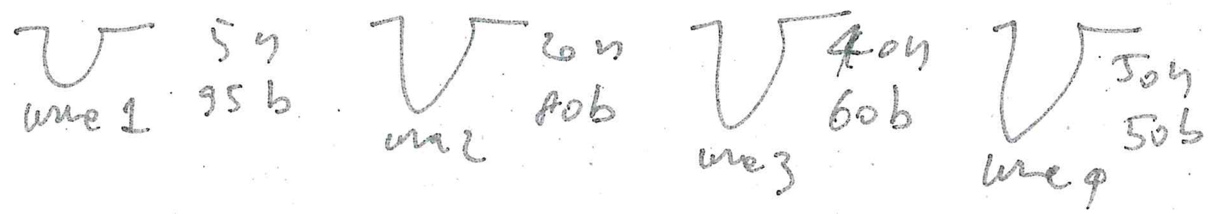
Compito del 28/06/22

1]



A_i = "scelta l'urna i" E = "una nera e 4 bianche in 5 estraz. con reins. (senza tener conto dell'ordine)"

chiesto $P(A_2 | E) = \dfrac{P(A_2)\, P(E|A_2)}{\sum_{i=1}^{4} P(A_i)\, P(E|A_i)}$ per Bayes

Prob. iniziali:

$P(A_i) = \frac{1}{4} \quad i = 1,2,3,4$

Verosimiglianze:

$P(E|A_1) = \binom{5}{1} 0.05^1\, 0.95^4 = 0.203266$

$P(E|A_2) = \binom{5}{1} 0.2^1\, 0.8^4 = 0.4096$

$P(E|A_3) = \binom{5}{1} 0.4^1\, 0.6^4 = 0.2592$

$P(E|A_4) = \binom{5}{1} 0.5^1\, 0.5^4 = 0.15625$

$P(A_2|E) = 0.398181$

2] $S_X = [0,1]$ $p_X(x) = c x^2$, $x \in S_X$, e 0 altrove

$c = ?$ $1 = \int_0^1 c x^2 dx = c\left[\frac{x^3}{3}\right]_0^1 = c\,\frac{1}{3} \Rightarrow c = 3$

$p_X(x) = \begin{cases} 3x^2, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$

$F_X(x) = ?$ $x \in (0,1)$

$F_X(x) = \int_0^x 3t^2 dt = x^3$

$$F_X(x) = \begin{cases} 0 & \text{se } x < 0 \\ x^3 & \text{se } x \in [0,1] \\ 1 & \text{se } x > 1 \end{cases}$$

$$E(X) = \int_0^1 x \cdot 3x^2\,dx = 3\left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}$$

$$E(X^2) = \int_0^1 x^2 \cdot 3x^2\,dx = 3\left[\frac{x^5}{5}\right]_0^1 = \frac{3}{5}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{3}{5} - \frac{3^2}{4^2} = \frac{3}{5} - \frac{9}{16} = \frac{16\cdot 3 - 9\cdot 5}{80} = \frac{48-45}{80}$$

$$= \frac{3}{80} = 0.0375$$

$T = 4X$: $S_T = [0,4]$ $\quad F_T(t) = P(T \le t) = P(4X \le t)$

$$= P\left(X \le \frac{t}{4}\right) = F_X\left(\frac{t}{4}\right) \Rightarrow F_T(t) = \begin{cases} 0 & \text{se } t < 0 \\ \left(\frac{t}{4}\right)^3 & \text{se } t \in [0,1] \\ 1 & \text{se } t > 4 \end{cases}$$

$P(T=1) = 0$

3] $X_1 \sim Exp\left(\frac{1}{6}\right)$ $\quad X_2 \sim Exp\left(\frac{1}{9}\right)$ $\quad X_3 \sim Exp\left(\frac{1}{12}\right)$ indip.

$T = \min(X_1, X_2, X_3)$

$S_T = [0, +\infty)$

$t>0$ $\quad F_T(t) = 1 - P(T>t) = 1 - P(X_1>t, X_2>t, X_3>t)$

$$= 1 - P(X_1>t)\,P(X_2>t)\,P(X_3>t)$$

$$= 1 - e^{-\frac{1}{6}t}\, e^{-\frac{1}{9}t}\, e^{-\frac{1}{12}t} = 1 - e^{-\left(\frac{1}{6}+\frac{1}{9}+\frac{1}{12}\right)t}$$

$$= 1 - e^{-\frac{1}{3}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)t} = 1 - e^{-\frac{1}{3}\frac{6+4+3}{12}t} = 1 - e^{-\frac{13}{36}t}$$

$t>0$ $\quad f_T(t) = \frac{13}{36} e^{-\frac{13}{36}t}$

terzo quartile di T è calc. di:

$$F_T(t) = \frac{3}{4} \Leftrightarrow 1 - e^{-\frac{13}{36}t} = \frac{3}{4} \Leftrightarrow \frac{1}{4} = e^{-\frac{13}{36}t}$$

cioè $-\frac{13}{36}t = \log\frac{1}{4}$ $\quad t = \frac{\log 4}{13/36} = \frac{36 \log 4}{13} \doteq 3.8$

$$P(3 \leq T \leq 6 \mid T > 3) = \frac{F_T(6) - F_T(3)}{1 - F_T(3)} = \frac{1 - e^{-\frac{13}{36}6} - (1 - e^{-\frac{13}{36}3}}{e^{-\frac{13}{36}\cdot 3}}$$

$$= \frac{e^{-\frac{13}{12}} - e^{-\frac{13}{6}}}{e^{-\frac{13}{12}}} = 1 - e^{-\frac{13}{6} + \frac{13}{12}}$$

$$= 1 - e^{-\frac{13}{12}} = 1 - 0.338465$$

$$= 0.66153$$

---

4) $M_Y(t) = e^{\frac{1}{2}t^2 - t}$ $\quad W = Y_1 - Y_2$ dove Y_i copie indip. di Y, i

$$M_W(t) = E(e^{tW}) = E(e^{t(Y_1 - Y_2)}) = E(e^{tY_1} e^{-t}$$

$$\underset{\text{indip.}}{=} E(e^{tY_1}) E(e^{-tY_2})$$

$$= e^{\frac{1}{2}t^2 - t} e^{\frac{1}{2}t^2 + t} = e^{t^2}$$

$$\frac{d}{dt} e^{t^2} = 2t e^{t^2} \qquad \frac{d^2}{dt^2} e^{t^2} = 2e^{t^2} + 4t^2 e^{t^2}$$

$$E(W) = \frac{d}{dt} M_W(t)\Big|_{t=0} = 0$$

$$E(W^2) = \frac{d^2}{dt^2} M_W(t)\Big|_{t=0} = 2$$

$$Var(W) = E(W^2) - E(W)^2 = 2 - 0^2 = 2.$$

5) Y_i i.i.d. $\sim N(1, 1)$. Mostrare che $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$

$\sim N(1, \frac{1}{n})$

$X \sim N(\mu, \sigma^2) \iff M_X(t) = e^{t\mu + \frac{1}{2}t^2\sigma^2}$

$$M_{\bar{Y}_n}(t) = E(e^{t\bar{Y}_n}) = E\left(\prod_{i=1}^{n} e^{\frac{t}{n}Y_i}\right)$$

$$\underset{\text{indip.}\ i=1}{=} \prod^{n} E(e^{\frac{t}{n}Y_i}) = M_{Y_1}\left(\frac{t}{n}\right)^n$$

$$= \left(e^{\frac{t}{n}\cdot 1 + \frac{1}{2}\frac{t^2}{n^2}\cdot 1}\right)^n = e^{t + \frac{1}{2}t^2\frac{1}{n}}$$

$\Rightarrow \bar{Y}_n \sim N(1, \frac{1}{n})$. Calcolare n=16 $\bar{Y}_{16} \sim N(1, \frac{1}{16})$

- $P(\bar{Y}_{16} > 1.5) = 1 - P(\bar{Y}_{16} \le 1.5) = 1 - \Phi\left(\frac{1.5-1}{1/4}\right)$
  $= 1 - \Phi(2) = 1 - 0.97725 = 0.0275$
- $P(\bar{Y}_{16} < 0.75) = \Phi\left(\frac{0.75-1}{1/4}\right) = \Phi(-1)$
  $= 1 - \Phi(1) = 1 - 0.84134 = 0.15866$
- novantesimo percentile di $\bar{Y}_{16}$
  $\mu + \sigma z_{0.90} = 1 + \frac{1}{4} z_{0.90} = 1 + \frac{1}{4} \times 1.28 = 1.32$

---

6) 100 pezzi 5 non idonei

$S_{100} \sim Bi(100, p)$ $p \in (0,1)$ ignoto mod. stat.

$\hat{p} = \frac{S_{100}}{100}$ $E(\hat{p}) = p$ $Var(\hat{p}) = \frac{p(1-p)}{100}$

$se = \sqrt{\frac{p(1-p)}{100}}$

Stime: • $\hat{p}_{100} = \frac{5}{100} = 0.05$
• $\hat{se} = \sqrt{\frac{0.05 \cdot 0.95}{100}} \doteq 0.02$