Chapter 4

TWO-CHANNEL ANALYSES

Noise is present in any measurement. Sometimes the desired signal is so strong that the presence of the noise has little effect on the analysis. But, more often than not, noise influences data analysis and, in some cases, the noise may overwhelm the desired signal. If possible, we'd like to design the measurement and the analysis to accentuate the signal[1] and minimize the noise.

Some measurements involve signals that are known and controlled. In these cases, the powerful synchronous averaging techniques should be considered. Typically, the more we know about the signal, the easier it will be to measure and isolate from a noisy background. The more intense the noise or the more severe the attenuation of the signal, the more profitable it is to design the test signal carefully.

Other measurements involve signals that are not under our control and, perhaps, not very well known. Passive measurements of aircraft or automobile noise are typical examples – the investigator rarely has much control over the source and rarely is there any possibility of synchronous processing. However, it is sometimes possible to put a sensor on or near the noise source and use another distant sensor to infer details about the propagation path. We can use knowledge of the source from the nearby sensor to help extract that source's "signature" from a noisy background at the other sensor.

The desired signal may be a sinusoidal signal or a collection of harmonically related tones. Or the signal could be distributed over frequency with a particular

[1] At times the "signal" is noise. For example, we might want to measure ambient acoustic noise or machinery noise. In this context, the unwanted component might be electromagnetic interference or digitizer noise. Here, we use "signal" for the desired component and "noise" for the unwanted component.

spectral shape. The nature of the signal (and the noise) should guide the selection of window function, sampling rate, record length, and so on.

A single measurement channel may be sufficient for the measurement in some cases. In other cases, multiple channels would be used to enhance a signal, reject noise, or determine spatial properties of the signal.

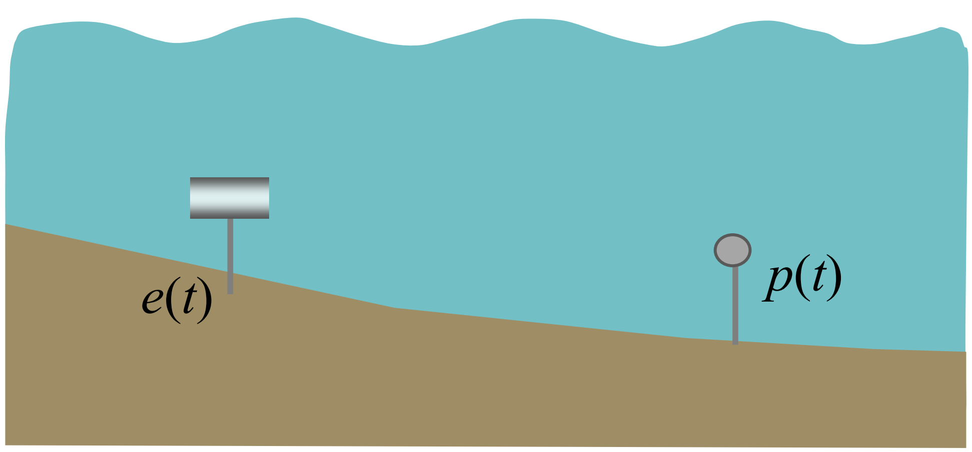
We'll continue to examine different situations and describe some of the options for measurement and analysis. This chapter addresses the case in which two "channels" of data are taken and there's some relationship between those two channels. Two critically important functions—the cross-spectral density and the coherence—are introduced.

Two-Channel Measurements

There are many cases in which a measurement involves two related channels of data. Here are a few common types:

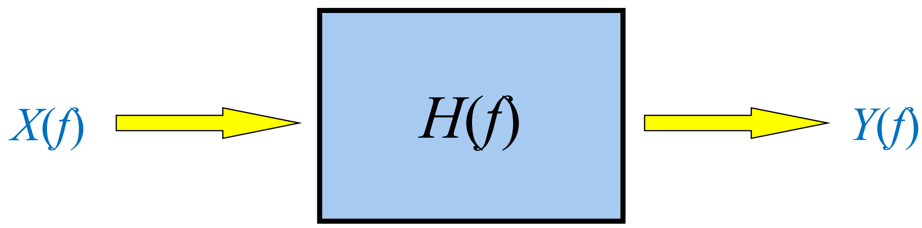
System transfer function

The transfer function measurement can take many forms. It could be measurement of a piece of hardware—a filter, a processor, or a data recorder—or it could be a transmission measurement, in air or underwater,



where we might drive a transducer with a voltage, *e(t)*, and record the received pressure, *p(t)*, some distance away.

If a device, a system, or even an environment has a “place” (e.g., a set of terminals, a port, or a specific point in space) that we can define as its input and a “place” that we can define as its output, it is often of value to find the relationship between the input and the output.



In the frequency domain, if the input linear spectrum is $X(f)$ and the output linear spectrum is $Y(f)$, the relationship between the input and the output is sometimes called the **Transfer Function**, $H(f)$, which is normally complex:

$$Y(f) = X(f) \cdot H(f) \tag{IV-1}$$

The inverse Fourier transform of $H(f)$ is a time-domain function called the **Impulse Response**, $h(t)$. The impulse response convolved with the input time function gives the output time function:

$$y(t) = x(t) \otimes h(t) \tag{IV-2}$$

The “x-in-circle” symbol stands for the convolution operation, an operation we’ll consider later.

Often, characterization in the frequency domain is more straightforward. If we can measure both the input and the output, then calculate the input and output linear spectra (X and Y), we can find the transfer function,

$$H(f) = \frac{Y(f)}{X(f)} \tag{IV-3}$$

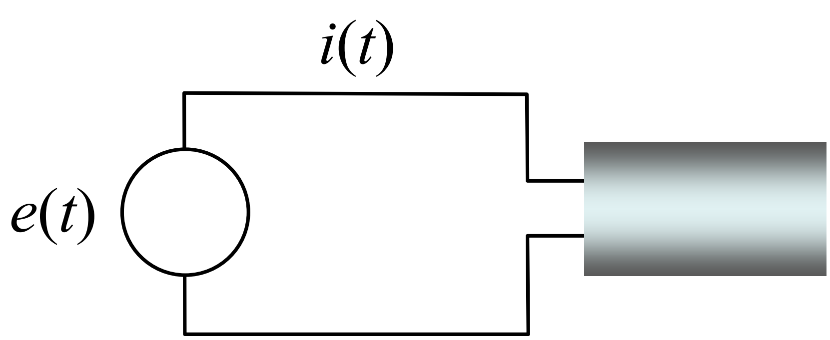
This transfer function, a complex quantity[2], is a characteristic of the system. If the system is linear, then, given the transfer function, we can find the system output for any input.

Impedance measurement

In other cases, we might be more interested in the relationship of two quantities at the same point. The mechanical impedance[3], for example, is the frequency-domain ratio of the linear spectrum of the force, *F*, on an object to its velocity, *V*. The electrical impedance is the frequency-domain ratio of the voltage, *E*, across a device to the current, *I*, through the device:

$$Z(f) = \frac{E(f)}{I(f)} \tag{IV-4}$$

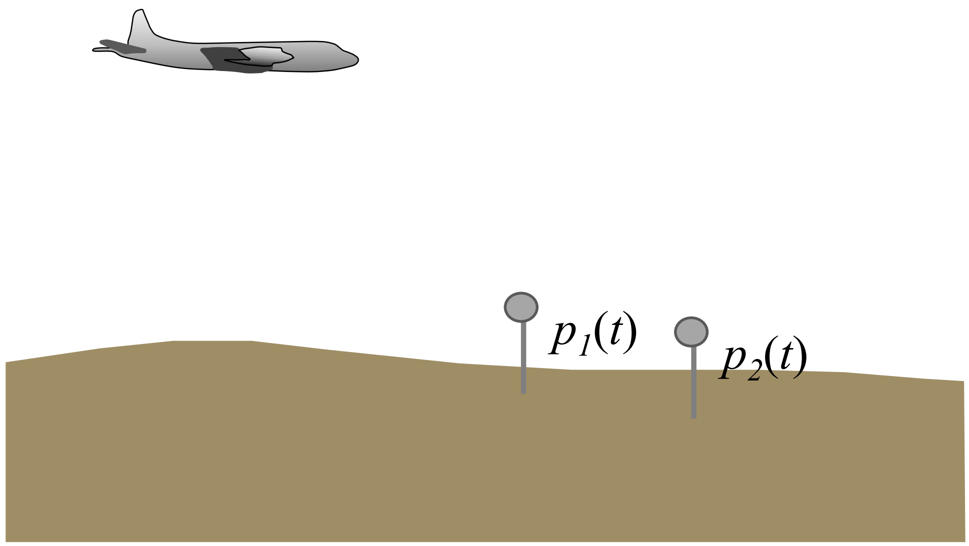
If we measure both *E* and *I*, we can calculate the impedance. The impedance is a characteristic of the device or system and, as long as the system is linear, we can then calculate the voltage that would be produced for any arbitrary current applied.



[2] It is *essential* to have both the magnitude and phase (or both the real and imaginary parts) of the transfer function. While the magnitude is more intuitive, the phase response can have dramatic effects on the time-domain waveforms.
[3] Often it is more appropriate to find the reciprocal of the impedance, which is called the admittance but that doesn't change any of the points made here.

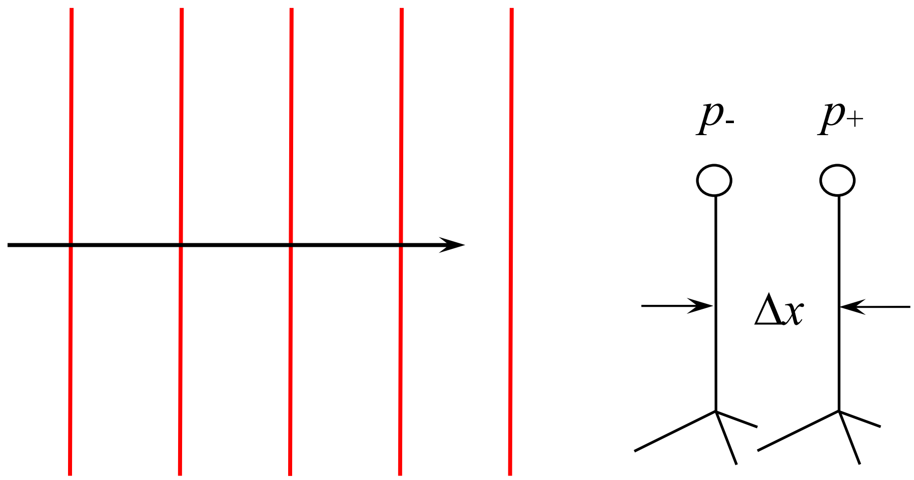
Acoustic field characterization

Yet another application of two-channel measurements is in characterizing an acoustic field. One sensor gives us the acoustic pressure but an acoustic field at a point is not completely determined by the pressure at that point. With two sensors, we can find the time delay between the two measurements,



and so, estimate a direction of arrival, or the acoustic particle velocity, or even the acoustic intensity. Or, with two sensors, we can "null" interference coming from a particular direction. (And, with many sensors, we can make arrays for locating sources or for imaging.)

For example, we could estimate the acoustic particle velocity, v, at some point by placing two microphones a short distance, Δx, apart. We would be measuring the acoustic pressure, p, at $x + \Delta x/2$ and $x - \Delta x/2$.



Newton's Law[4] (the linearized conservation of momentum equation) in the medium relates the pressure gradient to the fluid acceleration,

$$-\frac{dp}{dx} = \rho\frac{dv}{dt} = j\omega\rho v \qquad \text{(IV-5)}$$

where ρ is the density. The complex exponential form for the velocity allows replacement of the time derivative by $j\omega$. We can generate a "finite-difference" version of Newton's Law by approximating the spatial derivative on the left side,

$$-\frac{p(x+\Delta x/2) - p(x-\Delta x/2)}{\Delta x} = j\omega\rho v(x) \qquad \text{(IV-6)}$$

The estimate of particle velocity, $v(x)$, is then

$$v(x) = -\frac{p(x+\Delta x/2) - p(x-\Delta x/2)}{j\omega\rho\,\Delta x} = \frac{p_- - p_+}{j\omega\rho\,\Delta x} \qquad \text{(IV-7)}$$

where p_+ is $p(x + \Delta x/2)$ and p_- is $p(x - \Delta x/2)$, the two measured pressures.

[4] The equation that follows is $F = ma$ converted to a per-volume basis. The quantity on the left – the pressure gradient – is the force per unit volume, ρ is the mass per unit volume and the time derivative of velocity is acceleration.

In the particle velocity measurement (and in intensity measurements), the pressures at the two microphones are often nearly equal and the success of the measurement depends on extracting a small difference between them accurately. In practice this operation is fraught with peril. We can often do better with the following form of the equation for velocity:

$$v(x) \;=\; \frac{p_-}{j\,\omega\,\rho\,\Delta x}\left(1 \;-\; \frac{p_+}{p_-}\right) \qquad \text{(IV-8)}$$

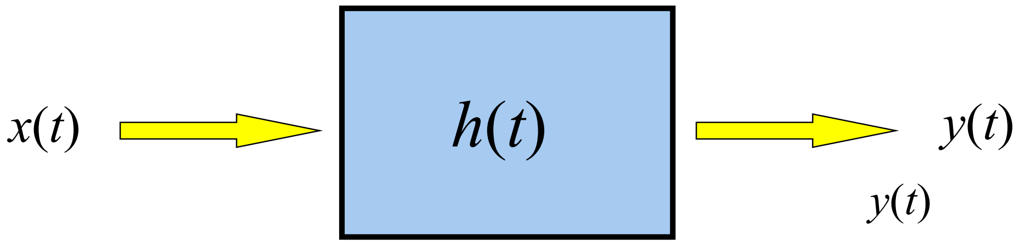
It is usually possible to measure the *ratio* of two quantities with considerably greater accuracy than it is to measure either quantity absolutely. In the equation above, we've replaced the difference of two absolute measurements by the difference between a constant (one) and the (more accurate) ratio of two measurements[5]. We'll discuss measuring ratios in more detail later (see the Interchange Method).

Often, then, an acoustic field characterization measurement can be expressed as the ratio of two linear spectra, the spectrum from the data at one microphone and the spectrum from the data at the other microphone.

Time Domain or Frequency Domain?

In each of the above examples, there are two channels of data and we'd like to find the relationship between these two channels. We can study this relationship in either the time domain or in the frequency domain. In the time domain, we have an input, x, an output, y, both functions of time. The transfer relationship between y and x is given by the impulse response, $h(t)$.

[5] The subtraction operation is the source of the greatest error. What we've done is replace subtraction of two separate measurements by subtraction of an accurately determined ratio from the integer, one.

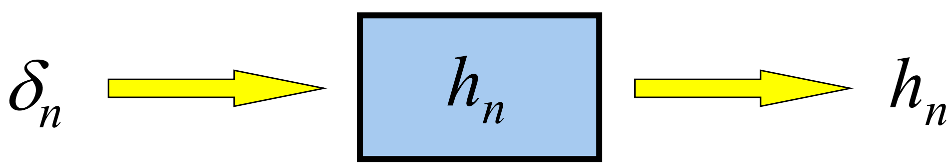


In mathematics, the relationship is a convolution

$$y(t) \;=\; x(t) \otimes h(t) \qquad \text{(IV-9)}$$

We'll cover convolution later; for now, accept it as a mathematical operation. Convolution is not easy to implement in an efficient manner and, if the objective is to find the impulse response, h, the equation above is difficult to solve for h.

Instead of trying to invert the convolution equation, the impulse response can be found by exciting the system with an impulse. While this can be challenging in a physical system, generating an impulse in the computational domain is simple and we'll find this to be an excellent method for testing numerical filters.



In taking this approach, it is critical to understand what an impulse is and how to construct it. The key to an impulse is that it must have only a single non-zero value and it must integrate to a non-dimensional value of 1.0. The "integration" in our sampled-signal domain is done by summation (*including the appropriate differential*):

$$\sum_{n=0}^{N-1} \delta_n \, \Delta t \;=\; 1 \qquad \text{(IV-10)}$$

Consequently, the impulse, δ_n, is

$$\delta_n = \left[\frac{1}{dt}, 0, 0, 0, \dots\right] \qquad \text{(IV-11)}$$

Notice that the discrete impulse[6] in the time domain has the units of (seconds)$^{-1}$.

Synchronized or Unsynchronized?

One critical distinction can help in the selection of an analysis procedure. In the first two examples above—the transfer function and the impedance measurement—there is an excitation (the input for the transfer function; the voltage for the impedance measurement) and a result (the output for the transfer function; the current for the impedance measurement). If you have control over the excitation, then you should consider a synchronous averaging process. If you don't have control over the excitation, then you may not be able to synchronize the averaging; unsynchronized ("ordinary") averaging is sometimes called *mean-square* or *root-mean-square* averaging. Even if you use random noise as the excitation, if you use exactly the same random-noise waveform repeated for each record, you can synchronize at the repetition rate of that waveform and still get the signal-to-noise improvement available from synchronous averaging.

If you need a known excitation with a large bandwidth and a uniform distribution of power over at least some range of frequency (i.e., "white" or nearly white), there are a number of options that still allow synchronous detection and averaging:

- an "impulse" – a very short, high-amplitude pulse
- a "chirp" – a fast sweep in frequency
- a repeating noise sequence – "pseudo-random" noise

[6] As tempting as it may be to use, instead, the Kronecker delta function, that choice is incorrect. Both the physical units and the values of subsequent operations would be wrong. We want the discrete equivalent of the Dirac delta function. If you are unfamiliar with the labels, Kronecker and Dirac, as applied to delta functions, ignore this footnote!

- a “maximum length sequence” – a special coded sequence that we’ll discuss later

Although disparaged as unsophisticated, a simple sine wave is often an excellent choice for an excitation waveform. With a single-frequency excitation, distortion (nonlinearity) is much easier to detect and data-acquisition channel gains can be set for optimum results. The total measurement time for a large number of single frequencies can be considerably longer than for a broadband excitation; however, high-quality results may be achievable even in relatively difficult circumstances. The simple sine wave excitation is often a smart choice for initial assessments even if another waveform will be used for the primary measurement.

For faster measurement over a range of frequency, chirps are often a good choice. The chirp still permits detecting nonlinearity and, in some environments, interference from reflections can be eliminated by limiting the time span of the record.

There are times when we may want to make a multi-channel measurement using noise (ambient noise, for example) and we may not be able to synchronize the processing. In those cases, we can take advantage of the properties of another function, the cross-spectral density (or “cross-spectrum”) [7].

The classical approach to random data analysis is to assume that you don’t have enough knowledge about the excitation to synchronize an average. We’ll work through this classical approach because it’s important to understand but it’s normally better to use a known excitation when possible.

[7] I will often write “auto-spectrum” for auto-spectral density and “cross-spectrum” for cross-spectral density. I may eventually decide that this is ambiguous and drop the short-hand notation. (In some communities, the phrase “auto-spectrum” refers to the “power spectrum” rather than the “spectral density”.)

The Cross-Spectrum and Ratios of Linear Spectra

If synchronous averaging is possible, then we can either average in the time domain and then find the linear spectra or we can average the linear spectra synchronously[8]. Both of these processes are described in a previous chapter. Once we've obtained sufficiently "clean" input and output linear spectra by synchronized averaging, we can find the ratio by direct division. In the synchronized case, finding the ratio of linear spectra is straightforward.

What recourse do we have if we can't perform synchronized averaging? We want a complex transfer function (or complex impedance, or complex particle velocity) so we could take a single record from both channels and divide the complex linear spectra. If the signal is far above the noise, this approach may be satisfactory but, in many cases, at least one channel may have significant unwanted noise. Averaging the spectral densities, G_{XX} and G_{YY}, can result in "clean" curves; however, the signal-to-noise ratio is not improved by ordinary averaging and the spectral densities are real so we couldn't determine anything about the phase relationship between X and Y from the ratio of those spectral densities.

If the averaging is not synchronous, the phase of X (or Y) changes from average to average. However, if the environment is stable and if there isn't too much noise, the *relative phase* between the signal components in X and Y should be stable from average to average. If we had some way of preserving this relative relationship, we could preserve the phase even with an unsynchronized average. The **cross-spectral density**, G_{XY},

$$G_{XY} = \frac{2}{T} X^* Y , \qquad \text{(IV-12)}$$

performs this function: it "tracks" the relative phase between X and Y. (Regarding the notation above, remember that at the points, $m = 0$ and $m = N/2$, the leading

[8] Unless there's a pressing reason to average synchronously in the frequency domain, your first choice should incline toward averaging synchronously in the time domain. It's simpler, it doesn't require as many *fft* operations, and it allows you to see (and listen to!) the clean time-domain waveform.

factor is one not two. I will often write the single-sided spectral density as $2/T$ times some product of linear spectra but I really mean that you should use the proper $1/T$ for those two special points.) Recall that

$$G_{XX} = \frac{2}{T} X^* X \qquad \text{and} \qquad G_{YY} = \frac{2}{T} Y^* Y\,. \tag{IV-13}$$

If we only had a single record, hence a single X and a single Y, then the complex ratio of Y to X would be identical to the ratio of the cross-spectrum, G_{XY}, to the auto-spectrum, G_{XX},

$$\frac{G_{XY}}{G_{XX}} \equiv \frac{Y}{X}. \tag{IV-14}$$

This case is trivial. But, if we are averaging, the two ratios are not equal. If we averaged the linear spectra X and Y without synchronization, then divided, the result would have little meaning. Without synchronization, the complex values of the linear spectrum average like noise – the resulting magnitude of the average bears little resemblance to the signal's actual magnitude.

However, if we average the *cross-spectrum*, G_{XY}, we preserve the relative phase between X and Y (instead of trying to preserve the absolute phases of X and Y separately—something that we can only do with synchronous averaging). We can see that this is true by examining the product of Y with the complex conjugate of X:

$$\begin{aligned} X &= \left|X\right| e^{j\phi_X} \\ Y &= \left|Y\right| e^{j\phi_Y} \\ X^* Y &= \left|X\right|\left|Y\right| e^{j\left(\phi_Y - \phi_X\right)} \end{aligned} \tag{IV-15}$$

The magnitude of the cross-spectrum is equal to the product of the magnitudes of the linear spectra and the phase of the cross-spectrum is equal to the phase difference.

Our estimate of the transfer function (for example) would be the ratio of the averaged cross-spectrum to the averaged auto-spectrum:

$$H(f) = \frac{\bar{G}_{XY}}{\bar{G}_{XX}} \qquad \text{(IV-16)}$$

This gives a complex result and it's also a meaningful estimate of the transfer function.

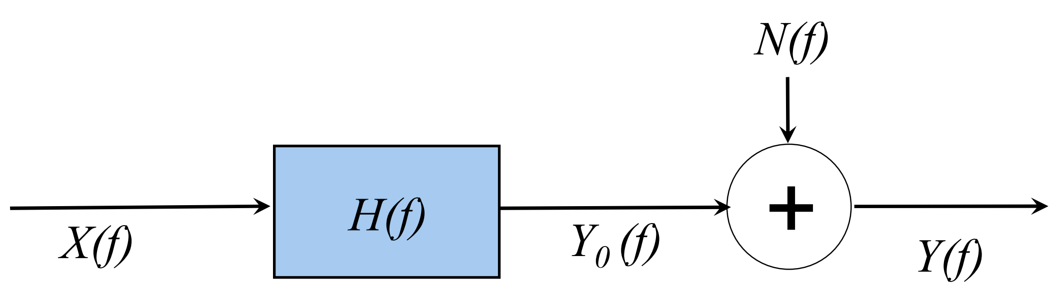
If you didn't know about the cross-spectral density, you might think that the best you could do with ordinary averaging is an estimate of the *magnitude* of the transfer function,

$$\left|H(f)\right| = \sqrt{\frac{\bar{G}_{YY}}{\bar{G}_{XX}}} \qquad \text{(IV-17)}$$

but phase is important and we don't want to lose the phase information.

Not only does the ratio of the averaged cross-spectral density to the averaged auto-spectral density give the complex transfer function, there is also an advantage with respect to signal-to-noise ratio. Suppose the system output is equal to the input times the transfer function but there is also an incoherent component of noise, $N(f)$, added to the output:

$$Y(f) = X(f) \cdot H(f) + N(f) \qquad \text{(IV-18)}$$



The auto-spectrum of the input is unaffected:

$$G_{XX} \;=\; \frac{2}{T} X^* X \tag{IV-19}$$

but the cross-spectrum of the input with the output is now,

$$G_{XY} \;=\; \frac{2}{T} X^* \left[X H \;+\; N \right] \;=\; \frac{2}{T} \left[X^* X H \;+\; X^* N \right] \tag{IV-20}$$

If we assume that the noise is not coherent with the input, the average of the second term in square brackets on the right will tend toward zero as the number of averages increases. Therefore, the averaged cross-spectrum, *with sufficient averaging*[9], approaches,

$$\bar{G}_{XY} \;\to\; \frac{2}{T} \left[\overline{X^* X H} \right] \tag{IV-21}$$

and the estimate of the transfer function approaches the actual value of the transfer function[10]:

$$\frac{\bar{G}_{XY}}{\bar{G}_{XX}} \;\to\; \frac{\overline{X^* X H}}{\overline{X^* X}} \;=\; H(f) \tag{IV-22}$$

Consequently, the transfer-function estimate using the cross-spectrum not only approaches the proper complex result, it also has some immunity to noise in the output channel. However, it's important to keep in mind that the average value of the product of two incoherent linear spectra decays toward zero as the reciprocal of the number of averages so the reduction of the noise term takes time. In addition, if the characteristics of the excitation, X, or the transfer function, H, change during the averaging process, the estimate of H will degrade even in the absence of noise.

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[9] We will discuss the all-important "sufficient averaging" criterion later.

[10] Throughout this section we will assume that the transfer function, H, is deterministic – that H has no random component. Consequently, it makes no difference whether we write the averaging overbar on H or not.

Exercise 4.1: Evaluating a Transfer Function with Averaged Spectra

Pick a sampling frequency and generate a random time series, x, 16 seconds long. The MatLab function, *randn*, will generate a normally distributed random series.

The complex frequency response of a simple "low-pass" filter is

$H_{LP} = 1/(1 + jf/f_0)$ where f_0 is the filter's characteristic frequency. Pick f_0 to be 0.1 times the sampling frequency. Apply this filter to x as follows: (a) find the linear spectrum, X; (b) take the positive-frequency part of that linear spectrum, find the corresponding frequencies, and multiply those linear spectrum points by H_{LP} evaluated at each frequency; (c) form the full linear spectrum from this new spectrum so that is has the proper complex-conjugate symmetry; and (d) transform back to the time domain to find the filtered time series, y. Note: in step (c) set the $f_s/2$ point to the real part of its actual value; it must be real to produce a real y. The output, y, should be real if you've done (c) correctly.

Find the linear spectra, X and Y, of the two time series, x and y. Plot the magnitude of the ratio Y/X and the magnitude of H_{LP} as functions of frequency. The results should be nearly identical. Next, using 0.5 second records, find the averaged spectra, G_{XX}, G_{YY}, and G_{XY}. (You don't need to use a window or overlap for these averages.) Plot the magnitude of the ratio G_{XY} / G_{XX} and the magnitude of H_{LP} as functions of frequency. Since x and y are completely correlated, all of these results should be nearly identical.

Now make a new y by adding another random time series to the old y. This simulates a noisy output. Use 0.5**randn* to generate the new random series so that its average amplitude is half that of x. Add this random series to the previous filtered series, y. Find the new averaged spectra, G_{YY}, and G_{XY}. Plot the square root of the ratio G_{YY} / G_{XX}, the magnitude of the ratio G_{XY} / G_{XX}, and the magnitude of H_{LP} as functions of frequency. The two results based on ratios of spectral densities should be different. Why? Also plot the phase (MatLab *angle* function) of the ratio G_{XY} / G_{XX} and the phase of H_{LP}. Although noisy, both the magnitude and phase of the filter response determined from the ratio G_{XY} / G_{XX} should be approximately correct. Try adding a time-domain window function and overlap to the averaging process.

The Coherence Function

There is a function called the **coherence function** that provides a measure of how well the averaging process succeeds in estimating the transfer function (or impedance, or pressure ratio, or...). Suppose that we really did know the output with no added noise. Call that output linear spectrum, Y_0, and the corresponding auto-spectral density, $G_{Y_0Y_0}$. Then the spectral densities—with noise and without—are,

$$\begin{aligned} G_{YY} &= \frac{2}{T}\left[X H + N\right]^*\left[X H + N\right] \quad \text{and} \\ G_{Y_0Y_0} &= \frac{2}{T}\left[X H\right]^*\left[X H\right] \end{aligned} \tag{IV-23}$$

With noise,

$$G_{YY} = \frac{2}{T}\left[X^* X H^* H + N^* N + X^* H^* N + X H N^*\right], \tag{IV-24}$$

which, if averaged long enough, approaches,

$$\bar{G}_{YY} \rightarrow \frac{2}{T}\left[\overline{X^* X}\,\overline{H^* H} + \overline{N^* N}\right], \tag{IV-25}$$

as long as the noise is not coherent with the input signal. The averaged auto-spectrum without noise is

$$\bar{G}_{Y_0Y_0} = \frac{2}{T}\left[\overline{X^* X}\,\overline{H^* H}\right] \tag{IV-26}$$

One measure of the impact of the uncorrelated noise might be the ratio of the spectral density with no noise to the spectral density with noise,

$$\frac{\bar{G}_{Y_0Y_0}}{\bar{G}_{YY}} = \frac{\left[\overline{X^* X}\,\overline{H^* H}\right]}{\left[\overline{X^* X}\,\overline{H^* H} + \overline{N^* N}\right]} \tag{IV-27}$$

Both numerator and denominator are real and the denominator is always the larger of the two so this ratio is always between zero and one.

We don't know the noise-free output but the averaged cross-spectrum does give an estimate for the noise-free transfer function. We can construct an expression identical to the ratio shown above as follows:

$$\begin{aligned}\frac{\left(\bar{G}_{XY}\right)^* \bar{G}_{XY}}{\bar{G}_{XX}\,\bar{G}_{YY}} &= \frac{\left[\overline{X^* X}\,\overline{X^* X}\,\overline{H^* H}\right]}{\left[\overline{X^* X}\right]\left[\overline{X^* X}\,\overline{H^* H} + \overline{N^* N}\right]} \\ &= \frac{\left[\overline{X^* X}\,\overline{H^* H}\right]}{\left[\overline{X^* X}\,\overline{H^* H} + \overline{N^* N}\right]}\end{aligned} \tag{IV-28}$$

This function has the special name, coherence function, and the usual symbol is the Greek gamma squared or γ^2:

$$\boxed{\gamma^2 = \frac{\left(\bar{G}_{XY}\right)^* \bar{G}_{XY}}{\bar{G}_{XX}\,\bar{G}_{YY}}} \tag{IV-29}$$

The coherence function ranges from zero to one and is a measure of how well the output can be expressed as a linear function of the input[11]. In order for the coherence to have any meaning, the spectra must be averaged. If you only took a single record and calculated the coherence you would get $\gamma^2 = 1$ no matter how noisy the system was. (Try substituting the single-record expressions for the auto- and cross-spectra into the equation for the coherence function. The result is one.)

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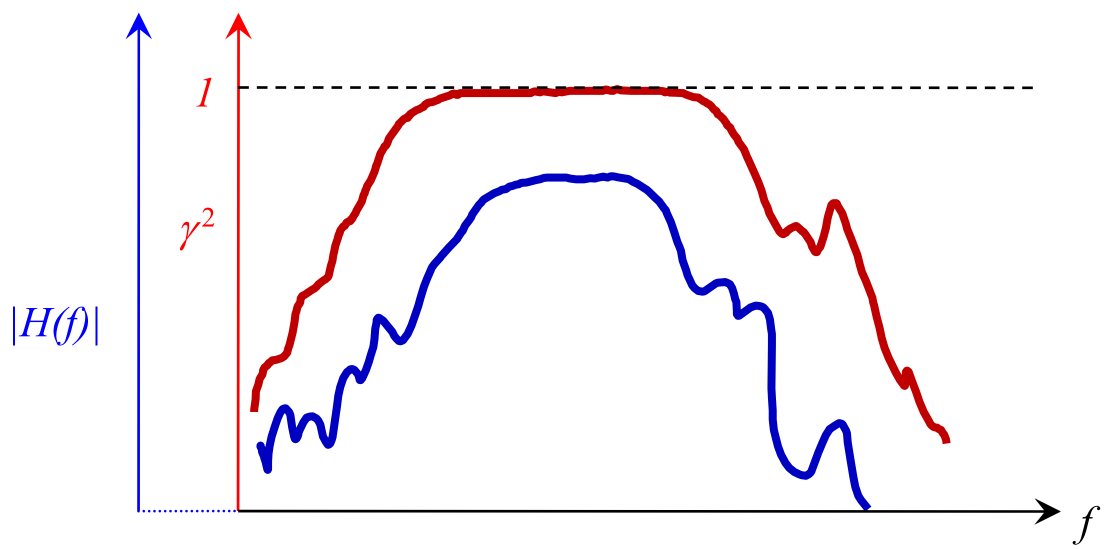
[11] The coherence is sometimes understood to be the degree to which two signals share a common component or have a common origin. This is a dangerous simplification. See the *Degradation of Coherence* section ahead.

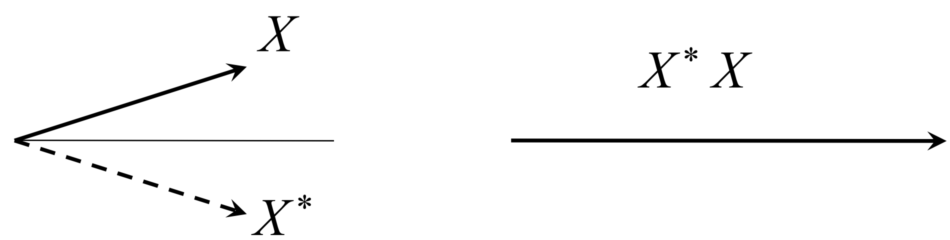
If you perform mean-square averaging in a two-channel measurement, you should always calculate the coherence function and examine the results. *If the coherence, γ^2, is not close to one, then the estimate of the transfer function (or impedance, or pressure ratio) is suspect*. In the figure below, the middle section of the transfer function (blue curve) would be reliable because the coherence (red curve) is nearly equal to one but we would expect larger uncertainty in the low- and high-frequency portions where the coherence drops off.



In some circumstances, you might want to see very low coherence. For example, you can check the independence of the channels of a two-channel data acquisition system by recording data with both inputs shorted (or connected to independent sources). Ideally, the coherence would be zero. Elevated coherence would indicate undesirable coupling ("cross-talk") between the channels that might interfere with subsequent measurements.

Characteristics of coherence and the cross-spectrum

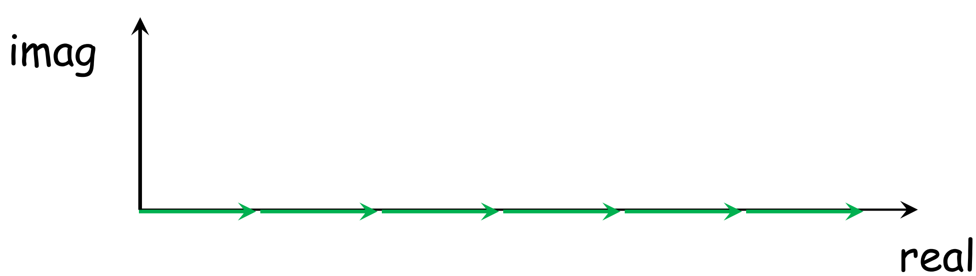
The coherence is based on the averaged cross-spectral density so, to understand coherence, it is necessary to understand the behavior of the cross-spectrum. As mentioned earlier, one of the key aspects of the cross-spectrum is that its phase is equal to the phase *difference* between the two channels. The auto-spectrum is the cross-spectrum between a signal and itself. The product of a linear spectrum with its complex conjugate has a magnitude equal to the magnitude squared of the linear spectrum and a phase of zero since there is no phase difference between a linear spectrum and itself[12]:



The first stage in averaging is summation of the individual spectral densities, which are all real numbers for the averaged auto-spectrum. This sum is in square brackets in the equation below,

$$\frac{T}{2}\overline{G}_{XX} = \frac{1}{N_{recs}}\left[\left(X^*X\right)_1 + \left(X^*X\right)_2 + \left(X^*X\right)_3 + \ldots + \left(X^*X\right)_{Nrecs}\right] \quad \text{(IV-30)}$$

Graphically, the vector sum proceeds along the real axis:



The figure above would correspond to the sum of the auto-spectral density vectors (less the $2/T$ factor), each evaluated at the same frequency. Each frequency would

[12] I will often show "vector" diagrams to illustrate points about the cross-spectral density. In such a diagram, only a single frequency of the cross-spectrum can be shown. The cross-spectrum may have *N* points; the diagram shows the behavior of one of those equivalent frequencies. A separate (and usually different) diagram would be necessary to show the behavior at any other frequency.

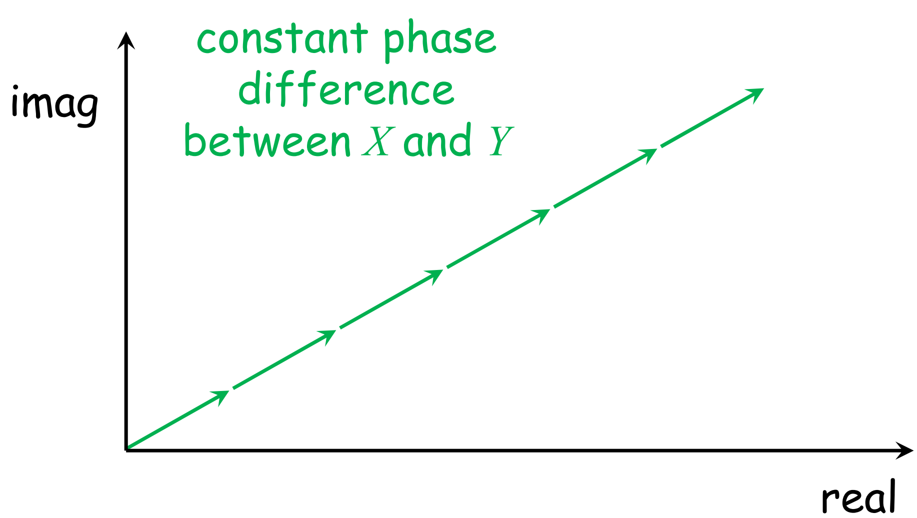
generate its own vector sum. For the auto-spectrum, each sum would be horizontal (zero phase) but the lengths could be different.

The magnitude of the cross-spectrum is equal to the product of the magnitudes of the two linear spectra and the phase of the cross-spectrum is equal to the phase difference between the linear spectra:

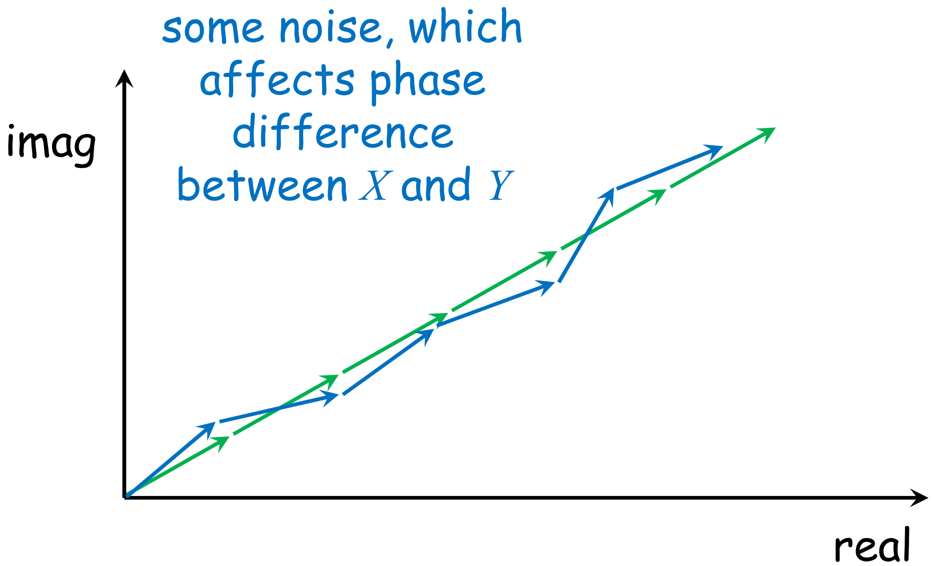
The averaged cross-spectrum also involves a sum but this time it is the sum of the individual cross-spectra, which are, in general, complex:

$$\frac{T}{2}\overline{G}_{XY} = \frac{1}{N_{recs}}\left[\left(X^*Y\right)_1 + \left(X^*Y\right)_2 + \left(X^*Y\right)_3 + \ldots + \left(X^*Y\right)_{Nrecs}\right] \quad \text{(IV-31)}$$

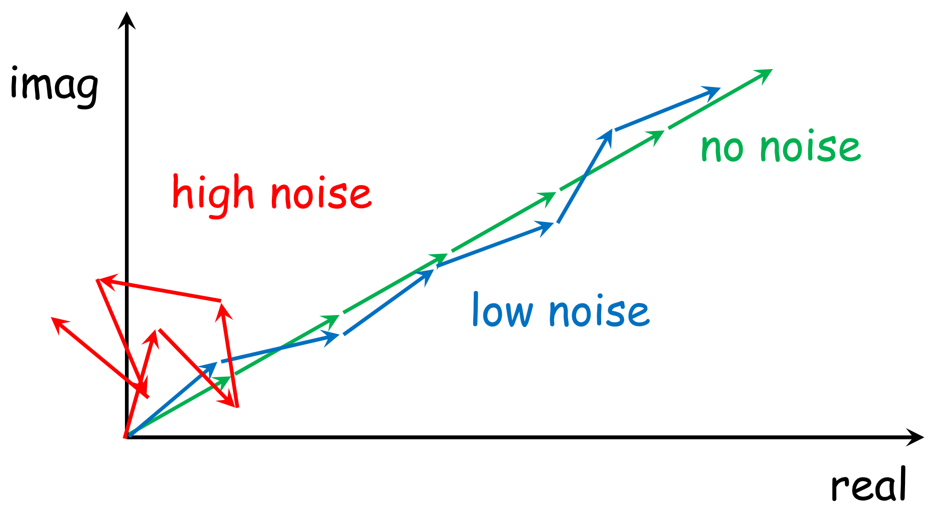
If the phase difference between X and Y is constant, then each individual vector will have the same angle and the sum proceeds along a straight line:



While the resulting line isn't horizontal, it is straight. If the phase difference is not constant, but does not deviate far from the average phase difference (because of the presence of some noise, for example), then the vector sum (blue vectors below) will deviate from a straight line:



With a high noise level, the vector sum may depart substantially from a straight line (red vectors below):



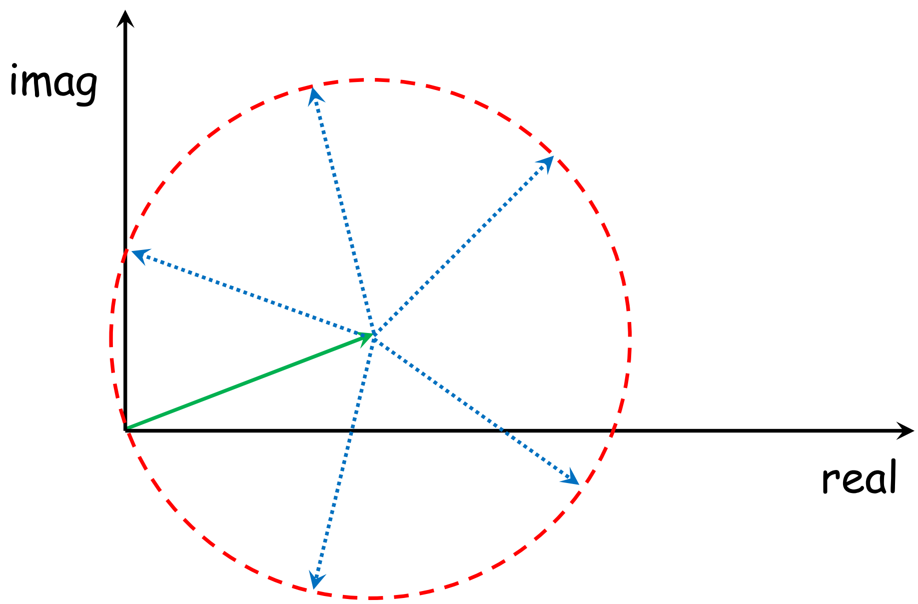
The no-noise case above represents the maximum length the vector sum can have. If the coherence is one, then the actual vector sum will reach that maximum length. If the coherence is zero, the magnitude of the actual vector sum would be zero.

Notice that the magnitude of the vector sum is always positive. Even with extremely high noise, the magnitude of the sum is unlikely to be exactly zero so the numerator of the coherence expression is unlikely to be zero. *For low coherence, the coherence estimate is biased*—the estimate from the averaging process always tends to be *higher* than the true value when the coherence is low.

Dependence of coherence on number of averages

As mentioned above, coherence does not reflect the true relationship of two signals unless sufficient averaging is done. If you try to compute coherence on a single record, you get a value of one even if the two channels are independent. While the coherence with no averaging (or one average to say this another way) is always one, a fair question is, "How many averages are necessary to make a reasonable estimate of the actual coherence?" Two? Six? Four hundred?

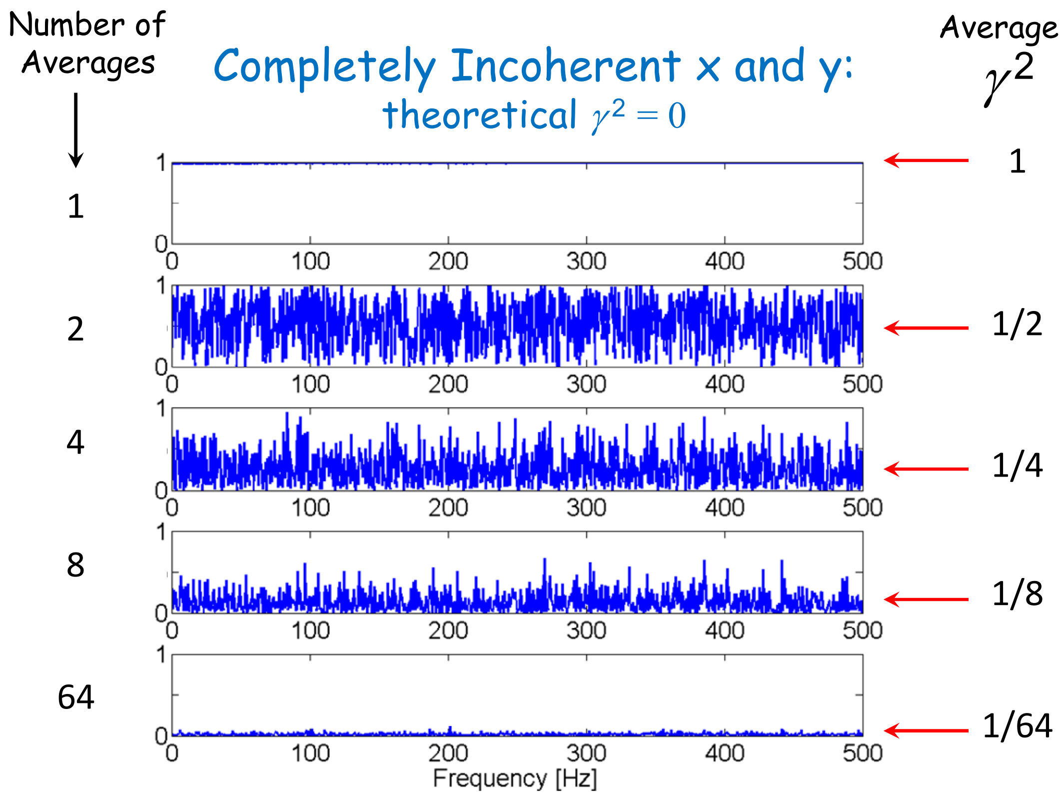
Consider taking two averages. That means, in the vector sum, adding two vectors. If the two channels are completely uncorrelated, then the two vectors would have random phase even though their magnitudes might be equal. The diagram below shows the potential for summing these two unrelated vectors. The first is the green vector. The second could be in any direction and several possible directions are shown by the blue, dotted vector.



The vector sum could wind up at any point on the red, dashed circle. Consequently, the magnitude of the sum could range from zero to twice the magnitude of either vector. With two records, the coherence would range randomly between zero and one with an average value of one-half.

If the two signals are completely independent, random signals, then the coherence (γ^2) estimated from the averaged cross- and auto-spectra approaches zero as the reciprocal of the number of records averaged. If you average only four records, the average coherence will be about 0.25 *even for completely uncorrelated signals*.

The following figure illustrates the approach to the true coherence value with increasing numbers of averages. To construct this diagram, I generated two signals as independent random time series. The theoretical coherence is zero. Then I computed the coherence from the averaged cross- and auto-spectra for various numbers of averages from one to 64. For small numbers of averages, the coherence has significant scatter but the average value across frequency is at least roughly equal to the reciprocal of the number of averages.



1) *Two uncorrelated signals*

The dependence of the coherence computation on the number of averages for *two independent, incoherent signals* is not difficult to analyze. Let these two channels be represented by their linear spectra:

$$X = A \quad ; \quad Y = B \quad . \tag{IV-32}$$

For example, A and B could be independent random-noise sequences. The double-sided[13] auto-spectra are

$$S_{XX} = \frac{A^* A}{T} \quad ; \quad S_{YY} = \frac{B^* B}{T} \tag{IV-33}$$

and the cross-spectrum is

[13] I'll use the double-sided spectral density here just to avoid the complication of the leading factor of the single-sided spectral density, which is 2 most of the time but 1 for the two "special" points. The essence of the argument does not change.

$$S_{XY} = \frac{A^* B}{T} \quad . \qquad \text{(IV-34)}$$

In the process of averaging, we add N of the spectra together (then divide by N). The auto-spectra are real numbers. If we say that, on average, the spectral density of A is σ_A^2, then the sum of N S_{XX}'s would be

$$\sum^N S_{XX} = N\,\sigma_A^2 \quad , \qquad \text{(IV-35)}$$

and the sum of N S_{YY}'s would be

$$\sum^N S_{YY} = N\,\sigma_B^2 \quad . \qquad \text{(IV-36)}$$

But, A and B are incoherent and the product A^*B is complex with random phase from average to average. Consequently, this sum increases as the square root[14] of N rather than as N:

$$\sum^N S_{XY} = \sqrt{N}\,\sigma_A\,\sigma_B \quad . \qquad \text{(IV-37)}$$

The coherence is

$$\gamma^2 = \frac{\left(\sum S_{XY}\right)^* \left(\sum S_{XY}\right)}{\left(\sum S_{XX}\right)\left(\sum S_{YY}\right)} \quad , \qquad \text{(IV-38)}$$

(all quantities would be divided by N in the averaging process but the common factor of N cancels out) or

[14] This is a classic result that I will not prove.

$$\gamma^2 \quad = \quad \frac{\left(\sqrt{N}\ \sigma_A\ \sigma_B\right)^2}{\left(N\ \sigma_A^2\right)\left(N\ \sigma_B^2\right)} \quad = \quad \frac{1}{N} \quad . \tag{IV-39}$$

For $N = 1$, the coherence is one and for N increasing to infinity, the coherence approaches zero. This is the proper limiting behavior but now we know how fast the coherence approaches zero for two incoherent signals. This behavior is shown in the previous figure.

The coherence estimate is *biased*: if the coherence is low, the estimate from the averaged auto- and cross-spectra is higher than the actual coherence. For completely independent signals, the true coherence would be zero but the coherence estimate is always positive so it can never have an average value of zero. This bias can be extremely important to account in calculations that depend on the coherence. We will encounter some of these situations shortly.

Notice also the relatively slow convergence to the true value. *This rather slow convergence to the final value is important to remember if you're designing some process that depends on averaging to "remove" incoherent components.* As discussed above, the averaged transfer function measurement (averaged cross-spectrum divided by averaged input auto-spectrum) has some immunity to noise in the system being measured. But, if only a few averages are taken, this immunity is not fully realized.

Also keep in mind that, because you may need a certain number of averages to approach the theoretical performance, the process is going to take a length of time equivalent to that number of averages. *If the desired signals are not stable over that length of time, then the process may not work nearly as well as you'd hoped.* For example, if you are trying to detect or track a moving source of sound, the changing propagation path (especially the changing phase relationship) can degrade the results if the source moves too far during the time required for processing.

2) *Two signals that share a common component*

The case in which there is partial correlation between the two channels is also interesting. Let these two channels be represented by their linear spectra,

$$X = C \quad ; \quad Y = C + B \quad , \tag{IV-40}$$

where C is the common component and B is the extra "noise." The auto-spectra are then

$$S_{XX} = \frac{C^* C}{T} \text{ , and} \tag{IV-41}$$

$$S_{YY} = \frac{C^* C + B^* B + B^* C + C^* B}{T} \quad , \tag{IV-42}$$

and the cross-spectrum is

$$S_{XY} = \frac{C^* C + C^* B}{T} \tag{IV-43}$$

The algebra is tedious in this case and I won't give further details; I'll just give the result. If we let the noise-to-signal power ratio be,

$$\alpha^2 = \frac{\sigma_B^2}{\sigma_C^2} \tag{IV-44}$$

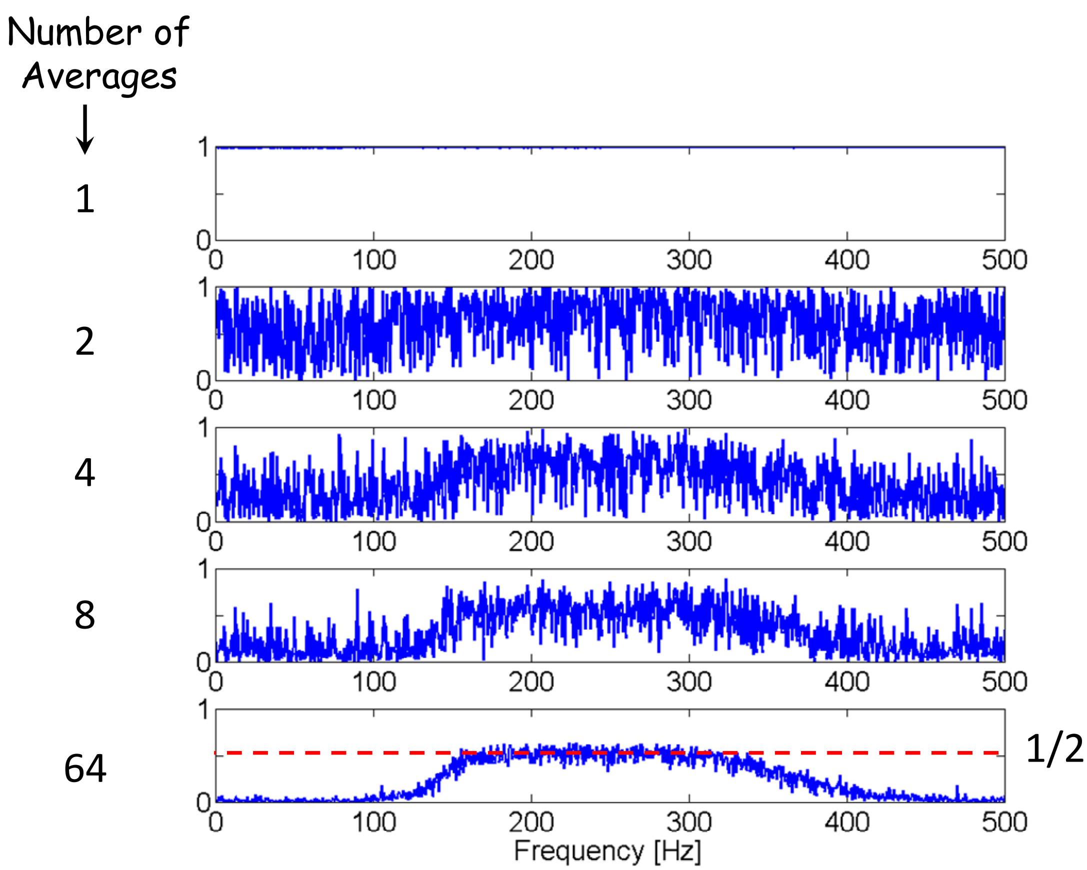
then the coherence after N averages is approximately,

$$\gamma^2 = \frac{1 + \alpha^2/N}{1 + \alpha^2}\left[1 - \frac{N\,\alpha^2}{5\left(1 + N^2\,\alpha^4\right)}\right]^2 \tag{IV-45}$$

This approximation is reasonably good for more than 4 averages. As N goes to infinity, the coherence approaches

$$\gamma^2 = \frac{1}{1 + \alpha^2} \quad . \tag{IV-46}$$

The figure below illustrates this convergence to the large-N expression above. The figure was constructed by calculating the coherence between two signals. Both signals contain a common random-noise component but one of the signals has an additional random component (the "noise") with the same power (*i.e.,* α is one). The common component is band limited (from about 150 to 350 Hz) to make it clear that the coherence approaches the theoretical value (of 0.5 for $\alpha = 1$) where there is a common component and zero in those portions of the spectrum that do not have the common component.



Exercise 4.2: Dependence of the Coherence on the Number of Averages

As you work your way through this course, you should be writing a series of MatLab m-functions for the important quantities. Once you are sure the m-function is working properly, you can use the function in more complicated assignments. Write MatLab m-functions to find the averaged spectra, G_{XX}, G_{YY}, and G_{XY}, and the coherence, γ^2. Each m-function should permit overlapping and windowing and should apply the appropriate window correction when necessary. Test them to be sure that the "integral" of the spectral density quantities equals the equivalent time-domain quantity.

Pick a sampling frequency and generate three separate random time series, a, b, and c, each 10 seconds long. The MatLab function, *randn*, will generate a normally distributed random series. Use these three time series to construct x and y: $x = a + c$, $y = a + b$. The two series, x and y, share a common component, a.

Write a script to use your m-functions to find the averaged spectra, G_{XX}, G_{YY}, and G_{XY}, and the coherence, γ^2. (You don't need to use a window or overlap for these averages.) Plot the coherence as a function of frequency for 1 average, for 10 averages, and for 100 averages. To what value is the coherence converging? Try to calculate the value you would expect for N approaching infinity for the same "power" (i.e., mean-square value) in a, b, and c.

Degradation of Coherence

The classical description of coherence is presented in the previous section. If the coherence is less than one, it is often assumed that there must be incoherent noise in the system. If that is true, then the signal-to-noise ratio can be computed from the measured coherence:

$$\frac{\bar{G}_{Y_0Y_0}}{\bar{G}_{NN}} = \frac{\gamma^2}{\left(1 - \gamma^2\right)} \qquad \text{(IV-47)}$$

where the numerator on the left side is the noise-free auto-spectral density of the output channel and the denominator on the left side is the auto-spectral density of the noise component.

In spite of the prevalence of this assumption in textbooks, coherence degradation may not be the result of incoherent noise. For example, if the "transfer function" is not constant over the duration of the measurement, then the coherence can be quite small even in the absence of noise. In this section, we'll take a closer look at the definition of coherence and its implications.

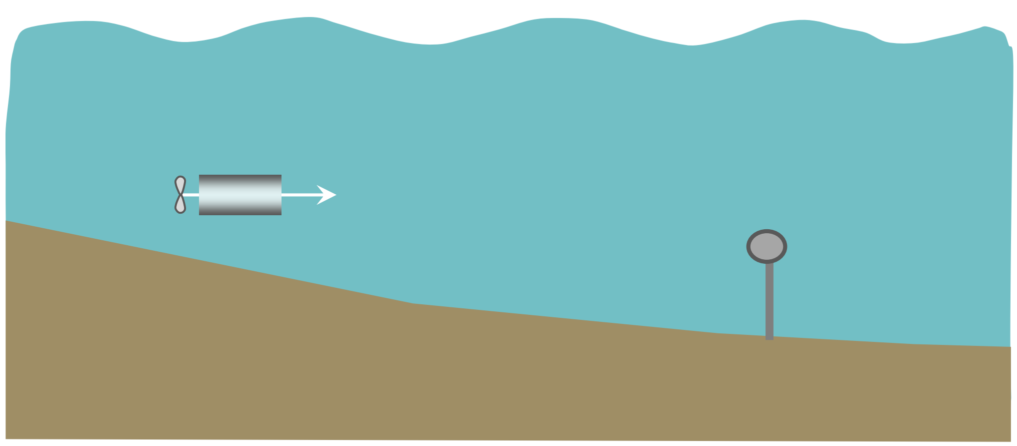
Coherence depends on the averaged cross-spectrum and on the two averaged auto-spectra. Recall that the phase of the cross-spectrum product is equal to the difference between the phase of X and the phase of Y:

$$G_{XY} = \frac{2X^* Y}{T} = \frac{2|X||Y|}{T} e^{j(\phi_Y - \phi_X)} \qquad \text{(IV-48)}$$

If this phase *difference* is stable (not changing with time), then in the averaging process, the individual vectors are all in the same direction and add to the maximum possible value. If the phase difference changes with time, then the vector sum is reduced in magnitude. Since the auto-spectrum products are always real, the denominator of the coherence function is independent of such phase changes.

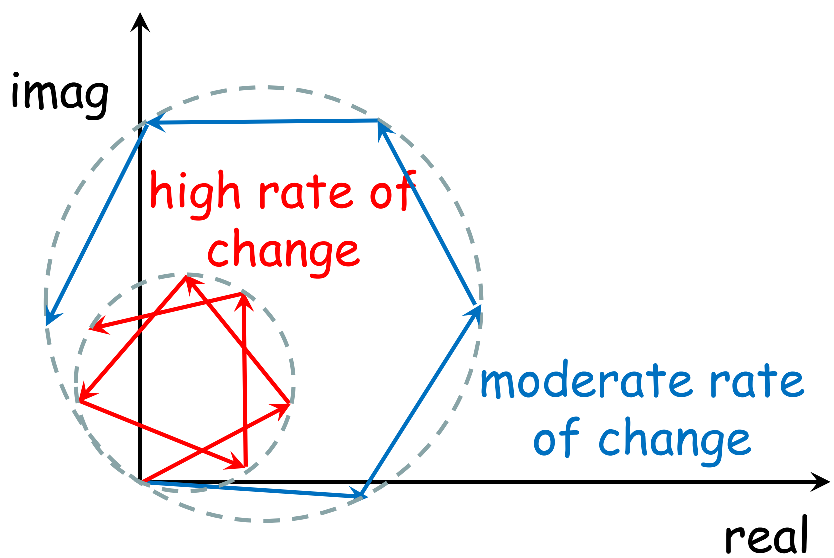
Coherence is related to the ratio of the actual length of the vector sum of the cross-spectral terms to the maximum possible length of that vector sum if all the vectors were aligned.

Anything that causes the relative phase between X and Y to change during the analysis interval causes the coherence to drop. It could be noise in either or both channels but it could also be a deterministic change. If, for example, there is relative motion between the sound (or vibration) source and the receiver, then, *even in the absence of noise*, the transfer function, which is determined by the propagation path, is changing and the coherence will degrade.

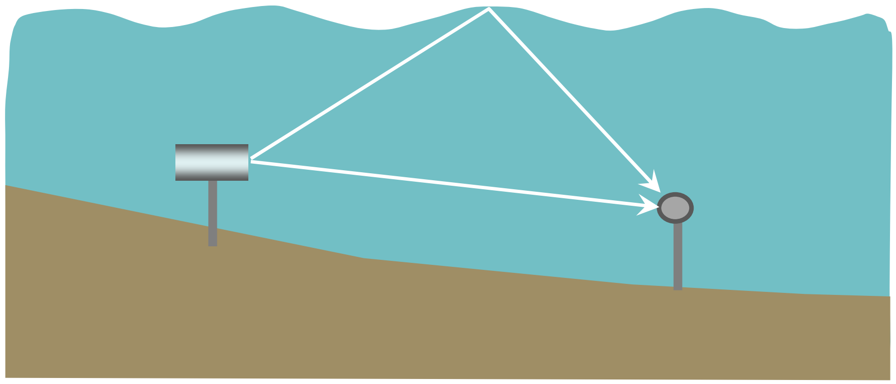


Even if the magnitude of the transfer function is not affected, the phase is directly proportional to the path length. If the path changes significantly relative to the acoustic wavelength over the averaging period then the relative phase between the source channel and the receiver channel changes with time and the cross-spectrum, instead of approaching a stable, constant value, decays with time. *This causes the coherence function to drop even though the relationship between the input and the output may be completely deterministic*.

For a constant rate of change of relative phase (a constant relative speed, for example), the relative phase is deterministic but it still impacts the vector sum as shown in the next figure. For a moderate rate of change (blue vectors), the angle of the cross-spectrum vector gradually increases and the vector sum has a smaller magnitude than if the vectors all had the same phase. The effect is even more dramatic for a higher rate of change (red vectors). The coherence can be quite small even if the noise is negligible!

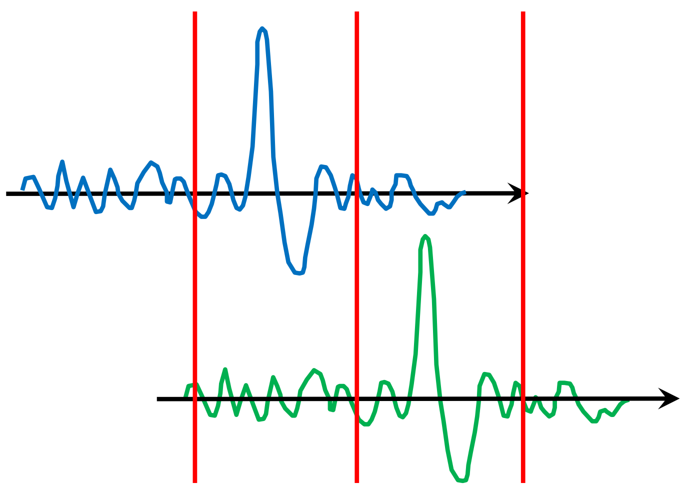


If the source and receiver are at fixed positions in space, the propagation path may still change with time. In measurements underwater, if part of the propagation path involves reflection from the ocean-air interface, the phase of that reflected path will vary with time because the interface is in motion.



For measurements in the atmosphere, changing winds and turbulence can cause changes in the propagation path leading to reduced coherence (even if the signal-to-noise ratio is high).

Another circumstance in which the coherence can be low without added noise occurs when there is a significant time difference between two paths. If the two channels of data are taken with two receivers some distance apart, the signal will arrive at different times.



If the analysis window (the record length, from one red line to the next in the figure) is too short, the signal may appear in one channel but not in the other (or only partially in the other). The coherence will be low but only because of the time mis-alignment. In this circumstance, it may be necessary to use longer records or to shift one of the channels in time to align the important signal component.

The traditional notions (a) that, if the coherence is less than one, there must be added noise and (b) that there is a clear relationship between the coherence and the signal-to-noise ratio are only valid under fairly restrictive assumptions. One must be extremely careful applying textbook results to real measurements[15].

With this caution in mind, we will, however, start with the usual textbook approach that assumes stationary signals and random, stationary noise, not because this is a useful approach but because the contrast between the behavior of real data and ideal data is critical to understand.

Evolution of the Cross-Spectrum

Synthetic, stationary signals

Consider the behavior of two "ideal" signals. For the following illustration, I generated two time-domain signals: (1) a white-noise signal (spectral density shown in black, left plot), and (2) the same white-noise signal but with an additional, independent, pink-noise component (blue, left plot). The common white-noise component represents the acoustic signal common to both sensors while the pink-noise[16] represents additional noise in one of the sensors. Because the additional-noise component is a function of frequency, the coherence between the two channels (right plot) is also a function of frequency—rather low coherence at low frequency and fairly high coherence at high frequency. The spectral densities (the auto-spectra) of the two signals and the coherence between them are shown below. These results were calculated using 287 one-thousand-second records—with a Hann window and 50% overlap—a simulated interval of 40 hours. These results were developed during an analysis of infrasound measurements so the frequency range is very low—0.005 to 20 Hz—however, the behavior is representative regardless of the frequency range.

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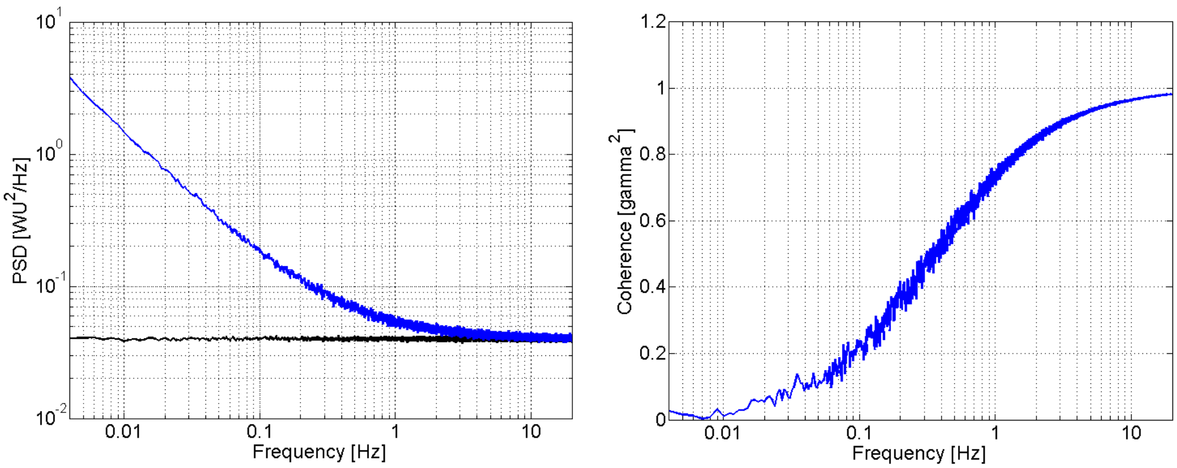
[15] Yes, you must even be careful applying the results in *this* textbook to real measurements!

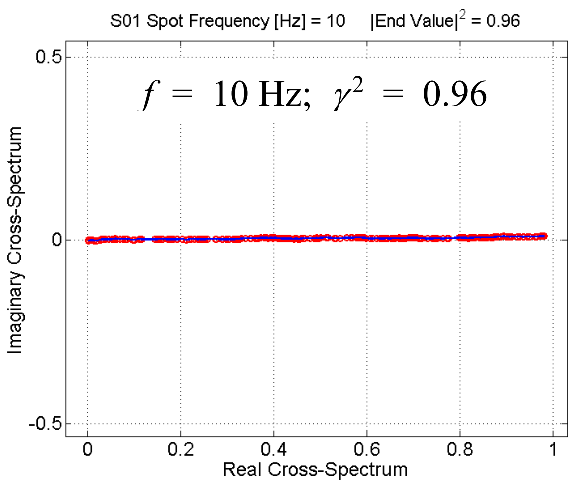
[16] There is no claim here that pink noise has special physical significance. Pink noise is used here solely to produce a coherence that is a simple function of frequency.

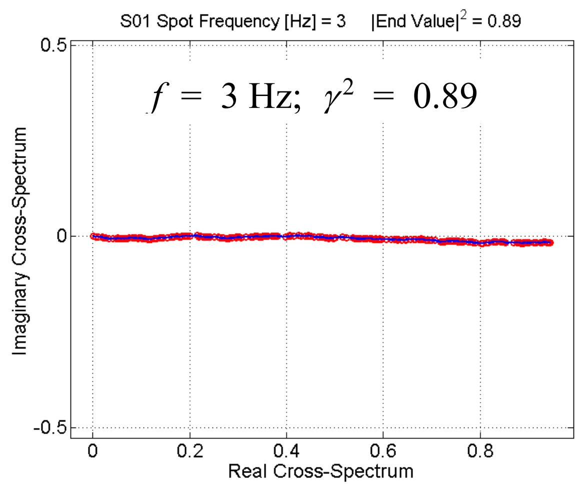


These two signals (in "working units" or WU here) illustrate the textbook relationship between coherence and signal-to-noise ratio. Given some value of coherence, the spectral density of the signal with noise should be the reciprocal of the coherence times the spectral density of the noise-free signal. For example, when the coherence is 0.2 (at about 0.1 Hz), the spectral density of the noisy signal is five times that of the noise-free signal (check this out on the left-hand plot). These signals are well-behaved: both the common component and the added noise component are stationary. Whereas the analysis shown spans 40 hours in simulated time, both the spectral densities and the coherence would look much the same, except for increased scatter, for a shorter time span taken anywhere in the original 40-hour interval.

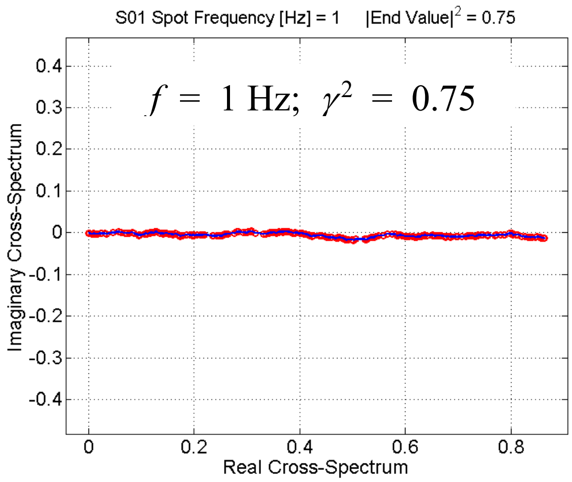
Since the common component is identical in both signals (which is equivalent to each channel having the same response in a transfer-function measurement) the relative phase between the two signal components should be zero. At 10 Hz, the signal-to-noise ratio is high and we should see a high coherence and an orderly summation of the record-by-record cross-spectrum values.

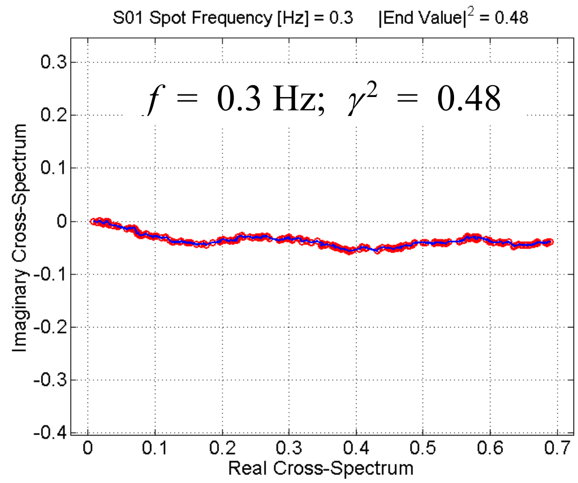
The evolution of the cross-spectral average can be visualized by plotting the individual cross-spectral values (at a specific frequency) as vectors added "tip-to-tail" in the complex plane. The following sequence of plots shows these cross-spectrum "trajectories" for different values of coherence. In each case, the "tip" of each cross-spectrum value is shown by the red circle and the "body" of the vector is shown as a short (sometimes very short) blue line segment. These plots have been normalized by the maximum possible summation magnitude so that the square of the final magnitude is roughly equal to the coherence. At 10 Hz (the first plot), the coherence is 0.96. All of the vectors are essentially horizontal (zero relative phase between the two channels). There are small variations in the lengths of the individual vectors but they are all adding in the same direction so the coherence is high. At 3 Hz, the coherence is 0.89. There are small variations in the directions (the relative phases) of the individual cross-spectral values but the overall progression is in a nearly straight, horizontal trajectory.



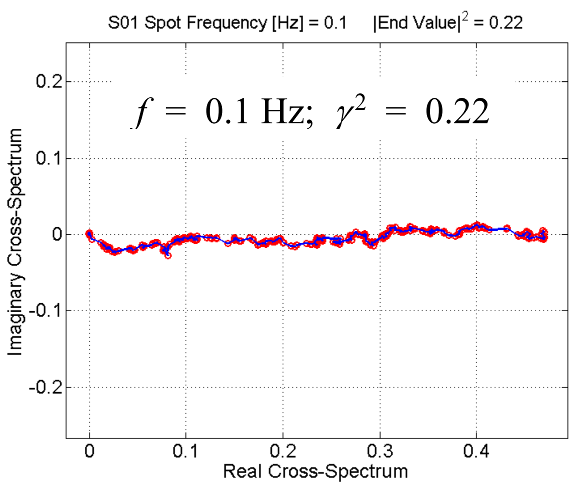


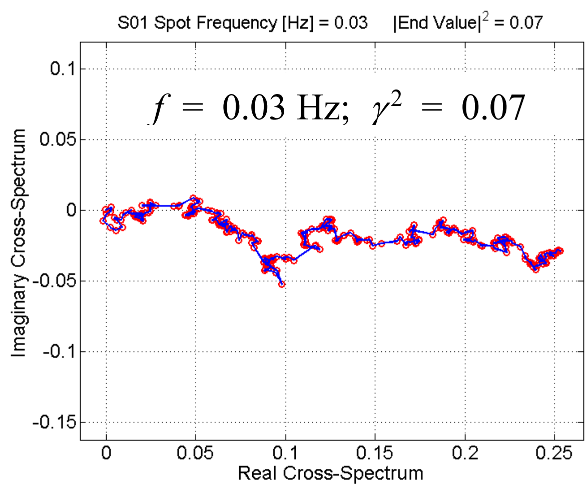
As the frequency decreases, the coherence decreases (because of our choice of pink noise as the contaminating component). For 1 Hz and 0.3 Hz, the variations in relative phase increase as do the wiggles in the overall trajectory.



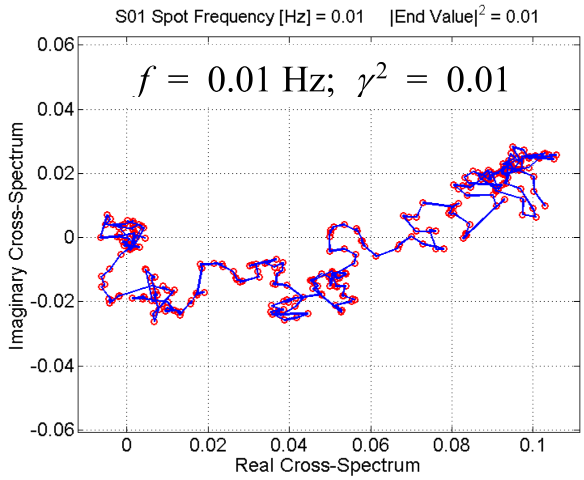


The trend continues for 0.1 Hz and 0.03 Hz. By 0.03 Hz, the coherence has dropped to 0.07 and the phases of the individual values of the cross-spectrum have a wide range. In places, the trajectory reverses direction.





The final plot (0.01 Hz with a coherence of 0.01) shows large variations in magnitude and direction for each step. Notice that the plots have been re-scaled in each case to show the trajectory more clearly. This last plot has an *x-y* range of about 0.1 in normalized cross-spectrum whereas the first plot in the series has an *x-y* range of about 1.0.



One key point to observe in the cross-spectral trajectory plots for the synthetic signals is the fact that the randomness—for any specific value of coherence—is fairly uniformly distributed over the entire trajectory. Analysis of a smaller section of the original signals would produce a similar coherence value regardless of the specific time interval of that subsection. This feature is a defining feature of stationarity. So far, the results are consistent with the usual textbook assumptions.

> Exercise 4.3: Trajectory of the Cross-Spectrum
>
> For the time series (either x or y) created for Exercise 4.2 and for 100 averages, try to construct the cross-spectral trajectory. This programming task is more challenging than the mean-square averaging but is a worthwhile exercise.

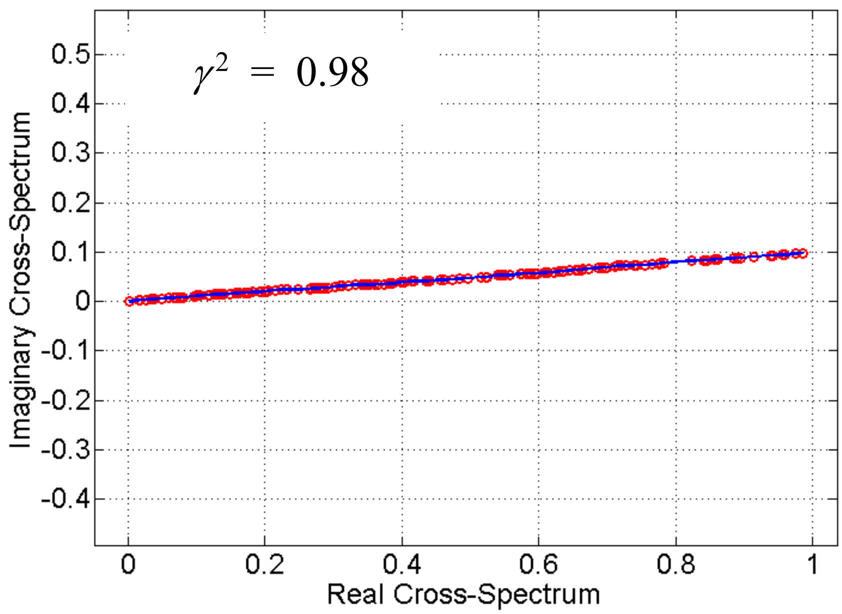
Stationarity is a convenient simplifying assumption in statistical analysis *but stationarity is the exception rather than the rule for many acoustic measurements*. Consequently, the assumption of stationarity can lead to flawed conclusions when applied to real measurements.

Evolution of the cross-spectrum: real measurements

Now that we have seen an example of ideal signals, we can examine a selection of real measurements to better appreciate the critical differences between "textbook" conditions and reality.

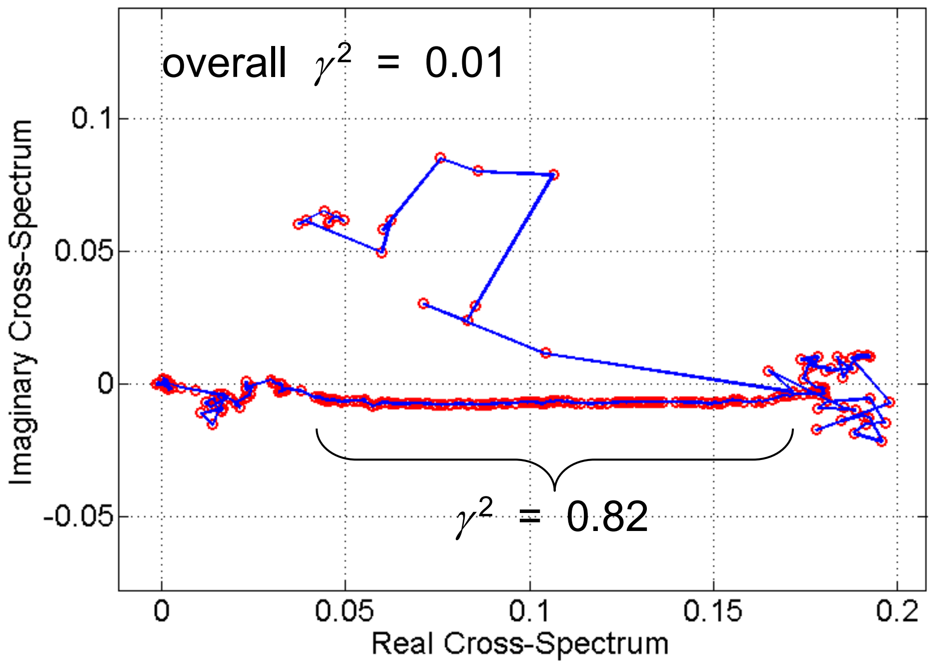
The following measurements were made with two infrasonic acoustic sensors outdoors.

With very little wind, the acoustic field dominates and the coherence between the two sensors is high. As for the synthetic signals, we expect that the cross-spectral trajectory should be a straight line with little side-to-side variation. We also expect that the angle of that trajectory will be equal to the relative phase between the two responses at the specific frequency of the trajectory. We see both of these features in the results shown below.



At the specific frequency for this cross-spectrum trajectory (0.6 Hz), there is a small phase difference between the two sensors, which appears as the shallow angle of the cross-spectrum trajectory. The trajectory is quite straight in keeping with the measured coherence of 0.98.

Especially at very low frequency, local wind can produce high levels of noise with substantial variability. Results for measurements taken under typical wind conditions are shown in the next figure where the evolution of the cross-spectrum for a 0.4 hour interval is shown. Notice the substantial variations in character of the trajectory. Starting from (0,0), there is a section of tightly packed randomness followed by a nearly straight section from one side of the plot to the other; then another tightly packed random section that is followed by a final section of relatively large, random steps. The component that is incoherent between the two sensors is clearly not stationary: its character (i.e., its statistics) changes with time. The coherence for this 0.4 hour interval is almost zero. However, if the time interval is trimmed so that only the straight-line section is included (a 0.22-hour segment), the coherence increases to 0.82.



The actual phase difference between the two sensors is close to zero as the straight-line section shows. If the entire 0.4-hour section were used, the apparent phase difference (the angle of the vector that connects the origin with the end point) would be about 45 degrees. While the coherence for the 0.4-hour interval—a value of 0.01—would tell us not to accept the 45-degree phase difference, examination of the cross-spectral trajectory shows that there is, in fact, a useable sub-section of data within that 0.4-hour interval. An assumption of stationarity would lead us to reject the measurement entirely; closer examination shows that much improved results are possible by careful selection of the measurement interval (and *less* averaging).

Real measurements are notable for non-stationary behavior. A value of coherence (for example) over a long interval is in no way representative of the quality of a sub-interval. By screening measurements over short intervals, it may be possible to extract acceptable data from measurements that seem hopeless.

Any approach for identification of "good" data segments must recognize the non-stationarity of the incoherent components. Statistical methods that rely on the assumption of stationarity will likely produce results that are misleading. Consequently, a viable approach should account for the time-evolution of the field.

Other Applications of Coherence

Coherent output power

The coherence function is not only useful as an indicator of the reliability of a measurement made by ordinary averaging. If we are sure that coherence degradation is the result of uncorrelated noise, the coherence function can also be used to estimate the power in the output that is coherent with the input. In the introduction to coherence, we supposed that we knew the noise-free output, Y_0, in order to make sense of the definition of coherence. We eventually constructed an expression for the coherence using quantities that are measureable but the relationship between the noise-free output and the actual output,

$$\frac{\bar{G}_{Y_0Y_0}}{\bar{G}_{YY}} \quad = \quad \frac{\left(\bar{G}_{XY}\right)^* \bar{G}_{XY}}{\bar{G}_{XX}\,\bar{G}_{YY}} \quad = \quad \gamma^2 \qquad \text{(IV-49)}$$

is still true. Once we've calculated the coherence function, we can find the "coherent output power", COP, which is the averaged spectral density, $\bar{G}_{Y_0Y_0}$,

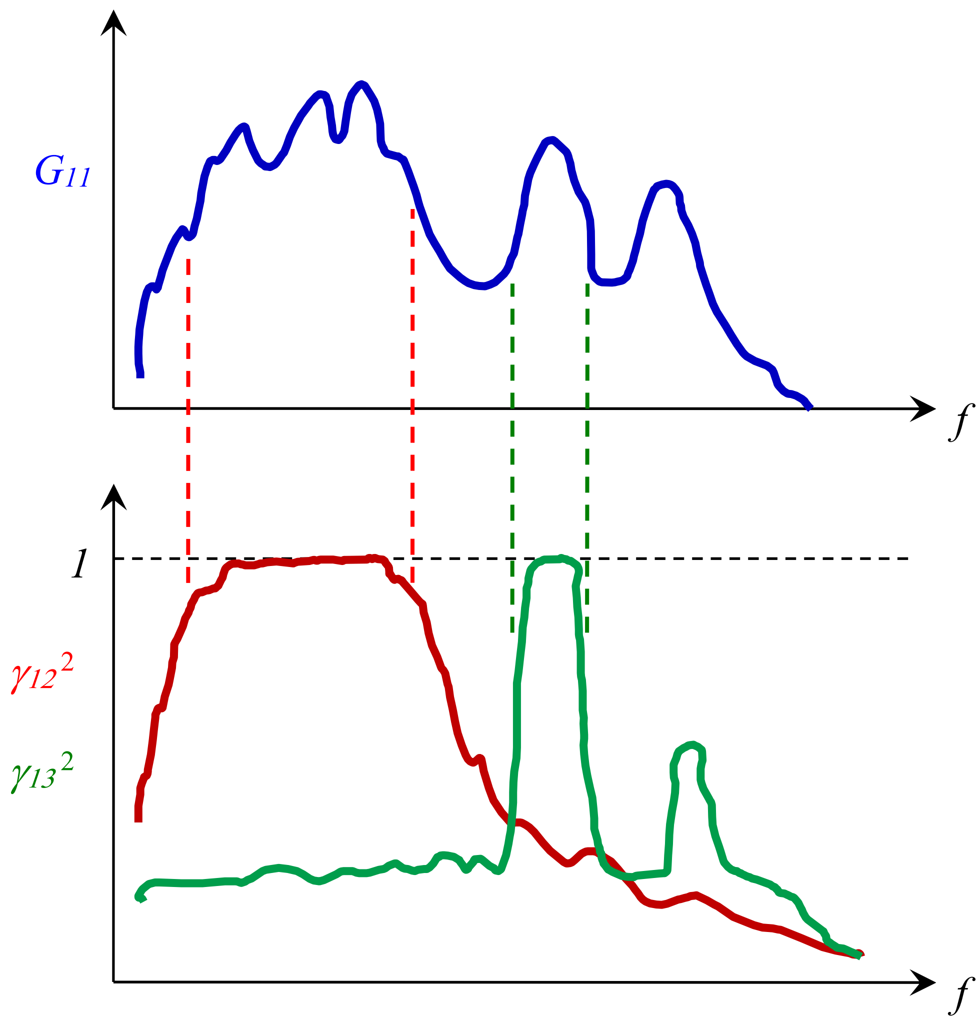
$$\boxed{\text{COP} \quad = \quad \bar{G}_{Y_0Y_0} \quad = \quad \gamma^2\,\bar{G}_{YY}} \qquad \text{(IV-50)}$$

Please notice that the COP is an auto-spectrum – that is, a quantity that gives the distribution in frequency of the magnitude-squared of Y_0. We learn nothing of the phase and we cannot reconstruct an equivalent coherent output waveform in the time domain. Even so, the coherent output power is useful for identification of the contribution of a particular noise generator to the overall noise as long as all components are stationary.

Also notice that the COP is a function of the value of coherence. You must keep in mind that, for small values of coherence, the coherence estimate is biased and the value for COP will be in error by this bias.

Coherent output power is often used to estimate the power resulting from particular sources of noise in a circumstance with more than one source. If we wanted to diagnose the noise in the driver's compartment of a large piece of heavy construction equipment, we could put an accelerometer on the engine and a microphone in the driver's compartment. Using the accelerometer signal as input and the microphone as "output" we could calculate the coherent output power. This would tell us how much of the compartment noise was coherent with the engine vibration. We'd repeat the same operation for, say, the compartment ventilation blower instead of the engine and, component-by-component, we could isolate the largest contributors to the compartment noise. This is the "*source identification*" problem. In this example, we're not interested in the time-domain waveform, just the acoustic power at the driver's location, so the COP is adequate.

Suppose, for example, that Channel 1 is the signal from a microphone in the driver's compartment (i.e., at the point of annoyance or health risk). Further suppose that Channel 2 is the signal from an accelerometer on the engine block and that Channel 3 is the signal from a microphone inside the ventilation system duct. By computing the coherences between Channels 1 and 2 (red, below) and between Channels 1 and 3 (green), we can see something of the relationship between the driver's compartment noise and the engine vibration or the ventilation blower noise.



In this example, the lower frequency part of the spectral density, G_{11}, in the driver's compartment is strongly coherent with the accelerometer signal (Channel 2), while the peak in the spectral density that is just into the upper half of the measurement band is strongly coherent with the ventilator noise (Channel 3). The coherent-output-power calculation would give the fractions of power associated with these two generating mechanisms but it can also be instructive to make a plot like the one above in which the coherences are plotted separately and compared to the target spectral density.

We can also calculate the "incoherent output power", IOP[17]:

[17] IOP is not a standard quantity but it's a logical companion to COP.

$$\text{IOP} \quad = \quad \left(1 \; - \; \gamma^2\right) \overline{G}_{YY} \qquad \text{(IV-51)}$$

If we replace the coherence function and the averaged auto-spectrum of the output by their equivalent expressions in terms of *X*, *Y*, and *N*, we find that the IOP is just the auto-spectral density of the incoherent noise:

$$\left(1 \; - \; \gamma^2\right) \overline{G}_{YY} \quad \equiv \quad \overline{G}_{NN} \qquad \text{(IV-52)}$$

As with COP, this estimation is only valid if the drop in coherence is the result of uncorrelated noise (as opposed to a changing transfer function, for example) and you should also recognize the potential for bias in the estimate of coherence.

Signal-to-noise ratio

Continuing with the supposition that the two channels, *X* and *Y*, share a common component and that one channel (*Y*, for example) also has a random-noise component, we can extract the ratio of signal power to noise power. In general, each channel will have its own transfer function, *H*, between incident acoustic pressure and voltage (or digitizer) output. If the common component is *S* and the noise component is *N*, the linear spectra for the two channels would be,

$$X \;=\; H_X\, S \qquad \text{and} \qquad Y \;=\; H_Y\left(S \;+\; N\right). \qquad \text{(IV-53)}$$

The averaged auto-spectra would be,

$$\overline{G_{XX}} \;=\; \frac{2}{T} H_X^* H_X \,\overline{S^* S} \qquad \text{and} \qquad \overline{G_{YY}} \;=\; \frac{2}{T} H_Y^* H_Y \left(\overline{S^* S} \;+\; \overline{N^* N}\right) \qquad \text{(IV-54)}$$

with the usual assumption that the averaging can be done over a sufficiently long time that averages of incoherent cross-products are negligible. With the same assumption, the averaged cross-spectrum would be,

$$\overline{G_{XY}} = \frac{2}{T} H_X^* H_Y \overline{S^* S} \tag{IV-55}$$

and the resulting coherence,

$$\gamma^2 = \frac{\left|\overline{S^* S}\right|^2}{\overline{S^* S}\left(\overline{S^* S} + \overline{N^* N}\right)} = \frac{1}{1 + \overline{N^* N}/\overline{S^* S}}. \tag{IV-56}$$

The averaged product, N^*N, is equivalent to the noise power in the time domain and the averaged product, S^*S, is equivalent to the signal power in the time domain. The ratio of signal power to noise power is then,

$$\frac{\overline{S^* S}}{\overline{N^* N}} = \frac{\gamma^2}{1 - \gamma^2}. \tag{IV-57}$$

It is critical to keep in mind that this connection between coherence and SNR is only valid when the underlying assumptions are satisfied. In particular, (a) if either the noise or the signal is not stationary, or (b) there is noise in the input also, this equation will not represent the noise-to-signal ratio. Notice also that (a) if the coherence is low, the result will depend directly on the value of coherence, which may have significant bias, and (b) if the coherence is high, the denominator will be small and sensitive to the specific value of coherence.

If you are depending on accurate values for COP, IOP, or signal-to-noise ratio, please examine the coherence carefully. If the coherence as a function of frequency has significant scatter, the estimates for quantities dependent on coherence will likely have substantial error.

Exercise 4.4: Signal to Noise Ratio

Refer back to Exercise 4.2. Modify the construction so that $x = a$ instead of $x = a + c$. (In other words, only y has an uncorrelated component.) Compute the coherence and the SNR. For that case, what should the SNR be based on what you know about x and y?

While the coherence, coherent output power, and signal-to-noise analysis are powerful tools in acoustic and vibration data analysis, they must be used cautiously. For example, make sure that you understand the signal path (and its stability) between the source to the receiver before you interpret γ^2 or COP.

Other diagnostic uses of coherence

Years ago, a team traveled to a number of locations across the country to measure the ambient floor vibrations in instrumentation-development facilities. Their intent was to characterize typical floor-vibration spectra and identify particularly quiet locations. After many measurements using a single accelerometer, they concluded that the background floor vibrations were virtually the same regardless of location. Years later, another investigator recognized that this team was really just measuring the self-noise of their accelerometer over and over! The sensor's self-noise floor was higher than the background vibration levels at any of the facilities.

If the team had used two accelerometers and computed the coherence between them during the measurement, they would have seen very low coherence—a clear indication that they were not measuring actual floor vibration. *Even if the end result seems to require only a single sensor, using two sensors can provide a valuable diagnostic of the measurement*.

Coherence can also be used to diagnose problems in data acquisition systems. For example, if two microphone signals of different levels are being recorded, it is possible for the stronger signal to "leak" (often by electromagnetic coupling internal to multi-channel acquisition systems) into the other channel. If you attach an inactive load in place of the microphone to the channel carrying the weaker signal, then compute the coherence between that channel and the active stronger channel, the coherence should be nearly zero. If the coherence is not near zero, inter-channel coupling ("cross-talk") may be indicated.

Spatial Coherence

Calculation of coherence requires two channels of data. In the special case in which these two channels are measurements of the same type at two different locations in the acoustic field, the result is often called spatial coherence. Examination of spatial coherence can reveal features of the acoustic field—either signal or noise—that cannot be resolved with a single sensor. In fact, even if a measurement only requires a single sensor, the use of two sensors can aid in the interpretation of that measurement.

Spatial coherence of a signal can tell us how far apart we can put sensors and still retain the identity or correlation of a signal between the two sensors. In the design of arrays of sensors intended to enhance signal detection or determine signal direction of arrival, the performance depends on correlation of the signal across the field of sensors. If the sensors are too far apart, variations in propagation conditions across the array can degrade the inter-sensor coherence and result in poorer than expected performance[18].

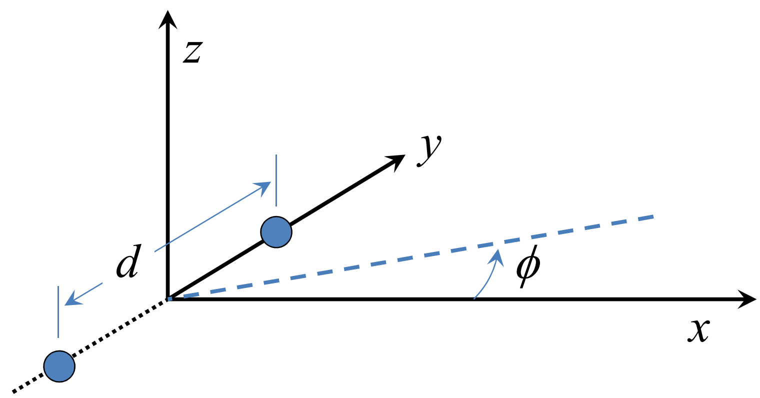
Spatial coherence of background noise can tell us that if we put sensors too close together, we may no longer be able to assume uncorrelated noise components. For example, if wind noise is limiting an outdoor acoustic measurement, the noise is associated with turbulence that has a distribution of eddy sizes. With a constant mean-flow speed, the large eddies produce lower frequency noise as an individual eddy takes longer to drift across the sensor. If two sensors are spaced closer than the dimension of a turbulent eddy, then, even though the pressure fluctuations are not acoustic, the coherence can be high at the corresponding frequency. An analysis process that depends on incoherent noise components might produce much worse results than predicted.

In this section, we will examine the consequences of a distribution of sources (or, equivalently, a moving source) with regard to the coherence between two sensors.

---

[18] Yet another difference between textbook results and results in practice.

Consider two sensors placed a distance, d, apart. To fix the coordinate system, let the y direction be the direction connecting the two sensors and let the x direction be perpendicular to y with both x and y in the horizontal plane. It is simpler to formulate the problem in polar coordinates where ϕ is the angle in the horizontal plane with respect to the x axis.



If the pressure amplitude of a plane-wave acoustic arrival is P_0 and k is the magnitude of the acoustic wave number vector, then the complex pressures at the two sensors are,

$$P_1 \;=\; P_0 e^{-j\frac{kd}{2}\sin\phi} \quad ; \quad P_2 \;=\; P_0 e^{j\frac{kd}{2}\sin\phi} \tag{IV-58}$$

The magnitude of k is $2\pi f/c$, where f is the frequency and c is the sound speed.

Both the frequency response and the coherence depend on the averaged auto- and cross-spectra. In calculating these averages, assume that the arrival angle varies over the averaging period (or, with the same effect, that there are simultaneous arrivals distributed over that range of angles). Under these conditions, an average over the span of arrival angles replaces the time average.

Since the coherence is a ratio of cross- and auto-spectra, the $1/T$ factor (or $2/T$ factor) in the spectral-density definition always cancels out; therefore, we can work directly with products of the complex linear spectra, P_1 and P_2. Except for that $1/T$ factor, the auto-spectra (before averaging) for Sensors 1 and 2 are,

$$P_1^* P_1 \;=\; P_0^2 \quad ; \quad P_2^* P_2 \;=\; P_0^2 \tag{IV-59}$$

The auto-spectra are independent of arrival angle so these quantities do not change with averaging. The cross-spectrum (as before, except for the $1/T$ factor[19]) for a single direction of arrival is,

$$P_1^* P_2 \quad = \quad P_0^2\, e^{j\,kd\sin\phi} \tag{IV-60}$$

If the direction of arrival, ϕ, is constant, the coherence is one since all of the cross-spectral vectors will have the same direction in the complex plane.

First, assume that the arrival-angle distribution is uniform over all horizontal angles (i.e., horizontally isotropic ambient). The averaged auto-spectra are identical to the single-record auto-spectra:

$$\overline{P_1^* P_1} \quad = \quad P_0^2 \qquad ; \qquad \overline{P_2^* P_2} \quad = \quad P_0^2 \tag{IV-61}$$

and the averaged cross-spectrum is the integral of the single-record cross-spectrum over all horizontal angles divided by the total range of angles:

$$\overline{P_1^* P_2} \quad = \quad \frac{P_0^2}{2\pi} \int_{-\pi}^{+\pi} e^{j\,kd\sin\phi}\, d\phi \tag{IV-62}$$

Expanding the integrand into real and imaginary parts gives,

$$\overline{P_1^* P_2} \quad = \quad \frac{P_0^2}{2\pi} \int_{-\pi}^{+\pi} \left[\cos\left(kd\sin\phi\right) \; + \; j\sin\left(kd\sin\phi\right)\right] d\phi \tag{IV-63}$$

The imaginary part of the integrand is an odd function of ϕ so the integral over the symmetric limits is zero. The remaining integral produces a Bessel function:

$$\overline{P_1^* P_2} \quad = \quad \frac{P_0^2}{2\pi} \int_{-\pi}^{+\pi} \cos\left(kd\sin\phi\right) d\phi \quad = \quad P_0^2\, J_0\left(kd\right) \tag{IV-64}$$

The coherence follows directly:

[19] T is the length of individual time records in the averaging process. Unless otherwise indicated, the $1/T$ factor is dropped from all spectral-density expressions for this discussion.

$$\gamma^2 = \frac{\left|\overline{P_1^* P_2}\right|^2}{\overline{P_1^* P_1}\,\overline{P_2^* P_2}} = \left[J_0(kd)\right]^2 \qquad \text{(IV-65)}$$

This result has little practical importance as horizontally isotropic ambient noise is unusual[20]. A more useful approach considers a limited angular distribution of ambient arrivals. At the cost of resorting to numerical integration, a weighting function, $W(\phi)$, can be introduced to restrict the angular range of arrivals. This function can either be "rectangular"—one for some range of arrival angles and zero elsewhere—or it can vary to taper the magnitude of arrivals over some span of angles. The averaged auto-spectrum for Sensor 1 becomes,

$$\overline{P_1^* P_1} = P_0^2 \frac{1}{2\pi}\int_{-\pi}^{+\pi} W(\phi)\,d\phi = P_0^2 W_0 \qquad \text{(IV-66)}$$

with a similar result for Sensor 2. The constant, W_0, is the average value of $W(\phi)$. The averaged cross-spectrum becomes:

$$\overline{P_1^* P_2} = \frac{P_0^2}{2\pi}\int_{-\pi}^{+\pi} W(\phi)\,e^{jkd\sin\phi}\,d\phi \qquad \text{(IV-67)}$$

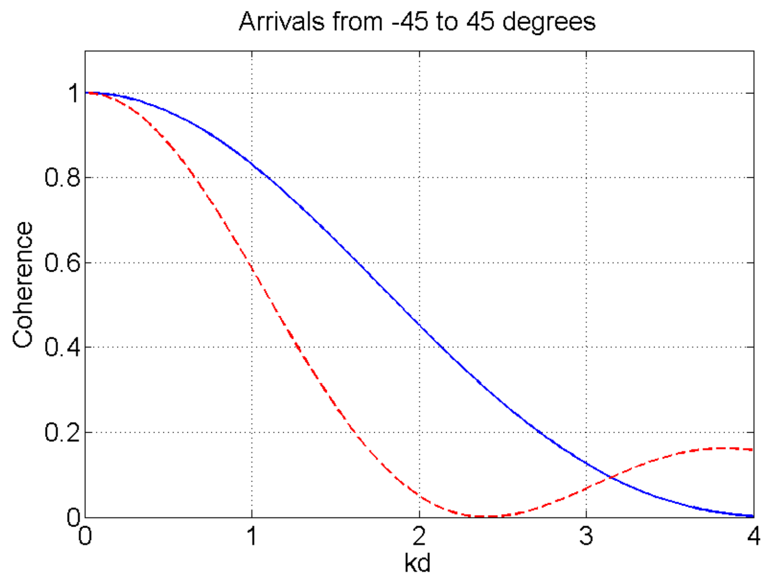
and the coherence is

$$\gamma^2 = \frac{1}{W_0^2}\left[\int_{-\pi}^{+\pi} W(\phi)\,e^{jkd\sin\phi}\,d\phi\right]^2 \qquad \text{(IV-68)}$$

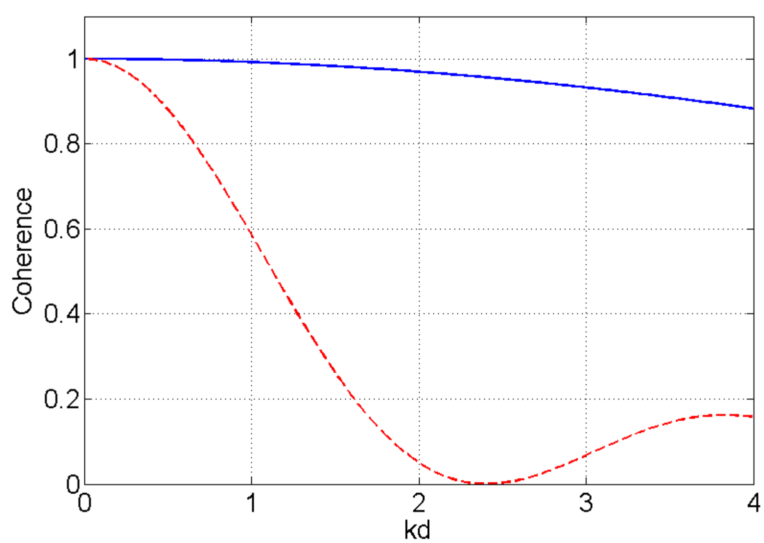
Fortunately, the integral in Eq. II-68 is well behaved with respect to numerical solution. The following plot shows the coherence for horizontally isotropic arrivals (Eq. IV-65, dashed red in the figure) and for arrivals confined to an angular range of ± 45 degrees with respect to the broadside direction (Eq. IV-68, blue). The broadside direction is the direction perpendicular to the line

[20] A result can also be obtained for noise that is isotropic in all three dimensions by extending the integration over all directions. This calculation is classic but is also of little practical importance.

connecting the two sensors—the x direction in our coordinate system. The horizontal scale is normalized frequency (or, equivalently, normalized spacing), kd.

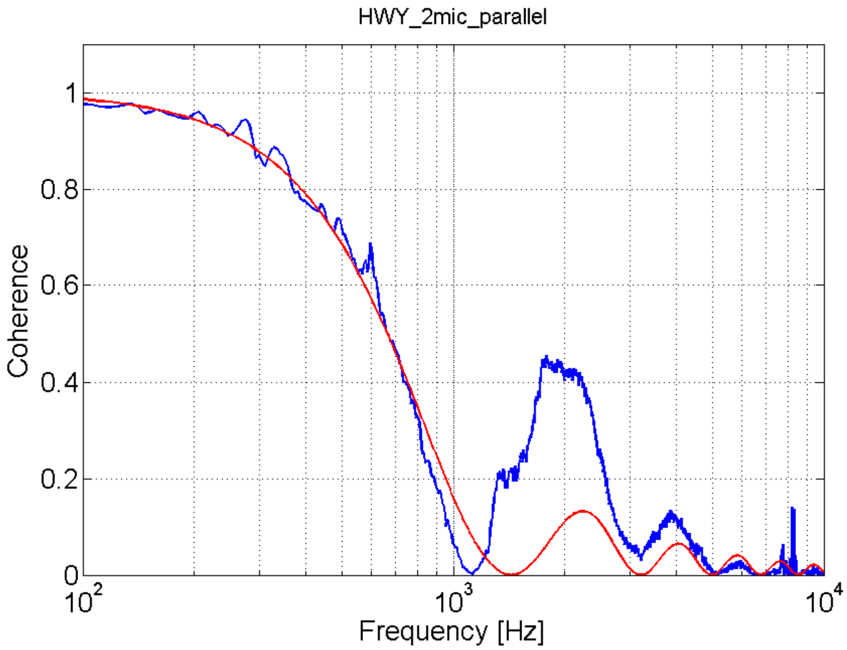


The coherence depends not only on the range of angles of arrival but also on the orientation of that angular range with respect to the microphone pair. The next plot (blue curve) shows the coherence for a ± 45 degree range with respect to the end-fire direction (the direction along the line connecting the two sensors – the y axis) along with the isotropic result (red dashed curve).

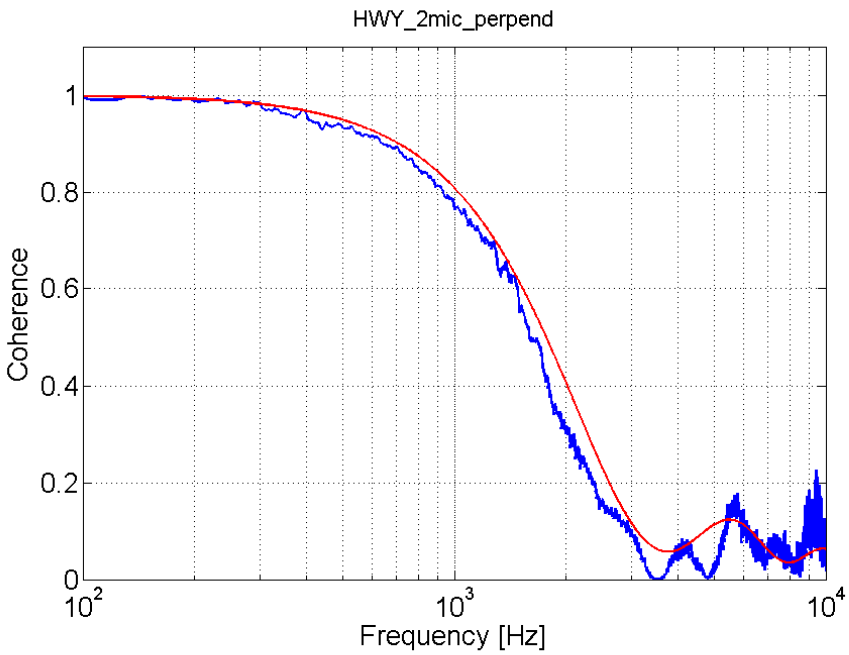


To illustrate the effect of angular distribution of sources on coherence, a measurement was made adjacent to a well-traveled highway. Two microphones were placed at the location shown by the yellow dot about 50 meters from the edge of the nearest lane.

The sensors were 10 cm apart and the primary acoustic excitation is traffic noise from the highway. The two figures below give the measured (blue) and theoretical (red) results for a sensor pair oriented parallel (first figure) to the roughly line source of traffic noise and for a sensor pair oriented perpendicular (second figure) to that line source.



In each case, the angular extent of the noise was taken to be the "visible" extent of the highway—roughly ±80 degrees to either side of the closest point of the highway.   Other than termination of the angular range at ±80 degrees, no weighting function was used. For a particular source at a distance, r, the received pressure would drop as $1/r$; however, the length of time that the source is in a fixed angular increment increases as r. The two effects cancel in the integrand.



With a few exceptions, the correspondence between theory and measurement is good. For the parallel orientation, the coherence peak near 2 kHz is much higher in the measurement than in the theoretical solution while for the perpendicular orientation, there appear to be more peaks in the measurement than in the model above a few kilohertz. These departures are likely the result of ground reflection, ground impedance, or departures from straight-line propagation.

Exercise 4.5: Coherence and Two-Channel Recordings

If you can record two channels (from two microphones) on your computer or if you have access to a wav-file recorder with two microphones, write a MatLab script to process a "stereo" file. Plot the averaged spectral densities of both channels and compute and plot the averaged coherence between the two channels. You can learn a great deal about coherence by recording sounds in many different environments.

If you are not able to record two channels, try to find a stereo file on the web (or on the class-resource site) and compute the averaged coherence. (Be aware that some "stereo" files on the web are actually a single waveform replicated in both channels! In that case, the coherence would be very close to one at all frequencies.)

Characteristics of an Excitation

While the tools described above can be applied to measurements of sound from sources over which you have no control, when you do have control over the sound source, the success of a such a measurement depends in part on the characteristics of the excitation (or input), $x(t)$. For example, you'd suspect that if the input signal doesn't have much power over a particular range of frequencies then an estimate of a transfer function might not be so good over that range of frequencies. Consequently, a signal that has a broad, reasonably flat spectral density is often used. There are a number of options:

- white noise
- a white noise sequence repeated (to allow synchronization)
- special pseudo-random sequences (we'll talk about these later)
- frequency chirps (see the next chapter for details)
- an impulse

While simple white noise is a popular option it does not permit synchronous averaging. All of the other choices do permit synchronization.

Later in this course, we'll discuss system noise and consider the so-called "dynamic range" of a measurement system (that is, the range between the smallest signal detectable in the system noise and the largest signal that can be recorded with reasonable linearity). Here, it's appropriate to mention one important aspect of system range in the context of input-signal selection. Suppose the system being evaluated has both resonances and anti-resonances—at some frequencies, the output may be large and at other frequencies the output may drop to small values. We have to set the data acquisition recorder's range to accommodate the highest output values without distortion. But, if we use an input signal with a flat spectrum, that means that, where the system response drops to small values, the output may drop down into the measurement system's noise floor and we won't be able to get an accurate measure at those frequencies. The coherence will warn us that there's a problem *but we have no way of correcting the problem if the excitation is white noise*.

There are several options for coping with the problem of the limitations imposed by the dynamic range of the measurement system. If we know enough about the general shape of the system under test, then we might be able to design an input signal with a spectrum that has more power over the frequency ranges with high system attenuation and less power over the frequency ranges with high system gain. For a system with narrow (i.e., high-Q) resonances, this can be impractical but for a test system in which the response variations are not concentrated in frequency, designing the input signal with an appropriately shaped spectrum can be effective.

Another possibility is to do the analysis in smaller ranges of frequency. Isolate the region around the resonances and measure those separately from the regions around the anti-resonances. Often the features of importance may be in the immediate vicinity of such features and narrowing the range of frequency of the analysis to those spectral windows can result in higher quality measurements. We might accomplish this by designing the input signal to have a relatively narrow-band spectrum around the frequencies of interest. (A frequency-swept pulse[21] or chirp, for example, with a relatively small frequency range.)

One of the best ways to make measurements of systems with a widely varying response over frequency is to use a sine wave as input, optimize the Channel 1 and Channel 2 acquisition amplitude ranges at each frequency, and average synchronously. After the measurement at that single frequency is made, the input sine wave frequency is changed, the input ranges are adjusted if necessary, and the measurement is repeated. This can be more time consuming than analysis with a broadband input but the ability to adjust the acquisition system's range for each frequency can be of great benefit when the system under test has a great deal of response variation. Commercial analyzers often implement this function as "Swept Sine" analysis (with "Autoranging"). The less commonly used but powerful "Lock-In Amplifier" performs this one-frequency-at-a-time analysis also.

Basic Questions Regarding Signals and Noise

When approaching a measurement problem, there are some basic questions that you should consider to guide your design of a measurement approach. The material in these notes presents a number of options for measurement and analysis but making a logical choice for the procedure will always depend on knowledge of the characteristics of the signal and the noise. The following list of questions is presented to stimulate thinking about these characteristics.

Questions about signals

How much do you know about the signal(s) that you are trying to measure?

- Can you control the signal?

[21] Measurement of systems with high Q (sharp resonances) is challenging case. If the source frequency is swept, the sweep must be slow enough that the system's transient response does not dominate the results. If the system is not given sufficient time to settle to steady state, the measurement may have substantial error.

- Do you know what the frequency content is?
- Do you know what the time waveform is?
- Is the signal concentrated in frequency? (sinusoidal or a collection of harmonics?)
- Is the signal limited in time? (a short burst?)
- Are the signal properties changing with time?
- Is the signal strongly correlated with some other observable phenomenon?

Questions about noise

How much do you know about the noise that will interfere with your measurement?

- Noise is always present!
- What is the nature of the ambient noise?
- What noise is introduced by your measurement system?
- Do you have control over the noise?
- Is the noise steady, quasi-steady, or bursting?
- Is the noise strongly correlated with some other observable phenomenon?

Uncertainty in Spectral Density, Coherence, and Frequency Response

Any real measurement has some degree of uncertainty and, while the theory of uncertainty, even with simplifying assumptions, can be complicated, the best estimates for measurement uncertainty often come from the measurement itself. A measurement with many points in a time series or many frequency points in a spectral density allows computation of percentile limits and these percentiles can give a more reliable indicator of uncertainty than theoretical analysis. For the theoretical normal distribution, one standard deviation encompasses 68.3% of the points, two standard deviations encompass 95.5% of the points, and three standard deviations encompass 99.7% of the points. Even if the distribution is not normal, defining uncertainty limits as, for example, the plus/minus limits that encompass 95% of the data has real, physical meaning; whereas the standard deviation may not.

If, however, we need to estimate uncertainty before making the measurement, we must appeal to statistics. It would be very poor practice, though, to take the statistical estimates as more accurate than the subsequent measurement!

Basic assumptions

Many (most?) measurements exhibit non-stationarity—the mean value and the variance can change rather dramatically over analysis time scales. Consequently, blind application of statistical analysis can produce misleading results. Given that warning, we can make some progress in prediction of uncertainty.

The non-stationarity of real signals makes application of statistical theory tenuous. Books and papers on the subject often begin their analyses with statements like, “Consider two sample time-history records from two stationary random processes...” or “Consider band-limited Gaussian noise with zero mean value...” Acoustic signals—especially interesting ones—often violate these simplistic assumptions. While some of these assumptions will be used here, the reader must recognize the limitations and further understand the critical need to select data records with great care; otherwise, the error estimates may be poor.

In this section, the statistical uncertainty will be treated as if the signal is stationary. This means that the results are only applicable if the measurements have been pre-screened to identify approximately stationary intervals. One screening process has been described in this chapter (Evolution of the Cross-Spectrum) and is based time-dependent behavior of the complex cross-spectrum—a powerful but little used tool.

Basic distribution functions

The spectral density is equivalent to the sum of the squares of equivalently filtered time-domain values. Consequently, the spectral density (the auto-spectrum, that is) values are always positive. The relevant statistical distribution for these spectral density values (if the underlying time series are normally distributed) is the chi-squared distribution. The frequency-response estimate is the ratio of an auto-spectrum and a cross-spectrum; it is complex with statistics that follow the F distribution [Otnes and Enochson, 1972]. For large numbers of averages (say, greater than 100), it is often possible to approximate these distributions by a normal distribution; however, we will examine the proper distributions first.

Confidence limits in spectral density

The pressure spectral density for a normally distributed random time series is chi-squared (χ^2) distributed. Consequently, the lower and upper confidence limits, α, to the normalized spectral density are [Otnes and Enochson, 1972]

$$\alpha_{PSD} = \frac{1}{n}\chi^2\left[n, \pm(1 - p)/2\right] \qquad \text{(IV-70)}$$

for the confidence fraction, p, (confidence percentage divided by 100) and n degrees of freedom. For all the cases considered here[22], $n = 2N$, where N is the

[22] For this analysis, assume that any record overlap is limited to preserve statistical independence from record to record. With excessive overlap, n would be less than $2N$.

number of averages. The number of degrees of freedom equals $2N$ because the real and imaginary parts count separately.

Bendat and Piersol (2000, Table 9.1) give an approximate form for the normalized error of a spectral-density estimate. Their expression is equivalent to one standard deviation with respect to the spectral density normalized to one:

$$\delta = \pm\sqrt{\frac{1}{N}}$$

The chi-squared distribution is simple enough to calculate (even Excel has a built-in chi-squared distribution function) that this crude approximation has little utility.

Confidence limits in frequency response

The frequency response is equal to the ratio of two spectral-density-like quantities so the relevant statistical distribution is the F distribution. Consequently, the lower and upper confidence limits, α_{FR}, to the normalized frequency response are [Otnes and Enochson, 1972]

$$\alpha_{FR} = 1 \pm \beta \qquad \text{(IV-71)}$$

where[23]

$$\beta = \sqrt{\frac{1}{n-2}\frac{\left(1-\gamma^2\right)}{\gamma^2}F\left[2,(n-2),q\right]} \qquad \text{(IV-72)}$$

Here, γ^2 is the coherence and q equals $1-p$. If the true frequency response is $H(f)$, then the confidence limits would be

$$\alpha_{FR}\,H(f) \qquad \text{(IV-73)}$$

using both values of α_{FR}. Notice that it is standard to use q rather than p as the third argument in F to specify the confidence fraction. For example, for 75% confidence limits, $p = 0.75$ and $q = 0.25$.

[23] Some care is required in interpreting the results from Otnes and Enochson; they express their result in terms of the auto-spectrum of the output divided by the auto-spectrum of the input, which is equal to the estimated response divided by γ^2. There also appears to be an extra factor of 2 in Otnes and Enochson, Eq. 9.98. When this factor is removed (as it has been removed in the equation above), the Otnes and Enochson result converges with the approximate result from Bendat and Piersol for large n.

Abramowitz and Stegun (1972) give tables for F and a general approximate form; however, for specific confidence limits, power-law approximations are considerably simpler and more accurate than the general approximate form. For 75% confidence limits, a least-squares power-law approximation to the values tabulated in Abramowitz and Stegun is

$$F\left[2, x, 0.25\right] \approx 1.386 + 3.05 \cdot x^{-1.15} \tag{IV-74}$$

and for 90% confidence limits,

$$F\left[2, x, 0.10\right] \approx 2.302 + 6.79 \cdot x^{-1.05} \tag{IV-75}$$

These expressions are accurate within 0.5% for $x > 10$ (i.e., for $n > 12$ or, equivalently, for more than 6 averages).

The significance of the confidence limits is clearer when the departure, δ, from the true value is considered instead of the normalized error:

$$\delta = \alpha_{FR} - 1 = \pm \beta \tag{IV-76}$$

The last expression on the right is approximately true for large n. Replacing n by $2N$, where N is the number of averages,

$$\delta \approx \pm \sqrt{\frac{\left(1 - \gamma^2\right)}{2 N \gamma^2} F\left[2, 2N, q\right]} \tag{IV-77}$$

where, again, the assumption of large n (large N) has been invoked. Here, we see that the squared error is proportional to the reciprocal of the number of averages and to $(1 - \gamma^2)/\gamma^2$. For coherence nearly equal to one, the statistical error can be quite small even with a modest number of averages[24]. For example, for 5% error ($\delta = 0.05$) with 75% confidence, 100 averages would be sufficient if $\gamma^2 > 0.85$ and only 25 averages would be necessary if $\gamma^2 > 0.96$. Furthermore, these confidence limits apply to every point individually in the spectrum. If averaging of several adjacent frequency values is permitted, the number of averages would be replaced

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[24] This is the key to the observation that the scatter in the frequency response is often far smaller than the scatter in a spectral density estimate made using the same measurements. The error in the spectral density does not have the one-minus-coherence factor.

by the product of the number of averages and the number of frequency bins in the smoothing.

Bendat and Piersol (2000, Eq. 9.90) give an approximate form for the normalized error in a frequency-response estimate. They give an expression that is equivalent to one standard deviation with respect to the frequency response normalized to one:

$$\delta_{BP} \quad = \quad \pm\sqrt{\frac{\left(1 \; - \; \gamma^2\right)}{2\,N\,\gamma^2}} \tag{IV-78}$$

Notice the similarity to Eq. IV-77 from the *F* distribution. To find the 75% confidence limits, multiply by 1.159; to find the 90% confidence limits, multiply by 1.645 (the factors appropriate for the standard deviation of a normal distribution).

The *F*-distribution (used in Eqs. IV-71,72) approximates the confidence limits for the frequency-response estimation process for both synthetic data and real measurements. If the real estimate is available, the confidence limits can be determined directly from that estimate since each frequency bin is a sample of the response magnitude.

In the absence of a measurement, the *F*-distribution (or the simpler effective standard deviation—Eq. IV-78—to replace the β) can be used to determine the required coherence for a certain number of averages. The simplest approach is to use the Bendat and Piersol (2000) approximation to the normalized standard error. Then the response has the following standard error

$$\left(1 \;\pm\; \delta_{BP}\right) H\left(f\right) \tag{IV-79}$$

This standard-error expression is useful for planning measurements. If the measurement has already been made, then the confidence limits can be obtained directly from the measurement by using a sliding window over frequency. *In practice, it is safer to find the uncertainty directly from the measurement since that procedure does not depend on the assumptions behind Eqs. IV-77 or 78.*

Semi-empirical statistics of sample coherence

There is a slight-of-hand in the expressions above. Whenever coherence appears, the statistician assumes that the true coherence is known and that known coherence is the basis for the uncertainty estimates for spectral density and frequency response. However, the true coherence is never known in a real measurement; only an estimate is available. Furthermore, coherence is inherently biased: since its value is restricted to the range of 0 to 1, a low coherence estimate is always biased high and a high coherence estimate is always biased low. This important issue is almost always ignored in classic treatments of spectral uncertainty.

In 1969, V. A. Benignus published a paper regarding biases and confidence limits in determination of coherence. His procedure seems to fit the results from simulated fairly well—certainly far better than fits based on assumption of a normal distribution. This procedure is described, with some modification, by Otnes and Enochson and is presented here, borrowing from both procedures for the best fit to simulations.

The procedure has three stages: (1) the coherence, which ranges from 0 to 1, is first transformed so that the transformed values have a nearly normal distribution; (2) the bias is removed (approximately); and (3) a corrected variance is calculated. The correction in the last step is based on curve fits to Monte-Carlo simulations in order to improve the fit for low values of coherence.

The sample coherence has the usual definition based on the averaged auto-spectra and on the averaged cross-spectrum

$$\hat{\gamma}^2 = \frac{\left|\overline{G_{XY}}\right|^2}{\overline{G_{XX}}\,\overline{G_{YY}}} \qquad \text{(IV-80)}$$

Let the number of averages be M.

(1) Transformation: as a variable, z, goes from 0 to infinity, the hyperbolic tangent of that variable goes from 0 to 1. This suggests the following transformation:

$$\begin{aligned} \gamma &= \tanh(z) \\ z &= \tanh^{-1}(\gamma) \end{aligned} \qquad \text{(IV-81)}$$

between the square root of coherence, γ, and the transform variable, z.

(2) Bias removal: for a finite number of averages, the coherence is inherently biased. For example, two completely incoherent signals generate a coherence that is always greater than zero except in the limit as the number of averages, N, goes to infinity. For completely incoherent signals, the true coherence is zero; whereas, the computed coherence is, on average, $1/N$. First, the sample coherence is transformed:

$$z = \tanh^{-1}\left(\sqrt{\hat{\gamma}^2}\right) \qquad \text{(IV-82)}$$

Then the average value of z is computed, the z-bias is removed, after which z is transformed back to coherence:

$$\gamma_{mean}^2 = \left[\tanh\left(\bar{z} + \frac{1}{2(N-1)}\right)\right]^2 \qquad \text{(IV-83)}$$

This step—the estimation of the true coherence—is not always straightforward in practice. If the coherence shows strong variations with frequency, then it may be more practical to make the estimate by smoothing the coherence[25] and using that smoothed estimate in place of the first term on the right in Eq. IV-83.

(3) The standard error (i.e., "one standard deviation") for the transform variable, z, is

$$\varepsilon[z] = \frac{A}{\sqrt{2(N-1)}} \qquad \text{(IV-84)}$$

[25] While smoothing is practical when making estimates of broadband noise, estimation of line components adds another complication: neighboring frequency bins cannot be used to smooth the coherence estimate. Most (all?) texts including Bendat and Piersol ignore the problems associated with estimation of levels and uncertainties for line components.

where A is an empirical correction developed by Benignus (1969) through Monte-Carlo simulation:

$$A = 1 - 0.004^{\left(1.6\gamma_{mean}^{2} + 0.22\right)} \qquad \text{(IV-85)}$$

The coherence used in A should be the true coherence, which is generally not known in a real measurements; however, the result from Eq. IV-85 is an adequate substitute. Note: If the true coherence is greater than 0.4, the correction, A, is within 1% of unity. The standard error in z is added and subtracted from the mean value of z and then these limits are transformed back to the corresponding coherence values.

It is particularly important to consider these corrections for measurements with small numbers of averages and coherence below 0.5. For example, Bendat and Piersol (3rd Ed., Eq. 9.82) give a simplified expression for the standard error of coherence,

$$\varepsilon\left[\hat{\gamma}^2\right] \approx \sqrt{\frac{2}{N}}\frac{1 - \gamma^2}{\gamma} \qquad \text{(IV-86)}$$

For 10 averages and a coherence of 0.5, this expression gives a standard error of 0.45, which implies that 68% of the time, the coherence would be between 0.05 and 0.95 and that 95% of the time, the coherence would be between -0.4 and 1.4!

References

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