Statistical Inference: the sign test

Duality between confidence intervals and hypothesis tests

- Tests and CIs really do the same thing, if you look at them the right way. They are both telling you something about a parameter, and they use same things about data.
- To illustrate, some data (two groups):

```
my_url <- "http://ritsokiguess.site/datafiles/duality.txt"
twogroups <- read_delim(my_url," ")
```

```
##
## -- Column specification ----------------------------
## cols(
##   y = col_double(),
##   group = col_double()
## )
```

The data

```
twogroups
```

y	group
10	1
11	1
11	1
13	1
13	1
14	1
14	1
15	1
16	1
13	2
13	2
14	2
17	2
18	2
19	2

95% CI (default)

```r
t.test(y ~ group, data = twogroups)
```

```
##
##  Welch Two Sample t-test
##
## data:  y by group
## t = -2.0937, df = 8.7104, p-value = 0.0668
## alternative hypothesis: true difference in means is not equ
## 95 percent confidence interval:
##  -5.5625675  0.2292342
## sample estimates:
## mean in group 1 mean in group 2
##        13.00000        15.66667
```

90% CI

```
t.test(y ~ group, data = twogroups, conf.level = 0.90)
```

```
##
##  Welch Two Sample t-test
##
## data:  y by group
## t = -2.0937, df = 8.7104, p-value = 0.0668
## alternative hypothesis: true difference in means is not equ
## 90 percent confidence interval:
##  -5.010308 -0.323025
## sample estimates:
## mean in group 1 mean in group 2
##        13.00000        15.66667
```

Hypothesis test

Null is that difference in means is zero:

```
t.test(y ~ group, mu=0, data = twogroups)
```

```
##
##  Welch Two Sample t-test
##
## data:  y by group
## t = -2.0937, df = 8.7104, p-value = 0.0668
## alternative hypothesis: true difference in means is not equ
## 95 percent confidence interval:
##  -5.5625675  0.2292342
## sample estimates:
## mean in group 1 mean in group 2
##        13.00000        15.66667
```

Comparing results

Recall null here is $H_0 : \mu_1 - \mu_2 = 0$. P-value 0.0668.

- 95% CI from -5.6 to 0.2, contains 0.
- 90% CI from -5.0 to -0.3, does not contain 0.
- At $\alpha = 0.05$, would not reject H_0 since P-value $>$ 0.05.
- At $\alpha = 0.10$, *would* reject H_0 since P-value $<$ 0.10.

Not just coincidence. Let $C = 100(1 - \alpha)$, so C% gives corresponding CI to level-α test. Then following always true. ($\iff$ means "if and only if".)

Reject H_0 at level α	$\iff$	$C\%$ CI does not contain H_0 value
Do not reject H_0 at level α	$\iff$	$C\%$ CI contains H_0 value

Idea: "Plausible" parameter value inside CI, not rejected; "Implausible" parameter value outside CI, rejected.

The value of this

- If you have a test procedure but no corresponding CI:
- you make a CI by including all the parameter values that would not be rejected by your test.
- Use:
  - $\alpha = 0.01$ for a 99% CI,
  - $\alpha = 0.05$ for a 95% CI,
  - $\alpha = 0.10$ for a 90% CI, and so on.

Testing for non-normal data

- The IRS ("Internal Revenue Service") is the US authority that deals with taxes (like Revenue Canada).
- One of their forms is supposed to take no more than 160 minutes to complete. A citizen's organization claims that it takes people longer than that on average.
- Sample of 30 people; time to complete form recorded.
- Read in data, and do t-test of $H_0 : \mu = 160$ vs. $H_a : \mu > 160$.
- For reading in, there is only one column, so can pretend it is delimited by anything.

Read in data

```
my_url <- "http://ritsokiguess.site/datafiles/irs.txt"
irs <- read_csv(my_url)
irs
```

Time
91
64
243
167
123
65
71
204
110
178
264

Test whether mean is 160 or greater

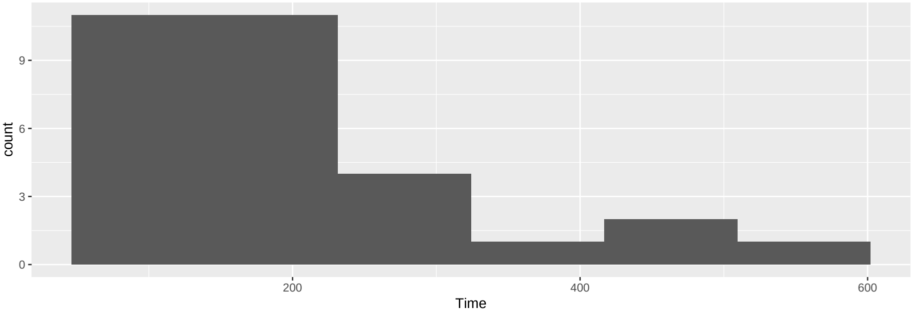
```
with(irs, t.test(Time, mu = 160,
                 alternative = "greater"))
```

```
##
##  One Sample t-test
##
## data:  Time
## t = 1.8244, df = 29, p-value = 0.03921
## alternative hypothesis: true mean is greater than 160
## 95 percent confidence interval:
##  162.8305      Inf
## sample estimates:
## mean of x
##  201.2333
```

Reject null; mean (for all people to complete form) greater than 160.

But, look at a graph

```
ggplot(irs, aes(x = Time)) + geom_histogram(bins = 6)
```



Comments

- Skewed to right.
- Should look at *median*, not mean.

The sign test

- But how to test whether the median is greater than 160?
- Idea: if the median really is 160 (H_0 true), the sampled values from the population are equally likely to be above or below 160.
- If the population median is greater than 160, there will be a lot of sample values greater than 160, not so many less. Idea: test statistic is number of sample values greater than hypothesized median.

Getting a P-value for sign test 1/3

- How to decide whether “unusually many” sample values are greater than 160? Need a sampling distribution.
- If H_0 true, pop. median is 160, then each sample value independently equally likely to be above or below 160.
- So number of observed values above 160 has binomial distribution with $n = 30$ (number of data values) and $p = 0.5$ (160 is hypothesized to be *median*).

Getting P-value for sign test 2/3

- Count values above/below 160:

```
irs %>% count(Time > 160)
```

Time > 160	n
FALSE	13
TRUE	17

- 17 above, 13 below. How unusual is that? Need a *binomial table*.

Getting P-value for sign test 3/3

- R function dbinom gives the probability of eg. exactly 17 successes in a binomial with $n = 30$ and $p = 0.5$:

```
dbinom(17, 30, 0.5)
```

```
## [1] 0.1115351
```

- but we want probability of 17 *or more*, so get all of those, find probability of each, and add them up:

```
tibble(x=17:30) %>%
  mutate(prob=dbinom(x, 30, 0.5)) %>%
  summarize(total=sum(prob))
```

total
0.2923324

Using my package smmr

- I wrote a package `smmr` to do the sign test (and some other things). Installation is a bit fiddly:
  - Install devtools with `install.packages("devtools")`
  - then install smmr:

```
library(devtools)
install_github("nxskok/smmr")
```

- Then load it:

```
library(smmr)
```

smmr for sign test

- smmr's function `sign_test` needs three inputs: a data frame, a column and a null median:

```
sign_test(irs, Time, 160)
```

```
## $above_below
## below above
##    13    17
##
## $p_values
##   alternative   p_value
## 1       lower 0.8192027
## 2       upper 0.2923324
## 3   two-sided 0.5846647
```

Comments (1/3)

- Testing whether population median *greater than* 160, so want *upper-tail* P-value 0.2923. Same as before.
- Also get table of values above and below; this too as we got.

Comments (2/3)

- P-values are:

Test	P-value
t	0.0392
Sign	0.2923

- These are very different: we reject a mean of 160 (in favour of the mean being bigger), but clearly *fail* to reject a median of 160 in favour of a bigger one.
- Why is that? Obtain mean and median:

```
irs %>% summarize(mean_time = mean(Time),
                  median_time = median(Time))
```

mean_time	median_time
201.2333	172.5

Comments (3/3)

- The mean is pulled a long way up by the right skew, and is a fair bit bigger than 160.
- The median is quite close to 160.
- We ought to be trusting the sign test and not the t-test here (median and not mean), and therefore there is no evidence that the "typical" time to complete the form is longer than 160 minutes.
- Having said that, there are clearly some people who take a lot longer than 160 minutes to complete the form, and the IRS could focus on simplifying its form for these people.
- In this example, looking at any kind of average is not really helpful; a better question might be "do an unacceptably large fraction of people take longer than (say) 300 minutes to complete the form?": that is, thinking about worst-case rather than average-case.

Confidence interval for the median

- The sign test does not naturally come with a confidence interval for the median.
- So we use the "duality" between test and confidence interval to say: the (95%) confidence interval for the median contains exactly those values of the null median that would not be rejected by the two-sided sign test (at $\alpha = 0.05$).

For our data

- The procedure is to try some values for the null median and see which ones are inside and which outside our CI.
- smmr has pval_sign that gets just the 2-sided P-value:

```
pval_sign(160, irs, Time)
```

```
## [1] 0.5846647
```

- Try a couple of null medians:

```
pval_sign(200, irs, Time)
```

```
## [1] 0.3615946
```

```
pval_sign(300, irs, Time)
```

```
## [1] 0.001430906
```

- So 200 inside the 95% CI and 300 outside.

Doing a whole bunch

- Choose our null medians first:

```
(d=tibble(null_median=seq(100,300,20)))
```

null_median
100
120
140
160
180
200
220
240
260
280
300

… and then

"for each null median, run the function `pval_sign` for that null median and get the P-value":

```
d %>% mutate(p_value = map_dbl(null_median,
                               ~ pval_sign(., irs, Time)))
```

null_median	p_value
100	0.0003249
120	0.0987371
140	0.2004884
160	0.5846647
180	0.8555356
200	0.3615946
220	0.0427739
240	0.0161248
260	0.0052229
280	0.0014300

Make it easier for ourselves

```
d %>%
  mutate(p_value = map_dbl(null_median,
                           ~ pval_sign(., irs, Time))) %>%
  mutate(in_out = ifelse(p_value > 0.05, "inside", "outside"))
```

null_median	p_value	in_out
100	0.0003249	outside
120	0.0987371	inside
140	0.2004884	inside
160	0.5846647	inside
180	0.8555356	inside
200	0.3615946	inside
220	0.0427739	outside
240	0.0161248	outside
260	0.0052229	outside
280	0.0014309	outside

confidence interval for median?

- 95% CI to this accuracy from 120 to 200.
- Can get it more accurately by looking more closely in intervals from 100 to 120, and from 200 to 220.

A more efficient way: bisection

- Know that top end of CI between 200 and 220:

```
lo=200
hi=220
```

- Try the value halfway between: is it inside or outside?

```
(try = (lo + hi) / 2)
```

```
## [1] 210
```

```
pval_sign(try,irs,Time)
```

```
## [1] 0.09873715
```

- Inside, so upper end is between 210 and 220. Repeat (over):

... bisection continued

```
lo = try
(try = (lo + hi) / 2)
```

```
## [1] 215
```

```
pval_sign(try, irs, Time)
```

```
## [1] 0.06142835
```

- 215 is inside too, so upper end between 215 and 220.
- Continue until have as accurate a result as you want.

Bisection automatically

- A loop, but not a `for` since we don't know how many times we're going around. Keep going while a condition is true:

```
lo = 200
hi = 220
while (hi - lo > 1) {
  try = (hi + lo) / 2
  ptry = pval_sign(try, irs, Time)
  print(c(try, ptry))
  if (ptry <= 0.05)
    hi = try
  else
    lo = try
}
```

The output from this loop

```
## [1] 210.00000000   0.09873715
## [1] 215.00000000   0.06142835
## [1] 217.50000000   0.04277395
## [1] 216.25000000   0.04277395
## [1] 215.62500000   0.04277395
```

- 215 inside, 215.625 outside. Upper end of interval to this accuracy is 215.

Using smmr

- smmr has function ci_median that does this (by default 95% CI):

```
ci_median(irs,Time)
```

```
## [1] 119.0065 214.9955
```

- Uses a more accurate bisection than we did.
- Or get, say, 90% CI for median:

```
ci_median(irs,Time,conf.level=0.90)
```

```
## [1] 123.0031 208.9960
```

- 90% CI is shorter, as it should be.

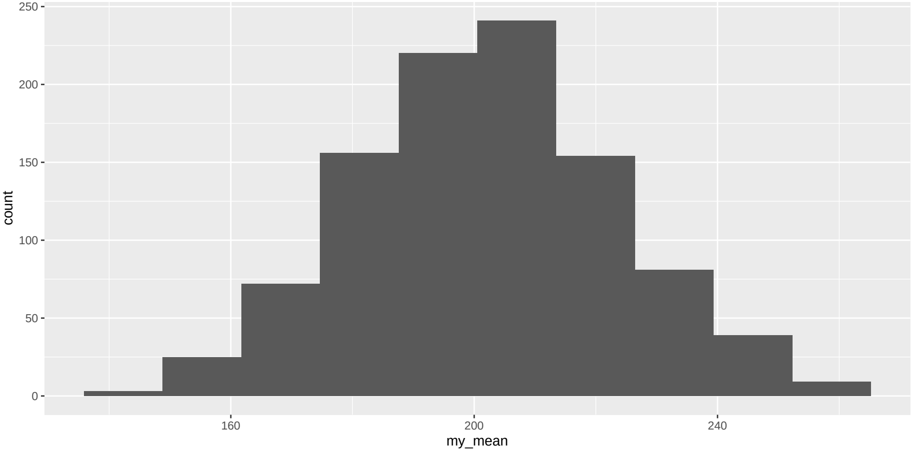
Bootstrap (optional)

- but, was the sample size (30) big enough to overcome the skewness?
- Bootstrap, again:

```
tibble(sim = 1:1000) %>%
  rowwise() %>%
  mutate(my_sample = list(sample(irs$Time, replace = TRUE))) %>%
  mutate(my_mean = mean(my_sample)) %>%
  ggplot(aes(x=my_mean)) + geom_histogram(bins=10) -> g
```

The sampling distribution

```
g
```



Comments

- A little skewed to right, but not nearly as much as I was expecting.
- The t-test for the mean might actually be OK for these data, *if the mean is what you want*.
- In actual data, mean and median very different; we chose to make inference about the median.
- Thus for us it was right to use the sign test.