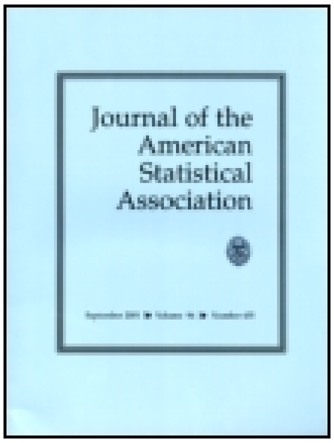

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A Test for Heteroscedasticity Based on Ordinary Least Squares Residuals

M. J. Harrison [a] & B. P. M. McCabe [b]
[a] Department of Economics , University of Dublin, Trinity College , Dublin , Ireland
[b] School of Economic Studies , University of Leeds , Leeds , England
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A Test for Heteroscedasticity Based on Ordinary Least Squares Residuals

M. J. HARRISON and B. P. M. McCABE*

Two variants of a test for heteroscedasticity based on ordinary least squares residuals are proposed: a bounds test and an "exact" test. Both variants are parametric and make use of tables of the F distribution. Under a variety of circumstances, the power of the bounds test compares favorably with the powers of the tests of Goldfeld and Quandt, Theil, and Harvey and Phillips. The computational simplicity of the bounds test compared with other tests makes it a particularly attractive practical procedure. The power of the "exact" test generally exceeds that of the other tests, but the "exact" test is rather difficult to apply in practice.

KEY WORDS: Heteroscedasticity; Least squares residuals; Bounds test; F distribution.

1. INTRODUCTION

The implications of heteroscedasticity for ordinary least squares estimation of linear regression models are well known. The importance of being able to detect heteroscedasticity is similarly widely recognized, and during recent years several test procedures have been developed. In the empirical econometrics literature, however, and in contrast to the concern with serial correlation, the results of tests for heteroscedasticity are rarely reported. That most empirical studies are based on time-series data is not a satisfactory explanation for this lack of reporting, for, as Harrison and McCabe (1975) and Epps and Epps (1977) pointed out, heteroscedasticity may often be a feature of time-series as well as cross-section models.

One plausible explanation may be that the omission results from the relatively heavy computational burden involved in applying the tests. For example, the test suggested by Goldfeld and Quandt (1965) requires the calculation of ordinary least squares residuals from separate regressions on two mutually exclusive subsets of the observations; the approach suggested by Theil (1971, pp. 214–218) requires the derivation of BLUS residuals from the least squares residuals; and the test suggested by Harvey and Phillips (1974) requires the calculation of a set of recursive residuals. Other procedures, such as those of Rutemiller and Bowers (1968), Glejser (1969), and Hedayat and Robson (1970), similarly require considerable computation to be carried out before their test statistics can be formed.

The purpose of this article is to propose a test for heteroscedasticity based on the direct use of the ordinary least squares residuals from a single regression on the complete set of observations. The test, which makes use of tables of the F distribution, has two variants: a simple bounds test and an "exact" test for use when the bounds test is inconclusive. Szroeter (1978) proposed a bounds test similar to that developed here. His test, however, makes use of the tables of the Durbin-Watson statistic for its critical values, which, unlike the tables used for the present test, are based on approximations to the true bounding distributions. His test is also more complicated to apply in practice than the present test. Furthermore, Szroeter considered neither the problem of inconclusiveness nor the power of his suggested test.

The theoretical details of the present test are given in Section 2. Section 3 is concerned with the power of the test and how it compares with the Goldfeld-Quandt, Theil, and Harvey-Phillips tests. Section 4 contains some concluding remarks.

2. DERIVATION OF THE TEST

Consider the general linear model

$$y = X\beta + u \ , \tag{2.1}$$

where y is an $n \times 1$ vector of observations on the dependent variable, X is an $n \times k$ matrix of observations on the k independent variables, β is a $k \times 1$ vector of parameters, and u is an $n \times 1$ vector of stochastic disturbances. One assumes that the rank of X is k and that the disturbances are distributed normally, independently of the explanatory variables and of one another, with zero mean and variance-covariance matrix $\sigma^2\Omega$, where σ^2 is a constant and Ω is a positive definite diagonal matrix of order n. Under the null hypothesis H_0 that the disturbances are homoscedastic, $\Omega = I_n$, the unit matrix of order n, and ordinary least squares yields the best linear unbiased estimate of β. Under the alternate hypothesis H_A that the disturbances are heteroscedastic, $\Omega \neq I_n$, and generalized least squares is necessary to obtain the best estimate of β. Therefore, testing H_0 against H_A is required.

* M.J. Harrison is Lecturer in Economics in the Department of Economics, University of Dublin, Trinity College, Dublin, Ireland. B.P.M. McCabe is Lecturer in Economic Statistics in the School of Economic Studies, University of Leeds, Leeds, England. The authors are grateful to Dr. R.C. Geary, Economic and Social Research Institute, Dublin; D. Bond, Planning and Research Branch, Department of Manpower Services, Belfast; the Theory and Methods Editor; and a referee for helpful comments on earlier drafts of this article.

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Goldfeld and Quandt (1972, Ch. 3) distinguished between constructive and nonconstructive testing procedures for this kind of situation. Constructive procedures, such as those of Rutemiller and Bowers (1968) and Glejser (1969), are designed to test for and estimate specific forms of heteroscedasticity. Consequently, if their use suggests rejection of H_0 and acceptance of H_A, an estimate of Ω is directly available for generalized least squares estimation of β. Nonconstructive procedures, such as those of Goldfeld and Quandt (1965), Theil (1971), and Harvey and Phillips (1974), are designed to establish the presence or absence of heteroscedasticity without regard to subsequent estimation. Indeed, if H_0 is rejected on the basis of a nonconstructive test and H_A contains only vague information about the nature of the heteroscedasticity—for example, that the disturbance variance is some increasing function of an independent variable—then efficient estimation of β is not possible. Only if H_A specifies the form of heteroscedasticity, hence Ω, precisely can generalized least squares estimation of β follow directly on the application of a nonconstructive test.

The test for heteroscedasticity proposed in this section is a nonconstructive test. The test is developed, as the test of Goldfeld and Quandt was originally, for the case of an alternative hypothesis specifying the precise nature of the heteroscedasticity. This approach allows the test procedure to be viewed as a means of ascertaining which of two models—an ordinary least squares or a specific generalized least squares interpretation of (2.1)—is appropriate. Like the Goldfeld-Quandt test and other nonconstructive procedures, however, the test may also be used in cases in which H_A contains less specific information about the nature of the heteroscedasticity.

The principal feature of the proposed test is that it is based on the direct use of the ordinary least squares residuals from a single regression on the complete sample of observations. Therefore, let the ordinary least squares regression equation fitted to the data be

$$y = X\hat{\beta} + e \ , \tag{2.2}$$

where $\hat{\beta}$ is the $k \times 1$ vector of parameter estimates, and e is the $n \times 1$ vector of regression residuals.

Like other nonconstructive tests, the present test uses a statistic given by a certain ratio of sums of squares of residuals. More specifically, this test is based on the ratio of the sum of squares of a given subset of the least squares residuals to the total residual sum of squares. Symbolically, let this ratio of quadratic forms be

$$b = \frac{e'Ae}{e'e} \ , \tag{2.3}$$

where A is an appropriate selector matrix of order $n \times n$, with $m(0 < m < n)$ ones and $n - m$ zeros on its principal diagonal and zeros elsewhere.

The choice of value for m, like the choice of the number of omitted observations in the Goldfeld-Quandt test, is an important practical consideration. This matter is examined, in relation to several specific alternative hypotheses, in Section 3 and commented on again in Section 4. For the moment to assume that m is approximately $n/2$ is sufficient.

The choice of the positions of the ones and zeros along the diagonal of A is determined in relation to the form of heteroscedasticity postulated in H_A. For example, if H_A postulates that the disturbance variance changes monotonically with time, then the positions of the ones and zeros could be such that A selects the first m of the chronologically ordered residuals; if H_A postulates that the variance is some increasing function of an independent variable, then the positions could be such that A selects the residuals corresponding to the smallest m values of that independent variable with which the variance is thought to be associated. Without loss of generality, A may be considered to have the form

$$A = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} , \tag{2.4}$$

where the least squares residuals have been ordered in an appropriate manner according to H_A.

The formulation of a test criterion for use with the statistic b and a given H_A requires a knowledge of the distribution of b. Unfortunately, both the numerator and denominator of b depend explicitly on the matrix X. Standard theoretical results exist, however, by which the numerator and denominator of b may be simultaneously reduced to their canonical forms.

Indeed, the form of the denominator of b in (2.3) was chosen with this reduction process in mind. More specifically, when the statistic is cast in the form of a ratio of quadratic forms in the true disturbances (see (2.5)), the matrices in the numerator and denominator must commute in order to reduce both quadratic forms simultaneously by a single orthogonal transformation (see Rao 1973, pp. 41–42). The statistic can then be shown to be bounded by two random variables that are independent of X. Furthermore, the distributions of these bounding variables are ascertainable. Therefore, the bounding distributions may be used to construct a test criterion. The derivation is similar to that used by Durbin and Watson (1950); most of the theoretical details are also expounded in Anderson (1971).

2.1 Bounds Test

Because $e = My = Mu$, where $M = I_n - X(X'X)^{-1}X'$ is idempotent, (2.3) may be rewritten as

$$b = \frac{u'MAMu}{u'Mu} \tag{2.5}$$

and by an appropriate orthogonal transformation as

$$b = \sum_{i=1}^{n-k} \lambda_i \xi_i^2 \Big/ \sum_{i=1}^{n-k} \xi_i^2 \ , \tag{2.6}$$

where the λ_i, $i = 1, 2, 3, \ldots, n-k$, are the eigenvalues of MAM arranged in descending order of magnitude, and the ξ_i, $i = 1, 2, 3, \ldots, n-k$, are, under the null hypothesis, independently distributed normal variates with zero mean and constant variance σ^2. Let the set of ordered λ_i be denoted by $\{\lambda_i\}$. This set of values, being also the eigenvalues of MA, satisfies the inequalities

$$\nu_{i+k} \leq \lambda_i \leq \nu_i \ , \quad i = 1, 2, 3, \ldots, n-k \ , \quad (2.7)$$

where $\{\nu_i\}$ is the set of ordered eigenvalues of A. Moreover, the form of A is such that $\{\nu_i\}$ simply comprises m ones and $n-m$ zeros. Therefore, from (2.6) and (2.7) it follows that

1. For $m > k$ and $n-m > k$, $\{\lambda_i\}$ consists of $m-k$ ones, $n-m-k$ zeros and k λ_i such that $0 \leq \lambda_i \leq 1$.
2. For $m \leq k$ and $n-m > k$, $\{\lambda_i\}$ consists of $n-m-k$ zeros and m λ_i such that $0 \leq \lambda_i \leq 1$.
3. For $m > k$ and $n-m \leq k$, $\{\lambda_i\}$ consists of $m-k$ ones and $n-m$ λ_i such that $0 \leq \lambda_i \leq 1$.
4. For $m \leq k$ and $n-m \leq k$, $\{\lambda_i\}$ consists entirely of λ_i such that $0 \leq \lambda_i \leq 1$.

In the remainder of the discussion attention is focused on case (1.) only, because it accords with what is considered most likely in practice and is the situation in which the test is expected to have most power. Results analogous to those for case 1. follow for the other cases, however.

By using (2.6) and setting all the k values of λ_i that are $0 \leq \lambda_i \leq 1$ to zero and one, respectively, it follows that $b_L \leq b \leq b_U$, where

$$b_L = \sum_{i=1}^{m-k} \xi_i^2 / \sum_{i=1}^{n-k} \xi_i^2 \quad \text{and} \quad b_U = \sum_{i=1}^{m} \xi_i^2 / \sum_{i=1}^{n-k} \xi_i^2 \ . \quad (2.8)$$

Moreover, b_L and b_U are attainable. They are attained when the columns of X coincide with, or are linearly independent linear combinations of, the eigenvectors associated with the k largest and the k smallest eigenvalues of A, respectively. One may easily show that the m eigenvectors corresponding to the m unit roots of A are orthogonal $n \times 1$ vectors of length 1, the first m elements of which are arbitrary and the remaining $n-m$ elements of which are zero. Similarly, the $n-m$ eigenvectors corresponding to the $n-m$ zero roots of A are orthogonal $n \times 1$ vectors of length 1, the first m elements of which are zero and the remaining elements of which are arbitrary. Therefore, it seems unlikely that in most situations encountered in practice b_L or b_U would in fact be attained.

Now both b_L and b_U are of the form $x_1/(x_1 + x_2)$, in which x_1 and x_2 are independent chi-squared variables. It follows (see Rao 1973, p. 167) that they are both distributed as beta variables. Specifically, their respective probability density functions are

$$B\left(b_L \,\middle|\, \frac{m-k}{2}, \frac{n-m}{2}\right)$$

and

$$B\left(b_U \,\middle|\, \frac{m}{2}, \frac{n-m-k}{2}\right), \quad (2.9)$$

where

$$B(\zeta \mid p, q) = \frac{1}{\beta(p, q)} \zeta^{p-1}(1-\zeta)^{q-1} \ ,$$

and $\beta(p, q)$ is the beta function with parameters p and q. Therefore, the test based on these bounds does not require special tables as critical values may be obtained directly from a published source such as Pearson's *Tables of the Incomplete Beta-Function* (1968).

Critical values may also be obtained from tables of the F distribution by some suitable transformation such as $\zeta = [1 + (q/p)F(2q, 2p)]^{-1}$, where ζ denotes a variable having a beta distribution of the first kind and $F(2q, 2p)$ a variable distributed as Fisher's F with degrees of freedom $2q$ and $2p$ in the numerator and denominator, respectively (see Kendall and Stuart 1969, pp. 150–151 and 377–380). More specifically, for a given level of significance, α, the critical values of b_L and b_U may be determined as

$$b_L^* = \left[1 + \frac{(n-m)F_\alpha^*(n-m, m-k)}{m-k}\right]^{-1}$$

and

$$b_U^* = \left[1 + \frac{(n-m-k)F_\alpha^*(n-m-k, m)}{m}\right]^{-1} ,$$

respectively, where $F_\alpha^*(\cdot\cdot)$ denotes the tabled α critical value of F, that is, the value at which the distribution function of F is $1-\alpha$ for the particular degrees of freedom specified in the brackets.

A suitable one-sided test criterion would be to reject H_0 if $b < b_L^*$ and not reject it if $b > b_U^*$, where b_L^* and b_U^* are the critical values of the bounding beta statistics for the required level of significance. Of course, if $b_L^* \leq b \leq b_U^*$, then the test is inconclusive. It is possible in such cases, however, to supplement the bounds test by an "exact" test based on either an approximation to the true distribution of b or an accurate calculation of the true distribution of b such as may be done by using the method of Imhof (1961).

For practical purposes, we propose that a beta distribution be fitted to b by using the method of moments along the lines suggested by Durbin and Watson (1951). This approach is based on the first two moments of b and is outlined in the following subsection. If greater accuracy is required, an approach based on the first four moments of b may be used, as suggested by Henshaw (1966, Section 3).

2.2 Beta Approximation

The moments of b, which depend on the matrix X, may be expressed in terms of the eigenvalues of MA. Under the null hypothesis, the mean and variance of b may be written as

$$E(b) = \frac{m - k + \sum_{i=1}^{k} \lambda_i}{n - k} = \bar{\lambda} ,$$

and

$$V(b) = \frac{2[(m-k)(1-\bar{\lambda})^2 + (n-m-k)\bar{\lambda}^2 + \sum_{i=1}^{k} (\lambda_i - \bar{\lambda})^2]}{(n-k)(n-k+2)} , \qquad (2.10)$$

respectively. These equations are derived from (2.6) by using (2.7) and the result of Geary (1933) that, under the null hypothesis, b is distributed independently of its own denominator so that the moments of b are ratios of the corresponding moments of numerator and denominator.

The mean and variance of a beta variable, ζ, with range (0, 1) and density function with parameters p and q are

$$E(\zeta) = \frac{p}{p + q}$$

and

$$V(\zeta) = \frac{pq}{(p + q)^2(p + q + 1)} , \qquad (2.11)$$

respectively.

The range of b is $[0, 1]$, although it is unlikely to reach the limits of its range in practice. Assuming that b has a beta distribution with density $B(b \mid p, q)$, the parameters p and q may be determined by solution of the equations

$$p + q = \frac{E(b)[1 - E(b)]}{V(b)} - 1$$

and

$$p = (p + q)E(b) , \qquad (2.12)$$

which are derived by using (2.11). Numerical values for $E(b)$ and $V(b)$ may, of course, be computed from the data by using (2.10). The computations are eased by the fact that all but k of the eigenvalues of MA are known, and one only has to evaluate the k λ_i that are $0 \leq \lambda_i \leq 1$. Details on the solution of this kind of eigenvalue problem are given in Wilkinson (1965). Alternatively, the moments of b may be calculated from the data by using the traces of the first two powers of the matrix MA (see Henshaw 1966, p. 649). The value of b, b^*, corresponding to the required significance level may be obtained from Pearson (1968) by using the tabled values of p and q nearest to the computed values, or if greater accuracy is required, by interpolation by using the computed values of p and q directly.

As the fitting of a beta distribution by this method is probably as laborious as the alternative tests cited in Section 1, we hope that the bounds test is sufficiently powerful to determine the significance or nonsignificance of the computed b values in many cases arising in practice. The question of power is considered in the following section.

3. POWER OF THE TEST

The statistic b, and the bounds b_L and b_U, are ratios of quadratic forms in normal variables. Therefore, for a given structure of the disturbance covariance matrix Ω, a given matrix X of observations on the independent variables, and a given level of significance, the probability of rejecting the null hypothesis H_0 can be calculated accurately by using the procedure of Imhof (1961). The Imhof method was used by Harvey and Phillips (1974) to compute powers for the Goldfeld-Quandt and Theil BLUS tests as well as for their own recursive test. In order to facilitate comparisons of the power of b with the power results reported by Harvey and Phillips, we decided to adopt a model similar to theirs for use in this study. Additional variants of the model were also considered, however.

3.1 Model

The basic model used was the simple linear regression model

$$y = \beta_0 1 + \beta_1 X_1 + u , \qquad (3.1)$$

where 1 is an $n \times 1$ vector all of the elements of which are unity, X_1 is an $n \times 1$ vector of observations on the single explanatory variable, β_0 and β_1 are scalar constants, and y and u are as previously defined.

Two basic forms of heteroscedasticity were used, namely,

$$\sigma_i^2 = \sigma^2 X_{1i} \quad \text{and} \quad \sigma_i^2 = \sigma^2 X_{1i}^2 , \qquad i = 1, 2, 3, \ldots, n . \qquad (3.2)$$

These forms accord with the usual assumption made about heteroscedasticity in econometric work, that is, that the disturbance variances σ_i^2 change monotonically with one of the independent variables or with a linear combination of them. As was intimated in Section 1, however, there may also be cases in which it is reasonable to postulate that the σ_i^2 vary monotonically with time. Accordingly, the following two additional forms of heteroscedasticity were also considered:

$$\sigma_i^2 = \sigma^2 i \quad \text{and} \quad \sigma_i^2 = \sigma^2 i^2 , \quad i = 1, 2, 3, \ldots, n . \qquad (3.3)$$

Because b is independent of the scale of the ξ_i, the value of σ^2 was, without loss of generality, taken to be unity. Let the four different alternative hypotheses corresponding to the disturbance covariance matrices implied by (3.2) and (3.3) be denoted by H_{Aj}, $j = 1, 2, 3, 4$, respectively.

Three variants of X_1—X_{1A}, X_{1B}, and X_{1C}—were used. The first two of these correspond to the variables reported by Harvey and Phillips, X_{1A} being generated from a uniform distribution with range (0, 20) and X_{1B} from a lognormal distribution with a coefficient of variation of 1.13. The generator used in the case of X_{1A} was the IBM FORTRAN subroutine RANDU, which uses the power residue method as discussed in IBM manual C20–8011. In the case of X_{1B}, random standard normal numbers, modified to yield a set from a normal population with standard deviation 10 and the required coefficient of variation, were taken from the tables published by the Rand Corporation (1950). The exponentials of this set of numbers were used to produce the X_{1B} series. The variant X_{1C} was chosen to be the dummy variable i, $i = 1, 2, 3, \ldots, n$, representing a pure (time) trend. In each case the sample size n was 20, again corresponding to the sample size reported on by Harvey and Phillips; the sample values of the three variables remained fixed throughout the power computations. The observations on X_{1A} and X_{1B} were arranged in ascending order of magnitude. Such ordering makes (2.4) the appropriate form for the matrix A for the alternative hypotheses corresponding to (3.2) as well as to (3.3). We decided to investigate several values of m. The values chosen were $m = 6, 8, 10, 12$, and 14.

Throughout the experiment one-sided tests at the nominal 5 percent level of significance were used, critical values being obtained from Pearson (1968). For each of the 60 combinations of Ω, X, and m, the power of b was calculated by using both b_L and b_U in order to obtain lower and upper limits. In selected cases a beta distribution was fitted to b by using the method outlined in Section 2.2 and an "exact" power was calculated.

The computations involved in applying the Imhof procedure were done by using the FORTRAN subroutine FQUAD given in Koerts and Abrahamse (1969, pp. 155–160). The Koerts and Abrahamse subroutine computes, by the Imhof method, the probability $P(Q \leq r)$, in which Q is a quadratic form in normal variables and r is a constant. For the purposes of this study, probabilities were required of the form $P(b \leq b_c)$, in which b_c is a constant denoting a particular critical value, and b is as defined in (2.5), that is, a ratio of quadratic forms. Therefore, before the use of the subroutine, the following transformation was carried out.

We have $u'M'AMu/u'Mu \leq b_c$. Therefore, $u'M'AMu - b_c u'Mu \leq 0$, or $u'(M'AM - b_c M)u \leq 0$. For the H_A case in which $E(uu' \mid X) = \sigma^2\Omega$, the additional transformation $v = \Omega^{-\frac{1}{2}}u$ yields independent normally distributed random variables with constant variance. Therefore, substitution for u gives $v'\Omega^{\frac{1}{2}}(M'AM - b_c M)\Omega^{\frac{1}{2}}v \leq 0$, which is of the required form.

The value of the probability $P[v'\Omega^{\frac{1}{2}}(M'AM - b_c M)\Omega^{\frac{1}{2}}v \leq 0]$ was actually computed by the subroutine in terms of the eigenvalues of $\Omega^{\frac{1}{2}}(M'AM - b_c M)\Omega^{\frac{1}{2}}$; these were evaluated by using the IBM subroutine EIGEN, which is based on the diagonalization method of Jacobi.

Finally, the values of the truncation error and integration error that were specified for use in the Koerts and Abrahamse subroutine were 1.0×10^{-4}. The machine used was the IBM 360 Model 44 at the University of Dublin.

3.2 Results

The main power results for b_L and b_U are given in Table 1. Several clear patterns of behavior of the power

1. Power of b at the .05 Significance Level

	m									
	6		8		10		12		14	
Variable	b_L	b_U	b_L	b_U	b_L	b_U	b_L	b_U	b_L	b_U
	a. H_{A1}: $\sigma_i^2 = \sigma^2 X_i$									
X_{1A}	.155	.537	.163	.491	.212	.472	.204	.444	.183	.423
X_{1B}	.268	.713	.438	.766	.460	.735	.449	.702	.345	.607
X_{1C}	.174	.588	.256	.581	.271	.550	.255	.509	.220	.472
	b. H_{A2}: $\sigma_i^2 = \sigma^2 X_i^2$									
X_{1A}	.439	.837	.532	.822	.512	.768	.478	.722	.414	.670
X_{1B}	.867	.990	.951	.993	.912	.976	.868	.954	.680	.853
X_{1C}	.636	.942	.714	.918	.676	.872	.599	.808	.498	.733
	c. H_{A3}: $\sigma_i^2 = \sigma^2 i$									
X_{1A}	.161	.569	.239	.564	.254	.534	.241	.496	.209	.460
X_{1B}	.170	.578	.256	.582	.275	.554	.262	.515	.226	.478
	d. H_{A4}: $\sigma_i^2 = \sigma^2 i^2$									
X_{1A}	.571	.921	.674	.904	.646	.859	.577	.796	.478	.721
X_{1B}	.647	.942	.726	.923	.689	.879	.615	.817	.509	.741

of the b test emerge. First, for a given form of variable and a given value of m, the power, as is to be expected, varies with the form of the alternative hypothesis H_A, being considerably greater for H_{A2} and H_{A4} than for H_{A1} and H_{A3}.

Second, for a given H_A and a given value of m, the power is, without exception, greater for the lognormal variable X_{1B} than for the uniform variable X_{1A}. In those cases in which the trend variable X_{1C} was used, that is, cases H_{A1} and H_{A2}, the power for X_{1C} lies between the powers for the other two variables.

Third, for a given form of variable and a given H_A, the power varies with the value of m. The pattern of variation is similar for all combinations of X_1 and H_A. For all variables, however, the highest power of b_L occurs for $m = 10$ in the case of H_{A1} and H_{A3}, while it occurs for $m = 8$ in the case of H_{A2} and H_{A4}. The highest power of b_U for all variables occurs when $m = 8$ for all forms of H_A. The maxima for b_L and b_U are relatively unpronounced in all cases.

Finally, the differences in the powers of b_U and b_L for the various combinations of X_1, H_A and m, are generally quite large, which suggests that the incidence of inconclusiveness of the bounds test is likely to be quite high in small samples. The differences tend to be smaller for $m = 8$, 10, and 12 than for $m = 6$ and 14. For X_{1B} and H_{A2}, they are relatively small for all values of m.

Table 2 has been compiled for comparative purposes by using information from Table 1, additional power results for b^*, and certain of the results of Harvey and Phillips (1974, p. 313) for three alternative tests, namely, the Goldfeld-Quandt (GQ), Theil (T), and Harvey-Phillips (HP) tests. The results for b_L and b_U are those for the values of m considered to be optimal for a given form of H_A, namely, $m = 10$ for H_{A1} and $m = 8$ for H_{A2}. Likewise for each of the alternative tests, the powers

2. *Comparison of Power of b and GQ, T, and HP*

Variable	b_L	b^*	b_U	GQ	T	HP
		a. H_{A1}: $\sigma_i^2 = \sigma^2 X_i$				
X_{1A}	.21	.33	.47	.25	.27	.26
X_{1B}	.46	.60	.74	.59	.61	.59
		b. H_{A2}: $\sigma_i^2 = \sigma^2 X_i^2$				
X_{1A}	.53	.69	.82	.62	.64	.62
X_{1B}	.95	.98	.99	.98	.90	.95

NOTE: m = 10 for case (a); m = 8 for case (b).

shown are those relating to the construction of the tests that Harvey and Phillips found was optimal for the four cases, that is, X_1, H_A combinations, examined.

Generally, the power of b_L is somewhat less than the powers of GQ, T, and HP. For X_{1B} and H_{A2}, however, the power of b_L, although less than that of GQ, is greater than the powers of T and HP. The power evaluated for b_U is in all cases greater, and in most cases markedly greater, than the powers of all three alternatives.

Given these results and given the possibility of inconclusiveness of the bounds test, the performance of b^* relative to that of the alternatives would seem to be of considerable interest. Therefore, for the cases considered in Table 2, beta approximations were carried out by using the approach outlined in Section 2.2. As can be seen, the power of b^* for the variable X_{1A} is superior to the powers of the alternatives both for case H_{A1} and H_{A2}. For X_{1B}, the power of b^* compares closely with all three alternatives in the case of H_{A1}, while in the case of H_{A2} it is similar to the power of GQ and higher than the powers of T and HP.

4. SUMMARY AND CONCLUSION

Two variants of a test for heteroscedasticity based on the use of ordinary least squares residuals have been proposed and compared for power with three well-known alternative procedures. The power of the bounds test, that is, b_L alone, while generally somewhat lower than the power of the Goldfeld-Quandt, Theil, and Harvey and Phillips tests, compares reasonably with them. On the other hand, the power of the "exact" test is at least as great as the powers of the alternatives in all the cases that have been considered.

Given the consistency of the patterns of behavior of the test that emerge from the results, the main conclusions drawn from the study probably carry over to other forms of heteroscedasticity. Therefore, in view of its practical simplicity relative to other procedures, the bounds test would seem eminently suitable as a first test for heteroscedasticity in single-equation least squares regression models. For practical purposes $n/2$ is suggested as a suitable value for m, particularly if nothing is known a priori about the precise form of the alternative hypothesis. If the bounds test proves to be inconclusive, an alternative test, or the "exact" variant of the b test, may then be used. In cases in which the values of the observations on the independent variable thought to be associated with the disturbance variance are fairly evenly spaced, the "exact" b test would seem to offer more power than the three alternative tests considered in this article.

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