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M303 - Linear Algebra
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Week 6 Assignment

1] $V = \mathbb{R}^2$ $\quad (x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2 + 5, y_1 + y_2 - 5)$

$k(x,y) = (kx, ky)$ "standard multiplication"

(a) $(2,3) \oplus (1,-1) = (2+1+5, 3-1-5) = \boxed{(8,-3)}$

(b) $\vec{0}$, $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v}$

Let $\vec{0} = (a,b)$, then

$(x,y) \oplus (a,b) = (x,y)$

$(x+a+5, y+b-5) = (x,y)$

$x + a + 5 = x$, $\quad y + b - 5 = y$

$a = -5$ $\qquad b = 5$

$\boxed{\vec{0} = (-5,5)}$

if $(x,y) = (10,10)$

$(10,10) \oplus (-5,5) = (10-5+5, 10+5-5)$

$= (10,10)$ ✓

(c) $\vec{v} = (x,y)$, $\quad -\vec{v}$, $\quad \vec{v} \oplus -\vec{v} = \vec{0}$

$(x,y) \oplus (a,b) = (-5,5)$

$(x+a+5, y+b-5) = (-5,5)$

$x + a + 5 = -5$, $\quad y + b - 5 = 5$

$a = -10 - x$ $\qquad b = 10 - y$

$\boxed{-\vec{v} = (-x-10, -y+10)}$

Addition Axioms

2]

1) closure under addition
2) commutativity : $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$
3) associativity : $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$
4) zero vector exists
5) additive inverse exists

Scalar Multiplication Axioms

6) closure under scalar multiplication
7) distributivity : $k(\vec{u} \oplus \vec{v}) = k\vec{u} \oplus k\vec{v}$
8) distributivity : $(k+m)\vec{v} = k\vec{v} \oplus m\vec{v}$
9) associativity : $k(m\vec{v}) = (km)\vec{v}$
10) identity : $1\vec{v} = \vec{v}$

Check Distributivity (7 & 8)

let $k = 2$, $\vec{u} = (1,0)$, $\vec{v} = (0,1)$

$k(\vec{u} \oplus \vec{v}) = k\vec{u} \oplus k\vec{v}$

$= 2((1,0) \oplus (0,1))$

$= 2(6,-4)$

$= (12,-8)$

$= 2(1,0) \oplus 2(0,1)$

$= (2,0) \oplus (0,2)$

$= (7,-3)$

$k(\vec{u} \oplus \vec{v}) \neq k\vec{u} \oplus k\vec{v}$

$\boxed{\therefore \text{ Axiom 7 fails}}$

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let $k=1$, $m=2$, $\vec{v} = (1,1)$

$(k+m)\vec{v} = k\vec{v} \oplus k\vec{m}$

$= 3(1,1)$

$= (3,3)$

$= 1(1,1) \oplus 2(1,1)$

$= (1,1) \oplus (2,2)$

$= (8,-2)$

$(3,3) \neq (8,-2)$

$\boxed{\therefore \text{ Axiom 8 Fails}}$

3| $k \otimes (x, y, z) = (k^2x, k^2y, k^2z)$

$k \otimes (\vec{u} \oplus \vec{v}) = k \otimes \vec{u} \oplus k \otimes \vec{v}$ (7)

$(k+m) \otimes \vec{v} = k \otimes \vec{v} \oplus m \otimes \vec{v}$ (8)

Test (7)

let $\vec{u} = (x_1, y_1, z_1)$, let $\vec{v} = (x_2, y_2, z_2)$

$k \otimes (\vec{u} \oplus \vec{v}) = k \otimes \vec{u} \oplus k \otimes \vec{v}$

$= k \otimes (x_1+x_2, y_1+y_2, z_1+z_2)$

$= (k^2(x_1+x_2), k^2(y_1+y_2), k^2(z_1+z_2))$

$= (k^2x_1+k^2x_2, k^2y_1+k^2y_2, k^2z_1+k^2z_2)$

$= (k^2x_1, k^2y_1, k^2z_1) \oplus (k^2x_2, k^2y_2, k^2z_2)$

$= (k^2x_1+k^2x_2, k^2y_1+k^2y_2, k^2z_1+k^2z_2)$

✓ $\therefore$ $\boxed{\text{Axiom 7 holds}}$

Test (8)

let $k=1$, $m=1$, $\vec{v} = (1,1,1)$

$(k+m) \otimes \vec{v} = k \otimes \vec{v} \oplus m \otimes \vec{v}$

$= 2 \otimes (1,1,1)$

$= (4,4,4)$

$= 1 \otimes (1,1,1) \oplus 1 \otimes (1,1,1)$

$= (1,1,1) \oplus (1,1,1)$

$= (2,2,2)$

x

$\therefore$ $\boxed{\text{Axiom 8 Fails}}$

4| $W_0 = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0 + a_1 + a_2 + a_3 = 0\}$

a) Zero polynomial in P_3 is:

$$p(x) = 0 + 0x + 0x^2 + 0x^3$$

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0$$

$$a_0 + a_1 + a_2 + a_3 = 0$$

✓ $\therefore$ the zero polynomial satisfies the condition and the zero vector is in W_0

b) Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be in W_0

Let $q(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ be in W_0

As $p \in W_0$: $a_0 + a_1 + a_2 + a_3 = 0$

As $q \in W_0$: $b_0 + b_1 + b_2 + b_3 = 0$

Their sum is: $p(x) + q(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3$

Check if sum of coeff. is 0, to confirm $p+q \in W_0$

$$(a_0+b_0) + (a_1+b_1) + (a_2+b_2) + (a_3+b_3) = 0$$

$$(a_0+a_1+a_2+a_3) + (b_0+b_1+b_2+b_3) = 0$$

$$0 = 0$$

Therefore $p+q \in W_0$ | So W_0 is closed under addition.

c) Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be in W_0

As $p \in W_0$, $a_0 + a_1 + a_2 + a_3 = 0$

$$k \cdot p(x) = ka_0 + ka_1x + ka_2x^2 + ka_3x^3$$

If $k \cdot p \in W_0$, then sum of coeff. is 0

$$ka_0 + ka_1 + ka_2 + ka_3 = 0$$

$$k(a_0 + a_1 + a_2 + a_3) = 0$$

$$k(0) = 0$$

$$0 = 0$$

✓

Therefore $k \cdot p \in W_0$

So W_0 is closed under scalar multiplication.

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5] $W_1 = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0 + a_1 + a_2 + a_3 = 1\}$

a) Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ $p \in W_0$, so $a_0 + a_1 + a_2 + a_3 = 1$
Let $q(x) = b_0 + b_1x^1 + b_2x^2 + b_3x^3$ $q \in W_0$, so $b_0 + b_1 + b_2 + b_3 = 1$

$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x_1 + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$

if $p + q \in W_1$, then sum of coeff. equals 1.

$(a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) = 1$
$(a_0 + a_1 + a_2 + a_3) + (b_0 + b_1 + b_2 + b_3) = 1$
$1 + 1 = 1$
$2 \neq 1$

Therefore $p + q \notin W_1$, and W_1 is not closed under addition

b) Let $p(x) = a_0 + a_1x_1 + a_2x^2 + a_3x^3$, $p \in W_1$ so $a_0 + a_1 + a_2 + a_3 = 1$
Let $k = 2$
If $k \cdot p \in W_1$, then sum of coeff. is 1

$k \cdot p(x) = ka_0 + ka_1x + ka_2x^2 + ka_3x^3$

$ka_0 + ka_1 + ka_2 + ka_3 = 1$
$k(a_0 + a_1 + a_2 + a_3) = 1$
$2(1) = 1$
$2 \neq 1$

Therefore, W_1 is not closed under scalor multiplication

b) $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

$\vec{v} = (17, 8, -1)$, $\vec{v_1} = (2, -1, 2)$, $\vec{v_2} = (1, 1, 5)$, $\vec{v_3} = (4, -3, 0)$

$$\vec{v} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$$

$$(17, 8, -1) = c_1(2, -1, 2) + c_2(1, 1, 5) + c_3(4, -3, 0)$$

$$(17, 8, -1) = (2c_1 + c_2 + 4c_3, -c_1 + c_2 - 3c_3, 2c_1 + 5c_2)$$

$$\begin{aligned} 2c_1 + c_2 + 4c_3 &= 17 \\ -c_1 + c_2 - 3c_3 &= 8 \\ 2c_1 + 5c_2 \quad &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 17 \\ -1 & 1 & -3 & 8 \\ 2 & 5 & 0 & -1 \end{array}\right] \xrightarrow{R_1 \cdot 1/2}$$

$$\begin{bmatrix} 1 & 1/2 & 2 & 17/2 \\ -1 & 1 & -3 & 8 \\ 2 & 5 & 0 & -1 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 + R_1}$$

$$\begin{bmatrix} 1 & 1/2 & 2 & 17/2 \\ 0 & 3/2 & -1 & 33/2 \\ 0 & 4 & -4 & -18 \end{bmatrix} \xrightarrow{R_2 \cdot 2/3}$$

$$\begin{bmatrix} 1 & 1/2 & 2 & 17/2 \\ 0 & 1 & -2/3 & 11 \\ 0 & 4 & -4 & -18 \end{bmatrix} \xrightarrow[R_3 - 4R_2]{R_1 - \frac{1}{2}R_2}$$

$-4 - 4(-\frac{2}{3})$

$-\frac{12}{3} + \frac{8}{3}$

$$\begin{bmatrix} 1 & 0 & 7/3 & 3 \\ 0 & 1 & -2/3 & 11 \\ 0 & 0 & -4/3 & -62 \end{bmatrix} \xrightarrow{R_3 \cdot -3/4}$$

$-18 - 4(11)$

$\cancel{-62}^{-31} \cdot -\frac{3}{\cancel{4}_{2}} = +\frac{93}{2}$

$$\begin{bmatrix} 1 & 0 & 7/3 & 3 \\ 0 & 1 & -2/3 & 11 \\ 0 & 0 & 1 & \frac{93}{2} \end{bmatrix} \xrightarrow[R_2 + \frac{2}{3}R_3]{R_1 - \frac{7}{3}R_3}$$

$3 - \frac{7}{\cancel{3}}\left(\frac{\cancel{93}^{31}}{2}\right) = -211/2$

$11 + \frac{\cancel{2}}{\cancel{3}}\left(\frac{\cancel{93}^{31}}{\cancel{2}}\right) = 42$

$$\begin{bmatrix} 1 & 0 & 0 & -211/2 \\ 0 & 1 & 0 & 42 \\ 0 & 0 & 1 & 93/2 \end{bmatrix}$$

$$\boxed{\therefore \vec{v} = -\frac{211}{2}\vec{v_1} + 42\vec{v_2} + \frac{93}{2}v_3}$$

7] $\vec{V} = 2 + x + x^2$

$\vec{V_1} = 1 + x$, $\vec{V_2} = 1 + x^2$, $\vec{V_3} = 1 + x + x^2$

$$\vec{V} = C_1\vec{V_1} + C_2\vec{V_2} + C_3\vec{V_3}$$

$$\vec{V} = C_1(1+x) + C_2(1+x^2) + C_3(1+x+x^2)$$

$$= C_1 + C_1x + C_2 + C_2x^2 + C_3 + C_3x + C_3x^2$$

$$2 + x + x^2 = (C_1 + C_2 + C_3) + (C_1x + C_3x) + (C_2x^2 + C_3x^2)$$

$$2 + x + x^2 = (C_1 + C_2 + C_3) + (C_1 + C_3)x + (C_2 + C_3)x^2$$

$$\begin{aligned} C_1 + C_2 + C_3 &= 2 \\ C_1 \quad\quad + C_3 &= 1 \\ C_2 + C_3 &= 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \cdot (-1)}$$

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_1 - R_2,\ R_3 - R_2}$$

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$C_1 = 1$, $C_2 = 1$, $C_3 = 0$

$$\boxed{\vec{V} = 1\vec{V_1} + 1\vec{V_2} + 0\vec{V_3}}$$

8|

$$\left.\begin{matrix}(2,4,-1)\\(1,4,9)\\(3,4,-11)\end{matrix}\right\}$$ For 3 vectors to span $\mathbb{R}^3$, they must be linearly independent

$\downarrow$

$$\begin{vmatrix}2 & 4 & -1\\1 & 4 & 9\\3 & 4 & -11\end{vmatrix}\begin{matrix}2 & 4\\1 & 4\\3 & 4\end{matrix}$$

if det. $= \emptyset$, then vectors are linearlly dependent & do not span $\mathbb{R}^3$

$$\left[2(4)(-11) + (4)(9)(3) + (-1)(1)(4)\right]$$
$$-\left[(4\cdot1\cdot-11) + (2\cdot9\cdot4) + (-1\cdot4\cdot3)\right]$$
$$= \left[-88 + 108 - 4\right] - \left[-44 + 72 - 12\right]$$
$$= 16 - 16$$
$$= \emptyset$$

Because the determinant is $\emptyset$, one of the vectors can be written as a linear combination of the other two. The 3 vectors lie in the same plane (a 2-D subspace).

Any linear combination of these 3 vectors will only produce vectors in this plane.

Since they only span a 2-D subspace (plane), they cannot span all of $\mathbb{R}^3$ which is 3-D.

9|

Week 6 Definitions

If W is a subset of the vector space V, then W is a linear subspace of V if W is itself a vector space under the operations defined in V.

If $S = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is a set of vectors in a vector space, then the span of S is the set of all linear combinations of the elements of S, that is

$$\text{span}(S) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots c_n\vec{v}_n : c_1, c_2, \dots, c_n \in \mathbb{R}\}$$

If S is a subset of V, then S is a spanning set of V, or spans V, if $\text{span}(S) = V$.