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9/27/25

M303 - Fall 2025

Exam 1

1

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 3 & 15 \\ 2 & -1 & 1 & -11 & 4 \\ 0 & 0 & 1 & -4 & 5 \\ 1 & 0 & 1 & -6 & 6 \end{array}\right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_4 - R_1}}$$

$4 - 30$

$-1 - 6$

$2 - 2 \quad -11 - 6 = -17$

$$\left[\begin{array}{ccccc} 1 & 3 & 1 & 3 & 15 \\ 0 & -7 & -1 & -17 & -26 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & -3 & 0 & -9 & -9 \end{array}\right] \xrightarrow{-\frac{1}{3}R_4}$$

$$\left[\begin{array}{ccccc} 1 & 3 & 1 & 3 & 15 \\ 0 & -7 & -1 & -17 & -26 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 1 & 0 & 3 & 3 \end{array}\right] \xrightarrow{R_2 \longleftrightarrow R_4}$$

$$\left[\begin{array}{ccccc} 1 & 3 & 1 & 3 & 15 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & -7 & -1 & -17 & -26 \end{array}\right] \xrightarrow{\substack{R_1 - 3R_2 \\ R_4 + 7R_2}}$$

$-17 + 21$

$-26 + 21 = -5$

$$\left[\begin{array}{ccccc} 1 & 0 & 1 & -6 & 6 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & -1 & 4 & -5 \end{array}\right] \xrightarrow{\substack{R_1 - R_3 \\ R_4 + R_3}}$$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & -11 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$w \quad x \quad y \quad$ free variable, z

let $z = t$

$w + 2t = 1$

$x + 3t = 3$

$y - 11t = 4$

$z = t$

> The system is consistent with infinitely many solutions with one free variable
>
> $w = 1 - 2t$
>
> $x = 3 - 3t$
>
> $y = 4 + 11t$
>
> $z = t$

2)

System 1

$$\left[\begin{array}{cccc} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable, y

let $y = t$

$x + 3t = -1 \rightarrow x = -3t - 1$

$y = t$

$z = -2$

The system is consistent with infinitely many solutions, with one free variable

$x = -3t - 1, \quad y = t, \quad z = -2$

System 2

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \qquad \begin{array}{l} x = 5 \\ y = 0 \\ z = -2 \end{array}$$

The system is consistent with one solution

$x = 5, \; y = 0, \; z = -2$

System 3

$$\left[\begin{array}{cccc} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad 0 \neq 1$$

The system is inconsistent, with no solution

because $0 \mathrel{!=} 1$

✓2

3| $AB^T = ?$

B^T $r \to c$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 \\ -1 & 1 & -3 \\ -1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2-1-2 & 3+1+8 & 0-3+2 \\ 2-1-3 & -3+1+12 & 0-3+3 \\ -2-3-1 & 3+3+4 & 0-9+1 \end{bmatrix}$$

$$AB^T = \boxed{\begin{bmatrix} -5 & 12 & -1 \\ -2 & 10 & 0 \\ -6 & 10 & -8 \end{bmatrix}}$$

✓ 3

4

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

$$A^{-1} = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \quad (-1)R_1 ✓$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{matrix} R_2 - R_1 ✓ \\ R_3 + R_1 ✓ \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right] \quad \tfrac{1}{2}R_2 ✓$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 5/2 & 1/2 & 1/2 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right] \quad \begin{matrix} R_1 + R_2 ✓ \\ R_3 - 2R_2 ✓ \end{matrix}$$

$-\frac{4}{2} + \frac{5}{2}$

$-1 - \frac{10}{2}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & -6 & -2 & -1 & 1 \end{array}\right] \quad -\tfrac{1}{6}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/3 & 1/6 & -1/6 \end{array}\right] \quad \begin{matrix} R_1 - \tfrac{1}{2}R_3 \\ R_2 - \tfrac{5}{2}R_3 \end{matrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & 5/12 & 1/12 \\ 0 & 1 & 0 & -1/3 & 1/12 & 5/12 \\ 0 & 0 & 1 & 1/3 & 1/6 & -1/6 \end{array}\right]$$

$\frac{1}{2} - \frac{1}{2}$

$-\frac{1}{2} - \frac{1}{6} = -\frac{4}{6}$

$\frac{1}{2} - \frac{1}{12}$

$\frac{5}{12}$

$\frac{1}{2} - \frac{5}{2}\left(\frac{1}{3}\right)$

$\frac{1}{2} - \frac{5}{6}$

$\frac{3}{6} - \frac{5}{6} = -\frac{1}{3}$

$\frac{1}{2} - \frac{5}{2}\left(\frac{1}{6}\right)$

$\frac{1}{2} - \frac{5}{12}$

$\frac{6}{12} - \frac{5}{12} = \frac{1}{12}$

$0 - \frac{5}{2}\left(-\frac{1}{6}\right)$

$\frac{5}{12}$

$$\boxed{A^{-1} = \begin{bmatrix} -2/3 & 5/12 & 1/12 \\ -1/3 & 1/12 & 5/12 \\ 1/3 & 1/6 & -1/6 \end{bmatrix}}$$

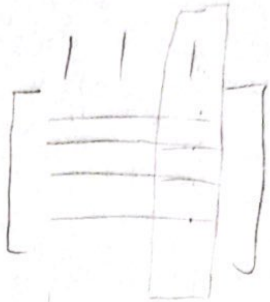
$$AA^{-1} = I\,? \quad \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} -2/3 & 5/12 & 1/12 \\ -1/3 & 1/12 & 5/12 \\ 1/3 & 1/6 & -1/6 \end{bmatrix}$$

$$\begin{bmatrix} 2/3 - 1/3 + 2/3 & -5/12 + 1/12 + 4/12 & -1/12 + 5/12 - 4/12 \\ -2/3 - 1/3 + 3/3 & 5/12 + 1/12 + 6/12 & 1/12 + 5/12 - 6/12 \\ 2/3 - 1 + 1/3 & -5/12 + 3/12 + 2/12 & -1/12 + 15/12 - 2/12 \end{bmatrix} \begin{matrix} ✓ \\ ✓ \\ ✓ \end{matrix}$$

✓ 14

5a| $R(4\times 2)$ $S(2\times 3)$

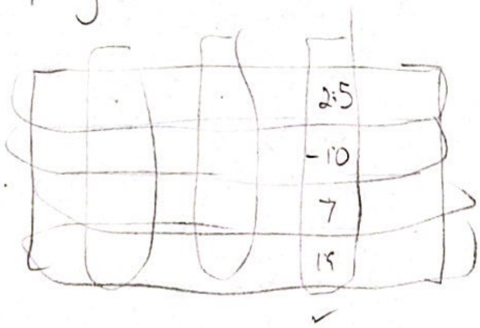
$$\boxed{RS : 4\times 3}$$



5b| 3rd col of RS

$$\boxed{-3\begin{bmatrix}1\\2\\3\\3\end{bmatrix} + 4\begin{bmatrix}7\\-1\\4\\7\end{bmatrix}}$$

$$\begin{bmatrix}1 & 7\\2 & -1\\3 & 4\\3 & 7\end{bmatrix}\begin{bmatrix}1 & 6 & -3\\5 & -5 & 4\end{bmatrix}$$



6|

$$f(x,y) = (x+3y, -2x, x-5y, -3y)$$

standard matrix

$$\boxed{A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 1 & -5 \\ 0 & -3 \end{bmatrix}} \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

4 x 2 2x1

$$f(3,-5) = \begin{bmatrix} 3-15 \\ -6+0 \\ 3+25 \\ 0+15 \end{bmatrix} = \boxed{\begin{bmatrix} -12 \\ -6 \\ 28 \\ 15 \end{bmatrix}}$$

$\sqrt{6}$

7]

$$\begin{bmatrix} a-1 & 6 \\ 4 & a+4 \end{bmatrix}$$

$$(a-1)(a+4) - 24 = \emptyset$$

$$a^2 + 3a - 4 - 24 = \emptyset$$

$$a^2 + 3a - 28 = \emptyset$$

$$(a+7)(a-4) = \emptyset$$

$$\boxed{a = -7 \quad \text{or} \quad a = 4}$$

make the matrix non-invertable (det = 0)

8]

$$M = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 3 & 0 & 4 & -1 \\ 1 & 2 & -2 & 0 \\ 3 & -4 & 2 & 7 \end{bmatrix} \xrightarrow{R_4 + 7R_2}$$

$$= \begin{bmatrix} 1^{+} & -2^{-} & 3^{+} & 0^{-} \\ 3^{-} & 0^{+} & 4^{-} & -1^{+} \\ 1^{+} & 2^{-} & -2^{+} & 0^{-} \\ 24^{-} & -4^{+} & 30^{-} & 0^{+} \end{bmatrix}$$

$3+21$

$-4+0$

$2+14$

$$= -1 \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & -2 \\ 24 & -4 & 30 \end{vmatrix} \xrightarrow{\substack{R_1 + R_2 \\ R_3 + 2R_2}}$$

$$= -1 \begin{vmatrix} 2^{+} & 0^{-} & 1^{+} \\ 1^{+} & 2^{+} & -2^{-} \\ 26^{+} & 0^{-} & 26^{+} \end{vmatrix}$$

$$= (-1)(2) \begin{vmatrix} 2 & 1 \\ 26 & 26 \end{vmatrix}$$

$$= (-1)(2)(52 - 26)$$

$$= (-1)(2)(26)$$

$$= \boxed{-52}$$

$$\begin{matrix} 1 & -2 & 3 & 1 & -2 \\ 1 & 2 & -2 & 1 & 2 \\ 3 & -4 & 2 & 3 & -4 \end{matrix}$$

$(4+12-12) - (-4+8+18)$

$4 - (22)$

-18

$$-1 \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & -2 \\ 3 & -4 & 2 \end{vmatrix} + 7 \begin{vmatrix} 1 & -2 & 3 \\ 3 & 0 & 4 \\ 1 & 2 & -2 \end{vmatrix}$$

$$\begin{matrix} 1 & -2 & 3 & 1 & -2 \\ 3 & 0 & 4 & 3 & 0 \\ 1 & 2 & -2 & 1 & 2 \end{matrix}$$

$(0 - 8 + 18) - (12 + 8)$

$10 - 20$

$= -10$

$-1(-18) + 7(-10)$

$18 - 70 = -52$ ✓

✓ 18

9]
$$\vec{a} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

a) $u_b = \frac{1}{|b|} b$

$$|b| = \sqrt{2^2 + (-3)^2 + (-1)^2} = \sqrt{4+9+1} = \sqrt{14}$$

$$u_b = \frac{1}{\sqrt{14}} \langle 2, -3, -1 \rangle$$
$$= \boxed{\left\langle \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}} \right\rangle}$$

b) $a \cdot b = |a| \, |b| \cos\theta$

$$\theta = \cos^{-1} \frac{a \cdot b}{|a| \, |b|}$$

$$a \cdot b = 8 + 6 - 1 = 13$$
$$|a| = \sqrt{4^2 + (-2)^2 + 1} = \sqrt{16+4+1} = \sqrt{21}$$

$$\theta = \cos^{-1} \frac{13}{\sqrt{21}\,\sqrt{14}} = \boxed{40.7^\circ}$$

$\underline{10|}$ $\vec{a} = \vec{w_1} + \vec{w_2}$

$\vec{w_1} \parallel \vec{b}$ $\quad$ $\vec{w_2} \perp \vec{b}$

$\vec{a} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$

$4+9+1$

$$\text{proj}_b a = \frac{a \cdot b}{b \cdot b} b$$

$$= \frac{13}{14} \langle 2, -3, -1 \rangle$$

$b \cdot b = 4 + 9 + 1$

$$\boxed{w_1 = \left\langle \frac{13}{7}, -\frac{39}{14}, -\frac{13}{14} \right\rangle}$$

$$\vec{w_2} = \vec{a} - \vec{w_1}$$

$$= \langle 4, -2, 1 \rangle - \left\langle \frac{13}{7}, -\frac{39}{14}, -\frac{13}{14} \right\rangle$$

$$= \left\langle \frac{28}{7}, -\frac{28}{14}, \frac{14}{14} \right\rangle - \left\langle \frac{13}{7}, -\frac{39}{14}, -\frac{13}{14} \right\rangle$$

$$\boxed{W_2 = \left\langle \frac{15}{7}, \frac{11}{14}, \frac{27}{14} \right\rangle}$$

$$\vec{a} = \vec{w_1} + \vec{w_2}$$

$$\langle 4, -2, 1 \rangle = \left\langle \frac{13}{7} + \frac{15}{7}, -\frac{39}{14} + \frac{11}{14}, -\frac{13}{14} + \frac{27}{14} \right\rangle$$

$$= \left\langle \frac{28}{7}, -\frac{28}{14}, \frac{14}{14} \right\rangle$$

$$= \langle 4, -2, 1 \rangle$$

$\sqrt{10}$

11a) $P(1,4,-5)$ $\quad 2x+3y+4z=6 \longrightarrow 2x+3y+4z-6=0$

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

$x_0=1$ $\quad a=2$
$y_0=4$ $\quad b=3$
$z_0=-5$ $\quad c=4$
$\quad d=-6$

$$D=\frac{|2(1)+3(4)+4(-5)-6|}{\sqrt{4+9+16}}$$

$$D=\frac{|2+12-20-6|}{\sqrt{29}}=\frac{12}{\sqrt{29}}=\boxed{\frac{12\sqrt{29}}{29}}$$

11b) $P(1,-15,3)$ $\quad \vec{n}=\langle 2,-1,8\rangle$

$x_0=1$ $\quad A=2$
$y_0=-15$ $\quad B=-1$
$z_0=3$ $\quad C=8$

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

$$\boxed{2(x-1)-1(y+15)+8(z-3)=0}$$

$2x-2-y-15+8z-24=0$
$2x-y+8z-41=0$
$\boxed{2x-y+8z=41}$ ✓

$ax+by+cz=d$
$2+15+24=d$
$\boxed{41=d}$

√11