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A robust modification of the Jarque–Bera test of normality

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Abstract

We propose a new robust Jarque–Bera (RJB) test utilizing a robust measure of variance. The RJB statistic is asymptotically χ_2^2-distributed and has equal or higher power than the JB test for several common alternatives to normality.

Keywords: Goodness-of-fit tests; Test of normality; Robust estimators of scale; Jarque–Bera test; Skewness; Kurtosis

JEL classification: C01; C10; C12; C20

1. Introduction

The Jarque–Bera (JB) test for normality (also known in statistics as the D'Agostino–Pearson or Bowman–Shenton test) is one of the most popular goodness-of-fit tests utilized in economics (Jarque and Bera, 1980; D'Agostino and Pearson, 1973; Bowman and Shenton, 1975). The JB test statistic is a sum of the sample standardized third and forth moments and asymptotically follows a χ_2^2-distribution with two degrees of freedom.

The sample moments are known to be very sensitive to outliers; and the sample variance is more affected by outliers than the mean (Stuart and Ord, 1987). Hence, the sample skewness and kurtosis in the JB test statistic are also sensitive to extreme observations. In this paper we propose a modification of the JB test utilizing a robust (resistant to outliers) estimate of spread, namely the average absolute deviation from the median (MAAD), in the denominators of skewness and kurtosis instead of the classical estimate of spread. This robust Jarque–Bera (RJB) test statistic asymptotically follows a χ_2^2-distribution. Simulation studies indicate that RJB shows higher or similar power in detecting heavy-tailed alternatives compared to the JB test.

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2. The Robust Jarque–Bera (RJB) test statistic

Let $X_1, X_2,\ldots, X_n$ be a sample of independent and identically distributed random variables. Let μ, v and σ be the population mean, median and standard deviation respectively. Let $\bar{X}$, M and s_n be the corresponding sample estimates of μ, v and σ. For any positive integer k define the k-th population central moment $\mu_k = E(X-\mu)^k$ and its sample estimate $\hat{\mu}\mathrm{k} = \sum_{\mathrm{i}=1}^{\mathrm{n}} (\mathrm{X_i} - \bar{\mathrm{X}})^{\mathrm{k}}$.

The Jarque and Bera (1980) test statistic (JB) is given by

$$\mathrm{JB} = \frac{n}{6}\left(\frac{\hat{\mu}_3}{\hat{\mu}_2^{3/2}}\right)^2 + \frac{n}{24}\left(\frac{\hat{\mu}_4}{\hat{\mu}_2} - 3\right)^2, \tag{1}$$

where $\hat{\mu}_3/\hat{\mu}_2^{3/2}$ is the sample skewness $\sqrt{b_1}$ and $\hat{\mu}_4/\hat{\mu}_2$ is the sample kurtosis b_2.

Under the null hypothesis of normality, the sample skewness and kurtosis are asymptotically independent and normally distributed with covariance matrix:

$$\sqrt{n}\begin{bmatrix} \hat{\mu}_3/\hat{\mu}_2^{3/2} \\ \hat{\mu}_4/\hat{\mu}_2 - 3 \end{bmatrix} \Rightarrow N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 & 0 \\ 0 & 24 \end{bmatrix}\right). \tag{2}$$

In this paper we utilize a robust estimate of spread which is less influenced by outliers, the average absolute deviation from the sample median (MAAD), in the denominators of the sample estimates of skewness and kurtosis. The MAAD is used to

Table 1
Power of the Shapiro–Wilk (SW) test, the Jarque–Bera (JB) test, the directional test s_n/J_n (SJ) and the robust Jarque–Bera (RJB) for α=0.05

n	Test	N(0,1)	t_3	t_5	Logistic	Double exponential	CN [a]	exp
30	SW	0.0527	0.4656	0.2512	0.1423	0.3605	0.3980	0.9673
	JB_{emp}	0.0508	0.5339	0.3076	0.1919	0.4116	0.4670	0.8288
	SJ_{emp}	0.0522	0.5596	0.3089	0.1941	0.5334	0.4242	0.5486
	RJB_{emp}	0.0529	0.5664	0.3228	0.2012	0.4884	0.4700	0.7517
50	SW	0.0513	0.6371	0.3579	0.1992	0.5194	0.5802	0.9998
	JB_{emp}	0.0470	0.6957	0.4351	0.2736	0.5610	0.6589	0.9799
	SJ_{emp}	0.0485	0.7291	0.4331	0.2755	0.7281	0.5879	0.7064
	RJB_{emp}	0.0481	0.7316	0.4573	0.2889	0.6574	0.6445	0.9394
100	SW	0.0495	0.8769	0.5522	0.3045	0.7913	0.8043	1.0000
	JB_{emp}	0.0487	0.9068	0.6355	0.3977	0.7906	0.8548	1.0000
	SJ_{emp}	0.0508	0.9316	0.6491	0.4140	0.9383	0.7962	0.9048
	RJB_{emp}	0.0507	0.9293	0.6603	0.4248	0.8783	0.8535	1.0000

The number of Monte Carlo simulations is 10,000.
[a] The contaminated normal (CN) distribution allows for more unusual observations than in a normal data by assuming the underlying density is a mixture of one normal distribution with another one with the same mean but larger variance (Tukey, 1960). Typically, the contaminating component forms a small fraction, e.g. .05 to .10, of the data and has a standard deviation 3 or 5 times that of the main one.

evaluate the fairness of tax assessments (Gastwirth, 1982) and is defined by

$$J_n = \frac{C}{n}\sum_{i=1}^{n} |X_i - M|, \quad C = \sqrt{\pi/2}. \tag{3}$$

The robust sample estimates of skewness and kurtosis are $\hat{\mu}_3/J_n^3$ and $\hat{\mu}_4/J_n^4$ respectively, which leads to the new robust Jarque–Bera (RJB) test statistic

$$\mathrm{RJB} = \frac{n}{C_1}\left(\frac{\hat{\mu}_3}{J_n^3}\right)^2 + \frac{n}{C_2}\left(\frac{\hat{\mu}_4}{J_n^4} - 3\right)^2, \tag{4}$$

where C_1 and C_2 are positive constants.

Theorem. *Let $X_1, X_2,\ldots, X_n \sim N(\mu, \sigma)$. Then*

$$\sqrt{n}\begin{bmatrix} \hat{\mu}_3/J_n^3 \\ \hat{\mu}_4/J_n^4 - 3 \end{bmatrix} \Rightarrow N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}\right), \tag{5}$$

where C_1 and C_2 are positive constants.

Proof. If $X_1, X_2,\ldots, X_n \sim N(\mu, \sigma)$, J_n is a consistent estimate of the population standard deviation σ and is asymptotically normally distributed (Gastwirth, 1982)

$$\lim_{n\to\infty} \mathrm{E}J_n = \sigma, \quad N\left(O, \sigma\sqrt{\frac{\pi}{2} - 1}\right), \quad n\to\infty. \tag{6}$$

Moreover, using the strong law large numbers, it can be shown that J_n converges to σ almost surely. Hence, the multivariate Slutsky Theorem (Lehmann, 2004) implies that the robust estimates of sample skewness and kurtosis, $\hat{\mu}_3/J_n^3$ and $\hat{\mu}_4/J_n^4$ respectively, are asymptotically jointly normally distributed and independent.[1] □

Remark. To obtain the constants C_1 and C_2, we need to find expressions for $\mathrm{E}J_n^3$, $\mathrm{E}J_n^4$, $\mathrm{E}J_n^6$ and $\mathrm{E}J_n^8$ for a finite sample size n, which can be calculated as suggested by Geary (1936). However, such calculations are quite tedious and are not of practical use since the convergence of estimators of kurtosis to the asymptotic normal distribution is very slow. Therefore, we obtain C_1 and C_2 from the Monte Carlo simulations. In particular, if one desires to preserve the nominal level of 0.05, we recommend C_1=6 and C_2=64.[2]

Corollary. *Under the null hypothesis of normality, the RJB test statistic asymptotically follows the χ_2^2-distribution with 2 degrees of freedom, i.e.*

$$\mathrm{RJB} \sim \chi_2^2.$$

Consequently, the one-sided rejection region is

$$\text{reject } H_0 \ : \text{normality, if } \mathrm{RJB} \geq \chi^2_{1-\alpha,2}, \tag{7}$$

where $\chi^2_{1-\alpha,2}$ is the upper α-percentile of the χ_2-distribution with 2 degrees of freedom.

3. Critical values and power of the new test

Our simulation study indicates that the asymptotic χ_2^2-approximation of critical values for both JB and RJB is not sufficiently accurate in small to moderate sized samples.[3] To improve the accuracy of the large sample approximation, Urzua (1996) suggested a modified standardization for $\sqrt{b_1}$ and b_2. As noted by Thadewald and Buning (2007), the difference in power between that modification and the usual JB test is minor. Therefore, we omit the Urzua–Jarque–Bera test from our study and use the Monte Carlo simulated empirical critical values for JB

[1] We omit the derivation of the variance–covariance matrix which is available from the authors.

[2] The values C_1 of 6 and C_2 of 64 should not be recalculated prior to applying the RJB test. Hence, the new test statistic takes the form $(n/6)[\hat{\mu}_3/J_n^3]^2+(n/64)[\hat{\mu}_4/J_n^4-3]^2$.

[3] Our study on the size of the JB and RJB tests indicates that typically RJB with χ_2^2-approximated critical values more accurately preserves the nominal 5% level than the JB test with the χ_2^2-approximated critical values.

and RJB. These simulated values provide a better approximation to the exact critical values than the limiting χ_2^2-distribution.[4]

We compare the power of the robust Jarque–Bera (RJB) test in respect to the classical Jarque–Bera (JB) test, the omnibus Shapiro–Wilk (SW) test (Shapiro and Wilk, 1965) and the directed SJ test (Gel et al., 2007) which focuses on detecting symmetric heavy-tailed alternatives. To ensure consistency of the comparative analysis, we utilize the Monte Carlo simulated critical values for the tests.

Table 1 presents the results of the simulation study. Overall, the new RJB test outperforms the SW, JB and SJ tests for moderately heavy-tailed alternatives, especially in small and moderate sample sizes. For very heavy-tailed symmetric alternatives, e.g. the double exponential distributions, RJB and JB are less powerful than the SJ test, which is not surprising as the SJ test is directed towards such alternatives. With the exception of small samples from the exponential distribution, the new RJB test has equal or greater power than the JB test. This loss in power for the exponential distribution is less than the gain in power for other common alternatives of interest.

4. Discussion

This paper presents a new robust Jarque–Bera (RJB) test utilizing a robust measure of scale in the denominators of sample skewness and kurtosis. The new RJB test statistic has an asymptotic χ_2^2-distribution. Thus, the χ_2^2-approximation for critical values is equally applicable to the JB and RJB tests, and RJB can be utilized just in the same standard procedure as the classical JB test. Similarly to JB, the approximated critical values for RJB are less accurate for small and moderate samples and the Monte Carlo simulated critical values are proposed to be utilized instead for both tests, especially for the cases on the border of significance. The simulation study demonstrates that the new RJB test is more powerful than the Jarque–Bera, Shapiro–Wilk and the directed SJ test in detecting moderately heavy-tailed departures from normality, especially in small and moderate samples. Overall, with the except of the exponential distribution, the RJB test performs at least as well as the JB test for all alternatives considered and all sample sizes.

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[4] The R-program for the robust Jarque–Bera (RJB) test with an option for empirical and approximated critical values is available in the R package *lawstat*.