COMPITO DEL 12/02/2018

ES. 1 $\mathcal{F} = \mathcal{F}(2, t, e_{max}, e_{min})$ con arrotondamento

- $\begin{cases} t = e_{min} + 1 \\ realmin = \frac{1}{32} \\ realmax = 62 \end{cases}$

$$realmin = B^{-e_{min}-1} = 2^{-e_{min}-1}$$

$$realmax = B^{e_{max}}(1 - B^{-t})$$
$$= 2^{e_{max}}\left(1 - \frac{1}{2^t}\right)$$

$$realmin = (0.\underbrace{10\ldots0}_{t})_2 \cdot 2^{-e_{min}} = 2^{-1} \cdot 2^{-e_{min}}$$

$$realmax = (0.\underbrace{11\ldots1}_{t})_2 \cdot 2^{e_{max}} =$$

$$= 2^{e_{max}} \cdot \sum_{i=1}^{t} 2^{-i} = 2^{e_{max}}\left(\sum_{i=0}^{t} 2^{-i} - 1\right)$$
$$= 2^{e_{max}}(2 - 2^{-t} - 1)$$

$$\sum_{i=0}^{t} 2^{-i} = \sum_{i=0}^{t}\left(\frac{1}{2}\right)^i = \frac{\left(\frac{1}{2}\right)^{t+1} - 1}{\frac{1}{2} - 1}$$
$$= \frac{1 - \left(\frac{1}{2}\right)^{t+1}}{1/2} = 2\left(1 - 2^{-t-1}\right)$$
$$= 2 - 2^{-t}$$

$$\begin{cases} t = e_{min} + 1 \\ realmin = \dfrac{1}{32} \\ real\,max = 62 \end{cases}$$

$$2^{-e_{min}-1} = \frac{1}{2^5} = 2^{-5}$$

$$\boxed{t = 5}$$

$$-e_{min} - 1 = -5$$
$$-e_{min} = -4$$
$$\boxed{e_{min} = 4}$$

$$2^{e_{max}}\left(1 - \frac{1}{2^t}\right) = 62$$
$$2^{e_{max}}\left(1 - \frac{1}{2^5}\right) = 62$$
$$2^{e_{max}}\,\frac{2^5 - 1}{2^5} = 62$$
$$2^{e_{max}}\,\frac{31}{32} = 62$$

$$2^{e_{max}} = \frac{\cancel{62}^{\,2} \cdot 32}{\cancel{31}_{\,1}} = 64 = 2^6$$

$$\boxed{e_{max} = 6}$$

$$\mathbb{F} = \mathbb{F}(2, \overset{t}{5}, \overset{e_{max}}{6}, \overset{e_{min}}{4})$$

• $|\mathbb{F}|$

$$\mathbb{F} = \{0\} \cup \left\{ \pm(0.d_1 d_2 \dots d_t)_2 \cdot 2^e \;\middle|\; \begin{array}{l} d_1 = 1, \\ d_i \in \{0,1\}, \\ -e_{min} \le e \le e_{max} \end{array} \right\}$$

$$|\mathbb{F}| = 1 + 2 \cdot 1 \cdot 2^{t-1}(e_{max} + e_{min} + 1)$$
$$= 1 + 2 \cdot 2^4 (6 + 4 + 1) = 1 + 2^5 \cdot 11 = 1 + 32 \cdot 11 = 1 + 352$$
$$= 353$$

$$\pm(0.d_1 d_2 \dots d_t)_B \cdot 2^e \qquad d_1 \neq 0 \qquad d_1 \in \{1, 2, \dots, B-1\}$$

- $\pm(0.d_1d_2\dots d_t)_2 \cdot 2^e$ $\qquad d_1 = 0 \quad , \quad e = -e_{min}$

$\qquad \pm(0.0d_2\dots d_t)_2 \cdot 2^{-e_{min}}$

$\cancel{2 \cdot 2^{t-1} = 2^t = 2^5 = 32}$

$2 \cdot (2^{t-1} - 1) = 2 \cdot (2^4 - 1) = 2 \cdot 15 = 30$

quelli positivi sono ~~16~~ 15

$(0.0d_2d_3d_4d_5)_2 \cdot 2^{-4}$

$(0.00001)_2 \cdot 2^{-4} = 2^{-5} \cdot 2^{-4} = 2^{-9}$

$$\boxed{K \cdot 2^{-9} \quad , K \in \{1, \dots, 15\}}$$

$(0.01111)_2 \cdot 2^{-4} =$

$= \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}\right) \cdot 2^{-4} =$

$= \frac{8+4+2+1}{32} \cdot 2^{-4} = \frac{15}{32} \cdot 2^{-4} = 15 \cdot 2^{-9}$

- $u = \begin{cases} B^{1-t} & \text{tronc.} \\ \frac{B^{1-t}}{2} & \text{arrot.} \end{cases}$

$B = 2$, arrot. $\quad t = 5$

$u = \frac{2^{1-t}}{2} = 2^{-t} = 2^{-5} = \frac{1}{32}$

- $x = (1.\overline{01})_2 \qquad \tilde{x} = fl(x) = fl((1.\overline{01})_2) = fl((0.1\overline{01})_2 \cdot 2)$

$y = (10.\overline{01})_2 \qquad \tilde{y} = fl(y)$

$= fl((0.1010\overline{01})_2 \cdot 2)$

$= (0.10101)_2 \cdot 2$

$\tilde{y} = fl(y) = fl((10.\overline{01})_2) = fl((0.10\overline{01})_2 \cdot 2^2) = fl((0.1001\overline{01})_2 \cdot 2^2)$

$= (0.10011)_2 \cdot 2^2$

- $x, y, \tilde{x}, \tilde{y}$: scrivere come frazioni in base 10

$$\tilde{x} = (0.10101)_2 \cdot 2 = (2^{-1} + 2^{-3} + 2^{-5}) \cdot 2 = 2^0 + 2^{-2} + 2^{-4} = 1 + \frac{1}{4} + \frac{1}{16}$$

$$= \frac{16+4+1}{16} = \frac{21}{16}$$

$$\tilde{y} = (0.10011)_2 \cdot 2^2$$

$$= (2^{-1} + 2^{-4} + 2^{-5}) \cdot 2^2 = (2^1 + 2^{-2} + 2^{-3}) = 2 + \frac{1}{4} + \frac{1}{8} = \frac{16+2+1}{8} = \frac{19}{8}$$

$$x = (1.\overline{01})_2 = \frac{(101)_2 - (1)_2}{(11)_2} = \frac{(100)_2}{(11)_2} = \frac{4}{3}$$

$$y = (10.\overline{01})_2 \qquad 2^2 y = 4y = (1001.\overline{01})_2 = (1001)_2 + \underbrace{(0.\overline{01})_2 + (10)_2} - (10)_2$$

$$4y = y + (1001)_2 - (10)_2$$

$$3y = (1001)_2 - (10)_2 = 1 + 8 - 2 = 7$$

$$y = \frac{7}{3}$$

$$\left|\frac{x - \tilde{x}}{x}\right| = \left|\frac{4}{3} - \frac{21}{16}\right| \cdot \frac{3}{4} = \left|1 - \frac{63}{64}\right| = \frac{64-63}{64} = \frac{1}{64} = 2^{-6}$$

$$\left|\frac{y - \tilde{y}}{y}\right| = \left|1 - \frac{\tilde{y}}{y}\right| = \left|1 - \frac{19}{8} \cdot \frac{3}{7}\right| = \left|1 - \frac{57}{56}\right| = \left|\frac{1-57}{56}\right| = \left|-\frac{1}{56}\right| = \frac{1}{56}$$

• $\tilde{z} = \tilde{x} \; fl(+) \; \tilde{y}$ e calcolare e t.c. $\frac{\tilde{z}}{2^{e+1}} < \text{realmin} < \frac{\tilde{z}}{2^e}$

$$\tilde{z} = fl(\tilde{x} + \tilde{y}) = fl((0.10101)_2 \cdot 2 + (0.10011)_2 \cdot 2^2)$$
$$= fl((0.010101)_2 \cdot 2^2 + (0.10011)_2 \cdot 2^2)$$
$$= fl((0.\underbrace{1110}11)_2 \cdot 2^2)$$
$$= (0.11110)_2 \cdot 2^2 = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right) \cdot 2^2$$
$$= \frac{8+4+2+1}{16} \cdot 2^2 = \frac{15}{16} \cdot 2^2 = \frac{15}{4}$$

$$\begin{array}{r} 0.010101 \; + \\ 0.10011 \\ \hline 0.111011 \end{array}$$

$$\frac{15}{2^{e+3}} = \frac{15}{4} \cdot \frac{1}{2^{e+1}} < \frac{1}{32} < \frac{15}{4} \cdot \frac{1}{2^e} = \frac{15}{2^{e+2}}$$

$$15 < \frac{2^{e+3}}{2^5} < 15 \cdot 2$$

$$15 < 2^{e+3-5} < 30$$

1 2 4 8 16 32

$$2^{e-2} = 16 = 2^4$$
$$e - 2 = 4$$
$$e = 6$$

Es. 2 $\quad y = f(x)$

- errore inerente $\quad \dfrac{f(x) - f(\tilde{x})}{f(x)}$

- $f(x) = \dfrac{2 \cdot e^x}{1 - x^2}$ $\qquad \operatorname{cond}_f(x) = \dfrac{|x| \cdot |f'(x)|}{|f(x)|}$

$$f'(x) = 2 \, \frac{e^x(1-x^2) - e^x(-2x)}{(1-x^2)^2} = 2e^x \, \frac{1-x^2+2x}{(1-x^2)^2}$$

$$\operatorname{cond}_f(x) = |x| \cdot \cancel{2} \cdot \cancel{e^x} \left| \frac{1-x^2+2x}{(1-x^2)^{\cancel{2}}} \right| \cdot \frac{\cancel{|1-x^2|}}{\cancel{2}\, \cancel{e^x}} = \frac{|x|\,|1-x^2+2x|}{|1-x^2|}$$

$$\lim_{x \to \pm\infty} \operatorname{cond}_f(x) = \lim_{x \to +\infty} \frac{|x|\,|1-x^2+2x|}{|1-x^2|} = +\infty$$

$$1 - x^2 = 0 \iff x^2 = 1 \implies x = 1 \text{ o } x = -1$$

$$\lim_{x \to \pm 1} \frac{|x|\,|1-x^2+2x|}{|1-x^2|} = +\infty$$

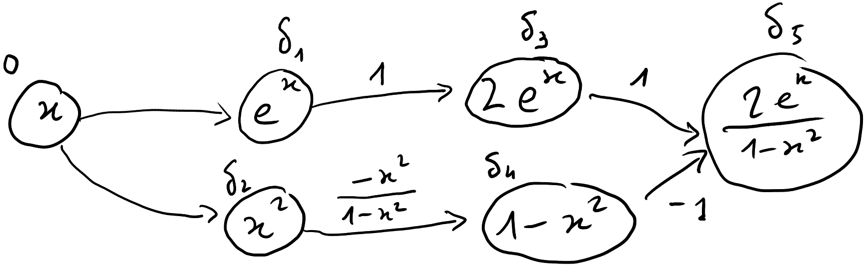
il calcolo di f è mal condizionato se x è vicino a 1 o a -1

o se $|x|$ è grande

- errore algoritmico $\qquad \dfrac{g(\tilde{x}) - \tilde{g}(\tilde{x})}{g(\tilde{x})}$

- $\dfrac{2e^x}{1-x^2} = e^x \left(\dfrac{1}{1-x} + \dfrac{1}{1+x} \right)$

$|\delta_i| \le u$



$h(x,y) = x+y$

$c_2(x,y) = \dfrac{y \cdot 1}{x+y}$

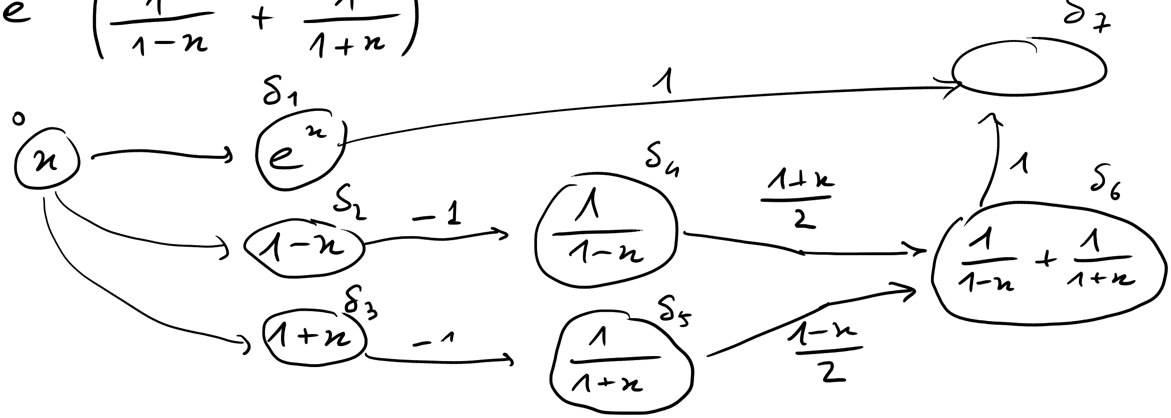
$$\varepsilon_{alg,1} = \delta_5 + \cancel{1}\,\delta_3 + \cancel{1}\,\delta_1 - \delta_4 - \frac{x^2}{1-x^2}\delta_2$$

$$|\varepsilon_{alg,1}| \le |\delta_5| + |\delta_3| + |\delta_1| + |\delta_4| + \left|\frac{x^2}{1-x^2}\right| |\delta_2|$$

$$\le 4u + \left|\frac{x^2}{1-x^2}\right| u$$

alg_1 è instabile (in avanti) se x è vicino a 1 o a -1

$$e^x\left(\frac{1}{1-x}+\frac{1}{1+x}\right)$$



$$\frac{\frac{1}{1-x}}{\frac{1}{1-x}+\frac{1}{1+x}}=\frac{1}{1-x}\cdot\frac{1}{\frac{1+x+1-x}{(1-x)(1+x)}}=\frac{1}{\cancel{1-x}}\cdot\frac{\cancel{(1-x)}(1+x)}{2}=\frac{1+x}{2}$$

$$|\delta_i|\le u$$

$$\varepsilon_{alg_2}=\delta_7+\delta_1+\delta_6+\frac{1+x}{2}(\delta_4-\delta_2)+\frac{1-x}{2}(\delta_5-\delta_3)$$

$$|\varepsilon_{alg_2}|\le|\delta_7|+|\delta_1|+|\delta_6|+\left|\frac{1+x}{2}\right|(|\delta_4|+|\delta_2|)+\left|\frac{1-x}{2}\right|(|\delta_5|+|\delta_3|)$$

$$\le 3u+\frac{|1+x|}{\cancel{2}}\cdot\cancel{2}u+\frac{|1-x|}{\cancel{2}}\cdot\cancel{2}u$$

$$=u\left(|1+x|+|1-x|+3\right)$$

alg_2 è instabile se $|x|$ è molto grande

COMPITO DEL 24/09/2018

Es. 1 $\quad \mathcal{F} = \mathcal{F}(2, t, p_{max}, p_{min})$ arrotond.

- t, p_{max}, p_{min}

$$\begin{cases} p_{max} = p_{min} \\ realmin = \frac{1}{32} \\ |\mathcal{F}| = 145 \end{cases}$$

realmax ?

$$realmin = B^{-p_{min}-1} = 2^{-p_{min}-1}$$

$$\begin{cases} p_{max} = p_{min} \\ 2^{-p_{min}-1} = 2^{-5} \\ 1 + 2 \cdot 2^{t-1}(p_{max} + p_{min} + 1) = 145 \end{cases}$$

$-p_{min} - 1 = -5 \qquad -p_{min} = -4 \qquad \boxed{\begin{aligned} p_{min} &= 4 \\ p_{max} &= 4 \end{aligned}}$

$$\cancel{1+}\, 2^t (2p_{min} + 1) = \cancel{145}\ 144$$

$$2^t \cdot 9 = 144 \qquad 2^t = 16 = 2^4 \qquad \boxed{t = 4}$$

$$realmax = B^{p_{max}}(1 - B^{-t}) = 2^{p_{max}}(1 - 2^{-t}) =$$
$$= 2^4(1 - 2^{-4}) = 2^4 - 1 = 16 - 1 = 15$$

- $u = \frac{2^{1-t}}{2} = 2^{-t} = 2^{-4} = \frac{1}{16}$

- - $x = (0.\overline{1011})_2$ $\qquad \tilde{x} = fl\left((0.1011\overline{1011})_2\right) = (0.1100)_2 \cdot 2^0$

  $y = (11.\overline{101})_2$ $\qquad \tilde{y} = fl\left((0.\underline{11}\overline{101})_2 \cdot 2^2\right) = (0.1111)_2 \cdot 2^2$

- $\tilde{z} = \tilde{x} \; fl(+) \; \tilde{y} = fl(\tilde{x} + \tilde{y}) = fl\left((0.001100)_2 \cdot 2^2 + (0.1111)_2 \cdot 2^2\right)$

  $= fl\left((1.0010)_2 \cdot 2^2\right) = fl\left((0.10010)_2 \cdot 2^3\right)$

  $= (0.1001)_2 \cdot 2^3$

$$\begin{array}{r} 0.0011 \; + \\ 0.1111 \; = \\ \hline 1.0010 \end{array}$$

- $\tilde{x} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ $\qquad \tilde{y} = 4\left(1 - \frac{1}{16}\right) = 4 \cdot \frac{15}{16} = \frac{15}{4}$

$$x = \frac{(1011)_2 - 0}{(1111)_2} = \frac{1+2+8}{1+2+4+8} = \frac{11}{15}$$

$$y = \frac{(11101)_2 - (11)_2}{(111)_2} = \frac{1+4+8+16-3}{1+2+4} = \frac{26}{7}$$

Es. 2

$$f(x) = e^{\frac{1+2x}{1-x^2}}$$

$$\begin{tikzcd}[column sep=large]
 & \overset{\delta_1}{2x} \arrow[r, "\frac{2x}{1+2x}"] & \overset{\delta_3}{1+2x} \arrow[dr, "1"] & & \\
\overset{0}{x} \arrow[ur] \arrow[dr] & & & \overset{\delta_5}{\frac{1+2x}{1-x^2}} \arrow[r, "\frac{1+2x}{1-x^2}"] & \overset{\delta_6}{e^{\cdots}} \\
 & \underset{\delta_2}{x^2} \arrow[r, "\frac{-x^2}{1-x^2}"] & \overset{\delta_4}{1-x^2} \arrow[ur, "-1"'] & &
\end{tikzcd}$$

$$h(x) = e^x$$

$$c = \frac{x e^x}{e^x} = x$$

$$\varepsilon_{alg} = \delta_6 + \frac{1+2x}{1-x^2}\left(\delta_5 + \delta_3 + \frac{2x}{1+2x}\delta_1 - \delta_4 - \frac{x^2}{1-x^2}\delta_2\right)$$

$$|\varepsilon_{alg}| \leq u + \left|\frac{1+2x}{1-x^2}\right|\left(3u + \left|\frac{2x}{1+2x}\right|u + \left|\frac{x^2}{1-x^2}\right|u\right) = \circledast$$

$$1+2x = 0 \iff 2x = -1 \iff x = -1/2$$

alg. è instabile x x è vicino a 1 o a -1

$$\circledast = u + 3u\left|\frac{1+2x}{1-x^2}\right| + u\left|\frac{1+2x}{1-x^2}\right|\left|\frac{2x}{1+2x}\right| + u\left|\frac{1+2x}{1-x^2}\right|\left|\frac{x^2}{1-x^2}\right|$$

$$\lim_{x\to -1/2} \circledast = u + 0 + u\frac{|2\cdot(-1/2)|}{|1-(-\frac{1}{2})^2|} + 0 = u + u\frac{1}{3/4} = u + \frac{4}{3}u = u\cdot\frac{7}{3}$$