MONASH University

ETC3250

Business Analytics

3. Regression

10 August 2015

Outline

Revision: Multiple regression

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \cdots + \beta_p X_{p,i} + e_i.$$

- Each $X_{j,i}$ is numerical and is called a "predictor".
- The coefficients $\beta_1, \dots, \beta_p$ measure the effect of each predictor after taking account of the effect of all other predictors in the model.
- Predictors may be transforms of other predictors. e.g., $X_2 = X_1^2$.
- The model describes a line, plane or hyperplane in the predictors.

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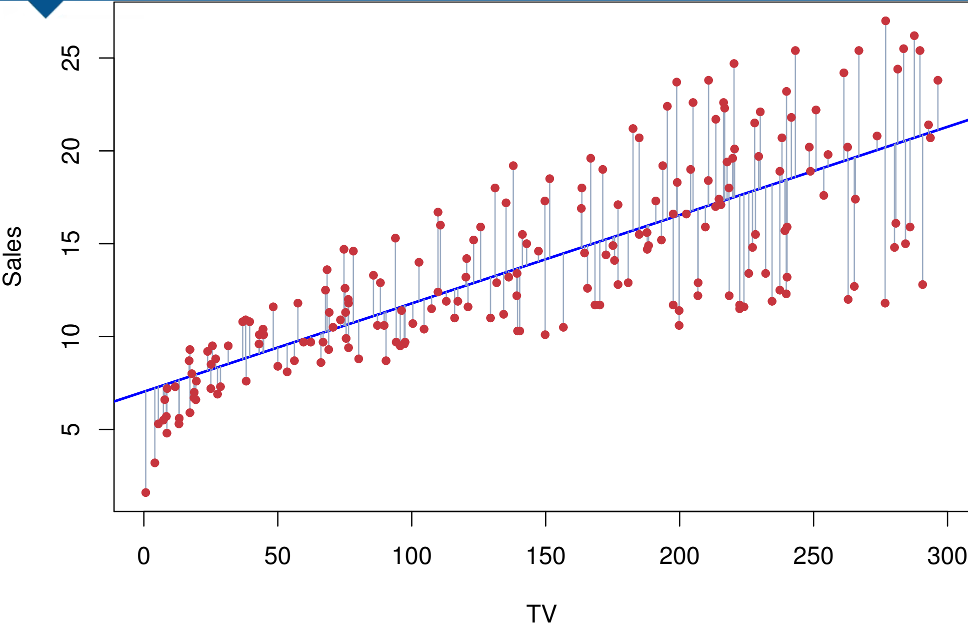

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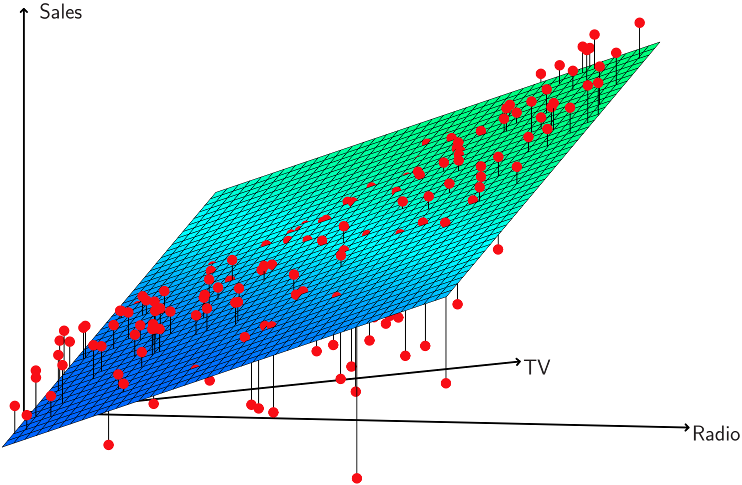

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- R^2
- se of coefficients
- residual standard error
- F statistic

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Important questions

1. Is at least one of the predictors useful in predicting the response?
2. Do all the predictors help to explain Y, or is only a subset of the predictors useful?
3. How well does the model fit the data?
4. Given a set of predictor values, what response value should we predict and how accurate is our prediction?

Credit scores

Banks score loan customers based on a lot of personal information. A sample of 500 customers from an Australian bank provided the following information.

Score	Savings	Income	Time current address	Time current job
	$'000	$'000	Months	Months
39.40	0.01	111.17	27	8
51.79	0.65	56.40	29	33
32.82	0.75	36.74	2	16
57.31	0.62	55.99	14	7
37.17	4.13	62.04	2	14
33.69	0.00	43.75	7	7
25.56	0.94	79.01	4	11
32.04	0.00	45.41	3	3
41.34	4.26	55.22	16	18
⋮	⋮	⋮	⋮	⋮

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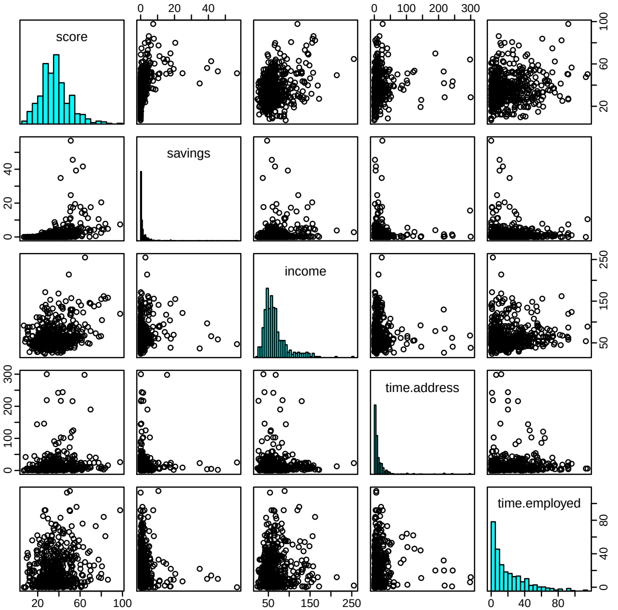
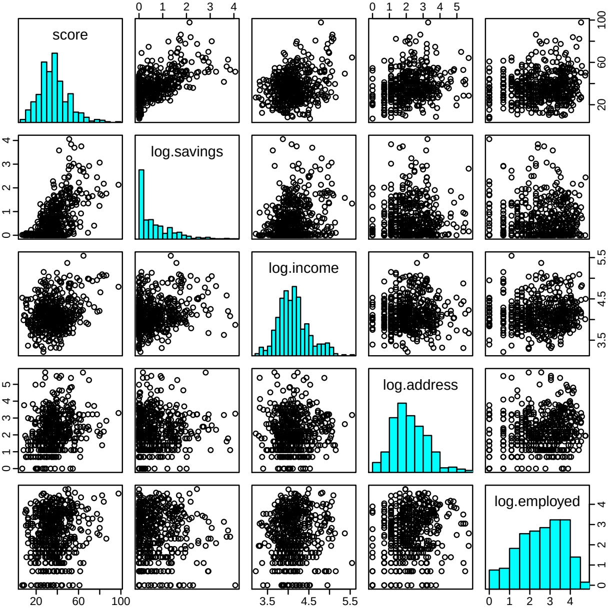
Can we use only savings, income, time @ address and time employed to predict the score of a new customer?

Credit scores



Credit scores

- Taking logarithms reduces the skewness in the predictor variables.
- Because of zeros, I used $\log(x + 1)$.



Credit scores

Proposed model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + e$$

$$\begin{aligned} \text{where}\quad Y &= \text{Credit score,} \\ X_1 &= \text{log savings,} \\ X_2 &= \text{log income,} \\ X_3 &= \text{log time at current address,} \\ X_4 &= \text{log time in current job,} \\ e &= \text{error.} \end{aligned}$$

Credit scores

```
lm(formula = score ~ log.savings + log.income + log.address +
    log.employed, data = creditlog)

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)    -0.219      5.231   -0.04   0.9667
log.savings    10.353      0.612   16.90  < 2e-16
log.income      5.052      1.258    4.02  6.8e-05
log.address     2.667      0.434    6.14  1.7e-09
log.employed    1.314      0.409    3.21   0.0014

Residual standard error: 10.2 on 495 degrees of freedom
Multiple R-squared: 0.47, Adjusted R-squared: 0.466
F-statistic:  110 on 4 and 495 DF,  p-value: <2e-16

    CV  AIC AICc  BIC   AdjR2
 104.7 2326 2326 2351 0.46582
```

Interpreting regression coefficients

- The ideal scenario is when the predictors are uncorrelated.
  - Each coefficient can be interpreted and tested separately.
- Correlations amongst predictors cause problems.
  - The variance of all coefficients tends to increase, sometimes dramatically.
  - Interpretations become hazardous – when X_j changes, everything else changes.
  - Predictions still work provided new X values are within the range of training X values.
- Claims of causality should be avoided for observational data.

Interactions

- An interaction occurs when the one variable changes the effect of a second variable. (e.g., spending on radio advertising increases the effectiveness of TV advertising).
- To model an interaction, include the product $X_1 X_2$ in the model in addition to X_1 and X_2.
- **Hierarchy principle**: If we include an interaction in a model, we should also include the main effects, even if the p-values associated with their coefficients are not significant. (This is because the interactions are almost impossible to interpret without the main effects.)

Residual patterns

- If a plot of the residuals vs any predictor in the model shows a pattern, then the relationship is nonlinear.
- If a plot of the residuals vs any predictor **not** in the model shows a pattern, then the predictor should be added to the model.
- If a plot of the residuals vs fitted values shows a pattern, then there is heteroscedasticity in the errors. (Could try a transformation.)

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Matrix formulation

$$Y_i = \beta_0 + \beta_1 X_{1,i} + \beta_2 X_{2,i} + \cdots + \beta_p X_{p,i} + e_i.$$

Let $\boldsymbol{Y} = (Y_1, \dots, Y_n)'$, $\boldsymbol{e} = (e_1, \dots, e_n)'$, $\boldsymbol{\beta} = (\beta_0, \dots, \beta_p)'$ and

$$\boldsymbol{X} = \begin{bmatrix} 1 & X_{1,1} & X_{2,1} & \dots & X_{p,1} \\ 1 & X_{1,2} & X_{2,2} & \dots & X_{p,2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & X_{1,n} & X_{2,n} & \dots & X_{p,n} \end{bmatrix}.$$

Then

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}.$$

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Matrix formulation

Least squares estimation

Minimize: $(\boldsymbol{Y} - \boldsymbol{X}\beta)'(\boldsymbol{Y} - \boldsymbol{X}\beta)$

Differentiate wrt β gives

$$\hat{\beta} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$

(The “normal equation”.)

$$\hat{\sigma}^2 = \frac{1}{n-k-1}(\boldsymbol{Y} - \boldsymbol{X}\hat{\beta})'(\boldsymbol{Y} - \boldsymbol{X}\hat{\beta})$$

Note: If you fall for the dummy variable trap, $(\boldsymbol{X}'\boldsymbol{X})$ is a singular matrix.

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Likelihood

If the errors are iid and normally distributed, then

$$\boldsymbol{Y} \sim \mathrm{N}(\boldsymbol{X}\beta, \sigma^2\boldsymbol{I}).$$

So the likelihood is

$$L = \frac{1}{\sigma^n (2\pi)^{n/2}} \exp\left(-\frac{1}{2\sigma^2}(\boldsymbol{Y}-\boldsymbol{X}\beta)'(\boldsymbol{Y}-\boldsymbol{X}\beta)\right)$$

which is maximized when $(\boldsymbol{Y}-\boldsymbol{X}\beta)'(\boldsymbol{Y}-\boldsymbol{X}\beta)$ is minimized.

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Optimal predictions

$$\hat{Y}^* = \mathrm{E}(Y^*|\boldsymbol{Y},\boldsymbol{X},\boldsymbol{X}^*) = \boldsymbol{X}^*\hat{\beta} = \boldsymbol{X}^*(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$

where $\boldsymbol{X}^*$ is a row vector containing the values of the regressors for the predictions (in the same format as $\boldsymbol{X}$).

Prediction variance

$$\mathrm{Var}(Y^*|\boldsymbol{Y},\boldsymbol{X},\boldsymbol{X}^*) = \sigma^2\left[1 + \boldsymbol{X}^*(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}^*)'\right]$$

- This ignores any errors in $\boldsymbol{X}^*$.
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