Statistical Inference: Power

Packages

```r
library(tidyverse)
```

Errors in testing

What can happen:

	Decision	
Truth	**Do not reject**	**Reject null**
Null true	Correct	Type I error
Null false	Type II error	Correct

Tension between truth and decision about truth (imperfect).

▶ Prob. of type I error denoted α. Usually fix α, eg. $\alpha = 0.05$.
▶ Prob. of type II error denoted β. Determined by the planned experiment. Low β good.
▶ Prob. of not making type II error called **power** $(= 1 - \beta)$. *High* power good.

Power

▶ Suppose $H_0 : \theta = 10$, $H_a : \theta \neq 10$ for some parameter θ.

▶ Suppose H_0 wrong. What does that say about θ?

▶ Not much. Could have $\theta = 11$ or $\theta = 8$ or $\theta = 496$. In each case, H_0 wrong.

▶ How likely a type II error is depends on what θ is:

    ▶ If $\theta = 496$, should be able to reject $H_0 : \theta = 10$ even for small sample, so β should be small (power large).

    ▶ If $\theta = 11$, might have hard time rejecting H_0 even with large sample, so β would be larger (power smaller).

▶ Power depends on true parameter value, and on sample size.

▶ So we play "what if": "if θ were 11 (or 8 or 496), what would power be?".

Figuring out power

▶ Time to figure out power is before you collect any data, as part of planning process.

▶ Need to have idea of what kind of departure from null hypothesis of interest to you, eg. average improvement of 5 points on reading test scores. (Subject-matter decision, not statistical one.)

▶ Then, either:
  ▶ "I have this big a sample and this big a departure I want to detect. What is my power for detecting it?"
  ▶ "I want to detect this big a departure with this much power. How big a sample size do I need?"

How to understand/estimate power?

▶ Suppose we test $H_0 : \mu = 10$ against $H_a : \mu \neq 10$, where μ is population mean.

▶ Suppose in actual fact, $\mu = 8$, so H_0 is wrong. We want to reject it. How likely is that to happen?

▶ Need population SD (take $\sigma = 4$) and sample size (take $n = 15$). In practice, get σ from pilot/previous study, and take the n we plan to use.

▶ Idea: draw a random sample from the true distribution, test whether its mean is 10 or not.

▶ Repeat previous step "many" times.

▶ "Simulation".

Making it go

▶ Random sample of 15 normal observations with mean 8 and SD 4:

```
x <- rnorm(15, 8, 4)
x
```

```
 [1] 14.487469  5.014611  6.924277  5.201860  8.852952 10.8
 [8] 11.165242  8.016188 12.383518  1.378099  3.172503 13.0
[15]  5.015575
```

▶ Test whether x from population with mean 10 or not (over):

…continued

```
t.test(x, mu = 10)
```

```
    One Sample t-test

data:  x
t = -1.8767, df = 14, p-value = 0.08157
alternative hypothesis: true mean is not equal to 10
95 percent confidence interval:
  5.794735 10.280387
sample estimates:
mean of x
 8.037561
```

▶ Fail to reject the mean being 10 (a Type II error).

or get just P-value

```
ans <- t.test(x, mu = 10)
str(ans)
```

```
List of 10
 $ statistic  : Named num -1.88
  ..- attr(*, "names")= chr "t"
 $ parameter  : Named num 14
  ..- attr(*, "names")= chr "df"
 $ p.value    : num 0.0816
 $ conf.int   : num [1:2] 5.79 10.28
  ..- attr(*, "conf.level")= num 0.95
 $ estimate   : Named num 8.04
  ..- attr(*, "names")= chr "mean of x"
 $ null.value : Named num 10
  ..- attr(*, "names")= chr "mean"
 $ stderr     : num 1.05
 $ alternative: chr "two.sided"
 $ method     : chr "One Sample t-test"
 $ data.name  : chr "x"
```

Run this lots of times

- without a loop!
- use `rowwise` to work one random sample at a time
- draw random samples from the truth
- test that $\mu = 10$
- get P-value
- Count up how many of the P-values are 0.05 or less.

In code

```r
tibble(sim = 1:1000) %>%
  rowwise() %>%
  mutate(my_sample = list(rnorm(15, 8, 4))) %>%
  mutate(t_test = list(t.test(my_sample, mu = 10))) %>%
  mutate(p_val = t_test$p.value) %>%
  count(p_val <= 0.05)
```

```
# A tibble: 2 x 2
# Rowwise:
  `p_val <= 0.05`      n
  <lgl>            <int>
1 FALSE              578
2 TRUE               422
```

We correctly rejected 422 times out of 1000, so the estimated power is 0.422.

Try again with bigger sample

```
tibble(sim = 1:1000) %>%
  rowwise() %>%
  mutate(my_sample = list(rnorm(30, 8, 4))) %>%
  mutate(t_test = list(t.test(my_sample, mu = 10))) %>%
  mutate(p_val = t_test$p.value) %>%
  count(p_val <= 0.05)
```

```
# A tibble: 2 x 2
# Rowwise:
  `p_val <= 0.05`       n
  <lgl>             <int>
1 FALSE               252
2 TRUE                748
```

Calculating power

▶ Simulation approach very flexible: will work for any test. But answer different each time because of randomness.

▶ In some cases, for example 1-sample and 2-sample t-tests, power can be calculated.

▶ `power.t.test`. Input `delta` is difference between null and true mean:

```
power.t.test(n = 15, delta = 10-8, sd = 4, type = "one.samp
```

```
     One-sample t test power calculation

              n = 15
          delta = 2
             sd = 4
      sig.level = 0.05
          power = 0.4378466
    alternative = two.sided
```

Comparison of results

Method	Power
Simulation	0.422
`power.t.test`	0.4378

▶ Simulation power is similar to calculated power; to get more accurate value, repeat more times (eg. 10,000 instead of 1,000), which takes longer.

▶ CI for power based on simulation approx. 0.42 ± 0.03.

▶ With this small a sample size, the power is not great. With a bigger sample, the sample mean should be closer to 8 most of the time, so would reject $H_0 : \mu = 10$ more often.

Calculating required sample size

▶ Often, when planning a study, we do not have a particular sample size in mind. Rather, we want to know how big a sample to take. This can be done by asking how big a sample is needed to achieve a certain power.

▶ The simulation approach does not work naturally with this, since you have to supply a sample size.
  ▶ For that, you try different sample sizes until you get power close to what you want.

▶ For the power-calculation method, you supply a value for the power, but leave the sample size missing.

▶ Re-use the same problem: $H_0 : \mu = 10$ against 2-sided alternative, true $\mu = 8$, $\sigma = 4$, but now aim for power 0.80.

Using power.t.test

▶ No n=, replaced by a power=:

```
power.t.test(power=0.80, delta=10-8, sd=4, type="one.sample
```

```
        One-sample t test power calculation

              n = 33.3672
          delta = 2
             sd = 4
      sig.level = 0.05
          power = 0.8
    alternative = two.sided
```

one-sided test?

```
power.t.test(power=0.80, delta=10-8, sd=4, type="one.sample
```

```
        One-sample t test power calculation
```

Power curves

▶ Rather than calculating power for one sample size, or sample size for one power, might want a picture of relationship between sample size and power.

▶ Or, likewise, picture of relationship between difference between true and null-hypothesis means and power.

▶ Called power curve.

▶ Build and plot it yourself.

Building it

▶ If you feed power.t.test a collection ("vector") of values, it will do calculation for each one.

▶ Do power for variety of sample sizes, from 10 to 100 in steps of 10:

```
ns <- seq(10,100,10)
ns
```

```
 [1]   10  20  30  40  50  60  70  80  90 100
```

▶ Calculate powers:

```
ans<- power.t.test(n=ns, delta=10-8, sd=4, type="one.sample")
ans
```

```
     One-sample t test power calculation

           n = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100
       delta = 2
          sd = 4
```

Building a plot (1/2)

▶ Make a data frame out of the values to plot:

```
d <- tibble(n=ns, power=ans$power)
d
```

```
# A tibble: 10 x 2
        n power
    <dbl> <dbl>
 1     10 0.293
 2     20 0.564
 3     30 0.754
 4     40 0.869
 5     50 0.934
 6     60 0.968
 7     70 0.985
 8     80 0.993
 9     90 0.997
10    100 0.999
```
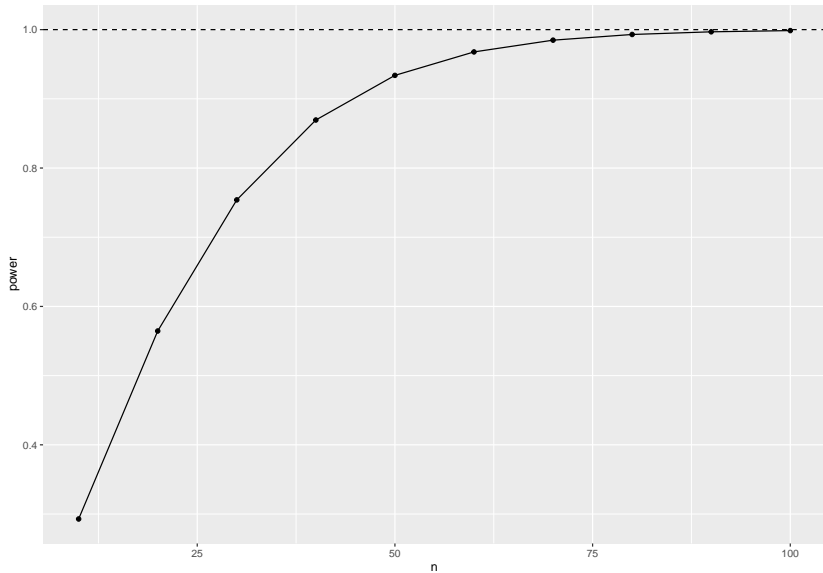
Building a plot (2/2)

▶ Plot these as points joined by lines, and add horizontal line at 1 (maximum power):

```
g  <-  ggplot(d, aes(x = n, y = power)) + geom_point() +
  geom_line() +
  geom_hline(yintercept = 1, linetype = "dashed")
```

The power curve
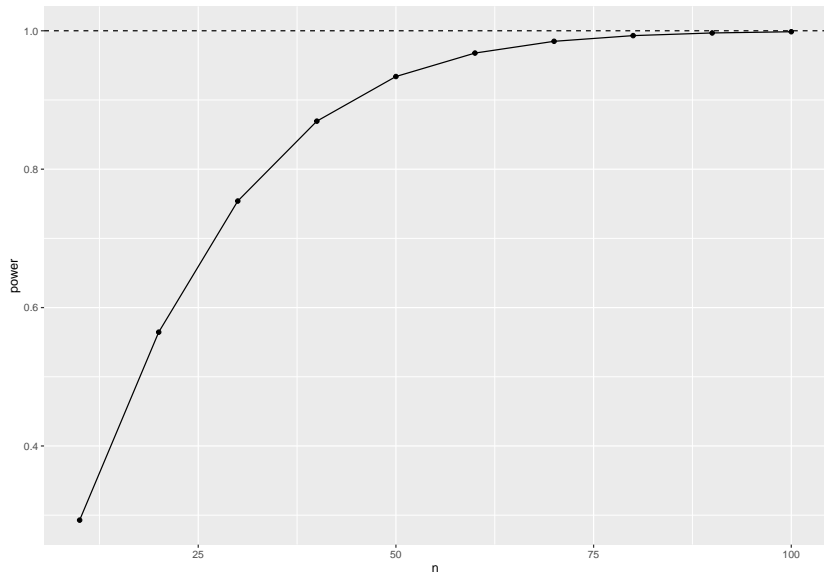
Another way to do it:

```r
tibble(n=ns) %>% rowwise() %>%
  mutate(power_output =
           list(power.t.test(n = n, delta = 10-8, sd = 4,
                             type = "one.sample"))) %>%
  mutate(power = power_output$power) %>%
  ggplot(aes(x=n, y=power)) + geom_point() + geom_line() +
    geom_hline(yintercept=1, linetype="dashed") -> g2
```

The power curve done the other way

Power curves for means

▶ Can also investigate power as it depends on what the true mean is (the farther from null mean 10, the higher the power will be).

▶ Investigate for two different sample sizes, 15 and 30.

▶ First make all combos of mean and sample size:

```
means <- seq(6,10,0.5)
means
```

```
[1]   6.0  6.5  7.0  7.5  8.0  8.5  9.0  9.5 10.0
```

```
ns <- c(15,30)
ns
```

```
[1] 15 30
```

```
combos <- crossing(mean=means, n=ns)
```

The combos

```
combos
```

```
# A tibble: 18 x 2
    mean    n
   <dbl> <dbl>
 1   6     15
 2   6     30
 3   6.5   15
 4   6.5   30
 5   7     15
 6   7     30
 7   7.5   15
 8   7.5   30
 9   8     15
10   8     30
11   8.5   15
12   8.5   30
13   9     15
14   9     30
15   9.5   15
16   9.5   30
17  10     15
18  10     30
```

Calculate and plot

▶ Calculate the powers, carefully:

```
ans <- with(combos, power.t.test(n=n, delta=10-mean, sd=4,
                                 type="one.sample"))
ans$power
```

```
 [1] 0.94908647 0.99956360 0.88277128 0.99619287 0.7707066(
 [7] 0.61513033 0.91115700 0.43784659 0.75396272 0.2721677
[13] 0.14530058 0.26245348 0.06577280 0.09719303 0.0250000(
```

Make a data frame to plot, pulling things from the right places:

```
d <- tibble(n=factor(combos$n), mean=combos$mean,
            power=ans$power)
d
```

```
# A tibble: 18 x 3
    n      mean  power
    <fct> <dbl>  <dbl>
 1 15       6    0.949
 2 30       6    1.00
 3 15       6.5  0.883
 4 30       6.5  0.996
 5 15       7    0.771
 6 30       7    0.978
 7 15       7.5  0.615
 8 30       7.5  0.911
 9 15       8    0.438
10 30       8    0.754
```
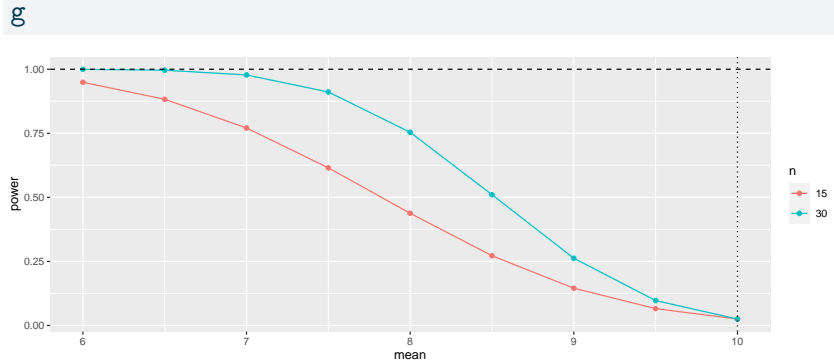
then make the plot:

```
g  <-  ggplot(d, aes(x = mean, y = power, colour = n)) +
  geom_point() + geom_line() +
  geom_hline(yintercept = 1, linetype = "dashed") +
  geom_vline(xintercept = 10, linetype = "dotted")
```

The power curves

Comments

▶ When mean=10, that is, the true mean equals the null mean, H_0 is actually true, and the probability of rejecting it then is $\alpha = 0.05$.

▶ As the null gets more wrong (mean decreases), it becomes easier to correctly reject it.

▶ The blue power curve is above the red one for any mean < 10, meaning that no matter how wrong H_0 is, you always have a greater chance of correctly rejecting it with a larger sample size.

▶ Previously, we had $H_0 : \mu = 10$ and a true $\mu = 8$, so a mean of 8 produces power 0.42 and 0.80 as shown on the graph.

▶ With $n = 30$, a true mean that is less than about 7 is almost certain to be correctly rejected. (With $n = 15$, the true mean needs to be less than 6.)

Two-sample power

▶ For kids learning to read, had sample sizes of 22 (approx) in each group
▶ and these group SDs:

```
kids %>% group_by(group) %>%
  summarize(n=n(), s=sd(score))
```

```
# A tibble: 2 x 3
  group     n     s
  <chr> <int> <dbl>
1 c        23  17.1
2 t        21  11.0
```

Setting up

▶ suppose a 5-point improvement in reading score was considered important (on this scale)
▶ in a 2-sample test, null (difference of) mean is zero, so `delta` is true difference in means
▶ what is power for these sample sizes, and what sample size would be needed to get power up to 0.80?
▶ SD in both groups has to be same in `power.t.test`, so take as 14.

Calculating power for sample size 22 (per group)

```
power.t.test(n=22, delta=5, sd=14, type="two.sample",
             alternative="one.sided")
```

```
     Two-sample t test power calculation

              n = 22
          delta = 5
             sd = 14
      sig.level = 0.05
          power = 0.3158199
    alternative = one.sided

NOTE: n is number in *each* group
```

sample size for power 0.8

```
power.t.test(power=0.80, delta=5, sd=14, type="two.sample",
             alternative="one.sided")
```

```
        Two-sample t test power calculation

               n = 97.62598
           delta = 5
              sd = 14
       sig.level = 0.05
           power = 0.8
     alternative = one.sided

NOTE: n is number in *each* group
```

Comments

▶ The power for the sample sizes we have is very small (to detect a 5-point increase).
▶ To get power 0.80, we need 98 kids in *each* group!