male and female college graduates. Then the natural estimator of the unknown population mean earnings for women, for example, is the average earnings of the female college graduates in the sample.

The same idea extends to the linear regression model. We do not know the population value of $\beta_{ClassSize}$, the slope of the unknown population regression line relating X (class size) and Y (test scores). But just as it was possible to learn about the population mean using a sample of data drawn from that population, so is it possible to learn about the population slope $\beta_{ClassSize}$ using a sample of data.

The data we analyze here consist of test scores and class sizes in 1999 in 420 California school districts that serve kindergarten through eighth grade. The test score is the districtwide average of reading and math scores for fifth graders. Class size can be measured in various ways. The measure used here is one of the broadest, which is the number of students in the district divided by the number of teachers—that is, the districtwide student–teacher ratio. These data are described in more detail in Appendix 4.1.

Table 4.1 summarizes the distributions of test scores and class sizes for this sample. The average student–teacher ratio is 19.6 students per teacher, and the standard deviation is 1.9 students per teacher. The 10th percentile of the distribution of the

TABLE 4.1 Summary of the Distribution of Student–Teacher Ratios and Fifth-Grade Test Scores for 420 K–8 Districts in California in 1999

			Percentile						
	Average	Standard Deviation	10%	25%	40%	50% (median)	60%	75%	90%
Student–teacher ratio	19.6	1.9	17.3	18.6	19.3	19.7	20.1	20.9	21.9
Test score	654.2	19.1	630.4	640.0	649.1	654.5	659.4	666.7	679.1

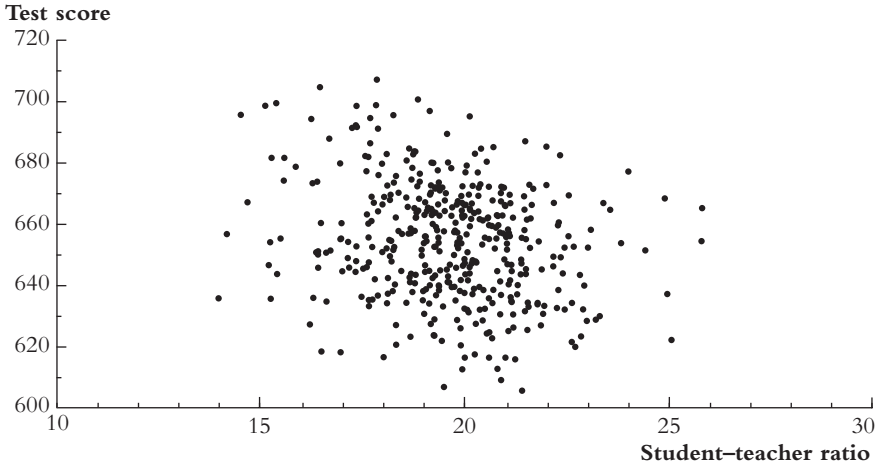
student–teacher ratio is 17.3 (that is, only 10% of districts have student–teacher ratios below 17.3), while the district at the 90th percentile has a student–teacher ratio of 21.9.

A scatterplot of these 420 observations on test scores and the student–teacher ratio is shown in Figure 4.2. The sample correlation is −0.23, indicating a weak negative relationship between the two variables. Although larger classes in this sample tend to have lower test scores, there are other determinants of test scores that keep the observations from falling perfectly along a straight line.

Despite this low correlation, if one could somehow draw a straight line through these data, then the slope of this line would be an estimate of $\beta_{ClassSize}$

FIGURE 4.2 Scatterplot of Test Score vs. Student–Teacher Ratio (California School District Data)

Data from 420 California school districts. There is a weak negative relationship between the student–teacher ratio and test scores: The sample correlation is −0.23.



based on these data. One way to draw the line would be to take out a pencil and a ruler and to "eyeball" the best line you could. While this method is easy, it is very unscientific, and different people will create different estimated lines.

How, then, should you choose among the many possible lines? By far the most common way is to choose the line that produces the "least squares" fit to these data—that is, to use the ordinary least squares (OLS) estimator.

The Ordinary Least Squares Estimator

The OLS estimator chooses the regression coefficients so that the estimated regression line is as close as possible to the observed data, where closeness is measured by the sum of the squared mistakes made in predicting Y given X.

As discussed in Section 3.1, the sample average, $\overline{Y}$, is the least squares estimator of the population mean, $E(Y)$; that is, $\overline{Y}$ minimizes the total squared estimation mistakes $\sum_{i=1}^{n}(Y_i - m)^2$ among all possible estimators m [see Expression (3.2)].

The OLS estimator extends this idea to the linear regression model. Let b_0 and b_1 be some estimators of β_0 and β_1. The regression line based on these estimators is $b_0 + b_1X$, so the value of Y_i predicted using this line is $b_0 + b_1X_i$. Thus the mistake made in predicting the i^{th} observation is $Y_i - (b_0 + b_1X_i) = Y_i - b_0 - b_1X_i$. The sum of these squared prediction mistakes over all n observations is

$$\sum_{i=1}^{n}(Y_i - b_0 - b_1X_i)^2. \tag{4.6}$$

The sum of the squared mistakes for the linear regression model in Expression (4.6) is the extension of the sum of the squared mistakes for the problem of estimating the mean in Expression (3.2). In fact, if there is no regressor, then b_1 does not enter Expression (4.6) and the two problems are identical except for the different notation [m in Expression (3.2), b_0 in Expression (4.6)]. Just as there is a unique estimator, $\overline{Y}$, that minimizes the Expression (3.2), so is there a unique pair of estimators of β_0 and β_1 that minimize Expression (4.6).

The estimators of the intercept and slope that minimize the sum of squared mistakes in Expression (4.6) are called the **ordinary least squares (OLS) estimators** of β_0 and β_1.

OLS has its own special notation and terminology. The OLS estimator of β_0 is denoted $\hat{\beta}_0$, and the OLS estimator of β_1 is denoted $\hat{\beta}_1$. The **OLS regression line**, also called the **sample regression line** or **sample regression function**, is the straight line constructed using the OLS estimators: $\hat{\beta}_0 + \hat{\beta}_1X$. The **predicted value** of Y_i

The OLS Estimator, Predicted Values, and Residuals

KEY CONCEPT 4.2

The OLS estimators of the slope β_1 and the intercept β_0 are

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} = \frac{s_{XY}}{s_X^2} \tag{4.7}$$

$$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1 \overline{X}. \tag{4.8}$$

The OLS predicted values $\hat{Y}_i$ and residuals $\hat{u}_i$ are

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i, \quad i = 1, \ldots, n \tag{4.9}$$

$$\hat{u}_i = Y_i - \hat{Y}_i, \quad i = 1, \ldots, n. \tag{4.10}$$

The estimated intercept ($\hat{\beta}_0$), slope ($\hat{\beta}_1$), and residual ($\hat{u}_i$) are computed from a sample of n observations of X_i and Y_i, $i = 1, \ldots, n$. These are estimates of the unknown true population intercept (β_0), slope (β_1), and error term (u_i).

given X_i, based on the OLS regression line, is $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$. The **residual** for the i^{th} observation is the difference between Y_i and its predicted value: $\hat{u}_i = Y_i - \hat{Y}_i$.

The OLS estimators, $\hat{\beta}_0$ and $\hat{\beta}_1$, are sample counterparts of the population coefficients, β_0 and β_1. Similarly, the OLS regression line $\hat{\beta}_0 + \hat{\beta}_1 X$ is the sample counterpart of the population regression line $\beta_0 + \beta_1 X$, and the OLS residuals $\hat{u}_i$ are sample counterparts of the population errors u_i.

You could compute the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ by trying different values of b_0 and b_1 repeatedly until you find those that minimize the total squared mistakes in Expression (4.6); they are the least squares estimates. This method would be quite tedious, however. Fortunately, there are formulas, derived by minimizing Expression (4.6) using calculus, that streamline the calculation of the OLS estimators.

The OLS formulas and terminology are collected in Key Concept 4.2. These formulas are implemented in virtually all statistical and spreadsheet programs. These formulas are derived in Appendix 4.2.

OLS Estimates of the Relationship Between Test Scores and the Student–Teacher Ratio

When OLS is used to estimate a line relating the student–teacher ratio to test scores using the 420 observations in Figure 4.2, the estimated slope is −2.28 and the estimated intercept is 698.9. Accordingly, the OLS regression line for these 420 observations is

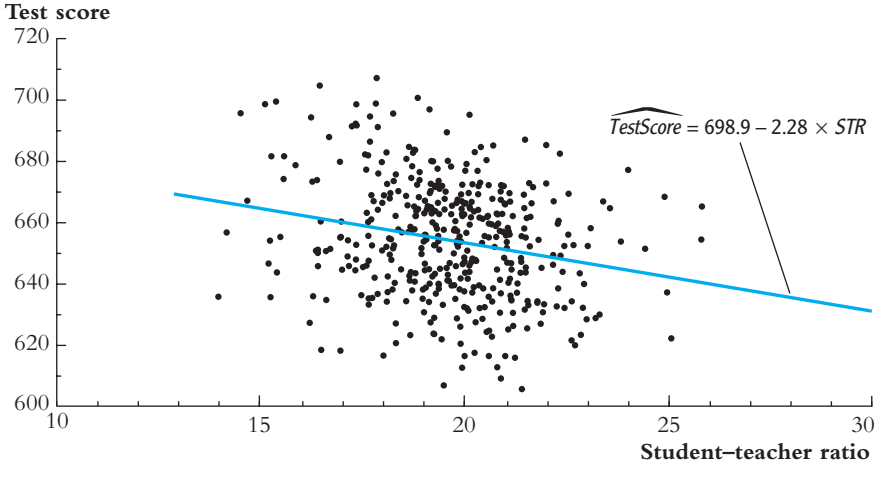
$$\widehat{TestScore} = 698.9 - 2.28 \times STR, \tag{4.11}$$

where *TestScore* is the average test score in the district and *STR* is the student–teacher ratio. The "^" over *TestScore* in Equation (4.11) indicates that it is the predicted value based on the OLS regression line. Figure 4.3 plots this OLS regression line superimposed over the scatterplot of the data previously shown in Figure 4.2.

The slope of −2.28 means that an increase in the student–teacher ratio by one student per class is, on average, associated with a decline in districtwide test scores by 2.28 points on the test. A decrease in the student–teacher ratio by two students per class is, on average, associated with an increase in test scores of 4.56 points $[= -2 \times (-2.28)]$. The negative slope indicates that more students per teacher (larger classes) is associated with poorer performance on the test.

FIGURE 4.3 The Estimated Regression Line for the California Data

The estimated regression line shows a negative relationship between test scores and the student–teacher ratio. If class sizes fall by one student, the estimated regression predicts that test scores will increase by 2.28 points.



It is now possible to predict the districtwide test score given a value of the student–teacher ratio. For example, for a district with 20 students per teacher, the predicted test score is $698.9 - 2.28 \times 20 = 653.3$. Of course, this prediction will not be exactly right because of the other factors that determine a district's performance. But the regression line does give a prediction (the OLS prediction) of what test scores would be for that district, based on their student–teacher ratio, absent those other factors.

Is this estimate of the slope large or small? To answer this, we return to the superintendent's problem. Recall that she is contemplating hiring enough teachers to reduce the student–teacher ratio by 2. Suppose her district is at the median of the California districts. From Table 4.1, the median student–teacher ratio is 19.7 and the median test score is 654.5. A reduction of two students per class, from 19.7 to 17.7, would move her student–teacher ratio from the 50th percentile to very near the 10th percentile. This is a big change, and she would need to hire many new teachers. How would it affect test scores?

According to Equation (4.11), cutting the student–teacher ratio by 2 is predicted to increase test scores by approximately 4.6 points; if her district's test scores are at the median, 654.5, they are predicted to increase to 659.1. Is this improvement large or small? According to Table 4.1, this improvement would move her district from the median to just short of the 60th percentile. Thus a decrease in class size that would place her district close to the 10% with the smallest classes would move her test scores from the 50th to the 60th percentile. According to these estimates, at least, cutting the student–teacher ratio by a large amount (two students per teacher) would help and might be worth doing depending on her budgetary situation, but it would not be a panacea.

What if the superintendent were contemplating a far more radical change, such as reducing the student–teacher ratio from 20 students per teacher to 5? Unfortunately, the estimates in Equation (4.11) would not be very useful to her. This regression was estimated using the data in Figure 4.2, and, as the figure shows, the smallest student–teacher ratio in these data is 14. These data contain no information on how districts with extremely small classes perform, so these data alone are not a reliable basis for predicting the effect of a radical move to such an extremely low student–teacher ratio.

Why Use the OLS Estimator?

There are both practical and theoretical reasons to use the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$. Because OLS is the dominant method used in practice, it has become the common language for regression analysis throughout economics, finance (see "The 'Beta' of a Stock" box), and the social sciences more generally. Presenting results

The "Beta" of a Stock

A fundamental idea of modern finance is that an investor needs a financial incentive to take a risk. Said differently, the expected return[1] on a risky investment, R, must exceed the return on a safe, or risk-free, investment, R_f. Thus the expected excess return, $R - R_f$, on a risky investment, like owning stock in a company, should be positive.

At first it might seem like the risk of a stock should be measured by its variance. Much of that risk, however, can be reduced by holding other stocks in a "portfolio"—in other words, by diversifying your financial holdings. This means that the right way to measure the risk of a stock is not by its *variance* but rather by its *covariance* with the market.

The capital asset pricing model (CAPM) formalizes this idea. According to the CAPM, the expected excess return on an asset is proportional to the expected excess return on a portfolio of all available assets (the "market portfolio"). That is, the CAPM says that

$$R - R_f = \beta(R_m - R_f), \tag{4.12}$$

where R_m is the expected return on the market portfolio and β is the coefficient in the population regression of $R - R_f$ on $R_m - R_f$. In practice, the risk-free return is often taken to be the rate of interest on short-term U.S. government debt. According to the CAPM, a stock with a $\beta < 1$ has less risk than the market portfolio and therefore has a lower expected excess return than the market portfolio. In contrast, a stock with a $\beta > 1$ is riskier than the market portfolio and thus commands a higher expected excess return.

The "beta" of a stock has become a workhorse of the investment industry, and you can obtain estimated betas for hundreds of stocks on investment firm websites. Those betas typically are estimated by OLS regression of the actual excess return on the stock against the actual excess return on a broad market index.

The table below gives estimated betas for seven U.S. stocks. Low-risk producers of consumer staples like Kellogg have stocks with low betas; riskier stocks have high betas.

Company	Estimated β
Verizon (telecommunications)	0.0
Wal-Mart (discount retailer)	0.3
Kellogg (breakfast cereal)	0.5
Waste Management (waste disposal)	0.6
Google (information technology)	1.0
Ford Motor Company (auto producer)	1.3
Bank of America (bank)	2.2

Source: finance.yahoo.com.

[1]The return on an investment is the change in its price plus any payout (dividend) from the investment as a percentage of its initial price. For example, a stock bought on January 1 for \$100, which then paid a \$2.50 dividend during the year and sold on December 31 for \$105, would have a return of $R = [(\$105 - \$100) + \$2.50]/\$100 = 7.5\%$.

using OLS (or its variants discussed later in this book) means that you are "speaking the same language" as other economists and statisticians. The OLS formulas are built into virtually all spreadsheet and statistical software packages, making OLS easy to use.

The OLS estimators also have desirable theoretical properties. They are analogous to the desirable properties, studied in Section 3.1, of $\overline{Y}$ as an estimator of the population mean. Under the assumptions introduced in Section 4.4, the OLS estimator is unbiased and consistent. The OLS estimator is also efficient among a certain class of unbiased estimators; however, this efficiency result holds under some additional special conditions, and further discussion of this result is deferred until Section 5.5.

4.3 Measures of Fit

Having estimated a linear regression, you might wonder how well that regression line describes the data. Does the regressor account for much or for little of the variation in the dependent variable? Are the observations tightly clustered around the regression line, or are they spread out?

The R^2 and the standard error of the regression measure how well the OLS regression line fits the data. The R^2 ranges between 0 and 1 and measures the fraction of the variance of Y_i that is explained by X_i. The standard error of the regression measures how far Y_i typically is from its predicted value.

The R^2

The **regression R^2** is the fraction of the sample variance of Y_i explained by (or predicted by) X_i. The definitions of the predicted value and the residual (see Key Concept 4.2) allow us to write the dependent variable Y_i as the sum of the predicted value, $\hat{Y}_i$, plus the residual $\hat{u}_i$:

$$Y_i = \hat{Y}_i + \hat{u}_i. \tag{4.13}$$

In this notation, the R^2 is the ratio of the sample variance of $\hat{Y}_i$ to the sample variance of Y_i.

Mathematically, the R^2 can be written as the ratio of the explained sum of squares to the total sum of squares. The **explained sum of squares (*ESS*)** is the sum of squared deviations of the predicted value, $\hat{Y}_i$, from its average, and the **total sum of squares (*TSS*)** is the sum of squared deviations of Y_i from its average:

$$ESS = \sum_{i=1}^{n} (\hat{Y}_i - \overline{Y})^2 \tag{4.14}$$

$$TSS = \sum_{i=1}^{n} (Y_i - \overline{Y})^2. \tag{4.15}$$

Equation (4.14) uses the fact that the sample average OLS predicted value equals $\overline{Y}$ (proven in Appendix 4.3).

The R^2 is the ratio of the explained sum of squares to the total sum of squares:

$$R^2 = \frac{ESS}{TSS}. \tag{4.16}$$

Alternatively, the R^2 can be written in terms of the fraction of the variance of Y_i *not* explained by X_i. The **sum of squared residuals**, or ***SSR***, is the sum of the squared OLS residuals:

$$SSR = \sum_{i=1}^{n} \hat{u}_i^2. \tag{4.17}$$

It is shown in Appendix 4.3 that $TSS = ESS + SSR$. Thus the R^2 also can be expressed as 1 minus the ratio of the sum of squared residuals to the total sum of squares:

$$R^2 = 1 - \frac{SSR}{TSS}. \tag{4.18}$$

Finally, the R^2 of the regression of Y on the single regressor X is the square of the correlation coefficient between Y and X (Exercise 4.12).

The R^2 ranges between 0 and 1. If $\hat{\beta}_1 = 0$, then X_i explains none of the variation of Y_i and the predicted value of Y_i is $\hat{Y}_i = \hat{\beta}_0 = \overline{Y}$ [from Equation (4.8)]. In this case, the explained sum of squares is zero and the sum of squared residuals equals the total sum of squares; thus the R^2 is zero. In contrast, if X_i explains all of the variation of Y_i, then $Y_i = \hat{Y}_i$ for all i and every residual is zero (that is, $\hat{u}_i = 0$), so that $ESS = TSS$ and $R^2 = 1$. In general, the R^2 does not take on the extreme values of 0 or 1 but falls somewhere in between. An R^2 near 1 indicates that the regressor is good at predicting Y_i, while an R^2 near 0 indicates that the regressor is not very good at predicting Y_i.

The Standard Error of the Regression

The **standard error of the regression (*SER*)** is an estimator of the standard deviation of the regression error u_i. The units of u_i and Y_i are the same, so the *SER* is a measure of the spread of the observations around the regression line, measured in the units of the dependent variable. For example, if the units of the dependent variable are dollars, then the *SER* measures the magnitude of a typical deviation

from the regression line—that is, the magnitude of a typical regression error—in dollars.

Because the regression errors $u_1, \ldots, u_n$ are unobserved, the *SER* is computed using their sample counterparts, the OLS residuals $\hat{u}_1, \ldots, \hat{u}_n$. The formula for the *SER* is

$$SER = s_{\hat{u}} = \sqrt{s_{\hat{u}}^2}, \text{ where } s_{\hat{u}}^2 = \frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{SSR}{n-2}, \tag{4.19}$$

where the formula for $s_{\hat{u}}^2$ uses the fact (proven in Appendix 4.3) that the sample average of the OLS residuals is zero.

The formula for the *SER* in Equation (4.19) is similar to the formula for the sample standard deviation of Y given in Equation (3.7) in Section 3.2, except that $Y_i - \overline{Y}$ in Equation (3.7) is replaced by $\hat{u}_i$ and the divisor in Equation (3.7) is $n - 1$, whereas here it is $n - 2$. The reason for using the divisor $n - 2$ here (instead of n) is the same as the reason for using the divisor $n - 1$ in Equation (3.7): It corrects for a slight downward bias introduced because two regression coefficients were estimated. This is called a "degrees of freedom" correction because two coefficients were estimated (β_0 and β_1), two "degrees of freedom" of the data were lost, so the divisor in this factor is $n - 2$. (The mathematics behind this is discussed in Section 5.6.) When n is large, the difference between dividing by n, by $n - 1$, or by $n - 2$ is negligible.

Application to the Test Score Data

Equation (4.11) reports the regression line, estimated using the California test score data, relating the standardized test score (*TestScore*) to the student–teacher ratio (*STR*). The R^2 of this regression is 0.051, or 5.1%, and the *SER* is 18.6.

The R^2 of 0.051 means that the regressor *STR* explains 5.1% of the variance of the dependent variable *TestScore*. Figure 4.3 superimposes this regression line on the scatterplot of the *TestScore* and *STR* data. As the scatterplot shows, the student–teacher ratio explains some of the variation in test scores, but much variation remains unaccounted for.

The *SER* of 18.6 means that standard deviation of the regression residuals is 18.6, where the units are points on the standardized test. Because the standard deviation is a measure of spread, the *SER* of 18.6 means that there is a large spread of the scatterplot in Figure 4.3 around the regression line as measured in points on the test. This large spread means that predictions of test scores made using only the student–teacher ratio for that district will often be wrong by a large amount.

What should we make of this low R^2 and large SER? The fact that the R^2 of this regression is low (and the SER is large) does not, by itself, imply that this regression is either "good" or "bad." What the low R^2 *does* tell us is that other important factors influence test scores. These factors could include differences in the student body across districts, differences in school quality unrelated to the student–teacher ratio, or luck on the test. The low R^2 and high SER do not tell us what these factors are, but they do indicate that the student–teacher ratio alone explains only a small part of the variation in test scores in these data.

4.4 The Least Squares Assumptions

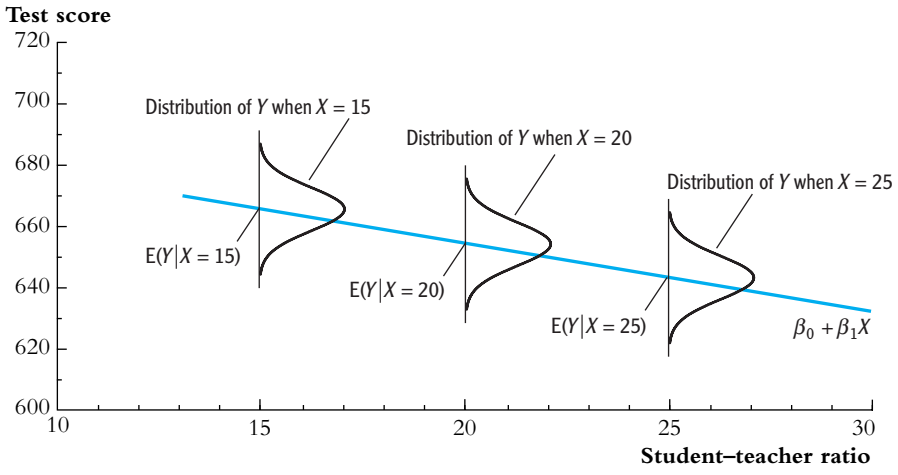
This section presents a set of three assumptions on the linear regression model and the sampling scheme under which OLS provides an appropriate estimator of the unknown regression coefficients, β_0 and β_1. Initially, these assumptions might appear abstract. They do, however, have natural interpretations, and understanding these assumptions is essential for understanding when OLS will—and will not—give useful estimates of the regression coefficients.

Assumption #1: The Conditional Distribution of u_i Given X_i Has a Mean of Zero

The first of the three **least squares assumptions** is that the conditional distribution of u_i given X_i has a mean of zero. This assumption is a formal mathematical statement about the "other factors" contained in u_i and asserts that these other factors are unrelated to X_i in the sense that, given a value of X_i, the mean of the distribution of these other factors is zero.

This assumption is illustrated in Figure 4.4. The population regression is the relationship that holds on average between class size and test scores in the population, and the error term u_i represents the other factors that lead test scores at a given district to differ from the prediction based on the population regression line. As shown in Figure 4.4, at a given value of class size, say 20 students per class, sometimes these other factors lead to better performance than predicted ($u_i > 0$) and sometimes to worse performance ($u_i < 0$), but on average over the population the prediction is right. In other words, given $X_i = 20$, the mean of the distribution of u_i is zero. In Figure 4.4, this is shown as the distribution of u_i being centered on the population regression line at $X_i = 20$ and, more generally, at other values x of X_i as well. Said differently, the distribution of u_i, conditional on $X_i = x$, has a mean of zero; stated mathematically, $E(u_i|X_i = x) = 0$, or, in somewhat simpler notation, $E(u_i|X_i) = 0$.

FIGURE 4.4 The Conditional Probability Distributions and the Population Regression Line



The figure shows the conditional probability of test scores for districts with class sizes of 15, 20, and 25 students. The mean of the conditional distribution of test scores, given the student–teacher ratio, $E(Y|X)$, is the population regression line. At a given value of X, Y is distributed around the regression line and the error, $u = Y - (\beta_0 + \beta_1 X)$, has a conditional mean of zero for all values of X.

As shown in Figure 4.4, the assumption that $E(u_i|X_i) = 0$ is equivalent to assuming that the population regression line is the conditional mean of Y_i given X_i (a mathematical proof of this is left as Exercise 4.6).

The conditional mean of u in a randomized controlled experiment. In a randomized controlled experiment, subjects are randomly assigned to the treatment group ($X = 1$) or to the control group ($X = 0$). The random assignment typically is done using a computer program that uses no information about the subject, ensuring that X is distributed independently of all personal characteristics of the subject. Random assignment makes X and u independent, which in turn implies that the conditional mean of u given X is zero.

In observational data, X is not randomly assigned in an experiment. Instead, the best that can be hoped for is that X is *as if* randomly assigned, in the precise sense that $E(u_i|X_i) = 0$. Whether this assumption holds in a given empirical application with observational data requires careful thought and judgment, and we return to this issue repeatedly.

Correlation and conditional mean. Recall from Section 2.3 that if the conditional mean of one random variable given another is zero, then the two random variables have zero covariance and thus are uncorrelated [Equation (2.27)]. Thus the conditional mean assumption $E(u_i|X_i) = 0$ implies that X_i and u_i are uncorrelated, or $\text{corr}(X_i, u_i) = 0$. Because correlation is a measure of linear association, this implication does not go the other way; even if X_i and u_i are uncorrelated, the conditional mean of u_i given X_i might be nonzero. However, if X_i and u_i are correlated, then it must be the case that $E(u_i|X_i)$ is nonzero. It is therefore often convenient to discuss the conditional mean assumption in terms of possible correlation between X_i and u_i. If X_i and u_i are correlated, then the conditional mean assumption is violated.

Assumption #2: (X_i, Y_i), $i = 1, \ldots, n$, Are Independently and Identically Distributed

The second least squares assumption is that (X_i, Y_i), $i = 1, \ldots, n$, are independently and identically distributed (i.i.d.) across observations. As discussed in Section 2.5 (Key Concept 2.5), this assumption is a statement about how the sample is drawn. If the observations are drawn by simple random sampling from a single large population, then (X_i, Y_i), $i = 1, \ldots, n$, are i.i.d. For example, let X be the age of a worker and Y be his or her earnings, and imagine drawing a person at random from the population of workers. That randomly drawn person will have a certain age and earnings (that is, X and Y will take on some values). If a sample of n workers is drawn from this population, then (X_i, Y_i), $i = 1, \ldots, n$, necessarily have the same distribution. If they are drawn at random they are also distributed independently from one observation to the next; that is, they are i.i.d.

The i.i.d. assumption is a reasonable one for many data collection schemes. For example, survey data from a randomly chosen subset of the population typically can be treated as i.i.d.

Not all sampling schemes produce i.i.d. observations on (X_i, Y_i), however. One example is when the values of X are not drawn from a random sample of the population but rather are set by a researcher as part of an experiment. For example, suppose a horticulturalist wants to study the effects of different organic weeding methods (X) on tomato production (Y) and accordingly grows different plots of tomatoes using different organic weeding techniques. If she picks the techniques (the level of X) to be used on the i^{th} plot and applies the same technique to the i^{th} plot in all repetitions of the experiment, then the value of X_i does not change from one sample to the next. Thus X_i is nonrandom (although the outcome Y_i is random), so the sampling scheme is not i.i.d. The results presented in this chapter developed for i.i.d. regressors are also true if the regressors are nonrandom. The case of a

nonrandom regressor is, however, quite special. For example, modern experimental protocols would have the horticulturalist assign the level of X to the different plots using a computerized random number generator, thereby circumventing any possible bias by the horticulturalist (she might use her favorite weeding method for the tomatoes in the sunniest plot). When this modern experimental protocol is used, the level of X is random and (X_i, Y_i) are i.i.d.

Another example of non-i.i.d. sampling is when observations refer to the same unit of observation over time. For example, we might have data on inventory levels (Y) at a firm and the interest rate at which the firm can borrow (X), where these data are collected over time from a specific firm; for example, they might be recorded four times a year (quarterly) for 30 years. This is an example of time series data, and a key feature of time series data is that observations falling close to each other in time are not independent but rather tend to be correlated with each other; if interest rates are low now, they are likely to be low next quarter. This pattern of correlation violates the "independence" part of the i.i.d. assumption. Time series data introduce a set of complications that are best handled after developing the basic tools of regression analysis, so we postpone discussion of time series data until Chapter 14.

Assumption #3: Large Outliers Are Unlikely

The third least squares assumption is that large outliers—that is, observations with values of X_i, Y_i, or both that are far outside the usual range of the data—are unlikely. Large outliers can make OLS regression results misleading. This potential sensitivity of OLS to extreme outliers is illustrated in Figure 4.5 using hypothetical data.

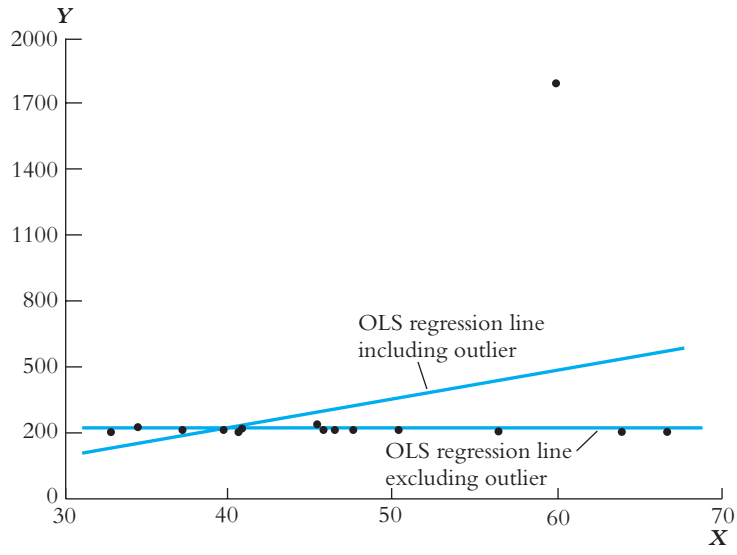
In this book, the assumption that large outliers are unlikely is made mathematically precise by assuming that X and Y have nonzero finite fourth moments: $0 < E(X_i^4) < \infty$ and $0 < E(Y_i^4) < \infty$. Another way to state this assumption is that X and Y have finite kurtosis.

The assumption of finite kurtosis is used in the mathematics that justify the large-sample approximations to the distributions of the OLS test statistics. For example, we encountered this assumption in Chapter 3 when discussing the consistency of the sample variance. Specifically, Equation (3.9) states that the sample variance is a consistent estimator of the population variance σ_Y^2 ($s_Y^2 \xrightarrow{p} \sigma_Y^2$). If $Y_1, \ldots, Y_n$ are i.i.d. and the fourth moment of Y_i is finite, then the law of large numbers in Key Concept 2.6 applies to the average, $\frac{1}{n}\sum_{i=1}^{n} Y_i^2$, a key step in the proof in Appendix 3.3 showing that s_Y^2 is consistent.

One source of large outliers is data entry errors, such as a typographical error or incorrectly using different units for different observations. Imagine collecting data on the height of students in meters, but inadvertently recording one student's

FIGURE 4.5 The Sensitivity of OLS to Large Outliers

This hypothetical data set has one outlier. The OLS regression line estimated with the outlier shows a strong positive relationship between *X* and *Y*, but the OLS regression line estimated without the outlier shows no relationship.



height in centimeters instead. This would create a large outlier in the sample. One way to find outliers is to plot your data. If you decide that an outlier is due to a data entry error, then you can either correct the error or, if that is impossible, drop the observation from your data set.

Data entry errors aside, the assumption of finite kurtosis is a plausible one in many applications with economic data. Class size is capped by the physical capacity of a classroom; the best you can do on a standardized test is to get all the questions right and the worst you can do is to get all the questions wrong. Because class size and test scores have a finite range, they necessarily have finite kurtosis. More generally, commonly used distributions such as the normal distribution have four moments. Still, as a mathematical matter, some distributions have infinite fourth moments, and this assumption rules out those distributions. If the assumption of finite fourth moments holds, then it is unlikely that statistical inferences using OLS will be dominated by a few observations.

Use of the Least Squares Assumptions

The three least squares assumptions for the linear regression model are summarized in Key Concept 4.3. The least squares assumptions play twin roles, and we return to them repeatedly throughout this textbook.

The Least Squares Assumptions

KEY CONCEPT
4.3

$$Y_i = \beta_0 + \beta_1 X_i + u_i, i = 1, \ldots, n, \text{ where}$$

1. The error term u_i has conditional mean zero given X_i: $E(u_i|X_i) = 0$;
2. $(X_i, Y_i), i = 1, \ldots, n$, are independent and identically distributed (i.i.d.) draws from their joint distribution; and
3. Large outliers are unlikely: X_i and Y_i have nonzero finite fourth moments.

Their first role is mathematical: If these assumptions hold, then, as is shown in the next section, in large samples the OLS estimators have sampling distributions that are normal. In turn, this large-sample normal distribution lets us develop methods for hypothesis testing and constructing confidence intervals using the OLS estimators.

Their second role is to organize the circumstances that pose difficulties for OLS regression. As we will see, the first least squares assumption is the most important to consider in practice. One reason why the first least squares assumption might not hold in practice is discussed in Chapter 6, and additional reasons are discussed in Section 9.2.

It is also important to consider whether the second assumption holds in an application. Although it plausibly holds in many cross-sectional data sets, the independence assumption is inappropriate for panel and time series data. Therefore, the regression methods developed under assumption 2 require modification for some applications with time series data. These modifications are developed in Chapters 10 and 14–16.

The third assumption serves as a reminder that OLS, just like the sample mean, can be sensitive to large outliers. If your data set contains large outliers, you should examine those outliers carefully to make sure those observations are correctly recorded and belong in the data set.

4.5 Sampling Distribution of the OLS Estimators

Because the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are computed from a randomly drawn sample, the estimators themselves are random variables with a probability distribution—the sampling distribution—that describes the values they could take over different possible random samples. This section presents these sampling distributions.

In small samples, these distributions are complicated, but in large samples, they are approximately normal because of the central limit theorem.

The Sampling Distribution of the OLS Estimators

Review of the sampling distribution of $\overline{Y}$. Recall the discussion in Sections 2.5 and 2.6 about the sampling distribution of the sample average, $\overline{Y}$, an estimator of the unknown population mean of Y, μ_Y. Because $\overline{Y}$ is calculated using a randomly drawn sample, $\overline{Y}$ is a random variable that takes on different values from one sample to the next; the probability of these different values is summarized in its sampling distribution. Although the sampling distribution of $\overline{Y}$ can be complicated when the sample size is small, it is possible to make certain statements about it that hold for all n. In particular, the mean of the sampling distribution is μ_Y, that is, $E(\overline{Y}) = \mu_Y$, so $\overline{Y}$ is an unbiased estimator of μ_Y. If n is large, then more can be said about the sampling distribution. In particular, the central limit theorem (Section 2.6) states that this distribution is approximately normal.

The sampling distribution of $\hat{\beta}_0$ and $\hat{\beta}_1$. These ideas carry over to the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ of the unknown intercept β_0 and slope β_1 of the population regression line. Because the OLS estimators are calculated using a random sample, $\hat{\beta}_0$ and $\hat{\beta}_1$ are random variables that take on different values from one sample to the next; the probability of these different values is summarized in their sampling distributions.

Although the sampling distribution of $\hat{\beta}_0$ and $\hat{\beta}_1$ can be complicated when the sample size is small, it is possible to make certain statements about it that hold for all n. In particular, the mean of the sampling distributions of $\hat{\beta}_0$ and $\hat{\beta}_1$ are β_0 and β_1. In other words, under the least squares assumptions in Key Concept 4.3,

$$E(\hat{\beta}_0) = \beta_0 \text{ and } E(\hat{\beta}_1) = \beta_1; \tag{4.20}$$

that is, $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased estimators of β_0 and β_1. The proof that $\hat{\beta}_1$ is unbiased is given in Appendix 4.3, and the proof that $\hat{\beta}_0$ is unbiased is left as Exercise 4.7.

If the sample is sufficiently large, by the central limit theorem the sampling distribution of $\hat{\beta}_0$ and $\hat{\beta}_1$ is well approximated by the bivariate normal distribution (Section 2.4). This implies that the marginal distributions of $\hat{\beta}_0$ and $\hat{\beta}_1$ are normal in large samples.

This argument invokes the central limit theorem. Technically, the central limit theorem concerns the distribution of averages (like $\overline{Y}$). If you examine the numerator in Equation (4.7) for $\hat{\beta}_1$, you will see that it, too, is a type of average—not a simple average, like $\overline{Y}$, but an average of the product, $(Y_i - \overline{Y})(X_i - \overline{X})$. As discussed

Large-Sample Distributions of $\hat{\beta}_0$ and $\hat{\beta}_1$

KEY CONCEPT 4.4

If the least squares assumptions in Key Concept 4.3 hold, then in large samples $\hat{\beta}_0$ and $\hat{\beta}_1$ have a jointly normal sampling distribution. The large-sample normal distribution of $\hat{\beta}_1$ is $N(\beta_1, \sigma^2_{\hat{\beta}_1})$, where the variance of this distribution, $\sigma^2_{\hat{\beta}_1}$, is

$$\sigma^2_{\hat{\beta}_1} = \frac{1}{n}\frac{\text{var}[(X_i - \mu_X)u_i]}{[\text{var}(X_i)]^2}. \tag{4.21}$$

The large-sample normal distribution of $\hat{\beta}_0$ is $N(\beta_0, \sigma^2_{\hat{\beta}_0})$, where

$$\sigma^2_{\hat{\beta}_0} = \frac{1}{n}\frac{\text{var}(H_i u_i)}{[E(H_i^2)]^2}, \text{ where } H_i = 1 - \left[\frac{\mu_X}{E(X_i^2)}\right]X_i. \tag{4.22}$$

further in Appendix 4.3, the central limit theorem applies to this average so that, like the simpler average $\overline{Y}$, it is normally distributed in large samples.

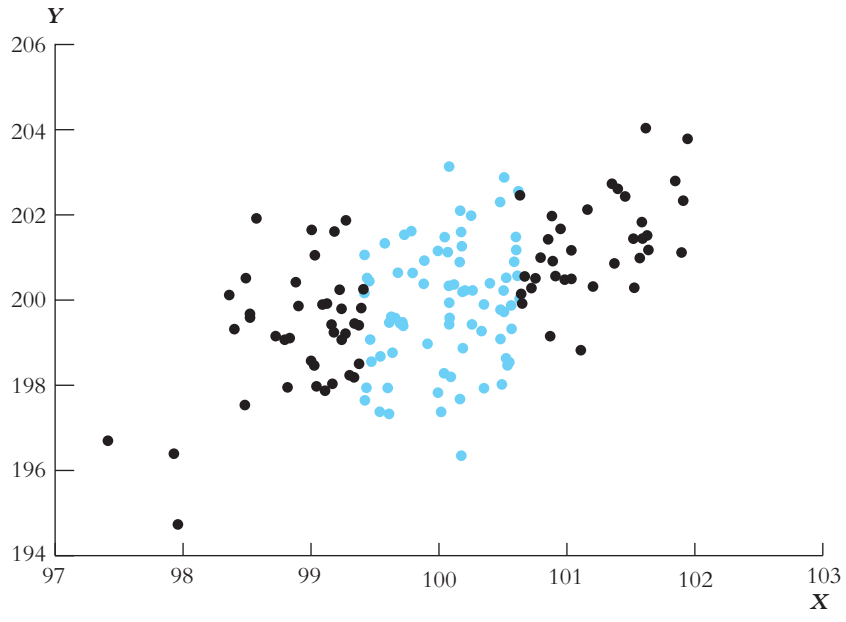
The normal approximation to the distribution of the OLS estimators in large samples is summarized in Key Concept 4.4. (Appendix 4.3 summarizes the derivation of these formulas.) A relevant question in practice is how large n must be for these approximations to be reliable. In Section 2.6, we suggested that $n = 100$ is sufficiently large for the sampling distribution of $\overline{Y}$ to be well approximated by a normal distribution, and sometimes smaller n suffices. This criterion carries over to the more complicated averages appearing in regression analysis. In virtually all modern econometric applications, $n > 100$, so we will treat the normal approximations to the distributions of the OLS estimators as reliable unless there are good reasons to think otherwise.

The results in Key Concept 4.4 imply that the OLS estimators are consistent—that is, when the sample size is large, $\hat{\beta}_0$ and $\hat{\beta}_1$ will be close to the true population coefficients β_0 and β_1 with high probability. This is because the variances $\sigma^2_{\hat{\beta}_0}$ and $\sigma^2_{\hat{\beta}_1}$ of the estimators decrease to zero as n increases (n appears in the denominator of the formulas for the variances), so the distribution of the OLS estimators will be tightly concentrated around their means, β_0 and β_1, when n is large.

Another implication of the distributions in Key Concept 4.4 is that, in general, the larger is the variance of X_i, the smaller is the variance $\sigma^2_{\hat{\beta}_1}$ of $\hat{\beta}_1$. Mathematically, this implication arises because the variance of $\hat{\beta}_1$ in Equation (4.21) is inversely proportional to the square of the variance of X_i: the larger is $\text{var}(X_i)$, the larger is the denominator in Equation (4.21) so the smaller is $\sigma^2_{\hat{\beta}_1}$. To get a better sense

FIGURE 4.6 The Variance of $\hat{\beta}_1$ and the Variance of X

The colored dots represent a set of X_i's with a small variance. The black dots represent a set of X_i's with a large variance. The regression line can be estimated more accurately with the black dots than with the colored dots.



of why this is so, look at Figure 4.6, which presents a scatterplot of 150 artificial data points on X and Y. The data points indicated by the colored dots are the 75 observations closest to $\overline{X}$. Suppose you were asked to draw a line as accurately as possible through *either* the colored or the black dots—which would you choose? It would be easier to draw a precise line through the black dots, which have a larger variance than the colored dots. Similarly, the larger the variance of X, the more precise is $\hat{\beta}_1$.

The distributions in Key Concept 4.4 also imply that the smaller is the variance of the error u_i, the smaller is the variance of $\hat{\beta}_1$. This can be seen mathematically in Equation (4.21) because u_i enters the numerator, but not denominator, of $\sigma^2_{\hat{\beta}_1}$: If all u_i were smaller by a factor of one-half but the X's did not change, then $\sigma_{\hat{\beta}_1}$ would be smaller by a factor of one-half and $\sigma^2_{\hat{\beta}_1}$ would be smaller by a factor of one-fourth (Exercise 4.13). Stated less mathematically, if the errors are smaller (holding the X's fixed), then the data will have a tighter scatter around the population regression line so its slope will be estimated more precisely.

The normal approximation to the sampling distribution of $\hat{\beta}_0$ and $\hat{\beta}_1$ is a powerful tool. With this approximation in hand, we are able to develop methods for making inferences about the true population values of the regression coefficients using only a sample of data.

4.6 Conclusion

This chapter has focused on the use of ordinary least squares to estimate the intercept and slope of a population regression line using a sample of n observations on a dependent variable, Y, and a single regressor, X. There are many ways to draw a straight line through a scatterplot, but doing so using OLS has several virtues. If the least squares assumptions hold, then the OLS estimators of the slope and intercept are unbiased, are consistent, and have a sampling distribution with a variance that is inversely proportional to the sample size n. Moreover, if n is large, then the sampling distribution of the OLS estimator is normal.

These important properties of the sampling distribution of the OLS estimator hold under the three least squares assumptions.

The first assumption is that the error term in the linear regression model has a conditional mean of zero, given the regressor X. This assumption implies that the OLS estimator is unbiased.

The second assumption is that (X_i, Y_i) are i.i.d., as is the case if the data are collected by simple random sampling. This assumption yields the formula, presented in Key Concept 4.4, for the variance of the sampling distribution of the OLS estimator.

The third assumption is that large outliers are unlikely. Stated more formally, X and Y have finite fourth moments (finite kurtosis). The reason for this assumption is that OLS can be unreliable if there are large outliers. Taken together, the three least squares assumptions imply that the OLS estimator is normally distributed in large samples as described in Key Concept 4.4.

The results in this chapter describe the sampling distribution of the OLS estimator. By themselves, however, these results are not sufficient to test a hypothesis about the value of β_1 or to construct a confidence interval for β_1. Doing so requires an estimator of the standard deviation of the sampling distribution—that is, the standard error of the OLS estimator. This step—moving from the sampling distribution of $\hat{\beta}_1$ to its standard error, hypothesis tests, and confidence intervals—is taken in the next chapter.

Summary

1. The population regression line, $\beta_0 + \beta_1 X$, is the mean of Y as a function of the value of X. The slope, β_1, is the expected change in Y associated with a one-unit change in X. The intercept, β_0, determines the level (or height) of the regression line. Key Concept 4.1 summarizes the terminology of the population linear regression model.

2. The population regression line can be estimated using sample observations $(Y_i, X_i), i = 1, \ldots, n$ by ordinary least squares (OLS). The OLS estimators of the regression intercept and slope are denoted $\hat{\beta}_0$ and $\hat{\beta}_1$.
3. The R^2 and standard error of the regression (SER) are measures of how close the values of Y_i are to the estimated regression line. The R^2 is between 0 and 1, with a larger value indicating that the Y_i's are closer to the line. The standard error of the regression is an estimator of the standard deviation of the regression error.
4. There are three key assumptions for the linear regression model: (1) The regression errors, u_i, have a mean of zero, conditional on the regressors X_i; (2) the sample observations are i.i.d. random draws from the population; and (3) large outliers are unlikely. If these assumptions hold, the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are (1) unbiased, (2) consistent, and (3) normally distributed when the sample is large.

Key Terms

linear regression model with a single regressor (158)
dependent variable (158)
independent variable (158)
regressor (158)
population regression line (158)
population regression function (158)
population intercept (158)
population slope (158)
population coefficients (158)
parameters (158)
error term (158)
ordinary least squares (OLS) estimators (162)
OLS regression line (162)
sample regression line (162)
sample regression function (162)
predicted value (162)
residual (163)
regression R^2 (167)
explained sum of squares (ESS) (167)
total sum of squares (TSS) (167)
sum of squared residuals (SSR) (168)
standard error of the regression (SER) (168)
least squares assumptions (170)

Review the Concepts

4.1 What is a linear regression model? What is measured by the coefficients of a linear regression model? What is the ordinary least squares estimator?

4.2 Explain what is meant by an error term. What assumptions do we make about an error term when estimating an ordinary least squares regression?

4.3 What is meant by the assumption that the paired sample observations of Y_i and X_i are independently and identically distributed? Why is this an important assumption for ordinary least-squares estimation? When is this assumption likely to be violated?

4.4 Distinguish between the R^2 and the standard error of a regression. How do each of these measures describe the fit of a regression?

Exercises

4.1 Suppose that a researcher, using data on class size (CS) and average test scores from 50 third-grade classes, estimates the OLS regression:

$$\widehat{TestScore} = 640.3 - 4.93 \times CS, R^2 = 0.11, SER = 8.7.$$

a. A classroom has 25 students. What is the regression's prediction for that classroom's average test score?

b. Last year a classroom had 21 students, and this year it has 24 students. What is the regression's prediction for the change in the classroom average test score?

c. The sample average class size across the 50 classrooms is 22.8. What is the sample average of the test scores across the 50 classrooms? (*Hint:* Review the formulas for the OLS estimators.)

d. What is the sample standard deviation of test scores across the 50 classrooms? (*Hint:* Review the formulas for the R^2 and SER.)

4.2 Suppose a random sample of 100 20-year-old men is selected from a population and that these men's height and weight are recorded. A regression of weight on height yields

$$\widehat{Weight} = -79.24 + 4.16 \times Height, R^2 = 0.72, SER = 12.6,$$

where *Weight* is measured in pounds and *Height* is measured in inches.

a. What is the regression's weight prediction for someone who is (i) 64 in. tall, (ii) 68 in. tall, (iii) 72 in. tall?

b. A man has a late growth spurt and grows 2 in. over the course of a year. What is the regression's prediction for the increase in this man's weight?

c. Suppose, that instead of measuring *Weight* and *Height* in pounds and inches, these variables are measured in centimeters and kilograms. What are the regression estimates from this new centimeter-kilogram regression? (Give all results, estimated coefficients, R^2, and *SER*.)

4.3 A regression of average weekly earnings (*AWE*, measured in dollars) on age (measured in years) using a random sample of college-educated full-time workers aged 25–65 yields the following:

$$\widehat{AWE} = 696.7 + 9.6 \times Age, R^2 = 0.023, SER = 624.1.$$

a. Explain what the coefficient values 696.7 and 9.6 mean.

b. The standard error of the regression (*SER*) is 624.1. What are the units of measurement for the *SER*? (Dollars? Years? Or is *SER* unit-free?)

c. The regression R^2 is 0.023. What are the units of measurement for the R^2? (Dollars? Years? Or is R^2 unit-free?)

d. What does the regression predict will be the earnings for a 25-year-old worker? For a 45-year-old worker?

e. Will the regression give reliable predictions for a 99-year-old worker? Why or why not?

f. Given what you know about the distribution of earnings, do you think it is plausible that the distribution of errors in the regression is normal? (*Hint:* Do you think that the distribution is symmetric or skewed? What is the smallest value of earnings, and is it consistent with a normal distribution?)

g. The average age in this sample is 41.6 years. What is the average value of *AWE* in the sample? (*Hint:* Review Key Concept 4.2.)

4.4 Read the box "The 'Beta' of a Stock" in Section 4.2.

a. Suppose that the value of β is greater than 1 for a particular stock. Show that the variance of $(R - R_f)$ for this stock is greater than the variance of $(R_m - R_t)$.

b. Suppose that the value of β is less than 1 for a particular stock. Is it possible that variance of $(R - R_f)$ for this stock is greater than the variance of $(R_m - R_t)$? (*Hint:* Don't forget the regression error.)

c. In a given year, the rate of return on 3-month Treasury bills is 2.0% and the rate of return on a large diversified portfolio of stocks (the S&P 500) is 5.3%. For each company listed in the table in the box, use the estimated value of β to estimate the stock's expected rate of return.

4.5 A researcher runs an experiment to measure the impact a short nap has on memory. 200 participants in the sample are allowed to take a short nap of either 60 minutes or 75 minutes. After waking up, each of the participants takes a short test on short-term recall. Each participant is randomly assigned one of the examination times, based on the flip of a coin. Let Y_i denote the number of points scored on the test by the i^{th} participant ($0 \ldots Y_i \ldots 100$), let X_i denote the amount of time for which the participant slept prior to taking the test ($X_i = 60$ or 75), and consider the regression model $Y_i = b_0 + b_1 X_i + u_i$.

a. Explain what the term u_i represents. Why will different students have different values of u_i?

b. What is $E(u_i|X_i)$? Are the estimated coefficients unbiased?

c. What concerns might you have about ensuring compliance among participants?

d. The estimated regression is $Y_i = 55 + 0.17\,X_i$.

i. Compute the estimated regression's prediction for the average score of participants who slept for 60 minutes before taking the test. Repeat for 75 minutes and 90 minutes.

ii. Compute the estimated gain in score for a participant who is given an additional 5 minutes on the exam.

4.6 Show that the first least squares assumption, $E(u_i|X_i) = 0$, implies that $E(Y_i|X_i) = \beta_0 + \beta_1 X_i$.

4.7 Show that $\hat{\beta}_0$ is an unbiased estimator of β_0. (*Hint:* Use the fact that $\hat{\beta}_1$ is unbiased, which is shown in Appendix 4.3.)

4.8 Suppose that all of the regression assumptions in Key Concept 4.3 are satisfied except that the first assumption is replaced with $E(u_i|X_i) = 2$. Which parts of Key Concept 4.4 continue to hold? Which change? Why? (Is $\hat{\beta}_1$ normally distributed in large samples with mean and variance given in Key Concept 4.4? What about $\hat{\beta}_0$?)

4.9 **a.** A linear regression yields $\hat{\beta}_1 = 0$. Show that $R^2 = 0$.

b. A linear regression yields $R^2 = 0$. Does this imply that $\hat{\beta}_1 = 0$?

4.10 Suppose that $Y_i = \beta_0 + \beta_1 X_i + u_i$, where (X_i, u_i) are i.i.d., and X_i is a Bernoulli random variable with $\Pr(X = 1) = 0.20$. When $X = 1, u_i$ is $N(0, 4)$; when $X = 0, u_i$ is $N(0, 1)$.

a. Show that the regression assumptions in Key Concept 4.3 are satisfied.

b. Derive an expression for the large-sample variance of $\hat{\beta}_1$. [*Hint:* Evaluate the terms in Equation (4.21).]

4.11 Consider the regression model $Y_i = \beta_0 + \beta_1 X_i + u_i$.

a. Suppose you know that $\beta_0 = 0$. Derive a formula for the least squares estimator of β_1.

b. Suppose you know that $\beta_0 = 4$. Derive a formula for the least squares estimator of β_1.

4.12 **a.** Show that the regression R^2 in the regression of Y on X is the squared value of the sample correlation between X and Y. That is, show that $R^2 = r_{XY}^2$.

b. Show that the R^2 from the regression of Y on X is the same as the R^2 from the regression of X on Y.

c. Show that $\hat{\beta}_1 = r_{XY}(s_Y/s_X)$, where r_{XY} is the sample correlation between X and Y, and s_X and s_Y are the sample standard deviations of X and Y.

4.13 Suppose that $Y_i = \beta_0 + \beta_1 X_i + \kappa u_i$, where κ is a nonzero constant and (Y_i, X_i) satisfy the three least squares assumptions. Show that the large sample variance of $\hat{\beta}_1$ is given by $\sigma_{\hat{\beta}_1}^2 = \kappa^2 \frac{1}{n} \frac{\text{var}[(X_i - \mu_X)u_i]}{[\text{var}(X_i)^2]}$. [*Hint:* This equation is the variance given in Equation (4.21) multiplied by κ^2.]

4.14 Show that the sample regression line passes through the point $(\overline{X}, \overline{Y})$.

Empirical Exercises

(Only two empirical exercises for this chapter are given in the text, but you can find more on the text website, **www.pearsonglobaleditions.com/Stock_Watson**.)

E4.1 On the text website, **www.pearsonglobaleditions.com/Stock_Watson**, you will find the data file **Growth**, which contains data on average growth rates from 1960 through 1995 for 65 countries, along with variables that are potentially related to growth. A detailed description is given in

Growth_Description, also available on the website. In this exercise, you will investigate the relationship between growth and trade.[1]

a. Construct a scatterplot of average annual growth rate (*Growth*) on the average trade share (*TradeShare*). Does there appear to be a relationship between the variables?

b. One country, Malta, has a trade share much larger than the other countries. Find Malta on the scatterplot. Does Malta look like an outlier?

c. Using all observations, run a regression of *Growth* on *TradeShare*. What is the estimated slope? What is the estimated intercept? Use the regression to predict the growth rate for a country with a trade share of 0.5 and with a trade share equal to 1.0.

d. Estimate the same regression, excluding the data from Malta. Answer the same questions in (c).

e. Plot the estimated regression functions from (c) and (d). Using the scatterplot in (a), explain why the regression function that includes Malta is steeper than the regression function that excludes Malta.

f. Where is Malta? Why is the Malta trade share so large? Should Malta be included or excluded from the analysis?

E4.2 On the text website, **www.pearsonglobaleditions.com/Stock_Watson**, you will find the data file **Earnings_and_Height**, which contains data on earnings, height, and other characteristics of a random sample of U.S. workers.[2] A detailed description is given in **Earnings_and_Height_Description**, also available on the website. In this exercise, you will investigate the relationship between earnings and height.

a. What is the median value of height in the sample?

b. i. Estimate average earnings for workers whose height is at most 67 inches.

ii. Estimate average earnings for workers whose height is greater than 67 inches.

[1]These data were provided by Professor Ross Levine of the University of California at Berkeley and were used in his paper with Thorsten Beck and Norman Loayza, "Finance and the Sources of Growth," *Journal of Financial Economics*, 2000, 58: 261–300.

[2]These data were provided by Professors Anne Case (Princeton University) and Christina Paxson (Brown University) and were used in their paper "Stature and Status: Height, Ability, and Labor Market Outcomes," *Journal of Political Economy*, 2008, 116(3): 499–532.

iii. On average, do taller workers earn more than shorter workers? How much more? What is a 95% confidence interval for the difference in average earnings?

c. Construct a scatterplot of annual earnings (*Earnings*) on height (*Height*). Notice that the points on the plot fall along horizontal lines. (There are only 23 distinct values of *Earnings*). Why? (*Hint:* Carefully read the detailed data description.)

d. Run a regression of *Earnings* on *Height*.

i. What is the estimated slope?

ii. Use the estimated regression to predict earnings for a worker who is 67 inches tall, for a worker who is 70 inches tall, and for a worker who is 65 inches tall.

e. Suppose height were measured in centimeters instead of inches. Answer the following questions about the *Earnings* on *Height* (in cm) regression.

i. What is the estimated slope of the regression?

ii. What is the estimated intercept?

iii. What is the R^2?

iv. What is the standard error of the regression?

f. Run a regression of *Earnings* on *Height*, using data for female workers only.

i. What is the estimated slope?

ii. A randomly selected woman is 1 inch taller than the average woman in the sample. Would you predict her earnings to be higher or lower than the average earnings for women in the sample? By how much?

g. Repeat (f) for male workers.

h. Do you think that height is uncorrelated with other factors that cause earning? That is, do you think that the regression error term, say u_i, has a conditional mean of zero, given *Height* (X_i)? (You will investigate this more in the *Earnings* and *Height* exercises in later chapters.)

APPENDIX

4.1 The California Test Score Data Set

The California Standardized Testing and Reporting data set contains data on test performance, school characteristics, and student demographic backgrounds. The data used here are from all 420 K–6 and K–8 districts in California with data available for 1999. Test scores are the average of the reading and math scores on the Stanford 9 Achievement Test, a standardized test administered to fifth-grade students. School characteristics (averaged across the district) include enrollment, number of teachers (measured as "full-time equivalents"), number of computers per classroom, and expenditures per student. The student–teacher ratio used here is the number of students in the district divided by the number of full-time equivalent teachers. Demographic variables for the students also are averaged across the district. The demographic variables include the percentage of students who are in the public assistance program CalWorks (formerly AFDC), the percentage of students who qualify for a reduced-price lunch, and the percentage of students who are English learners (that is, students for whom English is a second language). All of these data were obtained from the California Department of Education (www.cde.ca.gov).

APPENDIX

4.2 Derivation of the OLS Estimators

This appendix uses calculus to derive the formulas for the OLS estimators given in Key Concept 4.2. To minimize the sum of squared prediction mistakes $\sum_{i=1}^{n}(Y_i - b_0 - b_1X_i)^2$ [Equation (4.6)], first take the partial derivatives with respect to b_0 and b_1:

$$\frac{\partial}{\partial b_0}\sum_{i=1}^{n}(Y_i - b_0 - b_1X_i)^2 = -2\sum_{i=1}^{n}(Y_i - b_0 - b_1X_i) \text{ and} \tag{4.23}$$

$$\frac{\partial}{\partial b_1}\sum_{i=1}^{n}(Y_i - b_0 - b_1X_i)^2 = -2\sum_{i=1}^{n}(Y_i - b_0 - b_1X_i)X_i. \tag{4.24}$$

The OLS estimators, $\hat{\beta}_0$ and $\hat{\beta}_1$, are the values of b_0 and b_1 that minimize $\sum_{i=1}^{n}(Y_i - b_0 - b_1X_i)^2$, or, equivalently, the values of b_0 and b_1 for which the derivatives in Equations (4.23) and (4.24) equal zero. Accordingly, setting these derivatives equal to

zero, collecting terms, and dividing by n shows that the OLS estimators, $\hat{\beta}_0$ and $\hat{\beta}_1$, must satisfy the two equations

$$\overline{Y} - \hat{\beta}_0 - \hat{\beta}_1\overline{X} = 0 \text{ and} \tag{4.25}$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \hat{\beta}_0\overline{X} - \hat{\beta}_1\frac{1}{n}\sum_{i=1}^{n} X_i^2 = 0. \tag{4.26}$$

Solving this pair of equations for $\hat{\beta}_0$ and $\hat{\beta}_1$ yields

$$\hat{\beta}_1 = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i Y_i - \overline{X}\,\overline{Y}}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - (\overline{X})^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2} \text{ and} \tag{4.27}$$

$$\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1\overline{X}. \tag{4.28}$$

Equations (4.27) and (4.28) are the formulas for $\hat{\beta}_0$ and $\hat{\beta}_1$ given in Key Concept 4.2; the formula $\hat{\beta}_1 = s_{XY}/s_X^2$ is obtained by dividing the numerator and denominator in Equation (4.27) by $n - 1$.

APPENDIX

4.3 Sampling Distribution of the OLS Estimator

In this appendix, we show that the OLS estimator $\hat{\beta}_1$ is unbiased and, in large samples, has the normal sampling distribution given in Key Concept 4.4.

Representation of $\hat{\beta}_1$ in Terms of the Regressors and Errors

We start by providing an expression for $\hat{\beta}_1$ in terms of the regressors and errors. Because $Y_i = \beta_0 + \beta_1 X_i + u_i$, $Y_i - \overline{Y} = \beta_1(X_i - \overline{X}) + u_i - \overline{u}$, so the numerator of the formula for $\hat{\beta}_1$ in Equation (4.27) is

$$\begin{aligned}\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y}) &= \sum_{i=1}^{n}(X_i - \overline{X})[\beta_1(X_i - \overline{X}) + (u_i - \overline{u})] \\ &= \beta_1\sum_{i=1}^{n}(X_i - \overline{X})^2 + \sum_{i=1}^{n}(X_i - \overline{X})(u_i - \overline{u}).\end{aligned} \tag{4.29}$$

Now $\sum_{i=1}^{n}(X_i - \overline{X})(u_i - \overline{u}) = \sum_{i=1}^{n}(X_i - \overline{X})u_i - \sum_{i=1}^{n}(X_i - \overline{X})\overline{u} = \sum_{i=1}^{n}(X_i - \overline{X})u_i$, where the final equality follows from the definition of $\overline{X}$, which implies that $\sum_{i=1}^{n}(X_i - \overline{X})\overline{u} = [\sum_{i=1}^{n}X_i - n\overline{X}]\overline{u} = 0$. Substituting $\sum_{i=1}^{n}(X_i - \overline{X})(u_i - \overline{u}) = \sum_{i=1}^{n}(X_i - \overline{X})u_i$ into the final expression in Equation (4.29) yields $\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y}) = \beta_1\sum_{i=1}^{n}(X_i - \overline{X})^2 + \sum_{i=1}^{n}(X_i - \overline{X})u_i$. Substituting this expression in turn into the formula for $\hat{\beta}_1$ in Equation (4.27) yields

$$\hat{\beta}_1 = \beta_1 + \frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})u_i}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2}. \tag{4.30}$$

Proof That $\hat{\beta}_1$ Is Unbiased

The expectation of $\hat{\beta}_1$ is obtained by taking the expectation of both sides of Equation (4.30). Thus,

$$\begin{aligned} E(\hat{\beta}_1) &= \beta_1 + E\left[\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})u_i}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2}\right] \\ &= \beta_1 + E\left[\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})E(u_i|X_i, \ldots, X_n)}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2}\right] = \beta_1, \end{aligned} \tag{4.31}$$

where the second equality in Equation (4.31) follows by using the law of iterated expectations (Section 2.3). By the second least squares assumption, u_i is distributed independently of X for all observations other than i, so $E(u_i|X_1, \ldots, X_n) = E(u_i|X_i)$. By the first least squares assumption, however, $E(u_i|X_i) = 0$. It follows that the conditional expectation in large brackets in the second line of Equation (4.31) is zero, so that $E(\hat{\beta}_1 - \beta_1|X_1, \ldots X_n) = 0$. Equivalently, $E(\hat{\beta}_1|X_1, \ldots, X_n) = \beta_1$; that is, $\hat{\beta}_1$ is conditionally unbiased, given $X_1, \ldots, X_n$. By the law of iterated expectations, $E(\hat{\beta}_1 - \beta_1) = E[E(\hat{\beta}_1 - \beta_1 | X_1, \ldots, X_n)] = 0$, so that $E(\hat{\beta}_1) = \beta_1$; that is, $\hat{\beta}_1$ is unbiased.

Large-Sample Normal Distribution of the OLS Estimator

The large-sample normal approximation to the limiting distribution of $\hat{\beta}_1$ (Key Concept 4.4) is obtained by considering the behavior of the final term in Equation (4.30).

First consider the numerator of this term. Because $\overline{X}$ is consistent, if the sample size is large, $\overline{X}$ is nearly equal to μ_X. Thus, to a close approximation, the term in the numerator of Equation (4.30) is the sample average $\overline{v}$, where $v_i = (X_i - \mu_X)u_i$. By the first least squares assumption, v_i has a mean of zero. By the second least squares assumption, v_i is i.i.d. The variance of v_i is $\sigma_v^2 = [\text{var}(X_i - \mu_X)u_i]$, which, by the third least squares assumption, is nonzero and finite. Therefore, $\overline{v}$ satisfies all the requirements of the central limit theorem (Key Concept 2.7). Thus $\overline{v}/\sigma_{\overline{v}}$ is, in large samples, distributed $N(0, 1)$, where $\sigma_{\overline{v}}^2 = \sigma_v^2/n$. Thus the distribution of $\overline{v}$ is well approximated by the $N(0, \sigma_v^2/n)$ distribution.

Next consider the expression in the denominator in Equation (4.30); this is the sample variance of X (except dividing by n rather than $n - 1$, which is inconsequential if n is large). As discussed in Section 3.2 [Equation (3.8)], the sample variance is a consistent estimator of the population variance, so in large samples it is arbitrarily close to the population variance of X.

Combining these two results, we have that, in large samples, $\hat{\beta}_1 - \beta_1 \cong \overline{v}/\text{var}(X_i)$, so that the sampling distribution of $\hat{\beta}_1$ is, in large samples, $N(\beta_1, \sigma_{\hat{\beta}_1}^2)$, where $\sigma_{\hat{\beta}_1}^2 = \text{var}(\overline{v})/[\text{var}(X_i)]^2 = \text{var}[(X_i - \mu_X)u_i]/\{n[\text{var}(X_i)]^2\}$, which is the expression in Equation (4.21).

Some Additional Algebraic Facts About OLS

The OLS residuals and predicted values satisfy

$$\frac{1}{n}\sum_{i=1}^{n}\hat{u}_i = 0, \tag{4.32}$$

$$\frac{1}{n}\sum_{i=1}^{n}\hat{Y}_i = \overline{Y}, \tag{4.33}$$

$$\sum_{i=1}^{n}\hat{u}_i X_i = 0 \text{ and } s_{\hat{u}X} = 0, \text{ and} \tag{4.34}$$

$$TSS = SSR + ESS. \tag{4.35}$$

Equations (4.32) through (4.35) say that the sample average of the OLS residuals is zero; the sample average of the OLS predicted values equals $\overline{Y}$; the sample covariance $s_{\hat{u}X}$ between the OLS residuals and the regressors is zero; and the total sum of squares is the sum of squared residuals and the explained sum of squares. [The *ESS*, *TSS*, and *SSR* are defined in Equations (4.14), (4.15), and (4.17).]

To verify Equation (4.32), note that the definition of $\hat{\beta}_0$ lets us write the OLS residuals as $\hat{u}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i = (Y_i - \overline{Y}) - \hat{\beta}_1(X_i - \overline{X})$; thus

$$\sum_{i=1}^{n}\hat{u}_i = \sum_{i=1}^{n}(Y_i - \overline{Y}) - \hat{\beta}_1\sum_{i=1}^{n}(X_i - \overline{X}).$$

But the definitions of $\overline{Y}$ and $\overline{X}$ imply that $\sum_{i=1}^{n}(Y_i - \overline{Y}) = 0$ and $\sum_{i=1}^{n}(X_i - \overline{X}) = 0$, so $\sum_{i=1}^{n}\hat{u}_i = 0$.

To verify Equation (4.33), note that $Y_i = \hat{Y}_i + \hat{u}_i$, so $\sum_{i=1}^n Y_i = \sum_{i=1}^n \hat{Y}_i + \sum_{i=1}^n \hat{u}_1 = \sum_{i=1}^n \hat{Y}_i$, where the second equality is a consequence of Equation (4.32).

To verify Equation (4.34), note that $\sum_{i=1}^n \hat{u}_i = 0$ implies $\sum_{i=1}^n \hat{u}_i X_i = \sum_{i=1}^n \hat{u}_i(X_i - \overline{X})$, so

$$\begin{aligned}\sum_{i=1}^n \hat{u}_i X_i &= \sum_{i=1}^n [(Y_i - \overline{Y}) - \hat{\beta}_1(X_i - \overline{X})](X_i - \overline{X}) \\ &= \sum_{i=1}^n (Y_i - \overline{Y})(X_i - \overline{X}) - \hat{\beta}_1 \sum_{i=1}^n (X_i - \overline{X})^2 = 0, \end{aligned} \tag{4.36}$$

where the final equality in Equation (4.36) is obtained using the formula for $\hat{\beta}_1$ in Equation (4.27). This result, combined with the preceding results, implies that $s_{\hat{u}X} = 0$.

Equation (4.35) follows from the previous results and some algebra:

$$\begin{aligned} TSS &= \sum_{i=1}^n (Y_i - \overline{Y})^2 = \sum_{i=1}^n (Y_i - \hat{Y}_i + \hat{Y}_i - \overline{Y})^2 \\ &= \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 + \sum_{i=1}^n (\hat{Y}_i - \overline{Y})^2 + 2\sum_{i=1}^n (Y_i - \hat{Y}_i)(\hat{Y}_i - \overline{Y}) \\ &= SSR + ESS + 2\sum_{i=1}^n \hat{u}_i \hat{Y}_i = SSR + ESS, \end{aligned} \tag{4.37}$$

where the final equality follows from $\sum_{i=1}^n \hat{u}_i \hat{Y}_i = \sum_{i=1}^n \hat{u}_i(\hat{\beta}_0 + \hat{\beta}_1 X_i) = \hat{\beta}_0 \sum_{i=1}^n \hat{u}_i + \hat{\beta}_1 \sum_{i=1}^n \hat{u}_i X_i = 0$ by the previous results.