

## Statistical Inference: Power

# Packages

```
library(tidyverse)
```

# Errors in testing

What can happen:

Decision		
Truth	Do not reject	Reject null
Null true	Correct	Type I error
Null false	Type II error	Correct

Truth	Decision	
	Do not reject	Reject null
Null true	Correct	Type I error
Null false	Type II error	Correct

Tension between truth and decision about truth (imperfect).

- ▶ Prob. of type I error denoted  $\alpha$ . Usually fix  $\alpha$ , eg.  $\alpha = 0.05$ .
- ▶ Prob. of type II error denoted  $\beta$ . Determined by the planned experiment. Low  $\beta$  good.
- ▶ Prob. of not making type II error called **power** ( $= 1 - \beta$ ).  
*High power good.*

# Power

- ▶ Suppose  $H_0 : \theta = 10$ ,  $H_a : \theta \neq 10$  for some parameter  $\theta$ .
- ▶ Suppose  $H_0$  wrong. What does that say about  $\theta$ ?
- ▶ Not much. Could have  $\theta = 11$  or  $\theta = 8$  or  $\theta = 496$ . In each case,  $H_0$  wrong.
- ▶ How likely a type II error is depends on what  $\theta$  is:
  - ▶ If  $\theta = 496$ , should be able to reject  $H_0 : \theta = 10$  even for small sample, so  $\beta$  should be small (power large).
  - ▶ If  $\theta = 11$ , might have hard time rejecting  $H_0$  even with large sample, so  $\beta$  would be larger (power smaller).
- ▶ Power depends on true parameter value, and on sample size.
- ▶ So we play “what if”: “if  $\theta$  were 11 (or 8 or 496), what would power be?”.

## Figuring out power

- ▶ Time to figure out power is before you collect any data, as part of planning process.
- ▶ Need to have idea of what kind of departure from null hypothesis of interest to you, eg. average improvement of 5 points on reading test scores. (Subject-matter decision, not statistical one.)
- ▶ Then, either:
  - ▶ “I have this big a sample and this big a departure I want to detect. What is my power for detecting it?”
  - ▶ “I want to detect this big a departure with this much power. How big a sample size do I need?”

## How to understand/estimate power?

- ▶ Suppose we test  $H_0 : \mu = 10$  against  $H_a : \mu \neq 10$ , where  $\mu$  is population mean.
- ▶ Suppose in actual fact,  $\mu = 8$ , so  $H_0$  is wrong. We want to reject it. How likely is that to happen?
- ▶ Need population SD (take  $\sigma = 4$ ) and sample size (take  $n = 15$ ). In practice, get  $\sigma$  from pilot/previous study, and take the  $n$  we plan to use.
- ▶ Idea: draw a random sample from the true distribution, test whether its mean is 10 or not.
- ▶ Repeat previous step “many” times.
- ▶ “Simulation”.

# Making it go

- ▶ Random sample of 15 normal observations with mean 8 and SD 4:

```
x <- rnorm(15, 8, 4)
x
```

```
[1] 14.487469  5.014611  6.924277  5.201860  8.852952 10.8
[8] 11.165242  8.016188 12.383518  1.378099  3.172503 13.0
[15]  5.015575
```

- ▶ Test whether  $x$  from population with mean 10 or not (over):

...continued

```
t.test(x, mu = 10)
```

One Sample t-test

data: x

t = -1.8767, df = 14, p-value = 0.08157

alternative hypothesis: true mean is not equal to 10

95 percent confidence interval:

5.794735 10.280387

sample estimates:

mean of x

8.037561

► Fail to reject the mean being 10 (a Type II error).



or get just P-value

```
t.test(x, mu = 10)$p.value
```

```
[1] 0.0815652
```

## Run this lots of times

- ▶ without a loop!
- ▶ use `rowwise` to work one random sample at a time
- ▶ draw random samples from the truth
- ▶ test that  $\mu = 10$
- ▶ get P-value
- ▶ Count up how many of the P-values are 0.05 or less.

## In code

```
tibble(sim = 1:1000) %>%  
  rowwise() %>%  
  mutate(my_sample = list(rnorm(15, 8, 4))) %>%  
  mutate(t_test = list(t.test(my_sample, mu = 10))) %>%  
  mutate(p_val = t_test$p.value) %>%  
  count(p_val <= 0.05)
```

# A tibble: 2 x 2

# Rowwise:

	`p_val <= 0.05`	n
	<lgl>	<int>
1	FALSE	578
2	TRUE	422

We correctly rejected 422 times out of 1000, so the estimated power is 0.422.

## Calculating power

- ▶ Simulation approach very flexible: will work for any test. But answer different each time because of randomness.
- ▶ In some cases, for example 1-sample and 2-sample t-tests, power can be calculated.
- ▶ `power.t.test`. Input delta is difference between null and true mean:

```
power.t.test(n = 15, delta = 10-8, sd = 4, type = "one.sample")
```

One-sample t test power calculation

```
      n = 15
  delta = 2
     sd = 4
sig.level = 0.05
  power = 0.4378466
alternative = two.sided
```

## Comparison of results

Method	Power
Simulation	0.422
power.t.test	0.4378

- ▶ Simulation power is similar to calculated power; to get more accurate value, repeat more times (eg. 10,000 instead of 1,000), which takes longer.
- ▶ CI for power based on simulation approx.  $0.42 \pm 0.03$ .
- ▶ With this small a sample size, the power is not great. With a bigger sample, the sample mean should be closer to 8 most of the time, so would reject  $H_0 : \mu = 10$  more often.

## Calculating required sample size

- ▶ Often, when planning a study, we do not have a particular sample size in mind. Rather, we want to know how big a sample to take. This can be done by asking how big a sample is needed to achieve a certain power.
- ▶ The simulation approach does not work naturally with this, since you have to supply a sample size.
- ▶ For the power-calculation method, you supply a value for the power, but leave the sample size missing.
- ▶ Re-use the same problem:  $H_0 : \mu = 10$  against 2-sided alternative, true  $\mu = 8$ ,  $\sigma = 4$ , but now aim for power 0.80.

## Using power.t.test

- ▶ No n=, replaced by a power=:

```
power.t.test(power=0.80, delta=10-8, sd=4, type="one.sample")
```

One-sample t test power calculation

```
      n = 33.3672
delta = 2
    sd = 4
sig.level = 0.05
  power = 0.8
alternative = two.sided
```

- ▶ Sample size must be a whole number, so round up to 34 (to get at least as much power as you want).

## Power curves

- ▶ Rather than calculating power for one sample size, or sample size for one power, might want a picture of relationship between sample size and power.
- ▶ Or, likewise, picture of relationship between difference between true and null-hypothesis means and power.
- ▶ Called power curve.
- ▶ Build and plot it yourself.



## Building it

- ▶ If you feed `power.t.test` a collection (“vector”) of values, it will do calculation for each one.
- ▶ Do power for variety of sample sizes, from 10 to 100 in steps of 10:

```
ns <- seq(10,100,10)
ns
```

```
[1] 10 20 30 40 50 60 70 80 90 100
```

- ▶ Calculate powers:

```
ans <- power.t.test(n=ns, delta=10-8, sd=4, type="one.sample")
ans$power
```

```
[1] 0.2928286 0.5644829 0.7539627 0.8693979 0.9338976 0.9677886
[8] 0.9929987 0.9968496 0.9986097
```

## Building a plot (1/2)

- Make a data frame out of the values to plot:

```
d <- tibble(n=ns, power=ans$power)
d
```

```
# A tibble: 10 x 2
```

	n	power
	<dbl>	<dbl>
1	10	0.293
2	20	0.564
3	30	0.754
4	40	0.869
5	50	0.934
6	60	0.968
7	70	0.985
8	80	0.993
9	90	0.997
10	100	0.999

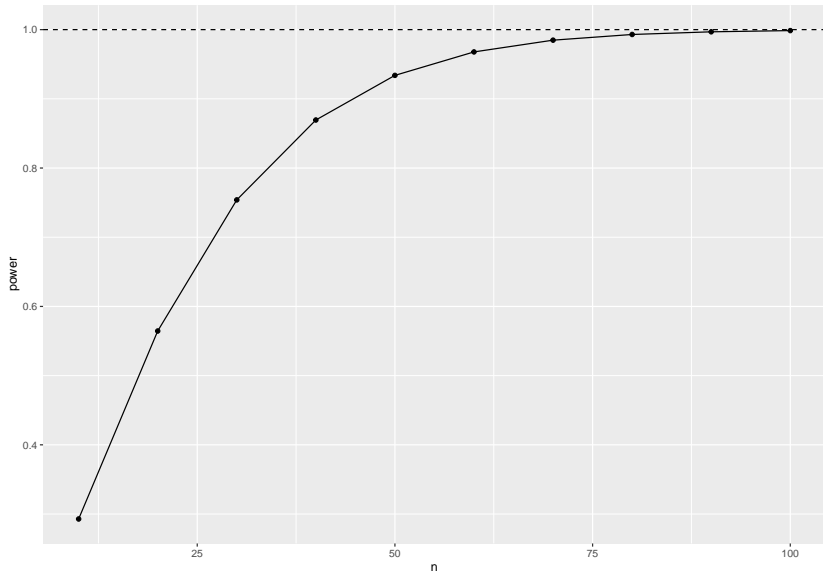
## Building a plot (2/2)

- Plot these as points joined by lines, and add horizontal line at 1 (maximum power):

```
g <- ggplot(d, aes(x = n, y = power)) + geom_point() +  
  geom_line() +  
  geom_hline(yintercept = 1, linetype = "dashed")
```

# The power curve

g

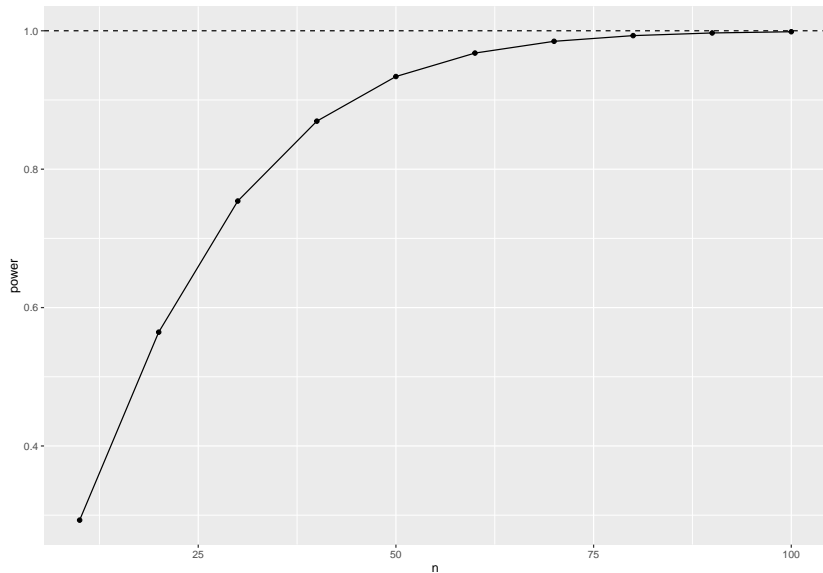


## Another way to do it:

```
tibble(n=ns) %>% rowwise() %>%  
  mutate(power_output =  
    list(power.t.test(n = n, delta = 10-8, sd = 4,  
                      type = "one.sample")))) %>%  
  mutate(power = power_output$power) %>%  
  ggplot(aes(x=n, y=power)) + geom_point() + geom_line() +  
    geom_hline(yintercept=1, linetype="dashed") -> g2
```

# The power curve done the other way

$g^2$



## Power curves for means

- ▶ Can also investigate power as it depends on what the true mean is (the farther from null mean 10, the higher the power will be).
- ▶ Investigate for two different sample sizes, 15 and 30.
- ▶ First make all combos of mean and sample size:

```
means <- seq(6,10,0.5)
```

```
means
```

```
[1] 6.0 6.5 7.0 7.5 8.0 8.5 9.0 9.5 10.0
```

```
ns <- c(15,30)
```

```
ns
```

```
[1] 15 30
```

```
combos <- crossing(mean=means, n=ns)
```

# The combos

```
combos
```

```
# A tibble: 18 x 2
```

	mean	n
	<dbl>	<dbl>
1	6	15
2	6	30
3	6.5	15
4	6.5	30
5	7	15
6	7	30
7	7.5	15
8	7.5	30
9	8	15
10	8	30
11	8.5	15
12	8.5	30
13	9	15
14	9	30
15	9.5	15
16	9.5	30
17	10	15
18	10	30



## Calculate and plot

► Calculate the powers, carefully:

```
ans <- with(combos, power.t.test(n=n, delta=10-mean, sd=4,  
                                type="one.sample"))
```

```
ans$power
```

```
[1] 0.94908647 0.99956360 0.88277128 0.99619287 0.77070660  
[7] 0.61513033 0.91115700 0.43784659 0.75396272 0.27216777  
[13] 0.14530058 0.26245348 0.06577280 0.09719303 0.02500000
```

Make a data frame to plot, pulling things from the right places:

```
d <- tibble(n=factor(combos$n), mean=combos$mean,  
            power=ans$power)
```

d

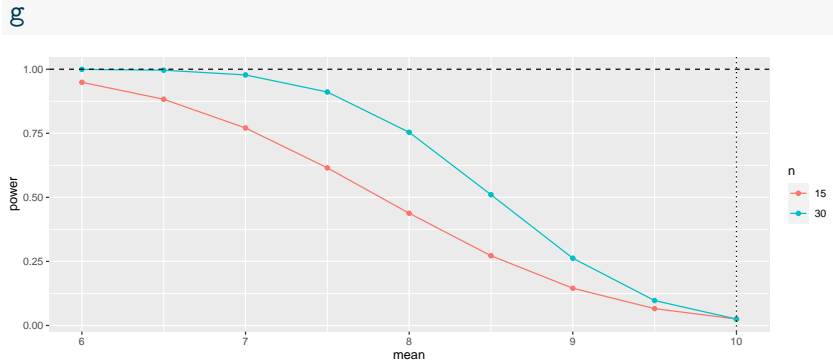
# A tibble: 18 x 3

	n	mean	power
	<fct>	<dbl>	<dbl>
1	15	6	0.949
2	30	6	1.00
3	15	6.5	0.883
4	30	6.5	0.996
5	15	7	0.771
6	30	7	0.978
7	15	7.5	0.615
8	30	7.5	0.911
9	15	8	0.438
10	30	8	0.754

then make the plot:

```
g <- ggplot(d, aes(x = mean, y = power, colour = n)) +  
  geom_point() + geom_line() +  
  geom_hline(yintercept = 1, linetype = "dashed") +  
  geom_vline(xintercept = 10, linetype = "dotted")
```

# The power curves



## Comments

- ▶ When mean=10, that is, the true mean equals the null mean,  $H_0$  is actually true, and the probability of rejecting it then is  $\alpha = 0.05$ .
- ▶ As the null gets more wrong (mean decreases), it becomes easier to correctly reject it.
- ▶ The blue power curve is above the red one for any mean  $< 10$ , meaning that no matter how wrong  $H_0$  is, you always have a greater chance of correctly rejecting it with a larger sample size.
- ▶ Previously, we had  $H_0 : \mu = 10$  and a true  $\mu = 8$ , so a mean of 8 produces power 0.42 and 0.80 as shown on the graph.
- ▶ With  $n = 30$ , a true mean that is less than about 7 is almost certain to be correctly rejected. (With  $n = 15$ , the true mean needs to be less than 6.)

## Two-sample power

- ▶ For kids learning to read, had sample sizes of 22 (approx) in each group
- ▶ and these group SDs:

```
kids %>% group_by(group) %>%  
  summarize(n=n(), s=sd(score))
```

```
# A tibble: 2 x 3  
  group      n      s  
  <chr> <int> <dbl>  
1 c      23  17.1  
2 t      21  11.0
```

## Setting up

- ▶ suppose a 5-point improvement in reading score was considered important (on this scale)
- ▶ in a 2-sample test, null (difference of) mean is zero, so  $\delta$  is true difference in means
- ▶ what is power for these sample sizes, and what sample size would be needed to get power up to 0.80?
- ▶ SD in both groups has to be same in power.t.test, so take as 14.

## Calculating power for sample size 22 (per group)

```
power.t.test(n=22, delta=5, sd=14, type="two.sample",  
             alternative="one.sided")
```

Two-sample t test power calculation

```
      n = 22  
    delta = 5  
      sd = 14  
sig.level = 0.05  
    power = 0.3158199  
alternative = one.sided
```

NOTE: n is number in *each* group



## sample size for power 0.8

```
power.t.test(power=0.80, delta=5, sd=14, type="two.sample",  
             alternative="one.sided")
```

Two-sample t test power calculation

```
      n = 97.62598  
delta = 5  
  sd = 14  
sig.level = 0.05  
  power = 0.8  
alternative = one.sided
```

NOTE: n is number in *each* group

## Comments

- ▶ The power for the sample sizes we have is very small (to detect a 5-point increase).
- ▶ To get power 0.80, we need 98 kids in *each* group!