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Some high-dimensional tests for a one-way MANOVA

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Abstract

A statistic is proposed for testing the equality of the mean vectors in a one-way multivariate analysis of variance. The asymptotic null distribution of this statistic, as both the sample size and the number of variables go to infinity, is shown to be normal. Thus, this test can be used when the number of variables is not small relative to the sample size. In particular, it can be used when the number of variables exceeds the degrees of freedom for error, a situation in which standard MANOVA tests are invalid. A related statistic, also having an asymptotic normal distribution, is developed for tests concerning the dimensionality of the hyperplane formed by the population mean vectors. The finite sample size performances of the normal approximations are evaluated in a simulation study.

AMS 2000 subject classification: 62E20; 62H15

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1. Introduction

We consider the comparison of mean vectors in a one-way completely randomized design. Suppose there are g groups and $\boldsymbol{x}_{i1}, \ldots, \boldsymbol{x}_{in_i}$ represents a random sample of $p \times 1$ vectors from the ith group, which has mean vector $\boldsymbol{\mu}_i$ and covariance matrix Σ. Standard multivariate analysis of variance procedures utilize the matrices

$$H = \sum_{i=1}^{g} n_i(\bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}})(\bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}})', \quad E = \sum_{i=1}^{g}\sum_{j=1}^{n_i}(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_i)(\boldsymbol{x}_{ij} - \bar{\boldsymbol{x}}_i)',$$

where

$$\bar{\boldsymbol{x}}_i = \sum_{j=1}^{n_i} \boldsymbol{x}_{ij}/n_i, \quad \bar{\boldsymbol{x}} = \sum_{i=1}^{g} n_i\bar{\boldsymbol{x}}_i/n, \quad n = \sum_{i=1}^{g} n_i.$$

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When sampling from multivariate normal distributions, the matrices E and H are independently distributed with $E \sim W_p(\Sigma, e, 0)$ and $H \sim W_p(\Sigma, h, \Delta)$, where $e = n - g$, $h = g - 1$,

$$\Delta = \sum_{i=1}^{g} n_i(\boldsymbol{\mu}_i - \bar{\boldsymbol{\mu}})(\boldsymbol{\mu}_i - \bar{\boldsymbol{\mu}})', \quad \bar{\boldsymbol{\mu}} = \sum_{i=1}^{g} n_i \boldsymbol{\mu}_i / n,$$

and the usual notation for the Wishart distribution is used. Tests of the hypothesis $\mathrm{H}_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_g$ are commonly based on the eigenvalues of HE^{-1}. For instance, the likelihood ratio test is based on the statistic

$$U_1 = |E|/|H + E| = \prod_{i=1}^{h} 1/\{1 + \phi_i(HE^{-1})\},$$

where we use the notation $\phi_1(A) \geqslant \cdots \geqslant \phi_p(A)$ to denote the ordered eigenvalues of a $p \times p$ matrix A. As e approaches infinity, $-\{e - \frac{1}{2}(p - h + 1)\} \log U_1$ converges in distribution to the chi-squared distribution with ph degrees of freedom. In addition, some exact percentage points for U_1 have been computed by Schatzoff [15], Pillai and Gupta [14], Mathai [13], Lee [12], and Davis [6].

In recent years, applications of multivariate analysis have involved an increasingly large number of variables p. In the context of the test for equal mean vectors described above, this presents a problem in that the likelihood ratio test is degenerate if $p > e$, and even if $p \leqslant e$, the exact critical values for the test have only been tabulated for small values of p. Further, the asymptotic result is based on asymptotic theory which has e going to infinity while p is fixed. Consequently, this approximation is not likely to be very accurate when p is of the same order of magnitude as e. In these situations, it would be better to use an inference procedure which is based on asymptotic theory as both e and p go to infinity. In particular, we would have e and p going to infinity with p/e converging to a constant $\gamma \in (0, \infty)$.

Other recent works on inferences in MANOVA in this high-dimensional setting include the following. Tonda and Fujikoshi [20] obtained the asymptotic null distribution of the likelihood ratio test when $p/e \to \gamma \in (0, 1)$ as did Fujikoshi [9] who also found the asymptotic null distributions for the Lawley–Hotelling trace and the Pillai trace statistics. Fujikoshi et al. [10] considered testing H_0 with the statistic

$$T_{np} = \frac{e\,\mathrm{tr}(H)/\mathrm{tr}(E) - h}{\sqrt{2h\{\mathrm{tr}(E^2)/\mathrm{tr}(E)^2 - e^{-1}\}}},$$

which they showed converges in distribution to a standard normal random variable when $p/e \to \gamma \in (0, \infty)$. Some additional statistics for testing H_0 can be found in Srivastava and Fujikoshi [19] and Srivastava [18].

The purpose of this paper is to propose an alternative statistic for testing H_0 that can be used when $p > e$. The asymptotic null distribution of this statistic, as $p/e \to \gamma \in (0, \infty)$, is shown to be normal. In addition, for those situations in which the hypothesis of equal mean vectors, H_0, is rejected, we consider tests concerning the dimensionality of the hyperplane formed by the population mean vectors. These tests can be based on a statistic, also having an asymptotic normal distribution, which is a generalization of the statistic proposed to test H_0. Some simulation results are given to assess the adequacy of the normal approximations.

2. A test for the equality of mean vectors

Tests for the equality of two mean vectors, that is, tests of our H_0 where $g = 2$, when the number of variables p is large have been developed by Chung and Fraser [5], Dempster [7,8], and Bai and Saranadasa [2]. In particular, Bai and Saranadasa [2] considered the statistic

$$M_{np} = (\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2)'(\bar{\boldsymbol{x}}_1 - \bar{\boldsymbol{x}}_2) - \frac{n_1 + n_2}{n_1 n_2 (n_1 + n_2 - 2)} \operatorname{tr}(E). \tag{1}$$

They showed that when $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$, under certain conditions, $M_{np}/\hat{\sigma}_{M_{np}}$ converges in distribution, as p, n_1, and n_2 all approach infinity, to the standard normal distribution, where

$$\hat{\sigma}_{M_{np}} = \left[\frac{2(n_1 + n_2)(n_1 + n_2 - 1)}{n_1^2 n_2^2 (n_1 + n_2 - 2)(n_1 + n_2 - 3)} \left\{ \operatorname{tr}(E^2) - (n_1 + n_2 - 2)^{-1} \operatorname{tr}(E)^2 \right\} \right]^{1/2}.$$

The statistic we construct in this section for testing $H_0 : \boldsymbol{\mu}_1 = \cdots = \boldsymbol{\mu}_g$ can be viewed as a generalization of the statistic given in (1).

Since $E(H) = h\Sigma + \Delta$, it follows that, under H_0, $E\{\operatorname{tr}(H)\} = h \operatorname{tr}(\Sigma)$. In addition, $E(E) = e\Sigma$, so a statistic having mean 0 if and only if the null hypothesis holds is given by

$$t_{np} = (n-1)^{-1/2} \left\{ h^{-1} \operatorname{tr}(H) - e^{-1} \operatorname{tr}(E) \right\}.$$

It is easy to show that, under H_0,

$$\sigma_{t_{np}}^2 = \operatorname{var}(t_{np}) = \frac{2}{he} \operatorname{tr}(\Sigma^2),$$

and so $t_{np}/\sigma_{t_{np}}$ will have mean 0 and variance 1 if the g population mean vectors are identical. Certainly other reasonable statistics for testing H_0 could be proposed. Our choice of t_{np} is sufficiently simple so that its asymptotic null distribution can be easily determined and it can be generalized to the tests of dimensionality considered in Section 3.

Our first result will establish the asymptotic normality of t_{np} under the following conditions.

Condition 1. $n_1 = n_{1k}, \ldots, n_g = n_{gk}$ and $p = p_k$ are all increasing functions of an index $k = 1, 2, \ldots$ such that $\lim_{k\to\infty} n_{ik} = \infty$, for $i = 1, \ldots, g$, $\lim_{k\to\infty} p_k = \infty$, $\lim_{k\to\infty} n_{ik}/n_{\cdot k} = \rho_i \in (0, 1)$, for $i = 1, \ldots, g$, and $\lim_{k\to\infty} p_k/n_{\cdot k} = \gamma \in (0, \infty)$, where $n_{\cdot k} = n_{1k} + \cdots + n_{gk}$.

Condition 2. For each k, the sample sum of squares and products matrices can be expressed as

$$E_k = \sum_{i=1}^{g} X_{ik}'(I_{n_{ik}} - n_{ik}^{-1} \mathbf{1}_{n_{ik}} \mathbf{1}_{n_{ik}}') X_{ik}$$

and

$$H_k = \sum_{i=1}^{g} n_{ik}^{-1} X_{ik}' \mathbf{1}_{n_{ik}} \mathbf{1}_{n_{ik}}' X_{ik} - n_{\cdot k}^{-1} \sum_{i=1}^{g} \sum_{j=1}^{g} X_{ik}' \mathbf{1}_{n_{ik}} \mathbf{1}_{n_{jk}}' X_{jk},$$

where $\mathbf{1}_{n_{ik}}$ is the $n_{ik} \times 1$ vector of 1's, the rows of the $n_{ik} \times p_k$ matrix X_{ik} are independently and identically distributed normal random vectors with mean vector $\boldsymbol{\mu}_{ik}$ and covariance matrix Σ_k, and $X_{1k}, \ldots, X_{gk}$ are independent of one another. Further, if we define

$$\Psi_k = \sum_{i=1}^{g} \boldsymbol{\mu}_{ik}\boldsymbol{\mu}_{ik}' - g^{-1}\sum_{i=1}^{g}\sum_{j=1}^{g} \boldsymbol{\mu}_{ik}\boldsymbol{\mu}_{jk}',$$

then the nonzero eigenvalues of $p_k^{-1}\Psi_k$ do not depend on k.

Condition 3. For $j = 1, 2,$

$$\lim_{k\to\infty} \frac{\operatorname{tr}\{(\Sigma_k^2)^j\}}{p_k} = \tau_j \in (0, \infty).$$

For notational convenience, the dependence of all parameters and statistics on k will be suppressed throughout the remainder of the paper. Note that under the conditions given above,

$$\lim \sigma_{t_{np}}^2 = 2h^{-1}\gamma\tau_1.$$

Also, Condition 2 implies that if $\operatorname{rank}(\Psi) = r$, then $\phi_i(\Psi)$ is $O(n)$ for $i = 1, \ldots, r$ and $\phi_i(\Delta)$ is $O(n^2)$ for $i = 1, \ldots, r$.

We now will find the asymptotic null distribution of t_{np} as n_i, $i = 1, \ldots, g$, and p go to infinity.

Theorem 1. *Under Conditions* 1–3, *if* $\phi_1(\Psi) = 0$, *then*

$$t_{np} \to^{\mathrm{d}} N(0, 2h^{-1}\gamma\tau_1),$$

where $N(\mu, \sigma^2)$ *denotes the normal distribution with mean* μ *and variance* σ^2.

Proof. Since t_{np} is invariant under transformations of the observation vectors by an orthogonal matrix, we may assume without loss of generality that Σ is diagonal. For $l = 1, \ldots, p$, let

$$T_{nl} = t_{nl} - t_{n,l-1} = (n-1)^{-1/2}\left(h^{-1}h_{ll} - e^{-1}e_{ll}\right),$$

where $t_{n0} = 0$, so that $t_{np} = \sum_{l=1}^{p} T_{nl}$. Here h_{ll} and e_{ll} denote the lth diagonal elements of H and E, respectively. If we define the set $\mathcal{F}_{n,l-1} = \{x_{ijk} : i = 1, \ldots, g; j = 1, \ldots, l-1; k = 1, \ldots, n_i\}$, where x_{ijk} is the jth component of $\boldsymbol{x}_{ik}$, then since Σ is diagonal

$$E(h_{ll}|\mathcal{F}_{n,l-1}) = E(h_{ll}) = h\sigma_{ll}$$

and

$$E(e_{ll}|\mathcal{F}_{n,l-1}) = E(e_{ll}) = e\sigma_{ll}$$

so that $E(T_{nl}|\mathcal{F}_{n,l-1}) = 0$. Consequently, for each n, $\{t_{nl}, l = 1, \ldots, p\}$ is a martingale and $T_{n1}, \ldots, T_{np}$ are martingale differences. As a result, the theorem will follow from Corollary 3.1 of Hall and Heyde [11, p. 58] if we can show that

$$\sum_l E\{T_{nl}^2 I(|T_{nl}| > \varepsilon)|\mathcal{F}_{n,l-1}\} \xrightarrow{\mathrm{p}} 0, \tag{2}$$

for all $\varepsilon > 0$, and

$$\sum_l E(T_{nl}^2|\mathcal{F}_{n,l-1}) \xrightarrow{\mathrm{p}} 2h^{-1}\gamma\tau_1. \tag{3}$$

Here $I(\cdot)$ denotes the indicator function. It is easily shown that

$$E(T_{nl}^2|\mathcal{F}_{n,l-1}) = E(T_{nl}^2) = \frac{2}{he}\sigma_{ll}^2.$$

Thus, using Conditions 1 and 3, we find that

$$\sum_l E(T_{nl}^2|\mathcal{F}_{n,l-1}) = \sum_l E(T_{nl}^2) = \frac{2}{he}\operatorname{tr}(\Sigma^2) \to 2h^{-1}\gamma\tau_1$$

thereby confirming (3). The Lindeberg condition given in (2) can be established by showing that the stronger Liapounov condition

$$\sum_l E(T_{nl}^4|\mathcal{F}_{n,l-1}) \xrightarrow{\text{p}} 0 \tag{4}$$

holds. Now

$$E\left\{\left(h^{-1}h_{ll} - e^{-1}e_{ll}\right)^4\right\} = \{12h^{-3}(h+4) + O(e^{-1})\}\sigma_{ll}^4,$$

so again using Conditions 1 and 3, we have

$$\sum_l E(T_{nl}^4|\mathcal{F}_{n,l-1}) = \sum_l E(T_{nl}^4) = \frac{\{12h^{-3}(h+4) + O(e^{-1})\}}{(n-1)^2}\operatorname{tr}(\Sigma^4) \to 0.$$

This establishes (4), and so the proof is complete. □

In order to use t_{nm} in practice, we will need to estimate $\sigma_{t_{np}}^2$, and this involves finding an estimator of $\operatorname{tr}(\Sigma^2)$. Now

$$E\{\operatorname{tr}(E)^2\} = 2e\operatorname{tr}(\Sigma^2) + e^2\operatorname{tr}(\Sigma)^2$$

and

$$E\{\operatorname{tr}(E^2)\} = (e^2+e)\operatorname{tr}(\Sigma^2) + e\operatorname{tr}(\Sigma)^2,$$

from which it follows that $E(a) = \operatorname{tr}(\Sigma^2)$, where

$$a = (e+2)^{-1}(e-1)^{-1}\left\{\operatorname{tr}(E^2) - e^{-1}\operatorname{tr}(E)^2\right\}. \tag{5}$$

An unbiased estimator of $\sigma_{t_{np}}^2$ is then given by $\hat{\sigma}_{t_{np}}^2 = 2h^{-1}a/e$. Further,

$$\begin{aligned}E\{\operatorname{tr}(E^2)^2\} = {} & (8e^3+20e^2+20e)\operatorname{tr}(\Sigma^4) + (16e^2+16e)\operatorname{tr}(\Sigma^3)\operatorname{tr}(\Sigma) \\ & +(e^4+2e^3+5e^2+4e)\operatorname{tr}(\Sigma^2)^2 + (2e^3+2e^2+8e)\operatorname{tr}(\Sigma^2)\operatorname{tr}(\Sigma)^2 \\ & +e^2\operatorname{tr}(\Sigma)^4,\end{aligned}$$

$$\begin{aligned}E\{\operatorname{tr}(E^2)\operatorname{tr}(E)^2\} = {} & (24e^2+24e)\operatorname{tr}(\Sigma^4) + (8e^3+8e^2+16e)\operatorname{tr}(\Sigma^3)\operatorname{tr}(\Sigma) \\ & +(2e^3+2e^2+8e)\operatorname{tr}(\Sigma^2)^2 + (e^4+e^3+10e^2)\operatorname{tr}(\Sigma^2)\operatorname{tr}(\Sigma)^2 \\ & +e^3\operatorname{tr}(\Sigma)^4,\end{aligned}$$

$$\begin{aligned}E\{\operatorname{tr}(E)^4\} = {} & 48e\operatorname{tr}(\Sigma^4) + 32e^2\operatorname{tr}(\Sigma^3)\operatorname{tr}(\Sigma) + 12e^2\operatorname{tr}(\Sigma^2)^2 + 12e^3\operatorname{tr}(\Sigma^2)\operatorname{tr}(\Sigma)^2 \\ & + e^4\operatorname{tr}(\Sigma)^4,\end{aligned}$$

and these lead to

$$\operatorname{var}(a) = (e+2)^{-2}(e-1)^{-2}\left[\{8e^3 + o(e^3)\}\operatorname{tr}(\Sigma^4) + \{4e^2 + o(e^2)\}\operatorname{tr}(\Sigma^2)^2\right].$$

As a result, it follows from Condition 3 that

$$\operatorname{var}(\hat{\sigma}^2_{t_{np}}) = \frac{4}{h^2e^2}\operatorname{var}(a)$$

converges to 0, and so $\hat{\sigma}^2_{t_{np}}$ converges in probability to $2h^{-1}\gamma\tau_1$. Thus, it follows from Theorem 1 that the asymptotic null distribution of $t^*_{np} = t_{np}/\hat{\sigma}_{t_{np}}$ is $N(0, 1)$.

3. A test for dimensionality

When H_0 is rejected, it may be useful to determine the dimension of the hyperplane formed by the population mean vectors, $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_g$. For instance, this dimension gives the number of discriminant functions necessary to describe differences among the groups. This dimension also corresponds to the number of positive eigenvalues of the matrix Δ and the matrix Ψ given in Condition 2. In this section, we consider a test of

$$\mathrm{H}_{0r} : \phi_r(\Psi) = 0, \quad \mathrm{H}_{1r} : \phi_r(\Psi) > 0. \tag{6}$$

Note that H_{01} is the hypothesis of equal mean vectors, H_0, discussed in the previous section. To determine the dimensionality, one would test (6) first with $r = 1$, then with $r = 2$, and continue until either H_{0r} is not rejected for some r or it is rejected for $r = \min(g-1, p)$.

When E is nonsingular, tests of H_{0r}, like tests of H_0, are commonly based on the eigenvalues of HE^{-1}. For example, Bartlett's [4] test uses the test statistic

$$U_r = \left\{n - 1 - \frac{1}{2}(p+g)\right\}\sum_{i=r}^{h}\log\{1 + \phi_i(HE^{-1})\}.$$

If p is fixed and H_{0r} holds, U_r converges in distribution to the chi-squared distribution with $(p-r+1)(g-r)$ degrees of freedom as e approaches infinity. In this section, we develop an alternative test that can be used when E is singular.

In testing H_{0r}, we will use the eigenvalues of $h_r^{-1}H - e^{-1}E$, where $h_r = h - r + 1$. In particular, we consider the statistic

$$u_{r,np} = (n-1)^{-1/2}\sum_{i=r}^{p}\phi_i(h_r^{-1}H - e^{-1}E).$$

This is a generalization of the statistic used to test H_0 in the previous section in that $t_{np} = u_{1,np}$. We are interested in the distribution of $u_{r,np}$ under Conditions 1–3 and the following additional condition.

Condition 4. As $p \to \infty$, $\lim \phi_1(\Sigma) = \rho < \infty$ and

$$\lim_{p\to\infty}\frac{\operatorname{tr}(\Sigma)}{p} = \tau_0 \in (0, \infty).$$

We first will show that treating the statistic $u_{r,np}$ as an $N(0, \sigma_*^2)$ random variable, where $\sigma_*^2 = 2(h_r e)^{-1}\,\mathrm{tr}(\Sigma^2)$, yields a test of H_{0r} that is conservative relative to the test of H_0 based on t_{np} and the result of Theorem 1. Note that if H_{0r} holds, then H can be written as $H = H_1 + H_2$, where $H_1 \sim W_p(\Sigma, h_r, 0)$ and $H_2 \sim W_p(\Sigma, r-1, \Delta)$, independently. For fixed H_2, let F be a $(p-r+1)\times p$ matrix satisfying $FH_2F' = 0$ and $FF' = I_{p-r+1}$. Since $FEF' \sim W_{p-r+1}(F\Sigma F', e, 0)$ and $FH_1F' \sim W_{p-r+1}(F\Sigma F', h_r, 0)$ independently, an application of Theorem 1 shows that the distribution of $(n-1)^{-1/2}\{h_r^{-1}\,\mathrm{tr}(FH_1F') - e^{-1}\,\mathrm{tr}(FEF')\}$ can be approximated by $N(0, \sigma_F^2)$, where $\sigma_F^2 = 2(h_r e)^{-1}\,\mathrm{tr}\{(F\Sigma F')^2\} < \sigma_*^2$. For a specified constant c, define the sets

$$B_1 = \left\{E, H_1, H_2 : (n-1)^{-1/2}\sum_{i=r}^{p}\phi_i(h_r^{-1}H - e^{-1}E) > c\right\},$$

$$B_2(H_2) = \left\{E, H_1 : (n-1)^{-1/2}\sum_{i=r}^{p}\phi_i(h_r^{-1}H - e^{-1}E) > c\right\},$$

$$B_3(H_2) = \left\{E, H_1 : (n-1)^{-1/2}\{h_r^{-1}\,\mathrm{tr}(FH_1F') - e^{-1}\,\mathrm{tr}(FEF')\} > c\right\}.$$

It follows from the Poincaré separation theorem (see, for example, [16, p. 111]) that $B_2(H_2) \subset B_3(H_2)$. As a result, for $c > 0$, we have

$$\begin{aligned} P(u_{r,np} > c) &= E_{H,E}\{I(B_1)\} = E_{H_2}\{E_{H_1,E}\{I(B_2(H_2))\}\} \\ &\leqslant E_{H_2}\{E_{H_1,E}\{I(B_3(H_2))\}\} \approx E_{H_2}\{1 - \Phi(c/\sigma_F)\} \\ &\leqslant E_{H_2}\{1 - \Phi(c/\sigma_*)\} = 1 - \Phi(c/\sigma_*), \end{aligned} \tag{7}$$

where $I(\cdot)$ denotes the indicator function and $\Phi(\cdot)$ is the standard normal distribution function.

We will show that, under certain conditions, $u_{r,np}$ does in fact converge in distribution to a normal random variable. Since $u_{r,np}$ is invariant under transformations of the observation vectors by an orthogonal matrix, we may assume without loss of generality that under H_{0r}, Δ is of the form

$$\Delta = \begin{pmatrix} \Delta_* & (0) \\ (0) & (0) \end{pmatrix},$$

where Δ_* is an $(r-1)\times(r-1)$ positive definite matrix. Similarly partition Σ and E as

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix}, \quad E = \begin{pmatrix} E_{11} & E_{12} \\ E_{12}' & E_{22} \end{pmatrix}.$$

For the partitioning of H, we will write

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12}' & H_{22} \end{pmatrix} = \begin{pmatrix} \Delta_*^{1/2}Y_1Y_1'\Delta_*^{1/2} & \Delta_*^{1/2}Y_1Y_2' \\ Y_2Y_1'\Delta_*^{1/2} & Y_2Y_2' \end{pmatrix},$$

where the columns of the $(r-1)\times h$ matrix Y_1 are independently distributed normal random vectors with covariance matrix $\Delta_*^{-1/2}\Sigma_{11}\Delta_*^{-1/2}$, $E(Y_1) = I_{h,1}$, and $I_{h,1}$ denotes the $(r-1)\times h$ matrix having 1 in the (i,i)th position, $i = 1, \ldots, r-1$, and zeros elsewhere. The columns of the $(p-r+1)\times h$ matrix Y_2 are independently distributed normal random vectors each with $\mathbf{0}$ mean vector and covariance matrix Σ_{22}. Partition Y_2 as $Y_2 = (Y_{21}, Y_{22})$, where Y_{22} is $(p-r+1)\times h_r$, so that $Y_{22}Y_{22}' \sim W_{p-r+1}(\Sigma_{22}, h_r, 0)$. Since $E_{22} \sim W_{p-r+1}(\Sigma_{22}, e, 0)$, it follows from the previous section that $t_* = (n-1)^{-1/2}\operatorname{tr}(h_r^{-1}Y_{22}Y_{22}' - e^{-1}E_{22})$ has mean 0 and variance

$$\sigma_{t_*}^2 = \frac{2}{h_r e}\operatorname{tr}(\Sigma_{22}^2). \tag{8}$$

Note that Conditions 3 and 4 guarantee that $\sigma_{t_*}^2 \to 2h_r^{-1}\gamma\tau_1$. Thus, under Conditions 1–4, we know from Theorem 1 that t_* converges in distribution to $N(0, 2h_r^{-1}\gamma\tau_1)$. We will next show that if we define $H_{22\cdot 1} = H_{22} - H_{12}'H_{11}^{-1}H_{12}$, then $(n-1)^{-1/2}\operatorname{tr}(h_r^{-1}H_{22\cdot 1} - e^{-1}E_{22})$ has this same asymptotic distribution.

Theorem 2. *Under Conditions* 1–4, *if* $\phi_r(\Psi) = 0$, *then*

$$(n-1)^{-1/2}\operatorname{tr}(h_r^{-1}H_{22\cdot 1} - e^{-1}E_{22}) \to^{\mathrm{d}} N(0, 2h_r^{-1}\gamma\tau_1).$$

Proof. Note that

$$\begin{aligned}
&(n-1)^{-1/2}\operatorname{tr}(h_r^{-1}H_{22\cdot 1} - e^{-1}E_{22}) - (n-1)^{-1/2}\operatorname{tr}(h_r^{-1}Y_{22}Y_{22}' - e^{-1}E_{22})\\
&\quad = (n-1)^{-1/2}h_r^{-1}\operatorname{tr}(H_{22\cdot 1} - Y_{22}Y_{22}')\\
&\quad = (n-1)^{-1/2}h_r^{-1}\operatorname{tr}(Y_{21}Y_{21}' - Y_2Y_1'(Y_1Y_1')^{-1}Y_1Y_2')\\
&\quad = (n-1)^{-1/2}h_r^{-1}\operatorname{tr}(Y_{21}'Y_{21} - Y_2'Y_2Y_1'(Y_1Y_1')^{-1}Y_1)\\
&\quad = (n-1)^{-1/2}h_r^{-1}\operatorname{tr}(I_{h,1}Y_2'Y_2I_{h,1}' - Y_2'Y_2J_{Y_1})\\
&\quad = g_1(J_{Y_1}, (n-1)^{-1/2}Y_2'Y_2),
\end{aligned}$$

where $J_{Y_1} = Y_1'(Y_1Y_1')^{-1}Y_1$. Since $\Delta_*^{-1/2}\Sigma_{11}\Delta_*^{-1/2} \to 0$, Y_1 converges in probability to $I_{h,1}$ and, hence, J_{Y_1} converges in probability to $J = I_{h,1}'I_{h,1}$. In addition, if $\beta = \lim n^{-1}\operatorname{tr}(\Sigma_{22}) = \gamma\tau_0$, then it can be shown that $J_{Y_2} = (n-1)^{1/2}\{(n-1)^{-1}Y_2'Y_2 - \beta I_h\}$ converges in distribution to a random matrix, say V, so by an application of Slutsky's theorem, we find that $g_1(J_{Y_1}, (n-1)^{-1/2}Y_2'Y_2) = g_1(J_{Y_1}, J_{Y_2})$ converges in distribution to $g_1(J, V) = 0$. This establishes that the asymptotic distribution of $(n-1)^{-1/2}\operatorname{tr}(h_r^{-1}H_{22\cdot 1} - e^{-1}E_{22})$ is the same as that of t_*, and so the proof is complete. □

Before deriving the asymptotic distribution of $u_{r,np}$, we will need the following result [17] regarding a $p \times p$ symmetric matrix A partitioned as

$$\begin{pmatrix} B & C \\ C' & D \end{pmatrix}, \tag{9}$$

where B is $q \times q$, D is $m \times m$, and C is $q \times m$.

Lemma 1. *Suppose A in* (9) *is nonnegative definite with B being positive definite while* $\operatorname{rank}(\hat{D}) = s$, *where $\hat{D} = D - C'B^{-1}C$. Let Q be any $m \times s$ matrix satisfying $Q'Q = I_s$ and $\hat{D} = Q\Lambda Q'$, where Λ is a diagonal matrix with the positive eigenvalues of $\hat{D}$ as its diagonal elements.*

Define $\hat{B}_* = B - CQ(Q'DQ)^{-1}Q'C'$ *and* $\hat{C}_* = -B^{-1}CQ\Lambda^{-1}$. *Then if* $\phi_1(\hat{D}) < \phi_q(\hat{B}_*)$,

$$0 \leqslant \sum_{i=1}^{m-s+k} \{\phi_{m-i+1}(\hat{D}) - \phi_{p-i+1}(A)\} \leqslant \frac{\phi^2_{s-k+1}(\hat{D})}{\{\phi^{-1}_{s-k+1}(\hat{D}) - \phi_q^{-1}(\hat{B}_*)\}} \sum_{i=1}^{k} \phi_i(\hat{C}_*'\hat{C}_*)$$

for $k = 1, \ldots, s$.

We now are ready to give the asymptotic distribution of $u_{r,np}$ under H_{0r}.

Theorem 3. *Under Conditions* 1–4, *if* $\phi_r(\Psi) = 0$, *then*

$$(n-1)^{-1/2} \sum_{i=r}^{p} \phi_i(h_r^{-1}H - e^{-1}E) \to^{\mathrm{d}} N(0, 2h_r^{-1}\gamma\tau_1).$$

Proof. Note that due to Theorem 2, our proof will be complete if we can show that

$$(n-1)^{-1/2} \left\{ \mathrm{tr}(h_r^{-1}H_{22\cdot 1} - e^{-1}E_{22}) - \sum_{i=r}^{p} \phi_i(h_r^{-1}H - e^{-1}E) \right\} = o_p(1). \tag{10}$$

Now it follows (see, for example, [16, Theorem 3.24]) that

$$\begin{aligned}\sum_{i=r}^{p} h_r^{-1}\phi_i(H) - \sum_{i=1}^{p-r+1} e^{-1}\phi_i(E) &\leqslant \sum_{i=r}^{p} \phi_i(h_r^{-1}H - e^{-1}E) \\ &\leqslant \sum_{i=r}^{p} h_r^{-1}\phi_i(H) - \sum_{i=r}^{p} e^{-1}\phi_i(E).\end{aligned}$$

Since when $r \leqslant p - r + 1$

$$\sum_{i=r}^{p} \phi_i(E) - \mathrm{tr}(E_{22}) \geqslant \sum_{i=r}^{p} \phi_i(E) - \sum_{i=1}^{p-r+1} \phi_i(E) \geqslant -(r-1)\phi_1(E)$$

and

$$\sum_{i=1}^{p-r+1} \phi_i(E) - \mathrm{tr}(E_{22}) \leqslant \sum_{i=1}^{p-r+1} \phi_i(E) - \sum_{i=r}^{p} \phi_i(E) \leqslant (r-1)\phi_1(E),$$

this then leads to

$$\begin{aligned}&h_r^{-1}\mathrm{tr}(H_{22\cdot 1}) - \sum_{i=r}^{p} h_r^{-1}\phi_i(H) - (r-1)e^{-1}\phi_1(E) \\ &\quad\leqslant \mathrm{tr}(h_r^{-1}H_{22\cdot 1} - e^{-1}E_{22}) - \sum_{i=r}^{p} \phi_i(h_r^{-1}H - e^{-1}E) \\ &\quad\leqslant h_r^{-1}\mathrm{tr}(H_{22\cdot 1}) - \sum_{i=r}^{p} h_r^{-1}\phi_i(H) + (r-1)e^{-1}\phi_1(E).\end{aligned} \tag{11}$$

Let $s = \mathrm{rank}(H_{22\cdot 1}) = \min(h_r, p - r + 1)$ and let Q be any $(p - r + 1) \times s$ matrix for which $Q'Q = I_s$ and $Q'H_{22\cdot 1}Q = \Lambda$ is diagonal with positive diagonal elements. Consider the set

$C = \{H : \phi_1(H_{22\cdot 1}) < \phi_{r-1}(\hat{H}_{11})\}$, where $\hat{H}_{11} = H_{11} - H_{12}Q(Q'H_{22}Q)^{-1}Q'H_{12}'$. Now $\phi_{r-1}^{-1}(\hat{H}_{11})$ is the largest eigenvalue of

$$\begin{aligned}\hat{H}_{11}^{-1} &= H_{11}^{-1} + H_{11}^{-1}H_{12}Q\Lambda^{-1}Q'H_{12}'H_{11}^{-1}\\ &= \Delta_*^{-1/2}(Y_1Y_1')^{-1}\Delta_*^{-1/2}\\ &\quad + \Delta_*^{-1/2}(Y_1Y_1')^{-1}Y_1Y_2'Q\Lambda^{-1}Q'Y_2Y_1'(Y_1Y_1')^{-1}\Delta_*^{-1/2}.\end{aligned}$$

Using properties of the eigenvalues of a matrix product (see, for example, [1]), this leads to

$$\begin{aligned}\phi_{r-1}^{-1}(\hat{H}_{11}) &\leqslant \phi_1(\Delta_*^{-1})\phi_1((Y_1Y_1')^{-1})\left\{1 + \phi_1(\Lambda^{-1})\,\mathrm{tr}(Y_2'QQ'Y_2Y_1'(Y_1Y_1')^{-1}Y_1)\right\}\\ &\leqslant \phi_{r-1}^{-1}(\Delta_*)\phi_{r-1}^{-1}(Y_1Y_1')\left\{1 + \phi_s^{-1}(H_{22\cdot 1})\,\mathrm{tr}(Y_2'QQ'Y_2J_{Y_1})\right\}.\end{aligned} \tag{12}$$

In addition, we find that

$$\phi_1(H_{22\cdot 1}) \leqslant \phi_1(Y_2Y_2') \leqslant \phi_1(\Sigma_{22})\phi_1(Z_2Z_2') \leqslant \phi_1(\Sigma_{22})\phi_1(Z_2'Z_2), \tag{13}$$

where $Y_2 = \Sigma_{22}^{1/2}Z_2$ so that the elements of the $(p - r + 1) \times h$ matrix Z_2 are independent and identically distributed standard normal random variables. Combining (12) and (13), we get

$$\begin{aligned}\phi_1(H_{22\cdot 1})\phi_{r-1}^{-1}(\hat{H}_{11}) &\leqslant \phi_1(\Sigma_{22})\phi_1(Z_2'Z_2)\phi_{r-1}^{-1}(\Delta_*)\phi_{r-1}^{-1}(Y_1Y_1')\\ &\quad \times\left\{1 + \phi_s^{-1}(H_{22\cdot 1})\,\mathrm{tr}(Y_2'QQ'Y_2J_{Y_1})\right\}.\end{aligned} \tag{14}$$

Now $\phi_{r-1}(Y_1Y_1') \to^{\mathrm{p}} 1$, $p^{-1}\phi_1(Z_2'Z_2) \to^{\mathrm{p}} 1$, $\phi_{r-1}(\Delta_*)$ is $O(n^2)$, and it is easily shown that $\phi_s(H_{22\cdot 1})$ is $O_p(n)$. Also it follows from Condition 4 that $\phi_1(\Sigma_{22})$ is $O(1)$, and so all that remains is to determine the order of the trace term in (14). Note that QQ' is the projection matrix of the column space of $Y_2(I_h - J_{Y_1})$, so we have

$$\begin{aligned}\mathrm{tr}(Y_2'QQ'Y_2J_{Y_1}) &= \mathrm{tr}\left(Y_2'Y_2(I - J_{Y_1})\{(I - J_{Y_1})Y_2'Y_2(I - J_{Y_1})\}^+(I - J_{Y_1})Y_2'Y_2J_{Y_1}\right)\\ &= \mathrm{tr}\Big((n-1)^{-1/2}Y_2'Y_2(I - J_{Y_1})\{(I - J_{Y_1})(n-1)^{-1}Y_2'Y_2(I - J_{Y_1})\}^+\\ &\quad \times(I - J_{Y_1})(n-1)^{-1/2}Y_2'Y_2J_{Y_1}\Big)\\ &= \mathrm{tr}\Big(J_{Y_2}(I - J_{Y_1})\{(I - J_{Y_1})(n-1)^{-1}Y_2'Y_2(I - J_{Y_1})\}^+\\ &\quad \times(I - J_{Y_1})J_{Y_2}J_{Y_1}\Big)\\ &= g_2(J_{Y_1}, J_{Y_2}, (n-1)^{-1}Y_2'Y_2).\end{aligned}$$

Since J_{Y_1} converges in probability to J, $(n-1)^{-1}Y_2'Y_2$ converges in probability to βI_h, and J_{Y_2} converges in distribution to V, it follows that $\mathrm{tr}(Y_2'QQ'Y_2J_{Y_1})$ converges in distribution to $g_2(J, V, \beta I) = \beta^{-1}\,\mathrm{tr}(V_{12}V_{12}')$, where the $h \times h$ random matrix V has been partitioned as

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}' & V_{22} \end{pmatrix}$$

with V_{12} being $(r-1) \times h_r$. This implies that $\mathrm{tr}(Y_2'QQ'Y_2J_{Y_1})$ is $O_p(1)$ and so we have shown that the right-hand side of (14) converges in probability to 0. That is, we have established that $P(C) \to 1$ and so attention can be restricted to this set. For $H \in C$, Lemma 1 implies that

$$0 \leqslant \operatorname{tr}(H_{22\cdot 1}) - \sum_{i=r}^{p} \phi_i(H) \leqslant \frac{\phi_1^2(H_{22\cdot 1})}{\{\phi_1^{-1}(H_{22\cdot 1}) - \phi_{r-1}^{-1}(\hat{H}_{11})\}} \operatorname{tr}(\Lambda^{-1} Q' H_{12}' H_{11}^{-2} H_{12} Q \Lambda^{-1}).$$

Using this in (11), we get

$$\begin{aligned} -(r-1)e^{-1}\phi_1(E) &\leqslant \operatorname{tr}(h_r^{-1} H_{22\cdot 1} - e^{-1} E_{22}) - \sum_{i=r}^{p} \phi_i(h_r^{-1} H - e^{-1} E) \\ &\leqslant \frac{h_r^{-1}\phi_1^2(H_{22\cdot 1})}{\{\phi_1^{-1}(H_{22\cdot 1}) - \phi_{r-1}^{-1}(\hat{H}_{11})\}} \operatorname{tr}(\Lambda^{-1} Q' H_{12}' H_{11}^{-2} H_{12} Q \Lambda^{-1}) \\ &\quad + (r-1)e^{-1}\phi_1(E) \\ &\leqslant \frac{h_r^{-1}\phi_1^2(H_{22\cdot 1})\phi_s^{-2}(H_{22\cdot 1})\phi_{r-1}^{-1}(H_{11})}{\{\phi_1^{-1}(H_{22\cdot 1}) - \phi_{r-1}^{-1}(\hat{H}_{11})\}} \operatorname{tr}(Y_2' Q Q' Y_2 J_{Y_1}) \\ &\quad + (r-1)e^{-1}\phi_1(E). \end{aligned} \tag{15}$$

Now $e^{-1}\phi_1(E) \leqslant \phi_1(\Sigma)e^{-1}\phi_1(E_*)$, where $E_* \sim W_p(I_p, e)$, and $e^{-1}\phi_1(E_*)$ converges in probability to $(1+\gamma^{1/2})^2$ [21] so $e^{-1}\phi_1(E)$ is $O_p(1)$. In addition, $\phi_{r-1}(H_{11})$ and $\phi_{r-1}(\hat{H}_{11})$ are $O_p(n^2)$, $\phi_1(H_{22\cdot 1})$, and $\phi_s(H_{22\cdot 1})$ are $O_p(n)$, whereas $\operatorname{tr}(Y_2' Q Q' Y_2 J_{Y_1})$ is $O_p(1)$. Consequently, both the lower bound and upper bound given in (15) are $O_p(1)$, and so this establishes (10). □

When using $u_{r,np}$ to test H_{0r}, we will need an estimator of its variance, and we see from Theorem 3 that any estimator can be used as long as it converges in probability to $2h_r^{-1}\gamma\tau_1$. For instance, in view of (7), the choice of $\hat{\sigma}_*^2 = 2(h_r e)^{-1} a$, where a is given in (5), would generally yield smaller significance levels than those produced by t_{np}. An alternative approach, which may produce better results for smaller sample sizes, is to use an estimate of $\sigma_{t_*}^2$ given in (8). In particular, we will use

$$\hat{\sigma}_{t_*}^2 = 2(h_r e)^{-1} a_r, \tag{16}$$

where

$$a_r = (e+2)^{-1}(e-1)^{-1}\left\{\operatorname{tr}((L'EL)^2) - e^{-1}\operatorname{tr}(L'EL)^2\right\},$$

and L is a $p \times (p-r+1)$ matrix whose columns form an orthonormal set of eigenvectors corresponding to the $p-r+1$ smallest eigenvalues of $h_r^{-1}H - e^{-1}E$. It was shown in Section 2 that $e^{-1}a$ converges in probability to $\gamma\tau_1$ and if the same is true of $e^{-1}a_r$, then it will follow that $\hat{\sigma}_{t_*}^2$ converges in probability to $2h_r^{-1}\gamma\tau_1$. Now

$$\begin{aligned} e^{-1}(a-a_r) = e^{-1}(e+2)^{-1}(e-1)^{-1}\Big[&\{\operatorname{tr}(E^2) - \operatorname{tr}((L'EL)^2)\} \\ &+ e^{-1}\{\operatorname{tr}(E)^2 - \operatorname{tr}(L'EL)^2\}\Big], \end{aligned} \tag{17}$$

and it is easily shown that

$$(r-1)\phi_p^2(\Sigma)\phi_p^2(E_*) \leqslant \operatorname{tr}(E^2) - \operatorname{tr}((L'EL)^2) \leqslant (r-1)\phi_1^2(\Sigma)\phi_1^2(E_*) \tag{18}$$

and

$$(r-1)\phi_p^2(\Sigma)\phi_p^2(E_*)\{(r-1)+2(p-r+1)\} \leqslant \operatorname{tr}(E)^2 - \operatorname{tr}(L'EL)^2 \leqslant (r-1)\phi_1^2(\Sigma)\phi_1^2(E_*)\{(r-1)+2(p-r+1)\}, \tag{19}$$

where $E_* \sim W_p(I_p, e, 0)$. Using (18) and (19) in (17), we obtain upper and lower bounds on $e^{-1}(a - a_r)$. Since $e^{-1}\phi_p(E_*)$ converges in probability to $(1-\gamma^{1/2})^{-2}$ [3] and $e^{-1}\phi_1(E_*)$ converges in probability to $(1+\gamma^{1/2})^2$, we find that both of these bounds converge in probability to 0 thereby confirming that $e^{-1}a_r$ converges in probability to $\gamma\tau_1$.

4. Some simulation results

Some simulation results were obtained so as to assess the effectiveness of the asymptotic normal distribution in approximating the actual null distributions of t_{np} and $u_{r,np}$. We restricted attention to the case in which $g = 3$ and $n_1 = n_2 = n_3$. Both p and n_i ranged over the values 4, 8, 16, 32, 64, and 128, and for each setting the significance level was estimated from 5000 simulations. The nominal significance level used was 0.05. Two different forms were used for the common covariance matrix, one being I_p, while the second had block diagonal structure with each block on the diagonal given by the 4×4 matrix $0.5I_4 + 0.5\mathbf{1}_4\mathbf{1}_4'$, where $\mathbf{1}_4$ is the 4×1 vector with each component equal to 1. Thus, this second covariance matrix used has $p/4$ of its eigenvalues equal to 2.5 while the remaining eigenvalues are all equal to 0.5.

Table 1 gives the estimated significance levels for the test of H_0 based on $t_{np}^* = t_{np}/\hat{\sigma}_{t_{np}}$ when $\Sigma = I_p$, while Table 2 tabulates the estimates when Σ has the block diagonal structure. The normal approximation consistently yields inflated significance levels, and this inflation is more pronounced in Table 2, that is, in the case in which the variables are correlated. These estimated significance levels are reasonably close to the 0.05 nominal level except when both p and n_i are very small. Table 3 has results for the statistic T_{np}, mentioned in Section 1, when $\Sigma = I_p$. Upon comparing Table 3 with Table 1, we find that T_{np} generally yields significance levels that are slightly more inflated than those of t_{np}^* if the sample sizes are not large.

Additional simulations were performed to compare the powers of t_{np}^* and T_{np} when $\Sigma = I_p$. Two of the populations had mean vectors of $\mathbf{0}$, while the third had a nonzero mean vector. In particular, the ith component of this third mean vector is equal to 0.5 when i is a multiple of 4 and 0 otherwise. The power estimates are given in Tables 4 and 5. The two tests seem to have very

Table 1
Estimated significance levels for t_{np}^* when $\Sigma = I_p$

p	n_i					
	4	8	16	32	64	128
4	0.092	0.075	0.070	0.074	0.063	0.064
8	0.090	0.065	0.065	0.070	0.064	0.065
16	0.080	0.066	0.061	0.059	0.059	0.065
32	0.072	0.059	0.064	0.061	0.061	0.059
64	0.063	0.063	0.057	0.054	0.061	0.053
128	0.064	0.057	0.055	0.054	0.054	0.057

Table 2
Estimated significance levels for t_{np}^{*} when Σ has block diagonal structure

p	n_i					
	4	8	16	32	64	128
4	0.118	0.093	0.088	0.079	0.075	0.074
8	0.110	0.088	0.076	0.076	0.070	0.070
16	0.095	0.081	0.074	0.073	0.069	0.067
32	0.083	0.072	0.072	0.069	0.068	0.066
64	0.081	0.060	0.068	0.057	0.062	0.066
128	0.076	0.057	0.070	0.055	0.057	0.061

Table 3
Estimated significance levels for T_{np} when $\Sigma = I_p$

p	n_i					
	4	8	16	32	64	128
4	0.100	0.079	0.072	0.074	0.064	0.064
8	0.101	0.071	0.067	0.070	0.064	0.065
16	0.089	0.069	0.062	0.060	0.059	0.065
32	0.082	0.063	0.067	0.061	0.061	0.059
64	0.074	0.066	0.058	0.055	0.061	0.053
128	0.077	0.062	0.057	0.056	0.055	0.057

Table 4
Estimated power for t_{np}^{*} when $\Sigma = I_p$

p	n_i					
	4	8	16	32	64	128
4	0.127	0.151	0.206	0.360	0.672	0.942
8	0.130	0.150	0.269	0.525	0.878	0.998
16	0.133	0.188	0.372	0.731	0.985	1.000
32	0.140	0.248	0.535	0.918	1.000	1.000
64	0.177	0.357	0.770	0.996	1.000	1.000
128	0.232	0.554	0.947	1.000	1.000	1.000

Table 5
Estimated power for T_{np} when $\Sigma = I_p$

p	n_i					
	4	8	16	32	64	128
4	0.140	0.158	0.209	0.362	0.673	0.942
8	0.142	0.155	0.271	0.526	0.878	0.998
16	0.147	0.197	0.377	0.733	0.985	1.000
32	0.160	0.257	0.540	0.919	1.000	1.000
64	0.202	0.369	0.777	0.996	1.000	1.000
128	0.256	0.567	0.949	1.000	1.000	1.000

Table 6
Estimated significance levels for $u_{2,np}$ when $\Sigma = I_p$

p	n_i					
	4	8	16	32	64	128
4	0.076	0.075	0.066	0.063	0.073	0.068
8	0.064	0.066	0.074	0.070	0.062	0.067
16	0.071	0.060	0.062	0.063	0.071	0.066
32	0.069	0.058	0.061	0.060	0.062	0.061
64	0.056	0.053	0.057	0.061	0.059	0.054
128	0.046	0.054	0.052	0.059	0.058	0.055

similar power properties; the power is slightly higher for T_{np} if n_i is not large, but this is attributable to the higher significance levels observed in Table 3. Finally, Table 6 has some estimated significance levels for the test of H_{0r} based on $u_{r,np}$ when $\Sigma = I_p$. In our simulations, we restricted attention to the case in which $r = 2$. In these simulations, the means for the three groups were the same as in the simulations for Tables 4 and 5, except the nonzero components were equal to 3 instead of 0.5. For the estimate of the variance of the normal distribution, we used (16). The significance levels for $u_{2,np}$, like those for t_{np}, are inflated, but less so than what was observed in Tables 1 and 2, especially when p or n_i is very small.

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References

[1] T.W. Anderson, S. Das Gupta, Some inequalities on characteristic roots of matrices, Biometrika 50 (1963) 522–524.

[2] Z.D. Bai, H. Saranadasa, Effect of high dimension: by an example of a two sample problem, Statist. Sinica 6 (1996) 311–329.

[3] Z.D. Bai, Y.Q. Yin, Limit of the smallest eigenvalue of a large dimensional sample covariance-matrix, Ann. Probab. 21 (1993) 1275–1294.

[4] M.S. Bartlett, Further aspects of the theory of multiple regression, Proc. Cambridge Philos. Soc. 34 (1938) 33–40.

[5] J.H. Chung, D.A.S. Fraser, Randomization tests for a multivariate two-sample problem, J. Amer. Statist. Assoc. 53 (1958) 729–735.

[6] A.W. Davis, On the differential equation for Meijer's $G^{p,0}_{p,p}$ function, and further tables of Wilk's likelihood ratio criterion, Biometrika 66 (1979) 519–531.

[7] A.P. Dempster, A high dimensional two sample significance test, Ann. Math. Statist. 29 (1958) 995–1010.

[8] A.P. Dempster, A significance test for the separation of two highly multivariate small samples, Biometrics 16 (1960) 41–50.

[9] Y. Fujikoshi, Multivariate analysis for the case when the dimension is large compared to the sample size, J. Korean Statist. Soc. 33 (2004) 1–24.

[10] Y. Fujikoshi, T. Himeno, H. Wakaki, Asymptotic results of a high dimensional MANOVA test and power comparisons when the dimension is large compared to the sample size, J. Japan Statist. Soc. 34 (2004) 19–26.

[11] P. Hall, C.C. Heyde, Martingale Limit Theory and its Applications, Academic Press, New York, 1980.

[12] Y.S. Lee, Some results on the distribution of Wilk's likelihood ratio criterion, Biometrika 59 (1972) 649–664.

[13] A.M. Mathai, On the distribution of the likelihood ratio criterion for testing linear hypotheses on regression coefficients, Ann. Inst. Statist. Math. 23 (1971) 181–197.

[14] K.C.S. Pillai, A.K. Gupta, On the exact distribution of Wilk's criterion, Biometrika 56 (1969) 109–118.

[15] M. Schatzoff, Exact distributions of Wilk's likelihood ratio criterion, Biometrika 53 (1966) 347–358.

[16] J.R. Schott, Matrix Analysis for Statistics, second ed., Wiley, New York, 2005.

[17] J.R. Schott, A high dimensional test for the equality of the smallest eigenvalues of a covariance matrix, J. Multivariate Anal. 97 (2006) 827–843.
[18] M.S. Srivastava, Multivariate theory for analyzing high dimensional data, J. Japan Statist. Soc. (2007), to appear.
[19] M.S. Srivastava, Y. Fujikoshi, Multivariate analysis of variance with fewer observations than the dimension, J. Multivariate Anal. 97 (2006) 1927–1940.
[20] T. Tonda, Y. Fujikoshi, Asymptotic expansion of the null distribution of LR statistic for multivariate linear hypothesis when the dimension is large, Comm. Statist. Theory Methods 33 (2004) 1205–1220.
[21] Y.Q. Yin, Z.D. Bai, P.R. Krishnaiah, On the limit of the largest eigenvalue of the large dimensional sample covariance matrix, Probab. Theory Related Fields 78 (1988) 509–521.