Q&A Assignment 1

Reinforcement Learning KU - Winter 2023/24

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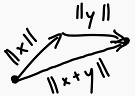
Mathematical Preliminaries

Norms

Definition

Let $\mathcal{V}$ be a vector space. Then the function $\|\cdot\| : \mathcal{V} \rightarrow \mathbb{R}$ is called a **norm on $\mathcal{V}$** if it satisfies the following properties:

1. $\|\mathbf{x}\| \geq 0, \quad \forall \mathbf{x} \in \mathcal{V}$ and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$ (non-negativity)
2. $\|\lambda \mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\|, \quad \forall \mathbf{x} \in \mathcal{V}, \forall \lambda \in \mathbb{R}$ (positive homogeneity)
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathcal{V}$ (triangle inequality)



Important Norms

For $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, we define

- The ℓ_2 norm $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
- The ℓ_1 norm $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$
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ℓ_p Norms

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \quad \text{with} \quad p \geq 1$$

- Actually, $\|\mathbf{x}\|_\infty = \lim_{p \to \infty} \|\mathbf{x}\|_p$

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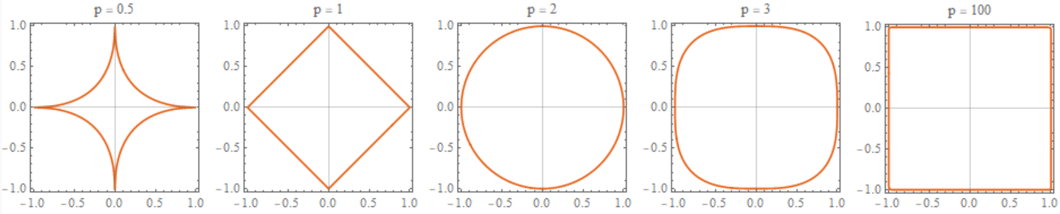

https://ekamperi.github.io/machine%20learning/2019/10/19/norms-in-machine-learning.html

Definition

Let $(\mathcal{V}, \|\cdot\|)$ be a **normed vector space** and $(v_n; n \geq 0)$ be a sequence of vectors in $\mathcal{V}$. Then v_n is called a **Cauchy Sequence** if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \text{ such that } \; \forall i, j \geq N : \; \|v_i - v_j\| < \varepsilon$$

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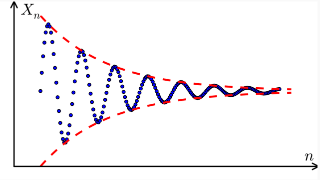
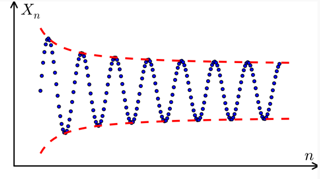


https://en.wikipedia.org/wiki/File:Cauchy_sequence_illustration.svg



https://en.wikipedia.org/wiki/File:Cauchy_sequence_illustration2.svg

Definition

Let $(\mathcal{V}, \|\cdot\|)$ be a **normed vector space**. A function $T : \mathcal{V} \to \mathcal{V}$ is called L-Lipschitz if for any $\mathbf{x}, \mathbf{y} \in \mathcal{V}$

$$\|T(\mathbf{x}) - T(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$$

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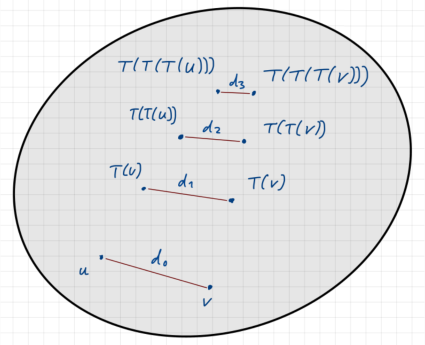

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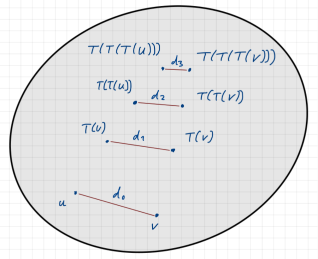
Banach's Fixed Point Theorem

Theorem !!

Let $(\mathcal{V}, \|\cdot\|)$ be a **Banach space** and $T : \mathcal{V} \to \mathcal{V}$ be a contraction. Then,

1. T has a **unique fixed point** $\mathbf{x}^*$ (i.e., $T(\mathbf{x}^*) = \mathbf{x}^*$)
2. Iterating T, i.e., $T(\cdots T(T(\mathbf{x}_0)))$, **converges to the fixed point** $\mathbf{x}^*$ **for any** $\mathbf{x}_0$.



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Numerical Demo in `python`

$$f(x) = \frac{1}{2}x + 1$$

$$|f(x) - f(y)| \leq L \cdot |x - y|$$

$$\left|\frac{1}{2}x + 1 - \left(\frac{1}{2}y + 1\right)\right| \leq L \cdot |x - y|$$

$$\left|\frac{1}{2}(x - y)\right| \leq L \cdot |x - y|$$

$$\frac{1}{2}|x - y| \leq L \cdot |x - y|$$

$$f(x) = \frac{1}{2}x^* + 1 \overset{!}{=} x^*$$

$$1 = \frac{1}{2}x^*$$

$$\underline{2 = x^*}$$

Proof of **Banach's Fixed Point Theorem**

Assume T is a contraction, i.e.,

$$\| T(x) - T(y) \| \leq L \cdot \| x - y \| \quad \forall x, y \in \mathbb{R}^n$$

(I) x_0, $T(x_0)$, $\underbrace{T(T(x_0))}_{T^2(x_0)}$, ...

$a_1 = x_0$

$a_2 = T(x_0)$

$\vdots$

$$\boxed{a_n = T(a_{n-1}),\ a_1 = x_0}$$

To show: $(a_n)_{n \in \mathbb{N}}$ is Cauchy

$\forall i, j \geq N$: $\| a_i - a_j \| < \varepsilon$

$$\| \underbrace{T^{i+1}(x_0)}_{} - \underbrace{T^i(x_0)}_{} \| \leq L \cdot \underbrace{\| T^i(x_0) - T^{i-1}(x_0) \|}_{\leq L \cdot \| T^{i-1}(x_0) - T^{i-2}(x_0) \|}$$

$$\| T(T^i(x_0)) - T(T^{i-1}(x_0)) \|$$

$$\leq L^2 \cdot \| T^{i-1}(x_0) - T^{i-2}(x_0) \|$$

$$\vdots$$

$$\leq L^i \| T(x_0) - x_0 \|$$

$$\boxed{\| T^{i+1}(x_0) - T^i(x_0) \leq L^i \| T(x_0) - x_0 \|} \quad (1)$$

Assume $i \geq j$

(e.g.: $\| a_4 - a_1 \| = \| (a_4 - a_3) + (a_3 - a_2) + (a_2 - a_1) \|$)

$$\| a_i - a_j \| = \| (a_i - a_{i-1}) + (a_{i-1} - a_{i-2}) + (a_{i-2} -) \dots + (a_{j+1} - a_j) \|$$

$$\overset{\Delta\text{-ineq.}}{\leq} \| a_i - a_{i-1} \| + \| a_{i-1} - a_{i-2} \| + \dots + \| a_{j+1} - a_j \|$$

(1)

$$\leq L^{i-1} \cdot \|T(x_0) - x_0\| + L^{i-2} \cdot \|T(x_0) - x_0\| + \dots + L^{j} \|T(x_0) - x_0\|$$

$$= \|T(x_0) - x_0\| \cdot \sum_{k=j}^{i-1} L^k$$

$$= \|T(x_0) - x_0\| \cdot \sum_{k=0}^{i-j-1} L^j \cdot L^k$$

$$= L^j \|T(x_0) - x_0\| \cdot \sum_{k=0}^{i-j-1} L^k$$

$(\|a_i - a_j\|)$

$$\leq L^j \cdot \|T(x_0) - x_0\| \cdot \sum_{k=0}^{\infty} L^k$$

$$= L^j \cdot \|T(x_0) - x_0\| \cdot \frac{1}{1-L} \overset{!}{<} \varepsilon$$

Pick $N \in \mathbb{N}$: $L^N < \underbrace{\frac{1-L}{\|T(x_0) - x_0\|} \cdot \varepsilon}_{\text{const w.r.t. } i,j}$

$\square$

(II) $\lim_{n \to \infty} a_n$ is a fixed point of T:

$$\lim_{n \to \infty} a_n = x^*$$

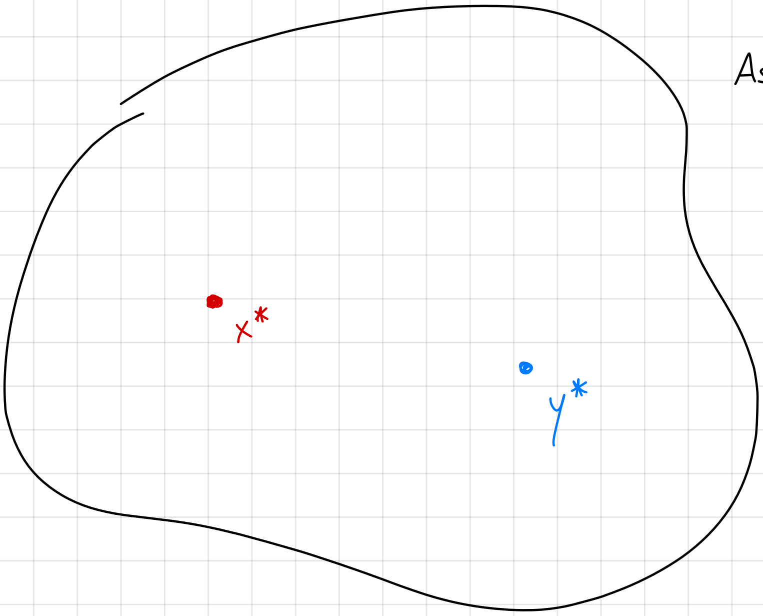
$$x^* = \lim_{n \to \infty} a_n = \lim_{n \to \infty} T(a_{n-1}) = T(\lim_{n \to \infty} a_{n-1})$$

$$= T(x^*)$$

$\Rightarrow x^*$ is a fixed point.

(III) Fixed point is unique.

Proof by contradiction:



Assume $\exists\, y^*, x^*$ such that

$$T(x^*) = x^*$$

$$T(y^*) = y^*$$

$$x^* \neq y^*$$

$$\| \underbrace{T(x^*)} - \underbrace{T(y^*)} \| \leq L \cdot \| x^* - y^* \|$$

$$\| x^* - y^* \| \leq L \cdot \| x^* - y^* \|$$

↯ (Because $L < 1$)

$\Rightarrow$ x^* is a unique fixed point.

Assignment 1 Questions

References I