

## The sign test

# Packages

```
library(tidyverse)  
library(smmr)
```

`smmr` is new. See later how to install it.

# Duality between confidence intervals and hypothesis tests

- ▶ Tests and CIs really do the same thing, if you look at them the right way. They are both telling you something about a parameter, and they use same things about data.
- ▶ To illustrate, some data (two groups):

```
my_url <- "http://ritsokiguess.site/datafiles/duality.txt"  
twogroups <- read_delim(my_url, " ")
```

# The data

```
twogroups
```

```
# A tibble: 15 x 2
```

	y	group
	<dbl>	<dbl>
1	10	1
2	11	1
3	11	1
4	13	1
5	13	1
6	14	1
7	14	1
8	15	1
9	16	1
10	13	2
11	13	2
12	14	2
13	17	2
14	18	2

## 95% CI (default)

for difference in means, group 1 minus group 2:

```
t.test(y ~ group, data = twogroups)
```

Welch Two Sample t-test

data: y by group

t = -2.0937, df = 8.7104, p-value = 0.0668

alternative hypothesis: true difference in means between gr

95 percent confidence interval:

-5.5625675 0.2292342

sample estimates:

mean in group 1 mean in group 2

13.00000

15.66667

## 90% CI

```
t.test(y ~ group, data = twogroups, conf.level = 0.90)
```

Welch Two Sample t-test

data: y by group

t = -2.0937, df = 8.7104, p-value = 0.0668

alternative hypothesis: true difference in means between groups

90 percent confidence interval:

-5.010308 -0.323025

sample estimates:

mean in group 1 mean in group 2

13.00000

15.66667

# Hypothesis test

Null is that difference in means is zero:

```
t.test(y ~ group, mu=0, data = twogroups)
```

Welch Two Sample t-test

data: y by group

t = -2.0937, df = 8.7104, p-value = 0.0668

alternative hypothesis: true difference in means between gr

95 percent confidence interval:

-5.5625675 0.2292342

sample estimates:

mean in group 1 mean in group 2

13.00000

15.66667

## Comparing results

Recall null here is  $H_0 : \mu_1 - \mu_2 = 0$ . P-value 0.0668.

- ▶ 95% CI from  $-5.6$  to  $0.2$ , contains  $0$ .
- ▶ 90% CI from  $-5.0$  to  $-0.3$ , does not contain  $0$ .
- ▶ At  $\alpha = 0.05$ , would not reject  $H_0$  since P-value  $> 0.05$ .
- ▶ At  $\alpha = 0.10$ , *would* reject  $H_0$  since P-value  $< 0.10$ .



## Test and CI

Not just coincidence. Let  $C = 100(1 - \alpha)$ , so  $C\%$  gives corresponding CI to level- $\alpha$  test. Then following always true. (Symbol  $\iff$  means “if and only if”.)

Test decision		Confidence interval
Reject $H_0$ at level $\alpha$	$\iff$	$C\%$ CI does not contain $H_0$ value
Do not reject $H_0$ at level $\alpha$	$\iff$	$C\%$ CI contains $H_0$ value

Idea: “Plausible” parameter value inside CI, not rejected;  
“Implausible” parameter value outside CI, rejected.

## The value of this

- ▶ If you have a test procedure but no corresponding CI:
- ▶ you make a CI by including all the parameter values that would not be rejected by your test.
- ▶ Use:
  - ▶  $\alpha = 0.01$  for a 99% CI,
  - ▶  $\alpha = 0.05$  for a 95% CI,
  - ▶  $\alpha = 0.10$  for a 90% CI, and so on.

## Testing for non-normal data

- ▶ The IRS (“Internal Revenue Service”) is the US authority that deals with taxes (like Revenue Canada).
- ▶ One of their forms is supposed to take no more than 160 minutes to complete. A citizen’s organization claims that it takes people longer than that on average.
- ▶ Sample of 30 people; time to complete form recorded.
- ▶ Read in data, and do  $t$ -test of  $H_0 : \mu = 160$  vs.  $H_a : \mu > 160$ .
- ▶ For reading in, there is only one column, so can pretend it is delimited by anything.

## Read in data

```
my_url <- "http://ritsokiguess.site/datafiles/irs.txt"
irs <- read_csv(my_url)
irs
```

```
# A tibble: 30 x 1
```

```
  Time
```

```
<dbl>
```

1	91
2	64
3	243
4	167
5	123
6	65
7	71
8	204
9	110
10	178

```
# i 20 more rows
```

## Test whether mean is 160 or greater

```
with(irs, t.test(Time, mu = 160,  
                  alternative = "greater"))
```

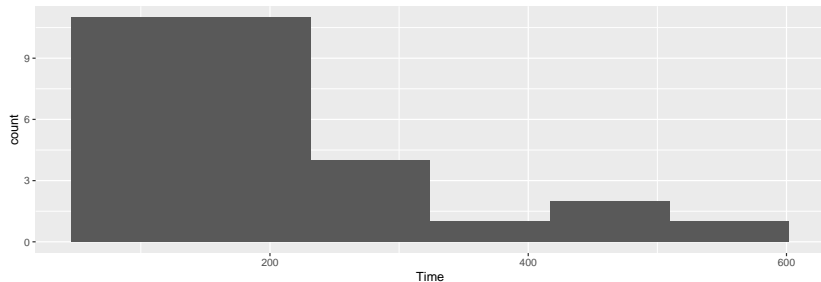
### One Sample t-test

```
data: Time  
t = 1.8244, df = 29, p-value = 0.03921  
alternative hypothesis: true mean is greater than 160  
95 percent confidence interval:  
 162.8305      Inf  
sample estimates:  
mean of x  
 201.2333
```

Reject null; mean (for all people to complete form) greater than 160.

But, look at a graph

```
ggplot(irs, aes(x = Time)) + geom_histogram(bins = 6)
```



## Comments

- ▶ Skewed to right.
- ▶ Should look at *median*, not mean.

# The sign test

- ▶ But how to test whether the median is greater than 160?
- ▶ Idea: if the median really is 160 ( $H_0$  true), the sampled values from the population are equally likely to be above or below 160.
- ▶ If the population median is greater than 160, there will be a lot of sample values greater than 160, not so many less. Idea: test statistic is number of sample values greater than hypothesized median.



## Getting a P-value for sign test 1/3

- ▶ How to decide whether “unusually many” sample values are greater than 160? Need a sampling distribution.
- ▶ If  $H_0$  true, pop. median is 160, then each sample value independently equally likely to be above or below 160.
- ▶ So number of observed values above 160 has binomial distribution with  $n = 30$  (number of data values) and  $p = 0.5$  (160 is hypothesized to be *median*).

## Getting P-value for sign test 2/3

- ▶ Count values above/below 160:

```
irs %>% count(Time > 160)
```

```
# A tibble: 2 x 2
  `Time > 160`      n
  <lgl>          <int>
1 FALSE           13
2 TRUE            17
```

- ▶ 17 above, 13 below. How unusual is that? Need a *binomial table*.

## Getting P-value for sign test 3/3

- ▶ R function `dbinom` gives the probability of eg. exactly 17 successes in a binomial with  $n = 30$  and  $p = 0.5$ :

```
dbinom(17, 30, 0.5)
```

```
[1] 0.1115351
```

- ▶ but we want probability of 17 *or more*, so get all of those, find probability of each, and add them up:

```
tibble(x=17:30) %>%  
  mutate(prob=dbinom(x, 30, 0.5)) %>%  
  summarize(total=sum(prob))
```

```
# A tibble: 1 x 1
```

```
total
```

```
<dbl>
```

```
1 0.292
```

or

```
1 - dbinom(17, 30, 0.5) # prob. of  $\leq 17$ 
```

## Using my package smmr

- ▶ I wrote a package `smmr` to do the sign test (and some other things). Installation is a bit fiddly:
  - ▶ Install `devtools` (once) with

```
install.packages("devtools")
```

- ▶ then install `smmr` using `devtools` (once):

```
library(devtools)  
install_github("nxskok/smmr")
```

- ▶ Then load it:

```
library(smmr)
```

## smmr for sign test

- ▶ smmr's function `sign_test` needs three inputs: a data frame, a column and a null median:

```
sign_test(irs, Time, 160)
```

```
$above_below
```

```
below above
```

```
13      17
```

```
$p_values
```

```
alternative    p_value
```

```
1         lower 0.8192027
```

```
2         upper 0.2923324
```

```
3    two-sided 0.5846647
```

## Comments (1/3)

- ▶ Testing whether population median *greater than* 160, so want *upper-tail* P-value 0.2923. Same as before.
- ▶ Also get table of values above and below; this too as we got.

## Comments (2/3)

- P-values are:

Test	P-value
$t$	0.0392
Sign	0.2923

- These are very different: we reject a mean of 160 (in favour of the mean being bigger), but clearly *fail* to reject a median of 160 in favour of a bigger one.
- Why is that? Obtain mean and median:

```
irs %>% summarize(mean_time = mean(Time),  
                  median_time = median(Time))
```

```
# A tibble: 1 x 2  
  mean_time median_time  
    <dbl>      <dbl>  
1    201.        172.
```

## Comments (3/3)

- ▶ The mean is pulled a long way up by the right skew, and is a fair bit bigger than 160.
- ▶ The median is quite close to 160.
- ▶ We ought to be trusting the sign test and not the t-test here (median and not mean), and therefore there is no evidence that the “typical” time to complete the form is longer than 160 minutes.
- ▶ Having said that, there are clearly some people who take a lot longer than 160 minutes to complete the form, and the IRS could focus on simplifying its form for these people.
- ▶ In this example, looking at any kind of average is not really helpful; a better question might be “do an unacceptably large fraction of people take longer than (say) 300 minutes to complete the form?”: that is, thinking about worst-case rather than average-case.



## Confidence interval for the median

- ▶ The sign test does not naturally come with a confidence interval for the median.
- ▶ So we use the “duality” between test and confidence interval to say: the (95%) confidence interval for the median contains exactly those values of the null median that would not be rejected by the two-sided sign test (at  $\alpha = 0.05$ ).

## For our data

- ▶ The procedure is to try some values for the null median and see which ones are inside and which outside our CI.
- ▶ `smmr` has `pval_sign` that gets just the 2-sided P-value:

```
pval_sign(160, irs, Time)
```

```
[1] 0.5846647
```

- ▶ Try a couple of null medians:

```
pval_sign(200, irs, Time)
```

```
[1] 0.3615946
```

```
pval_sign(300, irs, Time)
```

```
[1] 0.001430906
```

- ▶ So 200 inside the 95% CI and 300 outside.

## Doing a whole bunch

► Choose our null medians first:

```
(d <- tibble(null_median=seq(100,300,20)))
```

```
# A tibble: 11 x 1
```

```
  null_median
```

```
    <dbl>
```

1	100
2	120
3	140
4	160
5	180
6	200
7	220
8	240
9	260
10	280
11	300

... and then

“for each null median, run the function `pval_sign` for that null median and get the P-value”:

```
d %>% rowwise() %>%  
  mutate(p_value = pval_sign(null_median, irs, Time))
```

# A tibble: 11 x 2

# Rowwise:

	null_median	p_value
	<dbl>	<dbl>
1	100	0.000325
2	120	0.0987
3	140	0.200
4	160	0.585
5	180	0.856
6	200	0.362
7	220	0.0428
8	240	0.0161
9	260	0.00522

## Make it easier for ourselves

```
d %>% rowwise() %>%  
  mutate(p_value = pval_sign(null_median, irs, Time)) %>%  
  mutate(in_out = ifelse(p_value > 0.05, "inside", "outside"))
```

# A tibble: 11 x 3

# Rowwise:

	null_median	p_value	in_out
	<dbl>	<dbl>	<chr>
1	100	0.000325	outside
2	120	0.0987	inside
3	140	0.200	inside
4	160	0.585	inside
5	180	0.856	inside
6	200	0.362	inside
7	220	0.0428	outside
8	240	0.0161	outside
9	260	0.00522	outside
10	280	0.00143	outside
11	300	0.00143	outside

## confidence interval for median?

- ▶ 95% CI to this accuracy from 120 to 200.
- ▶ Can get it more accurately by looking more closely in intervals from 100 to 120, and from 200 to 220.

## A more efficient way: bisection

- ▶ Know that top end of CI between 200 and 220:

```
lo <- 200  
hi <- 220
```

- ▶ Try the value halfway between: is it inside or outside?

```
try <- (lo + hi) / 2  
try
```

```
[1] 210
```

```
pval_sign(try,irs,Time)
```

```
[1] 0.09873715
```

- ▶ Inside, so upper end is between 210 and 220. Repeat (over):

... bisection continued

```
lo <- try  
try <- (lo + hi) / 2  
try
```

```
[1] 215
```

```
pval_sign(try, irs, Time)
```

```
[1] 0.06142835
```

- ▶ 215 is inside too, so upper end between 215 and 220.
- ▶ Continue until have as accurate a result as you want.



## Bisection automatically

- ▶ A loop, but not a for since we don't know how many times we're going around. Keep going while a condition is true:

```
lo = 200
hi = 220
while (hi - lo > 1) {
    try = (hi + lo) / 2
    ptry = pval_sign(try, irs, Time)
    print(c(try, ptry))
    if (ptry <= 0.05)
        hi = try
    else
        lo = try
}
```

## The output from this loop

```
[1] 210.00000000    0.09873715  
[1] 215.00000000    0.06142835  
[1] 217.50000000    0.04277395  
[1] 216.25000000    0.04277395  
[1] 215.62500000    0.04277395
```

- 215 inside, 215.625 outside. Upper end of interval to this accuracy is 215.

## Using smmr

- ▶ smmr has function `ci_median` that does this (by default 95% CI):

```
ci_median(irs, Time)
```

```
[1] 119.0065 214.9955
```

- ▶ Uses a more accurate bisection than we did.
- ▶ Or get, say, 90% CI for median:

```
ci_median(irs, Time, conf.level=0.90)
```

```
[1] 123.0031 208.9960
```

- ▶ 90% CI is shorter, as it should be.

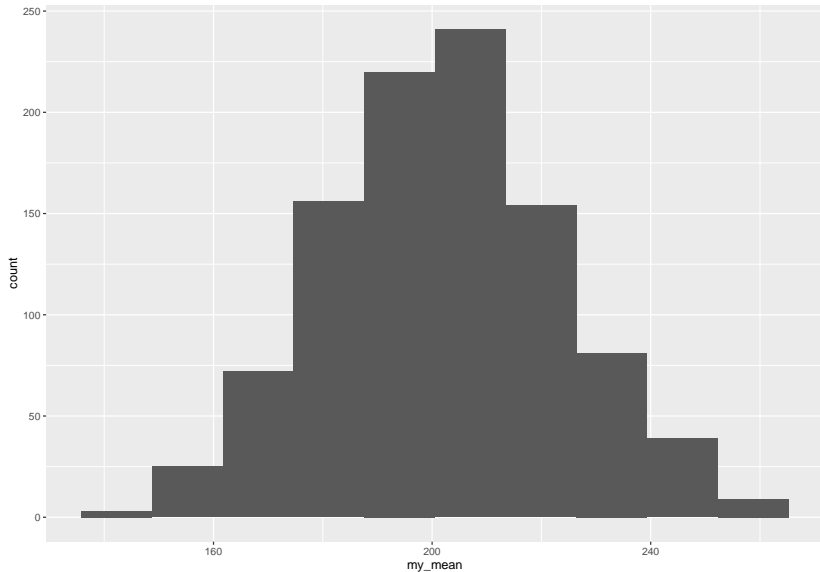
# Bootstrap

- ▶ but, was the sample size (30) big enough to overcome the skewness?
- ▶ Bootstrap, again:

```
tibble(sim = 1:1000) %>%  
  rowwise() %>%  
  mutate(my_sample = list(sample(irs$Time, replace = TRUE)))  
  mutate(my_mean = mean(my_sample)) %>%  
  ggplot(aes(x=my_mean)) + geom_histogram(bins=10) -> g
```

# The sampling distribution

g



## Comments

- ▶ A little skewed to right, but not nearly as much as I was expecting.
- ▶ The  $t$ -test for the mean might actually be OK for these data, *if the mean is what you want*.
- ▶ In actual data, mean and median very different; we chose to make inference about the median.
- ▶ Thus for us it was right to use the sign test.