2015-1

PONTIFICIA UNIVERSIDAD CATÓLICA DEL PERÚ
FACULTAD DE CIENCIAS SOCIALES
ESPECIALIDAD DE ECONOMÍA

ESTADÍSTICA INFERENCIAL
PRÁCTICA CALIFICADA 1

Ejercicio 1. **(5 puntos)**

Sea el vector aleatorio mixto (X, Y) con modelo probabilístico conjunto dado por

$$f_{X,Y}(x, y) = \begin{cases} 6y, & 0 < x + y < 1,\ 0 < x < 1,\ 0 < y < 1; \\ 0, & \text{en otro caso.} \end{cases}$$

a) Determine $E(XY)$. (2 puntos)

b) Determine el modelo marginal de X. (2 puntos)

Ejercicio 2. **(5 puntos)**

Sean X e Y tales que $Y \sim G(4; 2)$ y $X|Y = y \sim P(y)$, $\forall y > 0$.

a) Halle $E(XY)$. Previamente determine el modelo conjunto de X e Y. (2 puntos)

b) Halle el modelo condicional $Y|X = x$. (3 puntos)

Ejercicio 3. $f_Y(y)$ $f_{X|Y=y}$ **(4 puntos)**

X e Y son tales que $Y \sim \mathrm{B}(\alpha; \beta), 0 < y < 1$, y $X|Y = y \sim b(n, y)$. Determine $E(Y|X = x)$, $x = 0, \ldots, n$. Antes determine el modelo condicional de Y dado $X = x$.

Ejercicio 4. **(6 puntos)**

Considere el modelo de regresión lineal dado por

$$Y_j = \alpha + \beta x_j + \epsilon_j\,, \quad \text{para } j = 1, \ldots, n,$$

donde $\epsilon_1, \ldots, \epsilon_n$ son variables aleatorias tales que $E(\epsilon_j) = 0$, para $j = 1, \ldots, n.$, α y β son constantes desconocidas (para estimar) y $x_1, \ldots, x_n$ son constantes conocidas.

Como estimadores de los parámetros se proponen los siguientes:

$$\hat{\beta} = \sum_{j=1}^{n} b_j Y_j\,, \text{ donde } b_j = \frac{x_j - \bar{X}}{\sum_{i=1}^{n}(x_i - \bar{X})^2}\,, \text{ para } j = 1, \ldots, n.$$

$$\hat{\alpha} = \sum_{j=1}^{n} a_j Y_j\,, \text{ donde } a_j = \frac{1}{n} - b_j \bar{X}\,, \text{para } j = 1, \ldots, n.$$

a) Halle $E(Y_j)$, para $j = 1, \ldots, n$. (1 punto)

b) Halle expresiones simplificadas para $E(\hat{\beta})$ y $E(\hat{\alpha})$. (4 puntos)

c) Para $k \in \{1, \ldots, n\}$, se define $\hat{Y}_k = \hat{\alpha} + \hat{\beta} x_k$ determine una expresión simplificada para $E(\hat{Y}_k)$. (1 puntos)

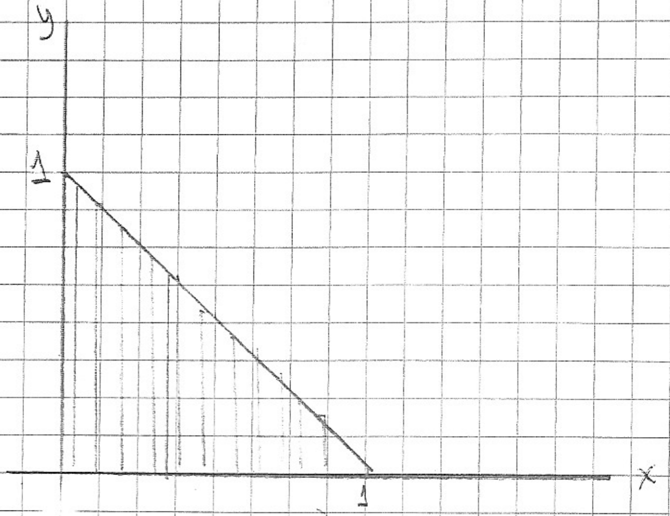
J.F. San Miguel, 11 de abril de 2015.

PC1

11 abril 2015

① $f_{x,y}(x,y) = \begin{cases} 6y & ; \quad 0 < x+y < 1 \quad ; \quad 0 < x < 1 \quad ; \quad 0 < y < 1 \\ 0 & ; \quad \text{otro caso} \end{cases}$

a) $E(xy)$



$$E(xy) = \int_0^1 \int_0^{1} xy\,6y\,dx\,dy = \int_0^1 6y^2 \left(\int_0^1 x\,dx\right) dy$$

$$\int_0^1 6y^2 \left(\left(\frac{x^2}{2}\right)\Big|_0^1\right) dy$$

$$\int_0^1 3y^2\,dy = 3\left(\frac{y^3}{3}\right)\Big|_0^1 = 1$$

0.5

b) Hallar $f_x(x)$

$$f_x(x) = \int_0^{1-x} 6y\,dy = 6\left(\frac{y^2}{2}\right)\Big|_0^{1-x} = \frac{6}{2}\left[(1-x)^2 - 0\right]$$

$$\boxed{f_x(x) = 3(1-x)^2}$$

1

2

② $Y \sim G(4,2)$; $X/_{Y=y} \sim P(y)$; $\forall y > 0$

$$f_Y(y) = \frac{(2)^4}{\Gamma(4)} \cdot x^3 e^{-2x} = \frac{16}{6} x^3 e^{-2x} = \frac{8}{3} x^3 e^{-2x}$$

$$f_{X/Y=y}(x) = e^{-y} \cdot \frac{y^x}{x!}$$

a) Hallar $E(XY)$; Determinar previamente modelo conjunto.

$$f(x,y) = f_Y(y) \cdot f_{X/Y=y}(x) = \left(\frac{8}{3} x^3 e^{-2x}\right)\left(e^{-y} \cdot \frac{y^x}{x!}\right) = \frac{8}{3} x^3 e^{-(2x+y)} \frac{y^x}{x!}$$

$$E(XY) = \int_0^{\infty}\int_{-\infty}^{+\infty} xy \, \frac{8}{3} x^3 e^{-(2x+y)} \frac{y^x}{x!} \, dx \, dy$$

¡Sumatoria!

$$E(XY) = \int_{-\infty}^{\infty} \frac{8}{3} x^4 e^{-2x} \left(\frac{1}{x!}\right)\left(\underbrace{\int_0^{\infty} y\, e^{-y} y^x \, dy}_{\text{gamma: } \alpha = x+1,\ \beta = 1}\right) dx$$

$$E(XY) = \int_{-\infty}^{\infty} \frac{8}{3} x^4 e^{-2x} \left(\frac{1}{x!}\right)\left(\int_0^{\infty} \frac{(1)^{x+1}}{(1)^{x+1}} \cdot \frac{\Gamma(x+1)}{\Gamma(x+1)} \cdot y\, e^{-y} y^x \, dy\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{8}{3} x^4 e^{-2x} \left(\frac{1}{x!}\right) \cdot \frac{\Gamma(x+1)}{(1)^{x+1}} \cdot \frac{x+1}{1} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{8}{3} x^4 e^{-2x} (x+1) \, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{8}{3} x^5 e^{-2x}\right) + \left(\frac{8}{3} x^4 e^{-2x}\right) dx$$

$$= \underbrace{\int_{-\infty}^{\infty} \frac{8}{3} x^5 e^{-2x} \, dx}_{\text{gamma: } \beta = 2,\ \alpha = 6} + \underbrace{\int_{-\infty}^{\infty} \frac{8}{3} x^4 e^{-2x} \, dx}_{\text{gamma: } \beta = 2,\ \alpha = 5}$$

$$= \int_{-\infty}^{\infty} \frac{8}{3} \left(\frac{64}{64}\right)\left(\frac{\Gamma(6)}{\Gamma(6)}\right) x^5 e^{-2x} dx + \int_{-\infty}^{\infty} \frac{8}{3} \left(\frac{32}{32}\right)\left(\frac{\Gamma(5)}{\Gamma(5)}\right) x^4 e^{-2x} \, dx$$

$$= \left(\frac{8}{3}\right)\left(\frac{1}{64}\right) \Gamma(6) + \frac{8}{3}\left(\frac{1}{32}\right) \Gamma(5)$$

$$= 5 + 2 = 7 = E(XY)$$

b) Hallar $Y/X = x$

$$f_{Y/X=x}(y) = \frac{f(x,y)}{f_X(x)}$$

pero por ley proporcionalidad

$$f_{Y/X=x}(y) \propto f(x,y)$$

$$f_{Y/X=x}(y) \propto e^{-y} y^x \rightsquigarrow \text{gamma: } \beta = 1,\ \alpha = x+1$$

$$\boxed{Y/X = x \sim G(x+1;\, 1)}$$

$$\boxed{f_{Y|X=x}(y) = \frac{(1)^{x+1}}{\Gamma(x+1)} \cdot y^x e^{-y}}$$

③ $Y \sim B(\alpha, B)$; $0 < y < 1$; $X|_{Y=y} \sim b(n, y)$

Hallar $E(Y/X=x)$, $x = 0, \dots; n$: Antes hallar $f_{Y/X=x}(y)$

⊙ $f_Y(y) = \frac{\Gamma(\alpha+B)}{\Gamma(\alpha)\,\Gamma(B)} \, y^{\alpha-1} (1-y)^{B-1}$

⊙ $f_{X|Y=y}(x) = \binom{n}{x} y^x (1-y)^{n-x}$

$\to$ $f(x,y) = f_Y(y) \cdot f_{X|Y=y}(x) = \frac{\Gamma(\alpha+B)}{\Gamma(\alpha)\,\Gamma(B)} \, y^{\alpha-1} (1-y)^{B-1} \cdot \binom{n}{x} y^x (1-y)^{n-x}$

$$f(x,y) = \frac{\Gamma(\alpha+B)}{\Gamma(\alpha)\,\Gamma(B)} \binom{n}{x} y^{x+\alpha-1} (1-y)^{B+n-x-1}$$

⊛ $f_{Y|X=x}(y) \propto f(x,y)$

$$f_{Y|X=x}(y) \propto \underbrace{y^{x+\alpha-1} (1-y)^{B+n-x-1}}_{\text{BetA :}}$$

$$\boxed{\begin{gathered} Y/_{X=x} \sim B(x+\alpha \,;\, B+n-x) \\ f_{Y|X=x}(y) = \frac{\Gamma(\alpha+B+n)}{\Gamma(x+\alpha)\,\Gamma(B+n-x)} \cdot (y)^{x+\alpha-1} (1-y)^{B+n-x-1} \end{gathered}}$$

identifica la distribución luego aplica la tabla de distribución

$$E(Y|X=x) = \int_0^1 y \, f_{Y|X=x}(y) \, dy$$

$$= \int_0^1 (y) \frac{\Gamma(\alpha+\beta+n)}{\Gamma(x+\alpha)\,\Gamma(\beta+n-x)} \, y^{x+\alpha-1} (1-y)^{\beta+n-x-1} \, dy$$

$$= \int_0^1 \frac{\Gamma(\alpha+\beta+n)}{\Gamma(x+\alpha)\,\Gamma(\beta+n-x)} \, \underbrace{y^{x+\alpha} (1-y)^{\beta+n-x-1}}_{\text{Beta} \sim B(x+\alpha-1\,;\,\beta+n-x)} \, dy$$

$$= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(x+\alpha)\,\Gamma(\beta+n-x)} \int_0^1 \frac{\Gamma(\alpha+\beta+n-1)}{\Gamma(\alpha+\beta+n-1)} \cdot \frac{\Gamma(x+\alpha-1)}{\Gamma(x+\alpha-1)} \cdot \frac{\Gamma(\beta+n-x)}{\Gamma(\beta+n-x)} \, y^{x+\alpha} (1-y)^{\beta+n-x-1} \, d$$

$$= \frac{\Gamma(\alpha+\beta+n)}{\Gamma(x+\alpha)\,\Gamma(\beta+n-x)} \cdot \frac{\Gamma(x+\alpha-1)\,\Gamma(\beta+n-x)}{\Gamma(\alpha+\beta+n-1)}$$

$$=$$

$$= \frac{(\alpha+\beta+n-1)!}{(x+\alpha-1)!\,\cancel{(\beta+n-x-1)!}} \cdot \frac{(x+\alpha-2)!\,\cancel{(\beta+n-x-1)!}}{(\alpha+\beta+n-2)!}$$

$$\boxed{E(Y|X=x) = \frac{\alpha+\beta+n-1}{x+\alpha-1}}$$

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4) $Y_j = \alpha + \beta X_j + \varepsilon_j \longrightarrow E(\varepsilon_j) = 0 \quad ; \quad j = 1, \ldots, n.$

- α y β constantes desconocidas.
- $X_1, \ldots, X_n$ constantes conocidas

$$\hat{\beta} = \sum_{j=1}^{n} b_j Y_j \quad ; \quad b_j = \frac{X_j - \bar{X}}{\sum_{i=1}^{n} (X_i - \bar{X})^2}$$

$$\hat{\alpha} = \sum_{j=1}^{n} a_j Y_j \quad ; \quad a_j = \frac{1}{n} - b_j \bar{X}$$

a) Hallar $E(Y_j)$

$$E(Y_j) = E(\alpha + \beta X_j + \varepsilon_j)$$

$$= E(\alpha) + E(\beta X_j) + \underbrace{E(\varepsilon_j)}_{0}$$

$$\boxed{E(Y_j) = \alpha + \beta X_j}$$

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b) $E(\hat{B}) = E\left(\sum_{j=1}^{n} \frac{(x_j - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \cdot (\alpha + B x_j + \varepsilon_j)\right)$

$= E\left(\frac{\sum (\cancel{x_j - \bar{x}})(\alpha + B x_j + \varepsilon_j)}{\sum (x_i - \bar{x})^{\cancel{2}}}\right) = E\left(\frac{\sum (\alpha + B x_j + \varepsilon_j)}{\sum x_i - \bar{x}}\right)$

$= \frac{\sum^{n} (\alpha + B x_j)}{\sum (x_i - \bar{x})}$

$= \frac{n\alpha + B \sum x_j}{-n\bar{x} + \sum x_i}$

$= \alpha$

$E(\hat{\alpha}) = E\left(\sum_{j=1}^{n} \left(\frac{1}{n} - b_j \bar{x}\right)(\alpha + B x_j + \varepsilon_j)\right)$

$E\left(\sum_{j=1}^{n} \left(\frac{1}{n} - \frac{(x_j \bar{x} - \bar{x}^2)}{\sum (x_j - \bar{x})^2}\right)(\alpha + B x_j + \varepsilon_j)\right)$

$E\left(\sum_{j=1}^{n} \left(\frac{\alpha}{n} + \frac{B x_j}{n} + \underbrace{\frac{\varepsilon_j}{n}}_{0} - \frac{\alpha \bar{x}(x_j - \bar{x})}{\sum (x_i - \bar{x})^2} - \frac{B \bar{x}(x_j)(x_j - \bar{x})}{\sum (x_i - \bar{x})^2} - \underbrace{\frac{\varepsilon_j (\bar{x})(x_j - \bar{x})}{\sum (x_i - \bar{x})^2}}_{0}\right)\right)$

$= \alpha + B\bar{x} - \alpha - B\bar{x} + B$

$= B$

c) $E(\hat{y}_k) = E(\hat{\alpha} + \hat{B} X_k)$

$\boxed{E(\hat{y}_k) = B + \alpha X_k}$