ADAPTIVE BACKSTEPPING UAV CONTROL

A Thesis

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Abstract

Adaptive Backstepping UAV Control

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This effort is aimed to extend the state of the art in the areas of adaptive reconfigurable flight control, specifically through implementation of Adaptive Backstepping control architecture for use in AeroMech Engineering's *Fury* Unmanned Aerial System (UAS).

I will be exploring an adaptive localized learning algorithm showing interconnects among lateral and longitudinal dynamics.

Backstepping is a systematic design approach [recursive design procedure] that allows the use of certain plant states to act as virtual controls for others.

Acknowledgements

Thank you...

Table of Contents

List of Tables **vii**

List of Figures **viii**

Nomenclature **ix**

1 Introduction **1**

1.1 Significance 2

1.2 Requirements and Outcomes 2

1.3 Thesis Outline 3

2 History **5**

2.1 Emergence of Adaptive Control 6

2.2 Emergence of Backstepping 7

3 Theory **10**

3.1 Aircraft Dynamics 10

3.2 System and Stability Concepts 28

3.3 Control Law Development 50

4 Design & Implementation **55**

4.1 Modeling 55

4.2 Simulation 55

4.3 Results 56

5 Conclusion **57**

List of Tables

3.1 Choices for State Variables 12

3.2 Direction Cosine Matrices for Plane Rotations 17

3.3 Stability Theorem Overview 42

List of Figures

2.1 WPAFB: Design Methods 9

2.2 Traditional Adaptive Control Flowchart 9

3.1 Air Vehicle Reference Frames 13

3.2 Axis Relationships: Body, Stability, and Wind Axes 14

3.3 Earth Reference Frames 15

3.4 Direction Cosine Matrix Visual 19

3.5 Body Axes, Forces, and Moments 22

3.6 Orientation of Gravity Vector with Respect to Body Axes 22

3.7 Operating Surface via Phase Portraits 33

3.8 Concepts of Stability 35

3.9 First Order Unstable Systems [1, Online Lecture Notes] 39

3.10 First Order Stable Systems [1, Online Lecture Notes] 39

3.11 First Order Asymptotically Stable Systems [1, Online Lecture Notes] 40

3.12 Equilibrium Point Classification for 2^{nd} Order Linear Systems [2] 41

3.13 Geometric Interpretation of Lyapunov's Stability Theorem. 44

3.14 Asymptotic Property 1: $\dot{\boldsymbol{f}}(t) \to 0 \nRightarrow \boldsymbol{f} \to \text{constant}$ 48

3.15 Asymptotic Property 2: $\boldsymbol{f} \to \text{constant} \nRightarrow \dot{\boldsymbol{f}}(t) \to 0$ 49

Nomenclature

α	Angle of Attack	deg
β	Angle of Sideslip	deg
χ	Heading Angle	deg
γ	Flight-Path Angle	deg
μ	Bank Angle	deg
ϕ	Roll Angle	deg
ψ	Yaw Angle	deg
θ	Pitch Angle	deg
u	Longitudinal Velocity	ft/s
V	Velocity	ft/s
v	Lateral Velocity	ft/s
w	Normal Velocity	ft/s

Chapter 1

Introduction

What's the best thesis? A done one.

Goal: Develop inner loop control law for "FCAS" using backstepping.

"Sometimes I tell people that I don't understand X. Sometimes they reply, 'Well, I clearly explained it on page 4.' That'd be the wrong answer. If I (or anyone else) tells you they don't understand X, they're saying you haven't explained X well enough. Rather than getting defensive, reexamine your explanation, see how you can rewrite to make it clearer. It's not the reader's fault if they don't understand what you're saying. Your job as a writer is to make it very clear where you're coming from. Telling me that I just missed the point won't score you any points. Take another look, see what's not obvious."
http://web.media.mit.edu/~intille/teaching/advising/draftAdvisorTips.htm

In all control techniques the goal is the same: to achieve a desired response even in the presence of external disturbances and sensor noise. It is the control engineer's responsibility to choose the appropriate design technique given the stability, performance, and cost requirements of the system.

As technology expands so does its complexity and the level of detail placed into the system model. Typically plants with nonlinearities and unknown or changing parameters are associated with this higher fidelity. These advances in control theory led to the development of adaptive control, the goal of which is to control plants with unknown or imperfect knowledge.

The focus of this thesis is to explore Backstepping, a relatively new control algorithm for non-linear systems. Its a subset of a popular modern adaptive control technique called Direct Adaptive Control (DAC) that utilizes Lyapunov synthesis to derive a stabilizing controller. Backstepping has applications in a broad spectrum of engineering disciplines, from electrical motors to jet engines, ship control to flight control just to name a few. It offers a systematically methodology for developing control schemes for non-linear systems. The main appeal is that useful nonlinearities do not have to be canceled in the control law thereby increasing robustness to modeling errors and decreasing control effort. The alternative, feedback linearization, transforms a non-linear system into an equivalent linear system by canceling all nonlinearities via its control law.

1.1 Significance

Adaptive flight control has been viewed as an enabling technology to deal with plants with highly nonlinear, time-varying, and/or uncertain dynamical characteristics. As UAS become more interactive with humans, providing safety for flight crews, it must be assured that they offer a level of reliability comparable to manned systems; this project strives to accomplish this goal, hence fostering the advancement of autonomous systems and user safety.

1.2 Requirements and Outcomes

International Traffic in Arms Regulations (ITAR) requires that information and material pertaining to defense and military related technologies may only be shared with US Persons.

The proprietary AeroMech Engineering *Fury UAS* simulation model will be substituted with a publicly available one for security as well as legal reasons. Proof of concept will be demonstrated by the dissimilarity in control effectiveness of these models. A concluding objective would include hardware implementation and flight test demonstration.

Make it clear that I'm only developing control law, maybe build block diagram with shading of provided and created blocks.

- Apply control architecture directly out of Barron Assoc. paper via Simulink → inner loop control law!
- Integrate with FCAS and fly with apprentice RC airplane
- Evaluate simulation, experiment in order to establish:
  - Stability
  - Performance
  - Flying Qualities

1.3 Thesis Outline

This thesis* will

- History
- Theory
  - Aircraft Dynamics (Modeling)
  - Control Law Development
    - Lyapunov

        * Backstepping

- Simulation
- Conclusion

review aircraft dynamics and supporting concepts essential to modeling and simulation. Next, results from Lyapunov theory then derive Backstepping theory and control law development. Two cases were modeled in simulink, a simple scalar nonlinear system and third order flight path angle controller. Each was selected to highlight a particular benefit of Backstepping, however they repetitively demonstrate the steps required to manipulate the plant and develop a control law. To close the simulation results will be discussed with respect to feedback linearization.

Chapter 2

History

See "Constructive Nonlinear Control: Progress in the 90's"

> Long before history began we men have got together apart from the women and done things. We had time.
>
> -CS Lewis

Introduce the concept of control, lead into adaptive control section.

(1) Emergence of Adaptive Control - Krstic (2) WPAFB: MVAR CONTROL DESIGN GUIDELINES

Design Methods Linear Quadratic Gaussian (LQG, H_2), Worst case L2 - induced norm problem (H∞), Worst case L∞ - induced norm problem (l1). "Introduction of these formal methods, together with use of state-space descriptions provided the first multivariable control design tools" (WL-TR-96-3099)

2.1 Emergence of Adaptive Control

Reference work from seminal papers on adaptive control: Bode lectures by Gunter Stein and Petar Kokotovic.

First implementations of feedback may be considered indirect attempts to control unknown plants

- "Even the most elementary feedback loops can tolerate significant uncertainties"

50's and 60's: Advances spawned more sophisticated systems

- Find Examples: Assume space-race drove this, Find aircraft that required adaptive control
- self-learning, self-optimizing, self-organizing, self-tuning, and adaptive control utilizing on-line identification or pattern recognition.
- Lyapunov techniques were not a major player in the control engineering until the early 1960's thanks to publications by Lur'e and a book by La Salle and Lefschetz* -Slotine and Lee

$>$ 60's: Theoretical: Stochastic and Dual Control

- gain-scheduling, fuzzy, neural, intelligent control

80's: Adaptive Linear Control or Traditional Adaptive Control

- Non-Linear Lyapunov-based control recently* achievable through recursive design procedures such as backstepping.
- "[Estimation-based designs are flexible and] achieved by treating the identifier as a separate module and guaranteeing its properties independent of the controller module."

Usually referred to as modular designs; can use gradient and least squares algorithms for parameter update laws.

Traditional control relies on the certainty equivalence principal: ”controller first designed as if all the plant parameters are known.”

- ”... control parameters are calculated by solving design equation for model matching, pole-zero placement, or optimality.”
- For each adaptive control scheme it is up to the designer to choose:
  - Filters
  - Design Coefficients
  - Initialization Rules
  - etc? CONTROL ENGINEER JOB...
- Adaptive Controls Tradeoff: Transient Performance vs. Robustness

2.2 Emergence of Backstepping

Development of backstepping, how it supports current requirements, and fits into current control law needs. Talk about how it is becoming a hot topic in the non-linear design world (an attempt to systematically approach non-linear control) and meets both aircraft and spacecraft control needs.

Adaptive Backstepping developed by Ioannis Kanellakopoulos [63] in collaboration with Petar Kokotovic, and Steve Morse “emerged as a confluence of the adaptive estimation idea on one side, and, on the other size, nonlinear control ideas expressed in works of [193], [12], [175], [85], [163].” Pg 11. Tuning functions were invented by Krstic [92,94] to reduce overparameterization.

Try to find actual implementation of backstepping in a UAV or some air/space craft

- Development –¿ Kokotovic and Kanellakopoulos
  - Feedback linearization / NDI first
- Examples and reference papers used
  - Kokotovic Jet Engine Controller and other examples listed in Hrkegrd's dissertation
  - Hrkegrd's work with ADMIRE model (Canard Fighter)
  - Farrell's flying wing work which my controller is implementing

..

- Future of backstepping, ie. its role in future nonlinear control architecture. Do some research on alternative control techniques.

Backstepping has applications in a broad spectrum of engineering disciplines, from electrical motors to jet engines, ship control to flight control just to name a few. It offers a systematically methodology for developing control schemes for non-linear systems. The main appeal is that useful non-linearities do not have to be cancelled in the control law thereby increasing robustness to modeling errors and decreasing control effort. The alternative, feedback linearization, transforms a non-linear system into an equivalent linear system by cancelling all non-linearities via its control law.

Table 1.1: Categorization of Design Methods

Formal Optimization Problems	
Basic Linear-Quadratic-Gaussian (LQG)	[Athans 1971]
LQG with Explicit/Implicit Model Following	[Asseo 1970, Tyler 1966]
Frequency Weighted LQG	[Gupta 1980]
Worst-case Induced L_2 Norm (H_∞)	[Zames 1983, Doyle 1989]
Worst-case Induced L_∞ Norm (ℓ_1)	[Dahleh 1987]
Mixed Criteria (H_2/H_∞)	[Zhou 1989, Rotea 1991, Yeh 1992]
Numerical Optimization Problems	
Fixed Structure LQ-Control	[Axsater 1966, Levine 1970, Stein 1971]
Fixed-Plus-Variable Gain LQ-Control	[VanDierendonck 1972]
Fixed Structure H_∞ Control	[Bernstein 1990]
Fixed Structure H_2/H_∞ Control	[Bernstein 1989, Ridgely 1992]
Q-Parameter Design (QDES)	[Boyd 1991]
μ-Synthesis via $D-K$-Iteration	[Doyle 1983, Stein 1991]
Frequency Domain Methods	
Diagonal Dominance/Inverse Nyquist Array	[Rosenbrock 1974]
Characteristic Loci	[Postlethwaite 1979]
Upper Triangular Structures	[Mayne 1973]
Singular-Value-Based Loop Shaping via Direct Inversion	[Hung 1982]
via LQG/LTR	[Stein 1987]
via H_∞	[McFarlane 1992]
via Dynamic Inversion	[Bugajski 1992b]
Quantitative Feedback Theory (QFT)	[Horowitz 1979]
Eigenstructure Assignment Methods	
via Full-State Feedback	[Andry 1983]
via Output Feedback	[Calvo-Ramon 1986, Sobel 1990]
via Quadratic Weights in LQ	[Harvey 1978]
via Numerical Optimization	[Garg 1989, Wilson 1990]
Fringe Methods	
Model Predictive Control (MPC, DMC, MAC)	[Morari 1989]
Covariance Control	[Skelton 1989]
Stochastic Parameters (maximum entropy)	[Hyland 1982]
Variable-Structure Control	[Utkin 1977]
Geometric Methods	[Wonham 1979]
Polynomial-Matrix Methods	[Peczkowski 1978, Wolovitch 1974]
Lyapunov-Based Methods	[Barmish 1985, Boyd 1989]

Figure 2.1: WPAFB: Design Methods

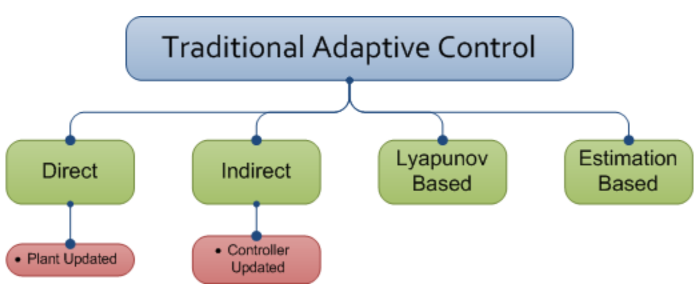


Figure 2.2: Traditional Adaptive Control Flowchart

Chapter 3

Theory

> The art of flight control design is to realize a solution that achieves an acceptable compromise among the evaluation criteria [Stability, Performance, and Flying Qualities] [3]

This chapter will...

3.1 Aircraft Dynamics

This section will introduce fundamental aircraft dynamics concepts and ultimately derive the equations of motion implemented in the backstepping control architecture. The aim is to establish a practical understanding of the equations of motion, reinforced by physical illustrations of key aspects to the derivation rather than pure mathematical formulation. The fallout allows the designer to both qualitatively *and* quantitatively evaluate the characteristic modes of motion, thereby providing an analytical playground for control design and performance evaluation via **handling qualities**[1]. Assumptions, hence consequent limitations of the derived equations, will be clearly identified with discussion immediately succeeding. No stone will be left unturned; equations will be derived from Newton's first principals and

[1]Reference handling qualities for unmanned aerial vehicles, this is a hot topic; degree of instability depends on what autopilot can handle...

include necessary supporting concepts. The philosophy of this approach and procedure itself is credited to work by [4] and [5].

3.1.1 Modeling

Establishing a way to mathematically describe the vehicle's dynamics is a necessity for any flight control architecture. As with any dynamic system, a set of differential equations may be used to calculate an object's position, velocity, and acceleration. Typically for complex systems — such as an airplane with flexible structure, rotating internal components, and variable mass — simplifying assumptions are applied in order to use Newton's Laws to derive vehicle dynamics. These assumptions lead to a direct appreciation of important factors that govern the vehicle dynamics. This level of understanding is an "implicit requirement for effective and efficient flight control system design activities. It affords a basic understanding of the vehicle/control system interactions and of the flight controller possibilities which are most likely to succeed." [5]

Assumption 3.1. Airframe is a rigid body.

"Rigid body models are described by six degrees of freedom and include forces and moments caused by gravity, aerodynamics, and propulsion." [6] The distances between any two points are fixed, hence forces acting between those points due to elastic deformation are absent. Consequently, the air vehicle may be modeled as an individual element of mass. In reality air vehicles diverge from the rigid body assumption in two ways: aeroelastic phenomena due to airframe structure deformation (such as wing bending due to air loads) and relative motion of components (engine, propeller, and control surfaces).

Under this assumption, the equations of motion may be decoupled into translational and rotational equations if the coordinate origin is chosen to coincide with the center of gravity. This yields a system description consisting of:

- 3 components of attitude to specify orientation relative to the gravity vector
- 3 components of velocity to specify translational kinetic energy
- 3 components of angular velocity to specify rotational kinetic energy
- 3 components of position to specify potential energy in earth's gravitational field

This implies that there are 12 equations necessary for control (list items 1–3) and navigation (list item 4). The implementation herein is concerned with control, therefore the **state-vector**[2] of this model will consist of 9 variables. Table 3.1 summarizes state variable choices, to be defined succeeding introduction of wind axes in subsection 3.1.2.

Table 3.1: Choices for State Variables

		Body		Flight-Path / Wind	
Kinetic Energy	Translational	Longitudinal Velocity	u	Velocity	V
		Lateral Velocity	v	Angle of Attack	α
		Normal Velocity	w	Sideslip Angle	β
	Rotational	Roll Rate	p	Roll Rate	p
		Pitch Rate	q	Pitch Rate	q
		Yaw Rate	r	Yaw Rate	r
Potential Energy	Attitudes	Euler Roll	ϕ	Heading Angle	χ
		Euler Pitch	θ	Flight-Path Angle	γ
		Euler Yaw	ψ	Bank Angle	μ
	Position	North Position	ξ	North Position	ξ
		East Position	η	East Position	η
		Altitude	h	Altitude	h

3.1.2 Reference Frames

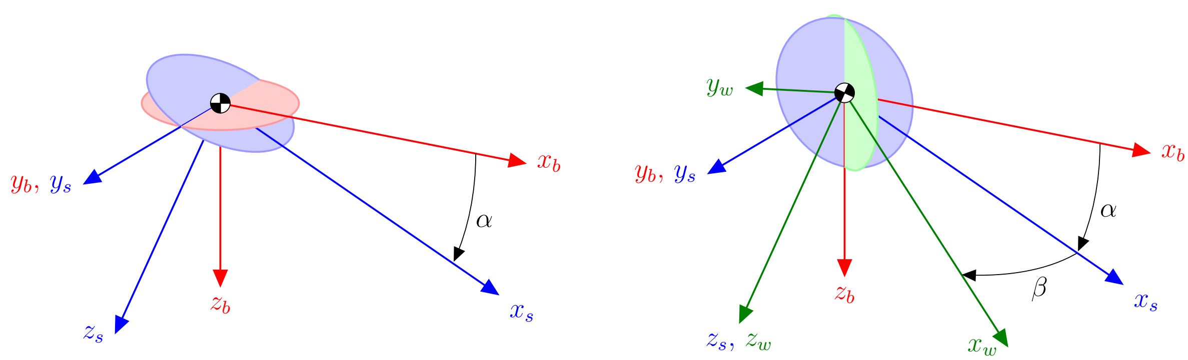
Just as language is needed to express thoughts, a reference frame is necessary to convey motion. The relationship between an object and the space it resides in is relative; choosing a

[2] The minimal set of system variables necessary to indicate the energy of the system, potential and kinetic, and its distribution at any given time.

reference frame, or coordinate system, enables an observer to describe the motion of an object over time. Selecting an appropriate reference frame can greatly simplify the description of this relationship.

Assumption 3.2. Earth is an inertial reference frame.

When earth is considered as an inertial frame of reference, one that is *fixed* or moving at a constant velocity (non-rotating and non-accelerating) relative to earth, it permits accurate short-term control and navigation analysis. Conversely, an inertial frame of reference is unacceptable for air vehicles that require long-term navigation, especially for high-speed flight, or extra-atmospheric operation; for most UAVs this assumption is fairly accurate however. As this situation dictates, there are numerous reference systems in aerospace applications. The frames applicable to the equations of motion derivation herein are: body, stability, wind or flight-path, and earth-centered-inertial or earth-fixed. Additionally, north-east-down or local-tangent-plane, vehicle-carried-vertical, and earth-centered-earth-fixed frames will be covered. All coordinate systems follow the right hand rule and are orthogonal.



(a) Body to Stability Axes

(b) Stability to Wind Axes

Figure 3.1: Air Vehicle Reference Frames

Body, stability, and wind axes are attached to the airframe at the center of gravity as depicted in Figure 3.1. By convention, body axis x_b points out the nose, y_b out the right wing, and z_b down the bottom of the aircraft. Stability axes are defined by a rotation of the body axes in the x_b-z_b plane by an angle-of-attack, α, that trims the air vehicle, ie. zero pitching moment; axis x_s points into the direction of steady flow, $y_s = y_b$, and z_s is perpendicular to the x_s-y_s plane in the direction following the right handed sign convention. Note that sideslip angle, β, is zero in stability axes. In wind, or flight-path, axes the x_w axis always points into the relative wind. This is defined by a rotation of the body axes through angle-of-attack and sideslip angle with $z_w = z_s$ and y_w following the right hand rule as shown in Figure 3.2:

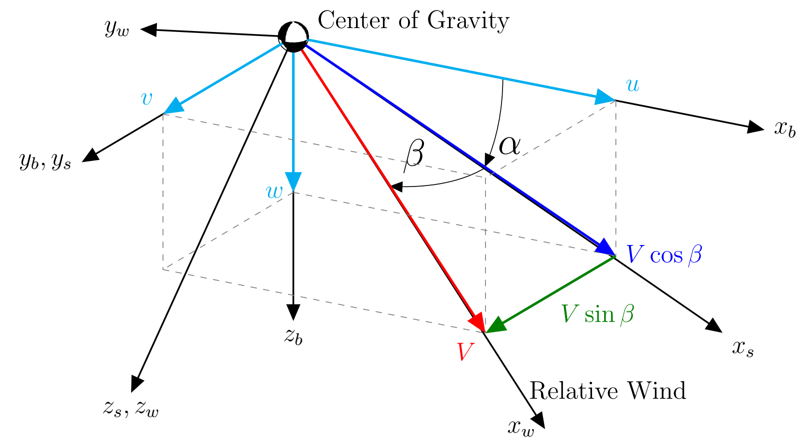


Figure 3.2: Axis Relationships: Body, Stability, and Wind Axes

The body state variables can be derived from the flight-path state variables as follows[6], recall Table 3.1:

$$u = V \cos\beta \cos\alpha \tag{3.1}$$

$$v = V \sin\beta \tag{3.2}$$

$$w = V \cos\beta \sin\alpha \tag{3.3}$$

$$\phi = \tan^{-1}\left(\frac{\cos\gamma \sin\mu \cos\beta - \sin\gamma \sin\beta}{-\cos\gamma \sin\mu \sin\alpha \sin\beta + \cos\gamma \cos\alpha \cos\mu - \sin\gamma \sin\alpha \cos\beta}\right) \tag{3.4}$$

$$\theta = \sin^{-1}\left(\cos\gamma \sin\mu \cos\alpha \sin\beta + \cos\gamma \cos\mu \sin\alpha + \sin\gamma \cos\alpha \cos\beta\right) \tag{3.5}$$

$$\psi = \tan^{-1}\left\{\frac{(\sin\mu \sin\alpha - \cos\alpha \cos\mu \sin\beta)\cos\chi + [\cos\gamma \cos\alpha \cos\beta - \sin\gamma(\sin\alpha \cos\mu + \sin\beta \cos\alpha \sin\mu)]\sin\chi}{-(\sin\mu \sin\alpha - \cos\alpha \cos\mu \sin\beta)\sin\chi + [\cos\gamma \cos\alpha \cos\beta - \sin\gamma(\sin\alpha \cos\mu + \sin\beta \cos\alpha \sin\mu)]\cos\chi}\right\} \tag{3.6}$$

Flight-path variables ca be derived from the body state variables as follows[6], recall Table 3.1:

$$V = \sqrt{u^2 + v^2 + w^2} \tag{3.7}$$

$$\alpha = \tan^{-1}\left(\frac{w}{u}\right) \tag{3.8}$$

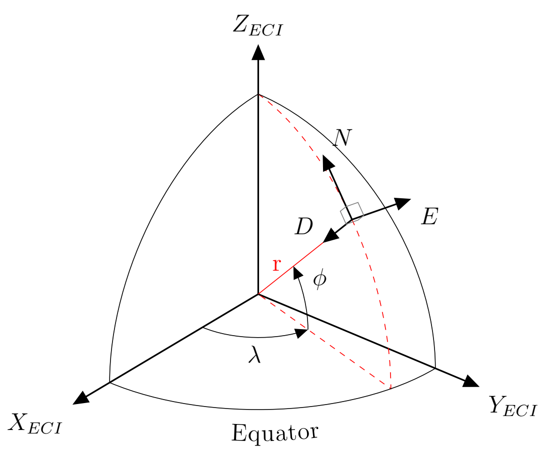
$$\beta = \sin^{-1}\left(\frac{v}{\sqrt{u^2 + v^2 + w^2}}\right) \tag{3.9}$$

$$\mu = \tan^{-1}\left[\frac{uv\sin\theta + (u^2 + w^2)\sin\phi\cos\theta - vw\cos\phi\cos\theta}{\sqrt{u^2 + v^2 + w^2}\,(w\sin\theta + u\cos\phi\cos\theta)}\right] \tag{3.10}$$

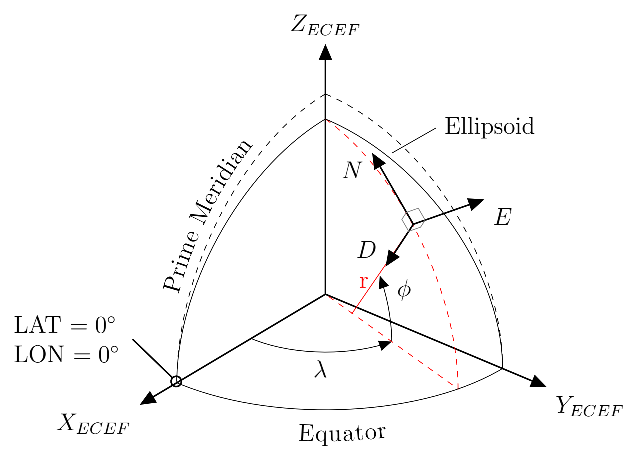
$$\gamma = \sin^{-1}\left(\frac{u\sin\theta - v\sin\phi\cos\theta - w\cos\phi\cos\theta}{\sqrt{\mathrm{u}^2 + v^2 + w^2}}\right) \tag{3.11}$$

$$\chi = \tan^{-1}\left[\frac{u\cos\theta\sin\psi + v\,(\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi) + w\,(\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi)}{u\cos\theta\cos\psi + v\,(\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) + w\,(\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi)}\right] \tag{3.12}$$

North East Down (NED), also known as Local Tangent Plane (LTP), is positioned on the surface of earth with it's origin vertically aligned to the aircraft's center of gravity. North is parallel to lines of longitude (λ), east is parallel to lines of latitude (ϕ), and down completes the right hand rule pointing into earth. Vehicle Carried Vertical (VCV) shares the NED orientation definition, with the exception of a shift in origin from Earth's surface to the vehicle's center of gravity, as the name suggests.



(a) Earth Centered Inertial (Fixed)



(b) Earth Centered Earth Fixed (Rotates)

Figure 3.3: Earth Reference Frames

The Earth Centered Inertial (ECI) frame is considered *fixed* in space with its origin at the center of Earth; it does not rotate with Earth and is oriented to suit the situation. Typically the z_{ECI} axis is aligned along Earth's spin axis pointing toward the North Pole. Consult [7], pg. 20, for alternative ECI orientations.

Earth Centered Earth Fixed (ECEF) is a non-inertial frame that rotates with earth. This reference system aligns x_{ECEF} to the intersection of the zero-longitude prime meridian and zero-latitude equator. y_s lies in the equatorial plane and z_{ECEF} points toward the Earth's North Pole. Note how the radius endpoint is not coincident with the center of the ellipsoid; this is because the radius emanates from a plane tangent to the ellipsoid surface.

3.1.3 Direction Cosine Matrices

If reference frames were languages, direction cosine matrices would be interpreters. It allows a vector's orientation to be expressed as components among relative coordinate systems. As the name suggests, rotations are achieved by defining a matrix of direction cosines[3] that relate *unit vectors* in one axis system to those in another, preserving the length of the rotated vector. The determination of matrix elements may be accomplished by inspection; McRuer [5] and Stevens [8] note several general properties for construction of these matrices:

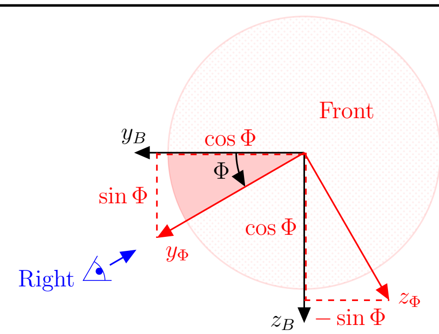
- "The one is always associated with the axis about which rotation occurs."
- "The remaining elements in the row and column containing the one are all zeros."
- The remaining main diagonal terms are the cosine of the angle of rotation.
- The remaining matrix elements contain the sine of the angle of rotation and are always symmetrically placed relative to the cosine terms; this is done so that zero rotation produces an identity matrix.
- "In the direct right-handed rotation the negative sign always appears in the row above the one (this is to be interpreted as the third row if the one is in the first)."
- "Changing the sign of the rotation angle yields the matrix transpose."

[3] Definition

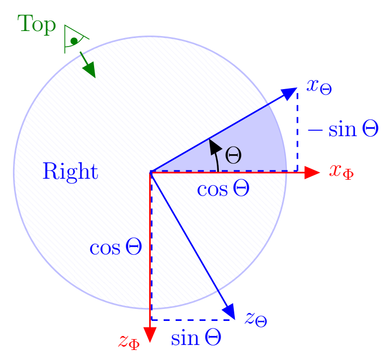
A coordinate rotation example from the body axis frame, $\boldsymbol{F_B}$, to north-east-down frame, $\boldsymbol{F_{NED}}$, is illustrated by three plane rotations in Table 3.2.

Table 3.2: Direction Cosine Matrices for Plane Rotations

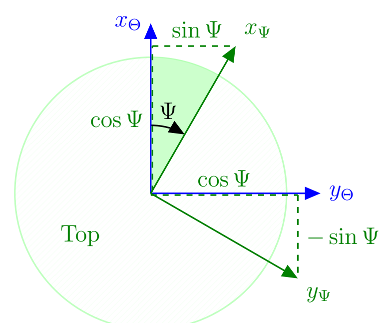
C_Φ	x_B	y_B	z_B
x_Φ	1	0	0
y_Φ	0	$\cos\Phi$	$\sin\Phi$
z_Φ	0	$-\sin\Phi$	$\cos\Phi$



C_Θ	x_Φ	y_Φ	z_Φ
x_Θ	$\cos\Theta$	0	$-\sin\Theta$
y_Θ	0	1	0
z_Θ	$\sin\Theta$	0	$\cos\Theta$



C_Ψ	x_Θ	y_Θ	z_Θ
x_Ψ	$\cos\Psi$	$\sin\Psi$	0
y_Ψ	$-\sin\Psi$	$\cos\Psi$	0
z_Ψ	0	0	1



As an example, the array C_Φ from Table 3.2 may be read either left to right or down as $\boldsymbol{y_\Phi} = \boldsymbol{x_B}\ 0 + \boldsymbol{y_B}\ \cos\Phi - \boldsymbol{z_B}\ \sin\Phi$ and $\boldsymbol{y_B} = \boldsymbol{x_\Phi}\ 0 + \boldsymbol{y_\Phi}\ \cos\Phi + \boldsymbol{z_\Phi}\ \sin\Phi$ respectively. The transpose of C_Φ, ie. C_Φ^T allows us to get to $\boldsymbol{x_\Phi}$, $\boldsymbol{y_\Phi}$, $\boldsymbol{z_\Phi}$ from $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, to be proven later. Doing the left to right read for the remaining rows and corresponding columns leads to a convenient matrix formulation of these equations:

$$\begin{bmatrix} \boldsymbol{x_\Phi} \\ \boldsymbol{y_\Phi} \\ \boldsymbol{z_\Phi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & \sin\Phi \\ 0 & -\sin\Phi & \cos\Phi \end{bmatrix} \begin{bmatrix} \boldsymbol{x_B} \\ \boldsymbol{y_B} \\ \boldsymbol{z_B} \end{bmatrix} \qquad \Longleftrightarrow \qquad \boldsymbol{F_\Phi} = C_\Phi \boldsymbol{F_B} \tag{3.13}$$

In this formulation C_Φ gets us to $\boldsymbol{x_B}$, $\boldsymbol{y_B}$, $\boldsymbol{z_B}$ from $\boldsymbol{x_\Phi}$, $\boldsymbol{y_\Phi}$, $\boldsymbol{z_\Phi}$. A variety of notations exist for direction cosine matrices, Stevens would write $C_{\boldsymbol{F_\Phi}/\boldsymbol{F_B}}$ instead of C_Φ which explicitly expresses the coordinate frame transformation in the subscript, ie. from $\boldsymbol{F_B}$ to $\boldsymbol{F_\Phi}$. Less trivial than notation are the properties which these rotation matrices possess:

1. *Orthogonality*

If we let Q be square, $n \times n$, matrix and suppose $Q^{-1} = Q^T$ then Q is orthogonal if and only if:

$$QQ^T = Q^TQ = I \tag{3.14}$$

where I is the identity matrix. This implies that the rotated vector length is unchanged. Alternatively, an orthogonal matrix may be defined as a square matrix with entries whose rows and columns are perpendicular and of unit length, ie. orthogonal unit vectors or orthonormal vectors.

2. *Non-Commutative*

Direction cosine matrices do not commute:

$$C_1C_2 \neq C_2C_1 \tag{3.15}$$

3. *Successive rotations may be described the by product of individual plane rotation matrices.*

The orientation of a three-dimensional coordinate frame to another may be obtained by a sequence of three successive rotations. By tradition, aerospace applications perform these transformations through right handed rotations in each coordinate planes, referred to earlier as plane rotations, in the Z-Y-X order [8]; alternate sign convention and planes of rotation exist in other fields, eg. 3D modeling in computer science. Right handed rotation about the z-axis is positive yaw, right handed rotation about the y-axis is positive pitch, and right handed rotation about the x-axis is positive roll. Order of rotation sequence is arbitrary, Figure 3.4 depicts a complete coordinate transformation in a X-Y-Z (Roll-Pitch-Yaw) manner. This rotation sequence is suitable for calculating aircraft attitudes with respect to the north-

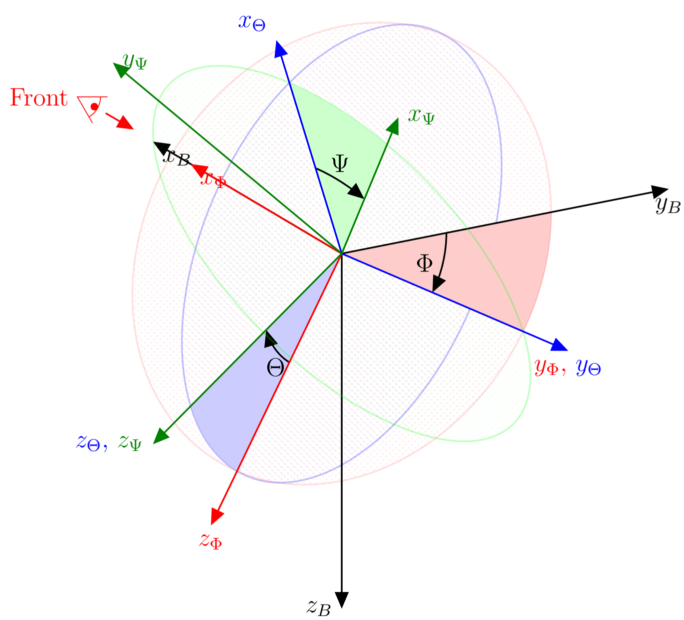


Figure 3.4: Direction Cosine Matrix Visual

east-down frame. These angles of rotation are called **Euler angles**. In terms of coordinate transformations

$$\boldsymbol{F_B} = (C_\Phi C_\Theta C_\Psi)\ \boldsymbol{F_{NED}} \tag{3.16}$$

where Equation 3.17 is the complete coordinate transformation from north-east-down to the body frame, ie. $C_{\boldsymbol{F_{NED}}/\boldsymbol{F_B}} = C_\Phi C_\Theta C_\Psi$

$$C_\Phi C_\Theta C_\Psi = \begin{bmatrix} \cos\Theta\cos\Psi & \cos\Theta\sin\Psi & -\sin\Theta \\ -\cos\Phi\sin\Psi + \sin\Phi\sin\Theta\cos\Psi & \cos\Phi\cos\Psi + \sin\Phi\sin\Theta\sin\Psi & \sin\Phi\cos\Theta \\ \sin\Phi\sin\Psi + \cos\Phi\sin\Theta\cos\Psi & -\sin\Phi\cos\Psi + \cos\Phi\sin\Theta\sin\Psi & \cos\Phi\cos\Theta \end{bmatrix} \tag{3.17}$$

3.1.4 Newton's Second Law

With Assumptions 3.1 & 3.2 in our front pocket, which is more accessible than the back pocket, we now have an idealized rigid body and live in a world suited for the application Newton's Laws. With this we can describe translational and rotational motion of the aircraft by its kinematic analogs: linear momentum, $\boldsymbol{p}$, and angular momentum, $\boldsymbol{H}$ respectively.

"By Newton's second law the time rate of change of linear momentum equals the sum of all *externally* applied forces, [$\boldsymbol{F}$].

$$\boldsymbol{F} = \frac{d}{dt}(\boldsymbol{p}) = \frac{d}{dt}(m\boldsymbol{V}) \tag{3.18}$$

and the rate of change of angular momentum is equal to the sum of all applied torques

$$\boldsymbol{M} = \frac{d}{dt}(\boldsymbol{H}) \tag{3.19}$$

These vector differential equations provide the starting point for a complete description of the rigid body motions of the vehicle." [5]

Assumption 3.3. Mass is considered constant

In most aerospace systems thrust is generated by an expenditure of vehicle mass; an exception being electric powered applications. Whether that trade in mass directly contributes to linear momentum or not needs to be considered. In the present propulsion case a heavy fuel piston engine turns a propeller, therefore the thrust generated may be considered an external force. Alternatively, if a jet engine were used there would be a component of thrust due to expulsion of vehicle mass[4].

TODO: Rectilinear acceleration eg, to get to expanded forms of Equation 3.18 and Equation 3.19 introduced in beginning of next two sections?

[4]McRuer [5] derives a modified extension of Equation 3.18 to take this into account.

Translational (Acceleration)

The goal is to derive equations for translation accelerations in the wind axis reference frame – $\dot{V}$, $\dot{\alpha}$, $\dot{\beta}$. Picking up where Equation 3.18 left off, along with Assumption 3.3, we may expand the expression to

$$\boldsymbol{F} = m\left[\frac{d}{dt}(\boldsymbol{V}) + \boldsymbol{\Omega} \times \boldsymbol{V}\right] \tag{3.20}$$

where $\boldsymbol{F}$ is the total force acing on the vehicle, m is the vehicle mass, $\boldsymbol{V}$ is the total vehicle velocity, and $\boldsymbol{\Omega}$ is the total vehicle angular velocity:

$$\boldsymbol{F} = \begin{bmatrix} \sum F_x \\ \sum F_y \\ \sum F_z \end{bmatrix} = \begin{bmatrix} F_{x_T} + F_{x_A} + F_{x_G} \\ F_{y_T} + F_{y_A} + F_{y_G} \\ F_{z_T} + F_{z_A} + F_{z_G} \end{bmatrix} \tag{3.21}$$

$$\boldsymbol{V} = [\,u,\, v,\, w\,]^T \tag{3.22}$$

$$\boldsymbol{\Omega} = [\,P,\, Q,\, R\,]^T \tag{3.23}$$

The elements of $\boldsymbol{F}$ are summations of externally applied propulsive (T), aerodynamic (A), and gravitational (G) forces respective to each body axis, (B). It will be assumed that the engine is mounted to align with body axes, therefore there are no thrust-angles or $F_{y_T} = F_{z_T} = 0$.

The body axis aerodynamic forces can be transformed to their equivalent stability axis components lift L, drag D, and side-force Y as Figure 3.5 indicates.

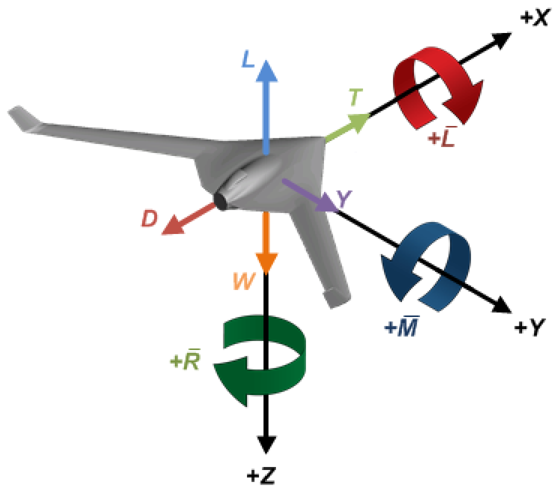


Figure 3.5: Body Axes, Forces, and Moments

$$F_{x_A} = -D\cos\alpha + L\sin\alpha \tag{3.24}$$

$$F_{y_A} = Y \tag{3.25}$$

$$F_{z_A} = -D\sin\alpha - L\cos\alpha \tag{3.26}$$

The gravitational forces can be resolved into body axis components such that

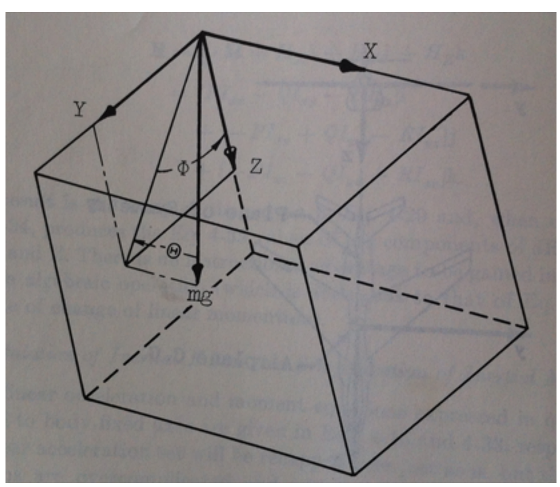


Figure 3.6: Orientation of Gravity Vector with Respect to Body Axes

$$F_{x_G} = -\, mg \sin\theta \tag{3.27}$$

$$F_{y_G} = \quad mg \sin\phi \cos\theta \tag{3.28}$$

$$F_{z_G} = \quad mg \cos\phi \cos\theta \tag{3.29}$$

Combining these we arrive at an expression that considers all external forces acting on the airframe.

$$\begin{bmatrix} \sum F_x \\ \sum F_y \\ \sum F_z \end{bmatrix} = \begin{bmatrix} F_{x_T} - D\cos\alpha + L\sin\alpha - mg\sin\theta \\ Y + mg\sin\phi\cos\theta \\ -D\sin\alpha - L\cos\alpha + mg\cos\phi\cos\theta \end{bmatrix} \tag{3.30}$$

By rearranging Equation 3.20 to solve for translational acceleration, $d\boldsymbol{V}/dt$, we can express body axis accelerations in terms of body axis forces, angular rates, and velocities:

$$\frac{d}{dt}(\boldsymbol{V}) = \frac{1}{m}\boldsymbol{F} - \boldsymbol{\Omega} \times \boldsymbol{V} \tag{3.31}$$

Substitution of Equations 3.22, 3.23, and 3.21 yields

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \frac{1}{m}\left(F_{x_T} + F_{x_A} + F_{x_G}\right) + Rv - Qw \\ \frac{1}{m}\left(F_{y_T} + F_{y_A} + F_{y_G}\right) + Pw - Ru \\ \frac{1}{m}\left(F_{z_T} + F_{z_A} + F_{z_G}\right) + Qu - Pv \end{bmatrix} \tag{3.32}$$

In order to express Equation 3.32 in the wind axis system, will need to use Equations 3.1, 3.2, and 3.3 as transforms

$$u = V\cos\beta\cos\alpha$$

$$v = V\sin\beta$$

$$w = V\cos\beta\sin\alpha$$

and Equations 3.7, 3.8, and 3.9

$$
\begin{aligned}
V &= \sqrt{u^2 + v^2 + w^2} \\
\alpha &= \tan^{-1}\left(\frac{w}{u}\right) \\
\beta &= \sin^{-1}\left(\frac{v}{\sqrt{u^2 + v^2 + w^2}}\right)
\end{aligned}
$$

Leads to $\dot{V}$, $\dot{\alpha}$, and $\dot{\beta}$ equations

Rotational (Acceleration)

The goal is to derive rotational acceleration equations – $\dot{p}$, $\dot{q}$, $\dot{r}$. Picking up where Equation 3.19 left off and substituting total angular momentum for the product of the moment of inertia matrix and rotational velocity vector, $\boldsymbol{H} = \boldsymbol{I\Omega}$, we may expand the expression to

$$\boldsymbol{M} = \left[\frac{d}{dt}(\boldsymbol{I\Omega}) + \boldsymbol{\Omega} \times \boldsymbol{I\Omega}\right] \tag{3.33}$$

where $\boldsymbol{M}$ is the total moment acing on the vehicle, $\boldsymbol{I}$ is the inertia matrix (alternatively referred to as tensor or dyad), and $\boldsymbol{\Omega}$ is the total vehicle angular velocity:

$$\boldsymbol{M} = \begin{bmatrix} \sum L \\ \sum M \\ \sum N \end{bmatrix} = \begin{bmatrix} L + L_T \\ M + M_T \\ N + N_T \end{bmatrix} \tag{3.34}$$

L, M ,and N are the total aerodynamic moments about the x_B, y_B, and z_B body axes with T subscript indicates moments induced by the power-plant.

$$\boldsymbol{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \tag{3.35}$$

Elements along the main diagonal are called the **moments of inertia** with respect to the x, y, and z axes respectively and are always positive. The off diagonal terms are referred to as the **products of inertia** and may be positive, negative, or zero; they are measures of the imbalance in mass distribution. Note that it is possible to orient the axes in such a way that the products of inertia are zero. In this case the diagonal terms are called the principal moments of inertia.

Assumption 3.4. The XZ plane is a plane of symmetry.

This is a very good approximation for most air vehicles, and leads to $I_{yz} = 0$ as well as $I_{xy} = 0$. If we assume that the inertia tensor is constant, as we did with mass in the translational acceleration derivation, then Equation 3.33 may be rewritten as

$$\frac{d}{dt}\boldsymbol{\Omega} = \boldsymbol{I}^{-1}\left(\boldsymbol{M} - \boldsymbol{\Omega} \times \boldsymbol{I\Omega}\right) \tag{3.36}$$

$$\frac{d}{dt}\boldsymbol{\Omega} = [\,\dot{p},\, \dot{q},\, \dot{r}\,]^T \tag{3.37}$$

$$\boldsymbol{I}^{-1} = \frac{1}{\det I}\begin{bmatrix} I_1 & I_2 & I_3 \\ I_4 & I_5 & I_6 \\ I_7 & I_8 & I_9 \end{bmatrix} \tag{3.38}$$

Leads to $\dot{p}$, $\dot{q}$, and $\dot{r}$ equations

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \dots \tag{3.39}$$

Attitudes

Leads to $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ equations, but I need to derive in terms of flight path components $\dot{\mu}$, $\dot{\gamma}$, and $\dot{\chi}$

$$\begin{bmatrix} \dot{\mu} \\ \dot{\gamma} \\ \dot{\chi} \end{bmatrix} = \dots \tag{3.40}$$

Earth Relative

Will not cover

3.1.5 Equations of Motion

Complete Set of Nonlinear EOM

$$
\begin{aligned}
\dot{\chi} &= \frac{1}{mV\cos\gamma}\left[D\sin\beta\cos\mu + Y\cos\beta\cos\mu + L\sin\mu\right. \\
&\quad \left. + T\left(\sin\alpha\sin\mu - \cos\alpha\sin\beta\cos\mu\right)\right] && \text{(3.41a)} \\
\dot{\gamma} &= \frac{1}{mV}\left[-D\sin\beta\sin\mu - Y\cos\beta\sin\mu + L\cos\mu\right. \\
&\quad \left. + T\left(\sin\alpha\cos\mu + \cos\alpha\sin\beta\sin\mu\right)\right] - \frac{g}{V}\cos\gamma && \text{(3.41b)} \\
\dot{V} &= \frac{1}{m}\left(-D\cos\beta + Y\sin\beta + T\cos\alpha\cos\beta\right) - g\sin\gamma && \text{(3.41c)}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mu} &= \frac{1}{mV}\left[D\sin\beta\tan\gamma\cos\mu + Y\cos\beta\tan\gamma\cos\mu + L\left(\tan\beta + \tan\gamma\sin\mu\right)\right. \\
&\quad \left. + T\left(\sin\alpha\tan\gamma\sin\mu + \sin\alpha\tan\beta - \cos\alpha\sin\beta\tan\gamma\cos\mu\right)\right] && \text{(3.41d)} \\
&\quad - \frac{g\tan\beta\cos\gamma\cos\mu}{V} + \frac{P\cos\alpha + R\sin\alpha}{\cos\beta} \\
\dot{\alpha} &= -\frac{1}{mV\cos\beta}\left(L + T\sin\alpha\right) + \frac{g\cos\gamma\cos\mu}{V\cos\beta} + Q && \text{(3.41e)} \\
&\quad - \tan\beta\left(P\cos\alpha + R\sin\alpha\right) \\
\dot{\beta} &= \frac{1}{mV}\left(D\sin\beta + Y\cos\beta - T\sin\beta\cos\alpha\right) + \frac{g\cos\gamma\sin\mu}{V} && \text{(3.41f)} \\
&\quad - R\cos\alpha + P\sin\alpha
\end{aligned}
$$

$$
\begin{aligned}
\dot{P} &= \left(c_1 R + c_2 P\right)Q + c_3\bar{L} + c_4\bar{N} && \text{(3.41g)} \\
\dot{Q} &= c_5 PR - c_6\left(P^2 - R^2\right) + c_7\bar{M} && \text{(3.41h)} \\
\dot{R} &= \left(c_8 P - c_2 R\right)Q + c_4\bar{L} + c_9\bar{N} && \text{(3.41i)}
\end{aligned}
$$

where [7] defines c terms as

$$
\begin{aligned}
\Gamma &= I_{xx}I_{zz} - I_{xz}^2 \\
\Gamma c_1 &= \left(I_{yy} - I_{zz}\right) I_{zz} - I_{xz}^2 \\
\Gamma c_2 &= \left(I_{yy} - I_{zz}\right) I_{zz} - I_{xz}^2 \\
\Gamma c_3 &= I_{zz} \\
\Gamma c_4 &= I_{xz} \\
c_5 &= \frac{I_{zz} - I_{xx}}{I_{yy}} \\
c_6 &= \frac{I_{xz}}{I_{yy}} \\
c_7 &= \frac{1}{I_{yy}} \\
\Gamma c_8 &= I_{xx}\left(I_{xx} - I_{yy}\right) + I_{xz}^2 \\
\Gamma c_9 &= I_{xx}
\end{aligned}
$$

3.2 System and Stability Concepts

The goal is to familiarize the reader with concepts required to prove stability for the backstepping control architecture. Proofs will not be included[5], but will be referenced immediately succeeding theorems. It is assumed that the reader has a good understanding of solution properties to ordinary differential equations such as existence, uniqueness, and continuity. Mathematical notation and terminology tied to these properties may be used in subsequent theorems and definitions without introduction. A great overview of mathematical preliminaries pertinent to nonlinear systems and [backstepping] control is included in [1, Chp.

[5]Consider including in appendix?

2]. The following standard mathematical abbreviation symbols shall be used:

$\forall$	"for all"
$\exists$	"there exists"
$\ni$	"such that"
$\in$	"in" or "belongs to"
$\subset$	"a subset of"
$\Rightarrow$	"implies"

Ultimately, this section condenses the philosophy, definitions, and theorems of [2, Chp. 3], [1, Chp. 3], [9, Chp. 3], [10, Chp. 2] and [11, Appendix A] in a brief and clear manner.

3.2.1 System Equations

The nonlinear differential equations summarized in subsection 3.1.5 may be reduced to the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}, t\right) \tag{3.42}$$

where $\dot{\boldsymbol{x}}$ is the $n \times 1$ derivative of the state vector with respect to time, $\boldsymbol{f}$ is an $n \times 1$ non-linear vector function expressing the six-degree of freedom rigid body equations, and $\boldsymbol{x}$ is the $n \times 1$ state-vector with respect to time. Additionally, the state-vector is defined as a set of real numbers, $\left(x_1, \ldots, x_n\right)^T$, contained in n-dimensional Euclidean space, denoted by the symbol $\mathbb{R}^n$, and is formally referred to as **state-space**. The parameter n is the **system order**[6] and refers to the number of first order differential equations required to represent an equivalent n^{th} order ordinary differential equation (ODE).

Equation 3.42 represents the closed-loop time-variant dynamics of a feedback control system, even though it does not explicitly contain a control input vector $\boldsymbol{u}$. This is because

[6]The highest derivative of the dependent variable with respect to the independent variable appearing in the equation.

the control input may be considered a function of state $\boldsymbol{x}$ and time t, therefore *disappearing* in the closed-loop dynamics. Showing this mathematically, if the plant dynamics are

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left(\boldsymbol{x}, \boldsymbol{u}, t\right) \tag{3.43}$$

and some control law $\boldsymbol{u}$ has been selected as

$$\boldsymbol{u} = \boldsymbol{g}\left(\boldsymbol{x}, t\right) \tag{3.44}$$

then the closed-loop dynamics are

$$\dot{\boldsymbol{x}} = \boldsymbol{f}\left[\boldsymbol{x}, \boldsymbol{g}\left(\boldsymbol{x}, t\right), t\right] \tag{3.45}$$

which can be rewritten in the form of Equation 3.42; since $\boldsymbol{g}(\boldsymbol{x}, t)$ is a function of the state $\boldsymbol{x}$, which is already represented in the expression, it may be discarded. Naturally, Equation 3.42 may also represent a system without control input, such as a freely moving spring-mass damper or pendulum. These examples, as with all physical systems, are time dependent.

As an aside, for a second order system ($n = 2$), solutions of an ODE may be realized in **phase-space**[7] as trajectories from $t = (0, \infty)$. Given a particular initial condition, an ODE may have several system trajectories. Continuity of $\boldsymbol{f}$, ie. $\lim_{x \to a} \boldsymbol{f}(x) = \boldsymbol{f}(a)$, ensures that there is at least one solution but does not ensure uniqueness of the solution. A stronger and therefore more frequently used condition, that guarantees *Lipschitz* continuity, may be used to prove existence *and* uniqueness as well as protect against the possibility of $\boldsymbol{f}(x)$ having an infinite slope, eg. a discontinuity.

Definition 3.2.1 (Lipschitz Condition). Khalil [1, §2.2]

If there exists a strictly positive Lipschitz constant L such that $\boldsymbol{f}(\boldsymbol{x}, t)$ satisfies the inequality,

$$\|\boldsymbol{f}(\boldsymbol{x}, t) - \boldsymbol{f}(\boldsymbol{y}, t)\| \leq L\|\boldsymbol{x} - \boldsymbol{y}\| \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{D}$$

then the function $\boldsymbol{f}(\boldsymbol{x}, t)$ is said to be *Lipschitz in* $\boldsymbol{x}$ for all points in the domain $\mathcal{D}$.

[7] A vector field of derivatives $[f_2(x_1, x_2),\ f_1(x_1, x_2)]$ at respective state variable $(x_2,\ x_1)$ locations that allows for visualization of the qualitative behavior of the system. Especially useful for classifying stability of equilibrium points

Further specifying conditions on the domain $\mathcal{D}$, over which the Lipschitz condition holds, imposes restrictions on input values for the function $\boldsymbol{f}(\boldsymbol{x}, t)$. A function is said to be *locally Lipschitz in* $\boldsymbol{x}$ if that for each point $\boldsymbol{x} \in \mathcal{D} \subset \mathbb{R}^n$ there exists a finite neighborhood $\mathcal{D}_0 \in \mathcal{D}$ such that the Lipschitz condition holds for all points in $\mathcal{D}_0$ with a corresponding Lipschitz constant L_0.

Theorem 3.2.1 (Global Existence and Uniqueness)**.** *Khalil [1, Thm 2.4]*
*Let $\boldsymbol{f}(\boldsymbol{x}, t)$ be piecewise continuous in t and **locally Lipschitz in $\boldsymbol{x}$** for all $t \geq t_0$ and all $\boldsymbol{x}$ in a domain $D \subset \mathbb{R}^n$ and let W be a **compact (closed and bounded) subset** of $\mathcal{D}$. If for $x_0 \subset W$ it is known that every solution of*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x_0} \qquad \forall t \geq t_0$$

*lies entirely in W. Then there is a **unique solution** that is defined for all $t \geq t_0$*

Proof: Refer to Khalil [1, Pg. 77] □

"The trick in applying Theorem 3.2.1 is in checking the assumption that every solution lies in a compact set without actually solving the differential equation." This concept is introduced in Lyapunov's stability definitions to come.

Definition 3.2.2 (Autonomous and Non-Autonomous Systems)**.** Slotine and Weiping [2]
The non-linear system $\dot{\boldsymbol{x}} = \boldsymbol{f}\,(\boldsymbol{x}, t)$ is said to be **autonomous** if $\boldsymbol{f}$ does not depend explicitly on time, ie. if the system's state equation can be written

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{3.46}$$

otherwise, the system is called **non-autonomous**.

Again, all real-world systems are non-autonomous, but "in practice system properties often change very slowly, so we can neglect their time variation without causing any practically meaningful error."[2] Most importantly, Definition 3.2.2 implies that solutions, or system trajectories, of autonomous ODEs are independent of initial time, thereby greatly simplifying stability analysis. In other words, stability does not depend on initial conditions!

Definition 3.2.3 (Equilibrium and Operating Points). Farrell and Polycarpou [11]
A system trajectory may correspond to only a single point $\boldsymbol{x}^*$, called an **equilibrium point**, if once $\boldsymbol{x}(t)$ is equal to $\boldsymbol{x}^*$ it remains equal to $\boldsymbol{x}^*$ for all time. For the non-autonomous system in Equation 3.42, the equilibrium points are the real roots (x-intercepts (for $n = 2$ systems)) of the differential equation, that is

$$\boldsymbol{f}\left(\boldsymbol{x}^*, t\right) = 0, \quad \forall t \geq 0$$

An **operating point** is a region of stability formally defined as "any state space location at which the system can be forced into equilibrium *by choice of control signal.*" For the generalized system containing control input $\boldsymbol{u}$ in Equation 3.44, the vectors $(\boldsymbol{x_0}, \boldsymbol{u_0})$ are operating points if

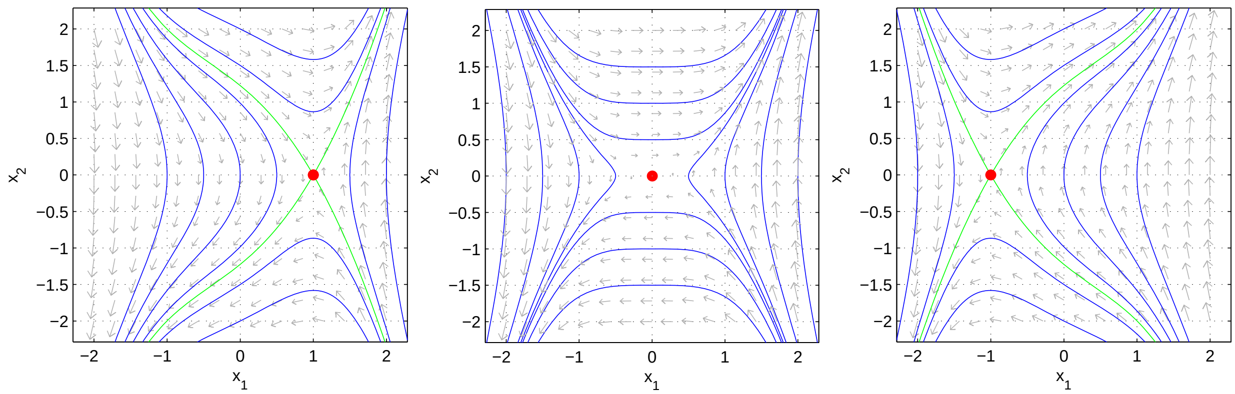
$$\boldsymbol{f}\left(\boldsymbol{x_0}, \boldsymbol{u_0}, t\right) = 0, \quad \forall t \geq 0$$

Varying the control input changes the operating point, implying that these points are not isolated. A collection of these points is called a surface of operating points, and is illustrated in the following example through multiple phase portraits.

Example 3.2.1 (Operating Points).
Given the system

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= x_1^3 + u \end{aligned}$$

and applying the operating point definition for an arbitrary control signal $\boldsymbol{u_0} = -1$, an operating point emerges at $\boldsymbol{x_0} = [1, 0]^T$. If the control input is varied, we can see the surface of operating points:



(a) $\boldsymbol{u_0} = -1, \quad \boldsymbol{x_0} = [1, 0]^T$ (b) $\boldsymbol{u_0} = 0, \quad \boldsymbol{x_0} = [0, 0]^T$ (c) $\boldsymbol{u_0} = 1, \quad \boldsymbol{x_0} = [-1, 0]^T$

Figure 3.7: Operating Surface via Phase Portraits

Visualization of trajectories, through phase portraits[8], provides an intuitive feel for the stability of an operating point. Red dots (●) are operating points, green lines (—) are stable or unstable manifolds, blue lines (—) are solution trajectories, and grey vectors ($\rightarrow$) indicate solution direction tangent to trajectories. Qualitative graphical insight is often more clear, simple, and useful than an analytical approach. You can see that the operating point moves along the x_1 axis as control input is varied and that trajectories flow away from these operating points, therefore they're unstable. The surface of operating points may be expressed as $\boldsymbol{x_0} = [a, 0]^T$ with $\boldsymbol{u_0} = -a^3$. This example proves that operating points are not isolated. Most importantly, the system could be forced to operate at any point on the surface if a stabilizing controller, $u = -x_1^3 - (x_1 - a) - x_2$, was selected.

As Figure 3.7 shows, the operating point is not always coincident with the state-space origin, ie. $\boldsymbol{x} = \boldsymbol{0}$. For the sake of notational and analytical simplicity one may translate the equilibrium point to the origin *without loss of generality* by redefining the state-vector. This allows one to analyze the local stability of the system, neglecting possible higher order terms $\mathcal{O}^3$ and above. Using notation from [10, Pg. 23]:

$$\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{x}^* \tag{3.47}$$

[8]Made using pplane8, http://math.rice.edu/~dfield/index.html

To prove this statement, substitute a reformulation of previous expression, $\boldsymbol{x} = \boldsymbol{z} + \boldsymbol{x}^*$, into Equation 3.42 as shown:

$$\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z} + \boldsymbol{x}^*, t) \tag{3.48}$$

It is clear that by substituting Equation 3.47 into Equation 3.48 one would arrive at a system equivalent to the original, ie. $\dot{\boldsymbol{z}} = \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$. "In addition $\boldsymbol{z} = 0$, the solution corresponding to the original system's equilibrium point $\boldsymbol{x} = \boldsymbol{x}^*$, is an equilibrium point of Equation 3.48; [recall Definition 3.2.3]. Therefore, instead of studying the behavior of the original system, Equation 3.42, in the neighborhood of $\boldsymbol{x}^*$, one can equivalently study behavior of the redefined system, Equation 3.48, in the neighborhood of the origin"[2] ; what was meant by, "without loss of generality."

In cases like aircraft trajectory control, the concept of stability about a point is not what we're concerned about. In the presence of atmospheric disturbances trajectory perturbations will arise, and it is the stability of *motion* that is important: Will the system remain on its desired trajectory if slightly perturbed away from it? Lyapunov synthesis will answer this question.

3.2.2 Stability Definitions

Backstepping control designs are constructed using Lyapunov stability theory. This is currently the most useful and general approach [2] to establishing stability for non-linear systems and may also be extended to linear systems. It was published in 1892 by Russian mathematician Alexandr Lyapunov in *The General Problem of Motion Stability* and provides two methods for stability analysis: indirect and direct. The first method[9], indirect or linearization, determines *local* stability properties; eigenvalues of a linear (approximate, hence indirect) system reveals stability in the immediate vicinity of an equilibrium point. The

[9] In my research I found conflicting evidence on what Lyapunov's first method actually was. [2] warns the reader that lineraization is sometimes incorrectly referred to as the first method, which should be Lyapunov's method of exponents. However, all other sources I found on the topic refer to the indirect method as the first method.

second method, direct, determines *regional* stability properties; the aim is to make the system act like a function whose time derivative guarantees some form of stability. Since our system equations are non-linear only Lyapunov's direct method will be covered.

A **stable system** is one that starts near a desired operating point and stays within a bound of that point ever after; unstable otherwise. Linear stability is evaluated in terms of a single equilibrium point, while non-linear stability is based on the idea of boundedness as these systems can have several isolated equilibrium points. This implies much more complex and unfamiliar behavior for non-linear systems, therefore requiring more refined stability concepts. This section will formally introduce these concepts in the Lyapunov sense, thereby providing the tools necessary to prove stability for backstepping control architectures.

To begin, let ϵ denote a spherical region defined by $\|\boldsymbol{x}\| < \epsilon$ in state-space and δ denote a spherical region that is generally within ϵ; δ is called a domain of attraction. Recall that

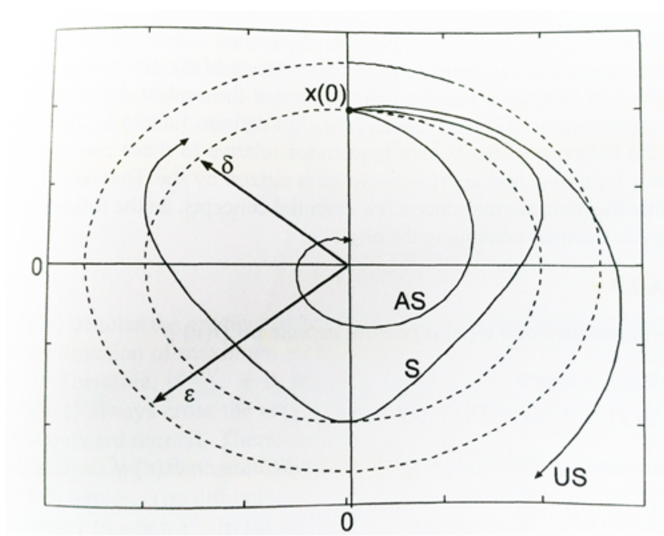


Figure 3.8: Concepts of Stability

equilibrium points may be translated to the origin by redefining the state-vector as shown in Equation 3.47. The theorems in this section will be presented as if the equilibrium point was translated to the origin; note that $\boldsymbol{z}$ notation will be dropped, and the usual $\boldsymbol{x}$ representation will be used to represent a system with redefined equilibrium point.

Definition 3.2.4 (Stability in the Sense of Lypaunov). Khalil [1, Def 3.2]

For the non-autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$$

where $x \in \mathbb{R}^n$, and $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x, the equilibrium point $\boldsymbol{x}^* = 0$ is

- **stable** if for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that

$$\|\boldsymbol{x}(t_0)\| < \delta \quad \Rightarrow \quad \|\boldsymbol{x}(t)\| < \epsilon \quad , \qquad \forall\, t \geq t_0 \geq 0 \tag{3.49}$$

- **uniformly stable** if for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$, *independent of* t_0 such that Equation 3.49 is satisfied
- **unstable** if not stable.
- **asymptotically stable** if it is stable and there exists a constant $c = c(t_0) > 0 \ni$

$$\lim_{t \to \infty} \boldsymbol{x}(t) = \boldsymbol{x}^* = 0, \qquad \forall\, \|\boldsymbol{x}(t_0)\| < c \tag{3.50}$$

- **uniformly asymptotically stable** if it is uniformly stable and there exists a constant $c > 0$ *independent of* t_0, such that Equation 3.50 is satisfied uniformly in t_0; that is, for each $\eta > 0$, there is $T = T(\eta) > 0$ such that

$$\|\boldsymbol{x}(t)\| < \eta \quad , \qquad \forall\, t \geq t_0 + T(\eta) \quad , \quad \forall\, \|\boldsymbol{x}(t_0)\| < c \tag{3.51}$$

- **globally uniformly asymptotically stable** if it is uniformly asymptotically stable and for each $\eta > 0$ and $c > 0$ there is $T = T(\eta, c) > 0$ such that

$$\|\boldsymbol{x}(t)\| < \eta \quad , \qquad \forall\, t \geq t_0 + T(\eta, c) \quad , \quad \forall\, \|\boldsymbol{x}(t_0)\| < c \tag{3.52}$$

- **exponentially stable** if for any $\epsilon > 0$ and some $\lambda > 0$ there exists $\delta = \delta(\epsilon) > 0 \ni$

$$\|\boldsymbol{x}(t_0)\| < \delta \quad \Rightarrow \quad \|\boldsymbol{x}(t)\| < \epsilon e^{-\lambda(t - t_0)} \quad , \quad \forall\, t > t_0 \geq 0 \tag{3.53}$$

I think that asymptotic stability uses c–η instead of ϵ–δ because its already used in the stability def and asymptotic stability requires stability

Again these definitions are developed with respect to a non-autonomous, or time-variant, system therefore include initial time, t_0. It is presented in this form for the sake of generality, as these definitions are also applicable to autonomous systems by the substitution $t_0 = 0$. It is clear through these definitions that Lyapunov evaluated stability by ensuring that solutions are not only bounded, but also that "the bound on the solution can be made as small as desired by restriction of the size of the initial condition."[11]. With regards to uniform stability, the additional stipulation over ordinary stability is that δ is independent of t_0; all properties are *uniform* if the system is time-invariant. If an equilibrium is asymptotically stable then it has a domain or region of attraction δ about the origin, a set of initial states $\boldsymbol{x}(t_0) = \boldsymbol{x_0}$ for which the limit holds. Stability properties are said to be *global* when the region of attraction is the whole space $\mathbb{R}^n$. You may also notice, that terms like asymptotic and global stability are foreign in classic controls sense; this is because all linear time-invariant (LTI) solutions are global and exponential. To conclude comments on Definition 3.2.4, the state vector of the exponentially stable system converges faster than an exponential function, where the positive number λ is the rate of exponential convergence[10]; exponential stability implies asymptotic stability, but not the other way 'round.

In real world systems stability will be affected by neglected or unknown parameters (non-parametric uncertainties), modeling errors (parametric uncertainties), and external disturbances. Uncertainties that do not affect system order may be treated as an additive term to the nominal system. Boundedness concepts from stability analysis in the Lyapunov sense may be applied to a reformulated system that takes into account said perturbations:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) + \boldsymbol{g}(\boldsymbol{x}, t) \tag{3.54}$$

[10] "By writing the positive constant ϵ as $\epsilon = e^{\lambda(t-t_0)}$ it is easy to see that after a time of $\tau_0 + (1/\lambda)$, the magnitude of the state vector decreases to less than 35% ($\approx e^{-1}$) of its original value; this is similar to the notion of a time-constant in a linear system. After $\tau_0 + (3/\lambda)$, the state magnitude $\|\boldsymbol{x}(t)\|$ will be less than 5% ($\approx e^{-3}$) of $\|\boldsymbol{x}(0)\|$" [2]

where $\boldsymbol{g}(\boldsymbol{x}, t)$ is the perturbation term. Typically we don't know $\boldsymbol{g}(\boldsymbol{x}, t)$ but we know something about its bounds, eg. upper bound on $\|\boldsymbol{g}(\boldsymbol{x}, t)\|$. Note that if $\boldsymbol{g}(\boldsymbol{x}, t) \neq 0$ the origin will not be an equilibrium point of the perturbed system, therefore if Equation 3.54's trajectory is kept arbitrarily close to an equilibrium point in the presence of **sufficiently small disturbances** then the system may be considered **total stability**. The following definition condenses this concept:

Definition 3.2.5. Slotine and Weiping [2, Defn. 4.13]
The equilibrium point $\boldsymbol{x}^* = \boldsymbol{x}(0) = 0$ for the unperturbed system in Equation 3.46, $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, is said to be totally stable if for every $\epsilon \geq 0$, two numbers exist δ_1 and δ_2 exist such that $\|\boldsymbol{x}(t_0)\| < \delta_1$ and $\|\boldsymbol{g}(\boldsymbol{x}, t)\| < \delta_2$ imply that every solution $\boldsymbol{x}(t)$ of the perturbed system Equation 3.54 satisfies the condition $\|\boldsymbol{x}(t_0)\| < \epsilon$

"Note that total stability is simply a local version (with small input) of BIBO (bounded-input bounded-output) stability." Furthermore, systems with uniform asymptotic stability, therefore exponentially stability as well, are capable of withstanding small disturbances through use of converse Lyapunov theorem

Theorem 3.2.2 (Total Stability)**.** *Slotine and Weiping [2, Thm. 4.14]*
If the equilibrium point $\boldsymbol{x}^* = \boldsymbol{x}(0) = 0$ for the unperturbed system in Equation 3.46, $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, is uniformly asymptotically stable, then it is totally stable.

Proof: Refer to Khalil [1, Chp. 5, Stability of Perturbed Systems] □

For systems up to second order ($n = 1, 2$), a two-dimensional[11] graphical analysis via vector fields and solution trajectories, called **phase portraits**, may be utilized to determine stability. First order systems may be represented as a vector field on the "x-axis": it dictates the velocity vector $\dot{\boldsymbol{x}}$ at each $\boldsymbol{x}$. "The behavior of the solution in the neighborhood of the

[11] However, technically speaking phase portraits may be drawn in three dimensional space. This is not very common, and can be complicated to generate as well as interpret.

origin can be determined by examining the sign of $\boldsymbol{f}(x)$. The $\epsilon - \delta$ requirement for stability is violated if $\boldsymbol{x}(\boldsymbol{f}(x)) > 0$ on either side of the origin."

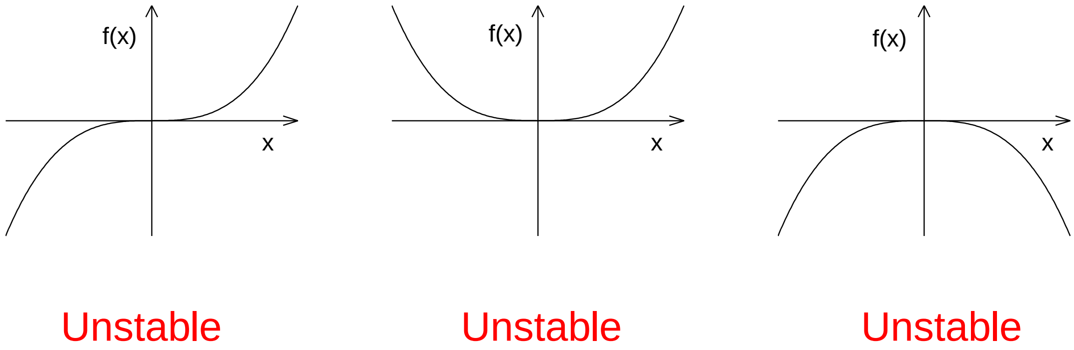


Figure 3.9: First Order Unstable Systems [1, Online Lecture Notes]

"The origin is stable if and only if $\boldsymbol{x}\boldsymbol{f}(x) \leq 0$ in some neighborhood of the origin"

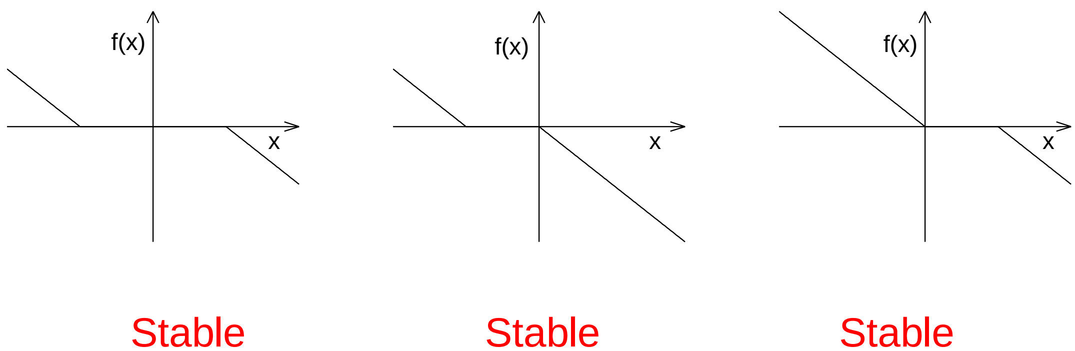
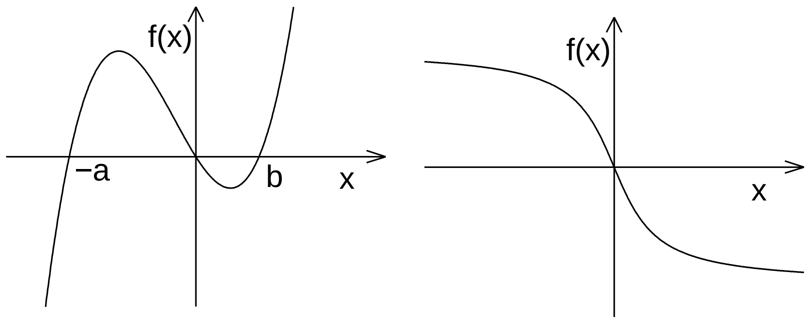


Figure 3.10: First Order Stable Systems [1, Online Lecture Notes]

"The origin is asymptotically stable if and only if $\boldsymbol{x}\boldsymbol{f}(x) < 0$ in some neighborhood of the origin"



(a) Asymptotically Stable (b) Globally Asymptotically Stable

Figure 3.11: First Order Asymptotically Stable Systems [1, Online Lecture Notes]

In two-dimensions, Figure 3.8 depicts stable (S), unstable (US), and asymptotically stable (AS) trajectories in phase-space with respect to ϵ and δ contours in Definition 3.2.4; the initial condition is $x(0)$, solid line is the trajectory $x(t)$, and origin is the equilibrium point x^*. This representation is a great conceptual tool, as it brings tangible meaning to the definitions. It may be used to analyze non-linear systems, especially useful for those with multiple equilibrium points, as around each point a linear system approximation is valid. In practice, many non-linear aircraft control architectures are linearized about a trim condition in order to apply classical frequency domain control techniques such as Bode or Root-Locus analysis. These are formalized and well understood techniques which use performance metrics, such as gain and phase margin, to ensure a particular degree of stability.

Figure 3.12 illustrates possible equilibrium point classifications in equivalent root-loci and phase-portraits:

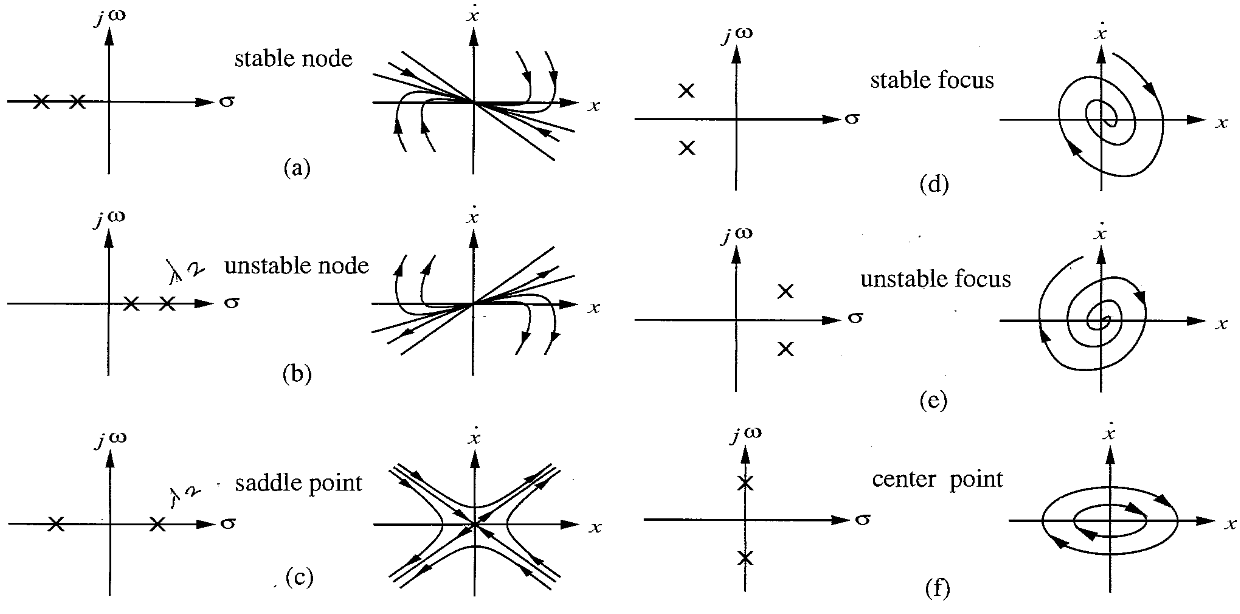


Figure 3.12: Equilibrium Point Classification for 2^{nd} Order Linear Systems [2]

Unfortunately however, most physical systems of interest are higher than second order and are therefore incapable of being displayed graphically. Lyapunov's direct method provides an analytical tool for this case.

3.2.3 Stability Analysis

Aleksandr Lyapunov realized that stability of an equilibrium point may be established if one can make the system act like a scalar, energy-like function, $V(x)$, and examine its time derivative along trajectories of the system. If this function's time derivative, $\dot{V}(x)$, is decreasing over time then it may be asserted that the system will eventually reach an equilibrium condition. Table 3.3 outlines the theorems that will be covered.

Table 3.3: Stability Theorem Overview

$\dot{\boldsymbol{x}} = \boldsymbol{f}(x)$	$V > 0$	$\dot{V} \leq 0$	S	Lyapunov's Direct Method	Thm. 3.2.3
	$V > 0$	$\dot{V} < 0$	AS		
	$V > 0$	$\dot{V} < 0$	GAS		
	$V \geq 0$	$\dot{V} \leq 0$	CONV	LaSalle	Thm. 3.2.4
	$V > 0$	$\dot{V} \leq 0$	GAS	Barbashin–Krasovskii	Cor. 3.2.1
$\dot{\boldsymbol{x}} = \boldsymbol{f}(x, t)$	$V \geq 0$	$\dot{V} \leq 0$	CONV	Barbalat's Lemma	Lem. 3.2.2
	$V > 0$	$\dot{V} \leq 0$	GUAS	LaSalle-Yoshizawa	Thm. 3.2.5

Before presenting stability theorems, some function related terminology must be introduced:

Definition 3.2.6. A scalar function $V(x)$ is

- **positive definite** if $V(0) = 0$ and $V(x) > 0$, $x \neq 0$
- **negative definite** if $V(0) = 0$ and $V(x) < 0$, $x \neq 0$
- **positive semidefinite** if $V(0) = 0$ and $V(x) \geq 0$, $x \neq 0$
- **negative semidefinite** if $V(0) = 0$ and $V(x) \leq 0$, $x \neq 0$
- **sign indefinite** if it is not any of the above
- **radially unbounded** if $V(x) \to \infty$ as $\|\boldsymbol{x}\| \to \infty$

Where $V(x)$ is assumed to be a scalar function on $\mathcal{D}$ into $\mathbb{R}$, $V : \mathcal{D} \to \mathbb{R}$, is continuously differentiable, and is defined in a domain $\mathcal{D} \subset \mathbb{R}^n$ that contains the origin $\boldsymbol{x} = 0$. The derivative of $V(x)$ along the system trajectory for the autonomous system in Equation 3.46 is obtained by the chain rule:

$$\dot{V}(\boldsymbol{x}) = \frac{d}{dt}V(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} = \frac{\partial V}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}} = \frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}) \tag{3.55}$$

The scalar function $V(\boldsymbol{x})$ has an implicit dependence on time and its derivative is dependent on the system's equation, therefore each system will have a different $\dot{V}(\boldsymbol{x})$. Not to mention that the form of $V(\boldsymbol{x})$ is anything but consistent, as will be covered later.

Theorem 3.2.3 (Lyapunov's Direct Method). *Khalil [1, Thm. 3.1]*

Let the origin be an equilibrium point, $\boldsymbol{x}^* = \boldsymbol{x}(0) = 0$, for an autonomous system, $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ and $V : \mathbb{R}^n \to \mathbb{R}$ a continuously differentiable, ***positive definite*** *function, then*

- *the origin is* ***stable*** *if*

$$\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) \leq 0 \quad , \quad \forall \, \boldsymbol{x} \in \mathcal{D} \tag{3.56}$$

- *the origin is* ***asymptotically stable*** *if*

$$\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) < 0 \quad , \quad \forall \, \{\boldsymbol{x} \in \mathcal{D} \,|\, \boldsymbol{x} \notin 0\} \tag{3.57}$$

Proof: Refer to Khalil [1, Pg. 101] □

Remark (Theorem 3.2.3) Because the system is time-invariant these equilibrium point properties are *uniform*, ie. uniformly stable and uniformly asymptotically stable. At its core, Lyapunov's direct method determines stability properties of $\boldsymbol{x}$ through a relationship between $\boldsymbol{f}$ and a positive definite function, $V(x, t)$. This theorem asserts that if the system loses energy over time it will eventually reach an equilibrium condition.

A function $V(\boldsymbol{x})$ *satisfying Theorem 3.2.3* is called a **Lyapunov function**, otherwise a **potential function**. It is essentially a model of system energy and therefore typically take the form of a quadratic, kinetic-energy like term. It is most easily understood from a geometric perspective through a two-dimensional phase-portrait. A Lyapunov function may be considered a series of concentric closed contours that make up a surface in $\mathbb{R}^3$; imagine looking down a funnel with numerous rings drawn on the inside that specify height, or volume, in that space. Formally, these contours, c , characterize level sets of the Lypaunov function $V(\boldsymbol{x}) = c$ for some $c > 0$, which is called a **Lyapunov or level surface**. Figure 3.13 demonstrates the funnel analogy, where Lypaunov surfaces are shown for decreasing values of c.

Is a Lyapunov surface just the slice c, or do those contours make up the surface in 3D space? Its weird how Khalil shows Fig 3.2 on the same plane, shouldn't it look like a funnel?

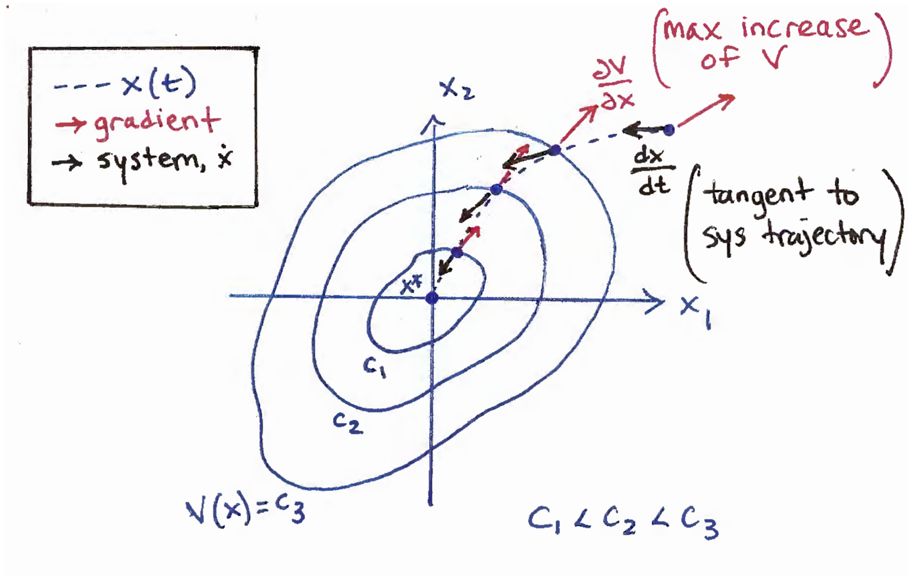


Figure 3.13: Geometric Interpretation of Lyapunov's Stability Theorem.

The partial derivative term in Equation 3.55 may be considered as the gradient of V with respect to $\boldsymbol{x}$, representing a vector pointing in the direction of maximum increase of V. The vector $d\boldsymbol{x}/dt$ represents the system dynamics, which could've equivalently been labeled $\boldsymbol{f}(\boldsymbol{x})$, and is tangent to the solution $\boldsymbol{x}(t)$. The condition imposed by Lyapunov, ie. $\dot{V}(\boldsymbol{x}) = (\partial V/\partial \boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}) \leq 0$, implies that solutions $\boldsymbol{x}(t)$ always cross the contours of V with an angle greater than or equal to 90° relative to the outward normal; if $d\boldsymbol{x}/dt$ points inward then system trajectories will always move to smaller and smaller values of V. $\dot{V}(\boldsymbol{x}) \leq 0$ does not ensure that a trajectory will get to the origin[12], but does imply the origin is stable, since by Definition 3.2.4: when a trajectory starts within δ (a level surface for this case), it will stay within ϵ.

When an equilibrium point is classified as asymptotically stable it requires the initial condition to be within some domain $\mathcal{D}$, but how big can this domain be? Establishing *global* asymptotic stability expands the region of attraction to the whole space $\mathbb{R}^n$ by an extra

[12]LaSalle's Invariance Principle [1, Thm. 3.4] may be used to prove convergence of a solution to the largest invariant set for all point within a region of attraction where $\dot{V}(\boldsymbol{x}) = 0$.

condition, radial unboundedness, as described previously in Definition 3.2.6.

Theorem 3.2.3 (continued) *Khalil [1, Thm. 3.2]*

Let the origin be an equilibrium point, $\boldsymbol{x}^* = \boldsymbol{x}(0) = 0$*, for an autonomous system,* $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ *and* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable, positive definite,* ***radially unbounded*** *function such that*

$$\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) < 0 \quad , \quad \forall\, \boldsymbol{x} \neq 0 \tag{3.58}$$

then the origin is ***globally asymptotically stable****.*

Proof: Refer to Khalil [1, Pg. 111] □

As shown, "the direct method of Lyapunov replaces [an] n-dimensional analysis problem that is difficult to visualize with a lower dimensional problem that is easy to interpret." [11] The large appeal of this method is that stability of an equilibrium point may be inferred without explicitly solving system equations. The difficulty is that choosing a Lyapunov function for complex systems is like [insert funny simile here]; these stability theorems are **non-constructive**, meaning that there is no systematic method for finding a V to satisfy stability requirements. In most cases they may be modeled as kinetic enegry functions, but when this doesn't work trial and error is often used.

Most importantly Lyapunov's theorem only give sufficient conditions for stability, it does not say whether the given conditions are also necessary. If a Lyapunov function does not satisfy the conditions for stability or asymptotic stability, then *no conclusion* can be made about the stability properties of the system. In other words, we cannot guarantee that as long as a trajectory starts within δ that *for all time* it will stay within ϵ; this is merely a definition, not a truth. However, there are theorems that make Lyapunov stability conditions necessary, called converse theorems[13], but these have a downside as almost always some knowledge of the solution is assumed.

[13] See reference [1, Sec. 3.6] for introduction to converse theorems.

For cases where $\dot{V}$ is only negative semi-definite, ie. $\dot{V} \leq 0$, global asymptotic stability may still be established through LaSalle's[14] Invariance Principle, an invariant set theorem:

Lemma 3.2.1 (Fundamental Property of Limit Sets). *Khalil [1, Lem. 3.1]*
If a solution $\boldsymbol{x}(t)$ to the autonomous system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is bounded and belongs to $\mathcal{D}$ for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover,

$$\lim_{t \to \infty} \boldsymbol{x}(t) = L^+ \tag{3.59}$$

Theorem 3.2.4 (LaSalle's Theorem). *Khalil [1, Thm. 3.4]*
Let $\Omega \subset \mathcal{D}$ be a compact set that is positively invariant for an autonomous system, $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(\boldsymbol{x}) \leq 0$ in Ω. Let $\mathcal{E}$ be the set of all points in Ω where $\dot{V}(\boldsymbol{x}) = 0$. Let $\mathcal{M}$ be the largest invariant set in $\mathcal{E}$. Then every solution starting in Ω approaches $\mathcal{M}$ as $t \to \infty$.

Proof: Refer to Khalil [1, Pg. 115] □

LaSalle's theorem also extends Lyapunov's theorem in two ways by *a*) providing an estimate to the region of attraction, δ, specified as any compact positively invariant set and *b*) allows Theorem 3.2.3 to be applied for cases where the system has an equilibrium set, ie. dynamic convergence or limit cycles, rather than a single equilibrium point.

Corollary 3.2.1 (Barbashin-Krasovskii). *Khalil [1, Cor. 3.2]*
Let the origin be an equilibrium point, $\boldsymbol{x}^* = \boldsymbol{x}(0) = 0$, for an autonomous system, $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ and $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable, positive definite, radially unbounded function such that

$$\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}) \leq 0 \quad , \quad \forall\, \boldsymbol{x} \tag{3.60}$$

*Let $\mathcal{S} = \{x \in \mathbb{R}^n \,|\, \dot{V}(\boldsymbol{x}) = 0\}$ and suppose that no solution can stay identically in $\mathcal{S}$, other than the trivial solution. Then the origin is **globally asymptotically stable**.*

[14] Joseph P. LaSalle received a mathematics doctoral degree in 1941 from CalTech and worked alongside Lefschetz at Brown University's [Lefschetz] Center for Dynamical Systems in the 1960's.

Remark (Corollary 3.2.1) When $\dot{V}(\boldsymbol{x}) \leq 0$ and $\mathcal{S} = \{0\}$, Corollary 3.2.1 coincides with Theorem 3.2.3. It is also referred to as the Krasovskii-LaSalle method in some textbooks. Mathematician Joseph P. LaSalle published this theorem in the west, unaware that it was earlier published in Russia; most likely attributed to a language barrier or lack of cooperation due to political tension of the 1950's when this theorem was developed.

It turns out that systems with uniformly asymptotically stable operating points are more robust to disturbances than those that are merely stable, especially for important for adaptive designs [1, Sec. 2.1]. In general, there are two tasks for any flight control system:

Definition 3.2.7. Control Tasks

- **Regulation**: Reference signal is constant – LaSalle's Theorem.
- **Tracking**: Reference signal is time-varying – Barbalat's Lemma.

Regulation is sometimes referred to as stabilization, examples being temperature control, aircraft altitude control, and position control of robot arms. If the controller has a tracking objective, it may be referred to as a tracker, examples being aircraft flight path following or making a robot hand draw circles. The goal in both these cases is to drive the tracking error between the reference signal and actual signal to zero. LaSalle's invariant set based theorem was developed for autonomous systems, therefore is applicable to regulation. For the tracking problem stability analysis is more difficult as the system is time-variant; it's harder to find a Lyapunov function, now dependent on both $\boldsymbol{x}$ and t, ie. $V(\boldsymbol{x}, t)$, that has a negative definite derivative. For convergence analysis, tools developed by LaSalle, Yoshizawa, and Barbalat are relied upon.

Theorem 3.2.5 (LaSalle–Yoshizawa). *Kokotovic et al. [10, Thm. 2.1/A.8]*
Let the origin be an equilibrium point, $\boldsymbol{x}^* = \boldsymbol{x}(0) = 0$*, for a non-autonomous system,* $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ *and suppose* $\boldsymbol{f}$ *is locally Lipschitz in* $\boldsymbol{x}$ *uniformly in* t*. Let* $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuously differentiable, positive definite, and radially unbounded function* $V = V(\boldsymbol{x}, t)$*,*

then

$$\dot{V}(\boldsymbol{x},t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},t) \leq -W(x) \leq 0 \quad , \quad \forall\, t \geq 0 \quad , \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n \tag{3.61}$$

Where $W(x)$ is a continuous function. Then all solutions of [Equation 3.61] are globally uniformly bounded and satisfy

$$\lim_{t\to\infty} W(\boldsymbol{x}(t)) = 0 \tag{3.62}$$

In addition, if $W(x) > 0$ ie. positive definite, then the equilibrium $\boldsymbol{x}^* = \boldsymbol{x}(0) = 0$ is ***globally uniformly asymptotically stable****.*

Proof: Refer to Kokotovic et al. [10, Appendix A, Pg. 492] □

A technical lemma by Barbalat usually precedes Theorem 3.2.5 which is a purely mathematical result inferred by asymptotic properties of functions and their derivatives:

- If $\dot{\boldsymbol{f}}(t) \to 0$ it does not imply that $\boldsymbol{f}(t)$ converges

  Example 3.2.2. As the derivative term converges to zero, the solution does not, for the system:

$$\boldsymbol{f}(t) = \sin(\log(t)) \quad ; \quad \dot{\boldsymbol{f}}(t) = (\cos(\log(t)))/t$$

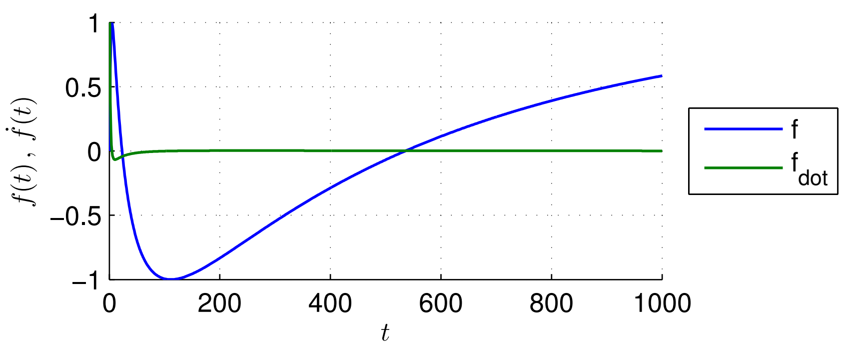


Figure 3.14: Asymptotic Property 1: $\dot{\boldsymbol{f}}(t) \to 0 \nRightarrow \boldsymbol{f} \to$ constant

- If $\boldsymbol{f}(t)$ converges it does not; imply that $\dot{\boldsymbol{f}}(t) \to 0$

  Example 3.2.3. As the solution tends to zero, the derivative is unbounded, for the system:

$$\boldsymbol{f}(t) = e^{-t}\sin^2(e^{2t}) \quad ; \quad \dot{\boldsymbol{f}}(t) = 2e^{t}\sin(2e^{2t}) - e^{-t}\sin^2(e^{2t})$$

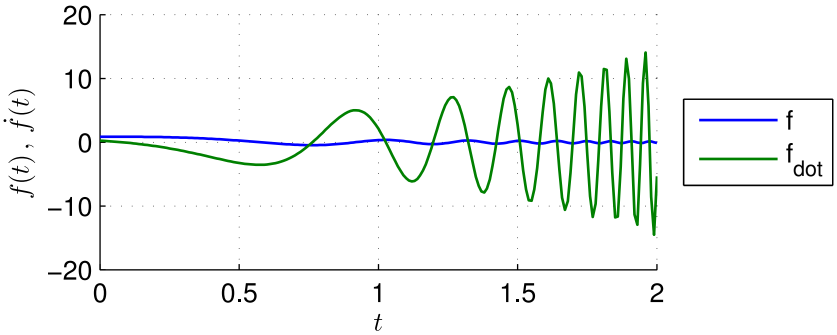


Figure 3.15: Asymptotic Property 2: $\boldsymbol{f} \to$ **constant** $\not\Rightarrow \dot{\boldsymbol{f}}(t) \to 0$

- If $\boldsymbol{f}(t)$ is lower bounded and decreasing, ie. $\dot{\boldsymbol{f}}(t) \leq 0$, then $\boldsymbol{f}(t)$ converges to a limit, but $\dot{\boldsymbol{f}}(t)$ **is not guaranteed to diminish if at all**.

An additional "smoothness" property on the Lyapunov derivative imposed by Barbalat's lemma guarantees that $\dot{\boldsymbol{f}}(t)$ actually converges to zero. This result ensures that a system will fulfill its tracking requirement.

Lemma 3.2.2 (Barbalat's Lemma)**.**
If it can be shown that the differentiable function is bounded, then it may be considered uniformly continuous and convergence may be established.

- ***Form 1:*** *Examine the Function's Derivative* *Slotine and Weiping [2]*

  "If the differentiable function $\boldsymbol{f}(t)$ has a finite limit as $t \to \infty$, and is such that $\ddot{\boldsymbol{f}}$ exists and is bounded, [ie. uniformly continuous], then:"

$$\lim_{t\to\infty} \dot{\boldsymbol{f}}(t) = 0$$

- ***Form 2:*** *"Lyapunov-Like Lemma"* *Slotine and Weiping [2]*

  If a scalar function $V(\boldsymbol{x}, t)$ satisfies the following conditions:

  - *$V(\boldsymbol{x}, t)$ is lower bounded*
  - *$\dot{V}(\boldsymbol{x}, t)$ is negative semi-definite*
  - *$\dot{V}(\boldsymbol{x}, t)$ is uniformly continuous in time (by proving $\ddot{V}$ is bounded)*

then $\dot{V}(\boldsymbol{x}, t) \to 0$ *as* $t \to \infty$.

- ***Form 3:*** *$\mathcal{L}_p$ Space Representation* *Farrell and Polycarpou [11, A.2.2.3]*

  Consider the function $\phi(t) : \mathbb{R}_+ \to \mathbb{R}$ *be in* $\mathcal{L}_\infty$, $d\phi/dt \in \mathcal{L}_\infty$, *and* $d\phi/dt \in \mathcal{L}_2$, *then*

$$\lim_{t\to\infty} \phi(t) = 0$$

- ***Form 4:*** *Initial Version, Barbalat 1959* *Khalil [1, Lemma 4.2]*

  Let $\phi(t) : \mathbb{R}_+ \to \mathbb{R}$ *be a uniformly continuous function. Suppose that* $\lim_{t\to\infty} \int_0^t \phi(\tau)d\tau$ *exists and is finite, then:*

$$\lim_{t\to\infty} \phi(t) = 0$$

Proof: Refer to Slotine and Weiping [2, Sec. 4.5.2, Pg. 124] □

In the next sections you will see how backstepping may be applied to control of nonlinear systems. The procedure involves choosing a $V(x)$ that keeps useful non-linearities and developing a stabilizing feedback control law.

3.3 Control Law Development

Hit high points in Slotine intro

> **Control Design Objective**
> "Given a physical system to be controlled and the specifications of its desired behavior, construct a feedback control law to make the closed-loop system display the desired behavior." [2]

Loosely speaking, the approach to non-linear control design is qualitative, while linear control is quantitative. Numerous analysis tools are available for linear control system design that have very specific metrics common in the controls community; in the time domain step response analysis yields terms like rise time; overshoot; and settling time, in the frequency-domain Bode analysis yields terms like phase margin; gain margin; and bandwidth, and root-locus / Nichols analysis represents a mixture of aforementioned methods. Through use of tools like these control system requirements may be imposed *systematically* on closed-loop controllers for linear systems, hence the term quantitative. For non-linear systems a frequency-domain description is not possible, therefore time-domain analysis of system response to a specified input is the typical treatment. Furthermore, non-linear systems often act in peculiar ways, expressing erratic behavior sensitive to even the smallest changes in initial condition and system parameters, or commands, (think chaotic systems, ie. Lorenz Attractor). This demonstrated inconsistency is a reason why a analysis tools are limited for non-linear systems, each system is unique and exhibited motion is dependent on inputs. As a result, for non-linear systems, *qualitative* requirements for desired behavior are specified within the operating region of interest.

The implications are that on the designers end a much deeper understanding of vehicle dynamics are necessary for non-linear system control development; one can't simply set and forget system equations as with linear systems and then rely on analysis tools to help tune a controller. In fact in backstepping designs if useful, ie. stabilizing, non-linearities are recognized then they may be retained, thereby reducing *a*) the overall control effort needed and *b*) the level of modeling fidelity. The problem is that experience drives these considerations, hence the motivation for including the equations of motion derivation in subsection 3.1.5. Other benefits and disadvantages of backstepping will be covered, but first non-linear control design considerations and options are briefly introduced.

The following characteristics, as outlined by Slotine and Weiping [2], are utilized by designers to establish and validate control system requirements:

- **Stability** *must be guaranteed for the nominal model, either in a local or global sense. The region of stability and convergence are also of interest.*
- **Accuracy and Speed of Response** *may be considered for some "typical" motion trajectories in the region of operation.*

- **Robustness** *is the sensitivity to effects which are not considered in the design, such as disturbances, measurement noise, unmodeled dynamics, etc.*
- **Cost** *of a control system is determined by the number and type of actuators, sensors, and computers necessary to implement it.*

Stability in the non-linear sense does not imply that a system is capable of handling *constant* disturbances; recall from Definitions 3.2.4 and 3.2.5 that stability is defined with respect to initial conditions. For example, a system is stable in the Lyapunov sense if a trajectory starts *within* δ and stays within ϵ; a persistent wind-shear – erratic disturbance due to thermals, downdraft, inversion layers, etc. – may shift the equilibrium point, therefore starting within the δ region does not mean you're starting within some bounds of the true equilibrium point. The effects of persistent disturbances are resolved through robustness techniques.

Lyapunov direct method based methods are to be evaluated. Feedback linearization, or nonlinear dynamic inversion, is the simplest, however it's a wasteful solution in that control laws are built to cancel all plant dynamics. When choosing a Lyapunov function in backstepping synthesis stabalizing terms in the dynamics, if recognized can be kept. (this is a problem I have with the research I've been doing... you must have the experience or intimate understanding of your dynamics model so that when you build your control laws you don't cancel out useful nonlinearities)

Need to figure out advantages and disadvantages of FBL...

"One of the main problems with applying feedback linearization techniques is that the process produces a system with the same **relative degree** as the original system, but usually with an order that is less. ... This process results in zero or internal dynamics, which are modes that are effectively rendered unobservable by the linearisation process. If the system is non-minimum phase, then the zero dynamics are unstable. The analogy with linear systems is that a zero-pole system is linearised into an all-pole system by selecting the pole-zero excess as the order of the approximating system. In order to produce linearised systems that have no internal dynamics, techniques which preserve the dynamic order of the system are needed."

There are two ways to apply Lyapunov's direct method based on where you begin. If a control law is hypothesized then a Lyapunov function needs to be found to validate that choice; the converse is the second way.

3.3.1 Backstepping

Mathematical Concept

Methodology Specific Techniques

- Mathematical Concept –¿ Lyapunov Stability Technique
- Methodology specific techniques (CFBS with Control Allocation)

Theorem

Simple Example

Cascaded Example

This should really be called something like, "intermediate example," but I want to highlight the difference between this eg. and the last: multiple loops where the commands are generated for loops after it (the heart of backstepping and exemplification of recursive nature)

3.3.2 Command Filtering

Avoids analytic computation of derivative, instead uses relatively simple low pass filter.

Insert figure showing generation of x_c and $\dot{x}_c$ by integrators only.

3.3.3 Control Allocation

v = Bu

pinv(B)*v = u

Least squares minimization via pseudo inverse, simplest technique.

Crucial to the robustness of the design, and attempted to implement simplest form of control allocation: not the focus of my thesis, and all the papers I read didn't fully address it, with the exception of Harkegard's dissertation.

Chapter 4

Design & Implementation

VALIDATION STEPS!

Will eventually test in hardware, at minimum hardware in loop at AME SIL. In the meanwhile and for the thesis I will be running a purely computer simulation.

4.1 Modeling

Describe simulink environment and any models referenced from AME (i.e the aerodynamics model for Fury based on windtunnel data / CFD analysis padded by Wurts)

Introduce 3 view of air vehicle

4.2 Simulation

List simulation stipulations: ie. back up assumptions (constant mass, deflections assumed to be measured, ideal actuators, constant density therefore altitude, what have you)

Possibly discuss simulation configuration –¿ ODE solver, considerations or lessons learned for those who follow. ie. dave suggested typecasting vars for future encapsulation into hardward (FCAS), library block usage, unit testing, initialization w/ memory block (avoid algebraic loops), how to send items to command workspae for plotting nicely, ...

4.2.1 Nominal

Plot time histories between command and response [$\gamma\mu V$]

4.2.2 Turbulence

Throw in atmospheric model to test turbulence? Currently model has constant rho... hmmm

4.2.3 Modeling Error / Unfamiliar Plant

Won't work if I don't have aero function approximations

4.2.4 Effector Failure

Won't work unless control allocation robust

4.3 Results

1. Here is a list item.
    (a) Here is a sub list item.
        i. Here is a sub sub list item.

Chapter 5

Conclusion

Could've covered...

1. adaptive aero coefficient modeling
2. linearization around operating points to evaluate PM/GM in classic controls sense
3. New(er) derivation for backstepping that uses tikanov's theorem to guaranteed GES.
4. hardware in the loop testing

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