Marking Sheet   Assignment 1   CS370 Winter 2013

Name: ....   ID:................

1.	(a) smallest, largest	(2)	________	
	(b) number of numbers	(1)	________	
	(c) fraction, distance	(2)	________	
	(d) fraction, distance	(2)	________	
				......./7
2.	(a) r1, r2, errors	(2)	________	
	(b) new r1	(2)	________	
	(c) improved algorithm	(2)	________	
	(d) improved r1, r2, errors	(2)	________	
	(e) improved r1, r2, errors	(2)	________	
				......./10
3.	stability analysis			
				......../5
4.	(a) quadratic interpolating spline	(2)	________	
	(b) plot	(2)	________	
				......../4
5.	not-a-knot spline			
				......../6
6.	(1) plots (2), code (2)	(4)	________	
	(2) plots (2), code (2)	(4)	________	
	(3) plots (2), code (6)	(8)	________	
				......../16

Total: (48)   ____________

1. (a) It is 0.1×2^{-1022}. $\blacksquare$

   (b) F has

$$
\begin{aligned}
|F| &= \underbrace{1}_{\text{for 0}} + \underbrace{1}_{\text{leading one}} \times \underbrace{2^{52}}_{\text{52 remaining digits}} \times \underbrace{(1022+1023)}_{\text{exponents of 2}} \\
&= 1 + 2025(2^{52})
\end{aligned}
$$

$\blacksquare$

   (c) Rewrite the inequality:

$$
1 \le x < 2 \iff 0.1 \times 2^1 \le x < 0.1 \times 2^2
$$

$$
\text{Fraction} = \frac{\underbrace{1}_{\text{leading one}} \times \underbrace{2^{52}}_{\text{52 remaining digits}} \times \underbrace{1}_{\text{one exponent}}}{|F|} \approx 0.0004938
$$

   The distance for consecutive numbers is $1 \times 2^{-53} \times 2 = 2^{-52}$ $\blacksquare$

   (d) Rewrite the inequality:

$$
\frac{1}{64} \le x < \frac{1}{32} \iff \frac{1}{2^6} \le x < \frac{1}{2^5} \iff 0.1 \times 2^{-5} \le x < 0.1 \times 2^{-4}
$$

$$
\text{Fraction} = \frac{1 \times 2^{52} \times 1}{|F|} \approx 0.0004938
$$

   The distance for consecutive numbers is $1 \times 2^{-53} \times 2^{-5} = 2^{-58}$ $\blacksquare$

2. (a) First caculate the parts:

$$
\begin{aligned}
b^2 &= 12345.4321 \approx 12345 \\
4ac &= 4 \times 1 \times 1.2121 = 4.8484 \\
b^2 - 4ac &= 12340.1516 \approx 12340 \\
\sqrt{b^2 - 4ac} &= 111.085552615990527825 \cdots \approx 111.09 \\
-b + \sqrt{b^2 - 4ac} &= -0.02 \\
-b - \sqrt{b^2 - 4ac} &= -222.2
\end{aligned}
$$

   Hence, $(fl(r_1), fl(r_2)) \approx (-0.01, -111.1)$. By calculator, $(r_1, r_2) \approx (-0.010910, -111.10)$. Hence, the errors are $(\delta_{r_1}, \delta_{r_2}) \approx (-0.083410, 0)$. $\blacksquare$

(b) Rationalize the numerator:

$$
\begin{aligned}
r_1 &= \frac{\sqrt{b^2 - 4ac}) - b}{2a} \\
&= \frac{(\sqrt{b^2 - 4ac}) - b) \times (\sqrt{b^2 - 4ac} + b)}{2a \times (\sqrt{b^2 - 4ac} + b)} \\
&= \frac{b^2 - 4ac - b^2}{2a \times (\sqrt{b^2 - 4ac} + b)} \\
&= \frac{-2c}{\sqrt{b^2 - 4ac} + b} \\
&= \frac{2c}{-b - \sqrt{b^2 - 4ac}}
\end{aligned}
$$

As required. ■

(c) Derive r_2 in term of r_1:

$$
\begin{aligned}
r_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(b + \sqrt{b^2 - 4ac})}{2a} \\
\implies &= \frac{-(b^2 - b^2 - 4ac)}{2a \times (b - \sqrt{b^2 - 4ac})} \\
&= \frac{2c}{b - \sqrt{b^2 - 4ac}} \\
\implies &= \frac{-2c}{-b + \sqrt{b^2 - 4ac}} \\
\implies &= \frac{-2c}{\frac{-b + \sqrt{b^2 - 4ac}}{2a} \times 2a} \\
\implies &= \frac{-c}{ar_1}
\end{aligned}
$$

Algorithm R.
if $b > 0$ then

$$
\begin{aligned}
r_2 &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\
r_1 &= \frac{c}{ar_2}
\end{aligned}
$$

else

$$
\begin{aligned}
r_1 &= \frac{\sqrt{b^2 - 4ac}) - b}{2a} \\
r_2 &= \frac{-c}{ar_1}
\end{aligned}
$$

end ■

(d)

$$r_2 = -111.1 \text{ (from before)}$$
$$r_1 = \frac{1.2121}{1 \times -111.1} = -0.010910$$

Hence, the errors $(\delta_{r_1}, \delta_{r_2}) \approx (0, 0)$. Unlike (a), the errors for (d) are close to 0. Hence, Algorithm R is a better algorithm than the one used in (a). ■

(e)

$$b^2 = 111.1^2 = 12343.21 \approx 12340$$
$$4ac = 4 \times 1 \times 1.212 \approx 4.848$$
$$b^2 - 4ac = 12335.152 \approx 12340$$
$$\sqrt{b^2 - 4ac} = 111.0855526 \cdots \approx 111.1$$
$$-b - \sqrt{b^2 - 4ac} = -222.2$$
$$r_2 = -111.1$$
$$r_1 = \frac{1.212}{1 \times -111.1} = -0.01090909 \cdots \approx 0.01090$$

Hence, the errors $(\delta_{r_1}, \delta_{r_2}) \approx (0, 0)$. Unlike (a), the errors for (d) are close to 0. Hence, Algorithm R is a better algorithm than the one used in (a). ■

3. Take the limit of p_n:

$$\lim_{n\to\infty} p_n = a \lim_{n\to\infty} 7^n + b \lim_{n\to\infty} \frac{1}{7^n}$$
$$= a \lim_{n\to\infty} 7^n + 0$$

But a, which is a constant based on p_0 and p_1, $a = \frac{1}{48}(7(\frac{1}{7}) - 1) = 0$. Hence the limit goes to zero. Therefore, the given computation is stable. ■

4. (a) Check whether $q(S) = \hat{f}(S)$ at $S1, S2, S3$, and continuity:

$$q(99.9) = 50(99.9 - 99.9)^2 = \hat{f}(99.9) = 0$$
$$q(100) = 50(100 - 99.9)^2 = 1 - 50(100 - 100.1)^2 = \hat{f}(100) = 0.5$$
$$q(100.1) = 1 - 50(100.1 - 100.1)^2 = \hat{f}(100.1) = 1$$

Hence q is continuous at $S1, S2, S3$ and contains the break-points. Now check the continuity of q':

$$q'(S) = \begin{cases} 0 & \text{if } S \leq 99.9 \\ 100(S - 99.9) & \text{if } 99.9 \leq S \leq 100 \\ -100(S - 100.1) & \text{if } 100 \leq S \leq 100.1 \\ 0 & \text{if } S \leq 99.9 \end{cases}$$

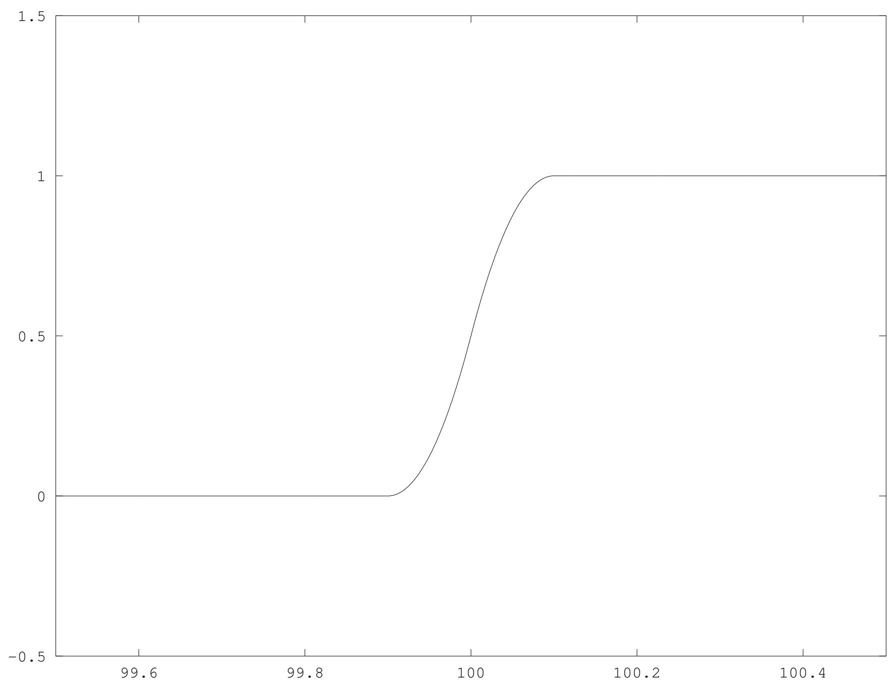
$$q'(99.9) = 100(99.9 - 99.9) = 0$$

$$q'(100) = 100(100 - 99.9) = -100(100 - 100.1) = 10$$

$$q'(100.1) = -100(100.1 - 100.1) = 0$$

Hence, q' is continuous at the break points. Also, since $\hat{f}'(S) = 0$ for all S, $q'(S_1) = 0 = \hat{f}'(S_1)$ and $q'(S_3) = 0 = \hat{f}'(S_3)$. Hence, the end conditions are satisfied. Therefore, q is a quadratic interpolating spline for $\hat{f}$ at S_1, S_2, S_3. ■

(b) The graph:



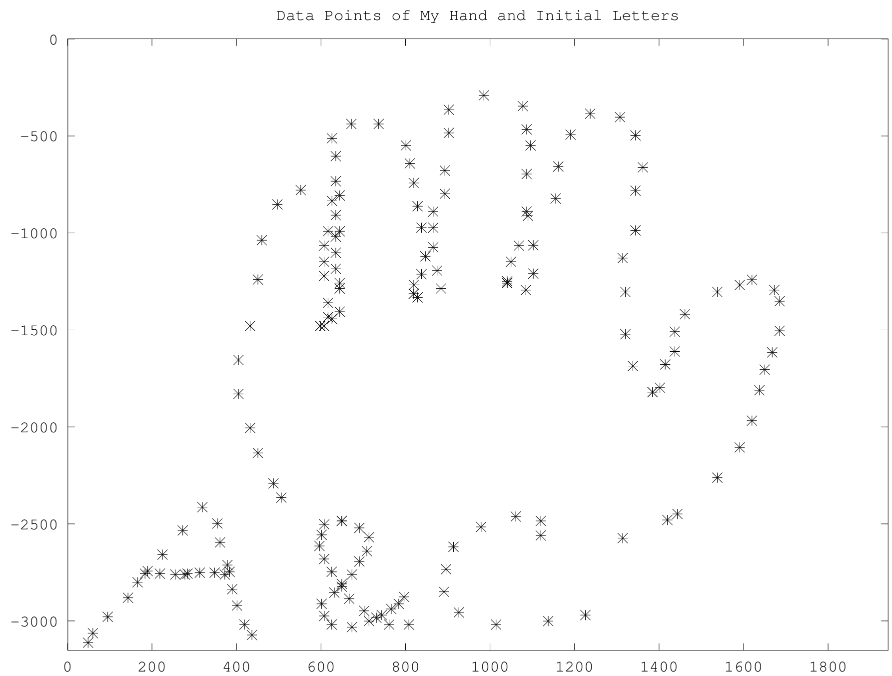
5. First, the end conditions of of not-a-knot cubic spline implies $S'''(x_1) = S'''(x_2)$ and $S'''(x_2) = S'''(x_3)$. Hence,

$$S'''(x_1) = S'''(x_2) = S'''(x_3)$$

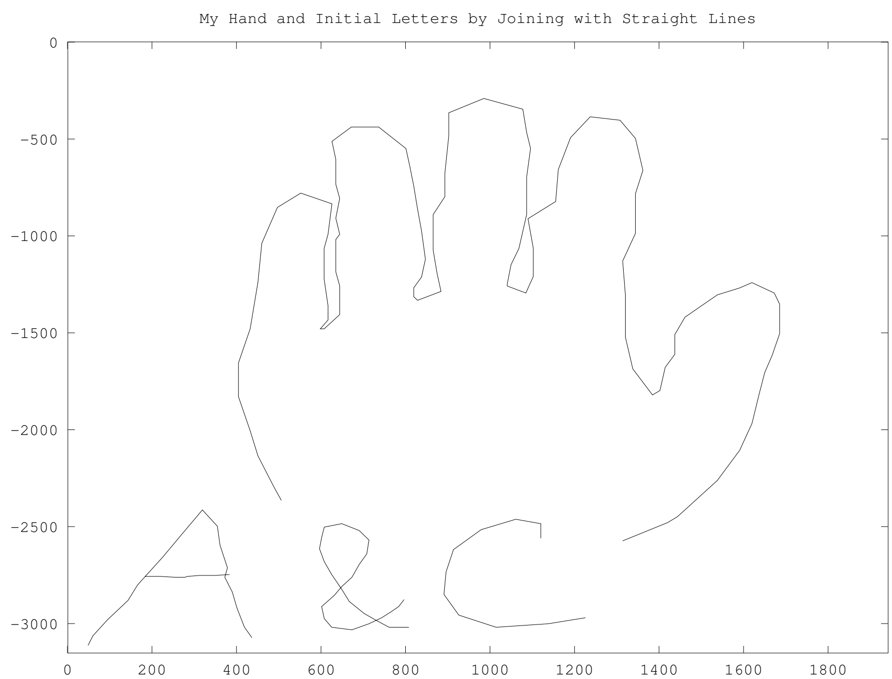
But $P'''(x) = 6$, a constant, which fits the end conditions of S. Secondly, P and its the first and second derivatives are continuous, because P is a polynomial (fact). Lastly, P maps all $(x_i, y_i), i \in \{1, 2, 3, 4\}$. Therefore, P is a not-a-knot spline interpolant for S. ■

6. The graphs:

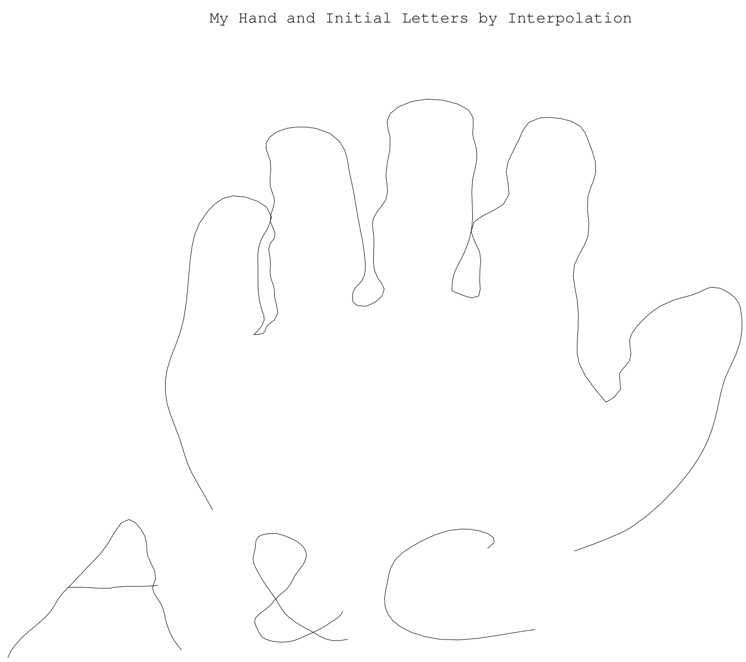
(a) The Data Points



(b) Data Points Connected With Straight Lines



(c) Data Points With Interpolation



The codes (assuming that the splines package is already being loaded):

- init.m (only used in extraction of data points)

```
data = imread('hand_final.png');
clf;
h = imagesc(data);
axis image;
[x, y] = ginput;
```

- findt.m (helper function for the generation of the t parameters)

```
function [t] = findt(x,y)
  len = length(x);
  t = zeros(1, len);
  for p = 2:length(x)
    prev = p-1;
    distance = sqrt((x(prev) - x(p))^2 + (y(prev)-y(p))^2);
    t(p) = t(prev) + distance;
  end
endfunction
```

- refinet.m (helper function for refining t)

```
function [tref] = refinet(t,fac)
  N = length(t);
  tref = zeros(1, fac*(N-1) + 1);
```

```
  for k = 1:N-1
    i = fac*(k-1)+1;
    dt = t(k+1)-t(k);
    for l = 0:fac-1
      tref(i+l) = t(k) + l*dt/fac;
    end
  end
  tref(fac*(N-1)+1) = t(N);
endfunction
```

- interpolate.m (helper function that returns the interpolated vectors)

```
function [tx ty] = interpolate(x,y,end_type,refine_fac)
  t = findt(x,y);
  ppx = csape(t,x,end_type);
  ppy = csape(t,y,end_type);
  tref = refinet(t,refine_fac);
  tx = ppval(ppx,tref);
  ty = ppval(ppy,tref);
endfunction
```

- a1q6_data.m (the data)

```
% initals data
x0 = [48.833 60.161 95.407 143.239 165.897 189.813 225.058 272.891 319.465 354.710 361.004 378.626 372.333 389.955 401.284 418.907 436.529];
y0 = [3112.3 3063.2 2979.2 2881.0 2800.7 2743.1 2658.2 2533.5 2413.6 2497.6 2595.8 2712.0 2761.1 2836.6 2920.7 3018.9 3072.7];
x1 = [183.52 218.76 255.27 277.93 284.22 313.17 348.42 383.66];
y1 = [2756.4 2756.4 2761.1 2761.1 2756.4 2751.6 2751.6 2746.9];
x2 = [649.26 607.72 601.43 596.39 607.72 625.34 649.26 666.88 702.13 731.08 761.29 807.86];
y2 = [2484.4 2502.3 2556.2 2613.8 2680.8 2746.9 2823.4 2885.7 2948.0 2983.9 3018.9 3018.9];
x3 = [796.53 783.95 766.32 743.67 713.46 673.18 625.34 607.72 601.43 631.64 649.26 673.18 690.80 708.42 713.46 690.80 649.26];
y3 = [2876.3 2912.2 2938.6 2969.8 3000.9 3032.1 3018.9 2974.5 2912.2 2854.6 2809.2 2761.1 2694.0 2640.2 2569.4 2520.3 2484.4];
x4 = [1120.03 1120.03 1060.87 979.05 913.60 895.97 890.94 926.18 1014.30 1137.66 1225.77];
```

```
y4 = [2559.9 2484.4 2461.7 2515.6 2618.5 2733.7 2849.8 2956.5 3018.9 3000.9 2969.8];

% hand data
x5 = [506.07 487.64 450.77 432.34 404.69 404.69 432.34 450.77 459.99 496.86 552.15 625.89 616.67 607.45 607.45 607.45 616.67 616.67 598.24];
y5 = [2364.5 2290.8 2134.1 2005.1 1830 1654.8 1479.7 1240.1 1037.3 853.01 779.28 834.58 991.26 1065 1147.9 1221.7 1359.9 1433.6 1479.7];
x6 = [598.24 607.45 625.89 644.32 644.32 644.32 635.1 635.1 635.1 644.32 635.1 644.32 635.1 635.1 625.89 671.97 736.48 801 810.21 819.43 828.65 837.86 847.08 837.86 819.43 819.43];
y6 = [1479.7 1479.7 1442.9 1406 1286.2 1258.5 1184.8 1101.9 1018.9 991.26 908.31 806.93 733.2 604.17 512.01 438.28 438.28 548.87 641.04 742.42 862.23 972.83 1120.3 1212.5 1267.8 1313.8];
x7 = [819.43 828.65 883.94 874.73 865.51 865.51 865.51 893.16 893.16 902.38 902.38 985.32 1077.5 1086.7 1095.9 1086.7 1086.7 1068.3 1049.8 1040.6 1040.6];
y7 = [1313.8 1332.3 1286.2 1194 1074.2 972.83 889.88 797.72 677.9 484.36 364.55 290.82 346.11 465.93 548.87 696.34 889.88 1065 1147.9 1249.3 1258.5];
x8 = [1040.6 1084.8 1102.4 1102.4 1089.8 1155.3 1161.6 1190.5 1237.1 1307.6 1344.1 1361.7 1344.1 1344.1 1313.9 1320.2 1320.2 1337.8 1384.4];
y8 = [1258.5 1294.6 1209.7 1063.3 911.27 822.51 657.26 492.96 385.31 403.26 496.74 661.98 781.9 986.81 1129.4 1304.1 1522.2 1686.5 1820.6];
x9 = [1384.4 1402 1414.6 1437.2 1437.2 1461.2 1537.9 1590.8 1619.8 1672.6 1685.2 1685.2 1667.6 1650 1637.4 1619.8 1590.8 1537.9 1443.5 1419.6 1313.9];
y9 = [1820.6 1797.9 1678 1611 1509 1419.3 1304.1 1268.2 1240.8 1294.6 1352.2 1504.3 1615.7 1704.4 1811.1 1967.9 2105.8 2261.6 2448.5 2479.7 2573.2];
axis_dim = [1 1942 -3152 1];
```

- a1q6a.m (plot of the data points)

```
clear
a1q6_data; % load data

plot(x0,-y0,'*', x1,-y1,'*', x2,-y2,'*', x3,-y3,'*', x4,-y4,'*', x5, -y5,'*', x6,-y6,'*', x7,-y7,'*', x8,-y8,'*', x9,-y9
```

```
   ,'*');

title('Data Points of My Hand and Initial Letters');
axis(axis_dim);
```

- a1q6b.m (plot of the data points that are connected by lines)

```
clear
a1q6_data; % load data

plot(x0,-y0, x1,-y1, x2,-y2, x3,-y3, x4,-y4, x5,-y5, x6,-y6,
   x7,-y7, x8,-y8, x9,-y9);

title('My Hand and Initial Letters by Joining with Straight
   Lines');
axis(axis_dim);
```

- a1q6c.m (plot of the interpolated data)

```
clear
type = 'not-a-knot';
type2 = 'complete';
fac = 3;

a1q6_data; % load data

% interpolate initials
[tx0 ty0] = interpolate(x0,y0,type,fac);
[tx1 ty1] = interpolate(x1,y1,type,fac);
[tx2 ty2] = interpolate(x2,y2,type,fac);
[tx3 ty3] = interpolate(x3,y3,type,fac);
[tx4 ty4] = interpolate(x4,y4,type,fac);

% interpolate hand
[tx5 ty5] = interpolate(x5,y5,type2,fac);
[tx6 ty6] = interpolate(x6,y6,type2,fac);
[tx7 ty7] = interpolate(x7,y7,type2,fac);
[tx8 ty8] = interpolate(x8,y8,type2,fac);
[tx9 ty9] = interpolate(x9,y9,type2,fac);

plot(tx0,-ty0, x1,-y1, tx2,-ty2, tx3,-ty3, tx4,-ty4, tx5,-ty5
   , tx6,-ty6, tx7,-ty7, tx8,-ty8, tx9,-ty9);
title('My Hand and Initial Letters by Interpolation');
axis(axis_dim);
axis off;
```