HoTT as a logical framework for the Minimalist Foundation

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CCC 2021, Birmingham, 22 September 2021

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- Martin-Löf type theory is an example of intensional theory.
- It enjoys a lot of nice computational properties:
    - Church-Rosser
    - Decidability of Type-Checking
    - Normalisation
- It allows for different styles of definitions (inductive, inductive-recursive).

△ B.Nordström, K.Petersson & J.M.Smith, *Programming in Martin-Löf's type theory*, (1990).

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There exists also an extensional version of Martin-Löf type theory:

- Reflection rule for propositional equality

$\Longrightarrow$ extensional rules break normalization.

$\Longrightarrow$ type checking is not decidable.

$\Longrightarrow$ extensional MLTT is inconsistent with formal CT

$$(\forall f \in \mathbb{N} \to \mathbb{N})(\exists e \in \mathbb{N})(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(T(e, x, y) \& U(y) =_{\mathbb{N}} f(x))$$

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The Minimalist Foundation

Ideally, a constructive foundation should integrate a user-friendly language for doing mathematics with a formal-intensional level supporting computer-assisted proof checking and well-behaved computational features.

More formally, the intensional setting should satisfy the *Proofs-as-Programs* paradigm:

$$\mathbf{T} \text{ satisfies P-as-P iff } \mathbf{T} \text{ is consistent with AC and CT}$$

where AC is $(\forall x \in A)(\exists y \in B)R(x,y) \longrightarrow (\exists f : A \to B)(\forall x \in A)R(x, f(x))$

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- The Minimalist Foundation (MF) consists of:
  - an intensional level **mTT** which is a language for computer formalized proofs (and has all the desirable computational properties)
  - an extensional level **emTT** which is a kind of predicative local set theory and thus a practical language for developing math.
- **emTT** is a fragment of the internal language of the quotient completion of the intensional level.

△M.E.Maietti, A minimalist two-level foundation for constructive mathematics, *APAL*, (2009)

⟹ **emTT** is interpreted via a quotient model in **mTT**.

⟹ extensional sets = quotients of intensional sets = Bishop's setoids.

⟹ Informally, moving from **mTT** to **emTT** amounts to an abstraction process, while moving from **emTT** to **mTT** via the setoid model corresponds to restoring computational information.

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A third level for program extraction:

- Kleene Realizability interpretation for **mTT**.

$$\textbf{emTT} \xrightarrow{\textit{Quotient}} \textbf{mTT} \xrightarrow{\textit{Realizability}} \text{computational level.}$$

The actual foundation consists just of the first two levels!

△ H.Ishihara, M.E.Maietti, S.Maschio & T.Streicher, Consistency of the intensional level of the Minimalist Foundation with Church's Thesis and the Axiom of Choice, *Arch. for Math.Log.*, (2018)

△ M.E.Maietti, S.Maschio, M.Rathjen, A Realizability Semantics for Inductive Formal Topologies, Church's Thesis and Axiom of Choice, *LMCS*, (2021) A third level for program extraction:

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- Another important feature of MF is that it constitutes the common core among the most relevant existing foundations.
- MF is compatible with classical as well as constructive theories (and with predicative and impredicative ones).
- A theory $\mathcal{T}$ is compatible with another theory $\mathcal{S}$ iff there is a translation $\phi : \mathcal{T} \to \mathcal{S}$ which preserves the meaning of logical and set-theoretical constructors.

$\Longrightarrow$ Our aim is to show that MF is compatible with HoTT $\Longleftarrow$

HoTT can be regarded as a logical framework for MF

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- HoTT is an extension of Intensional MLTT with the Univalence Axiom and with Higher Inductive Types (HIT).
- Stratification of types in h-levels:
  - Types of h-level 1 are called mere propositions or h-propositions
  - Types of h-level 2 are called h-sets
- The hierarchy of h-levels is cumulative.

$\Longrightarrow$ computational interpretation of HoTT $\Rightarrow$ cubical type theory.

Sterling & Angiuli (2021): cubical TT enjoys normalisation!

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cf. *prop as monotypes* in △ M.E.Maietti, Modular Correspondence between Dependent Type Theories and Categories including Pretopoi and Topoi, *MSCS*, (2005)

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- a type $\mathcal{H}$ is an instance of HIT if it has constructors not only for its elements, but also for elements of its identity type.
- Quotient sets can be formalized as HIT.

- Let $R : A \to A \to \text{hProp}$ be an equivalence relation. Then we can form the quotient of A as follows:
    - $[]{:}A \to A/R$
    - For all $a, b : A$, $R(a, b) \to [a] =_{A/R} [b]$
    - For all $x, y : A/R$ and $p, q : x =_A y$, then $p = q$, i.e. A/R is an h-set.

- Coquand, Huber & Mortberg (2019) $\to$ cubical model of HoTT +HIT with closure under universes level

$\Longrightarrow$ The quotient lives in the same universe as the carrier set.

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- The Minimalist Foundation has four sorts:

collections, sets, propositions, small propositions.

$$\begin{tikzcd}
\mathit{S\text{-}prop} \arrow[r, hook] \arrow[d, hook] & \mathit{Set} \arrow[d, hook] \\
\mathit{Prop} \arrow[r, hook] & \mathit{Col}
\end{tikzcd}$$

- Thanks to the new machinery available in HoTT we can define the constructors for all the four sorts of MF.
- Further, we can interpret both the levels of MF in HoTT.
- The translation will make use just of the first two levels of the homotopical hierarchy, namely h-prop and h-sets.

- In **mTT** we have the usual judgement forms:
  i) A is a type
  ii) $A = B$ (the types A and B are definitionally equal)
  iii) $a : A$ (a is a term of type A)
  iv) $a = b : A$ (a and b are definitionally equal terms of type A)

$\Longrightarrow$ but type ranges over $\{$*s-prop, prop, set, col*$\}$.

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The translation of **mTT** in HoTT works as follows:

- s-prop $\rightsquigarrow$ h-prop in $\mathcal{U}_0$
- prop $\rightsquigarrow$ h-prop in $\mathcal{U}_i$ with $i > 0$
- set $\rightsquigarrow$ h-set in $\mathcal{U}_0$
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$\Longrightarrow$ Let $()^*$ be the mapping from **mTT** to HoTT. Then this mapping preserves the derivability of judgements.

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- In **emTT** are available some constructors and rules which are not in **mTT**:

  - Quotient of the collection of s-props under equiprovability, denoted by $\wp(\mathbf{1})$.
  - Propositions are mono $\implies$ proof-irrelevant.
  - η-rules for sets and congruence rules for all constructors.
  - reflection rule for propositional equality.
  - Quotient sets (which are effective)

- The translation has to account for these additional elements.
- Main problem: **emTT** is an extensional type theory, hence the relation of judgemental equality is undecidable.
- We have to convert judgemental equalities into propositional ones in a intensional type theory.
- Undecidable equalities $\rightsquigarrow$ decidable equalities.

- The clauses of the translation for judgements containing judgemental equalities need to be modified.
- $\equiv \rightsquigarrow Id$
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Summing up:

- The translation works thanks to the new machinery available in HoTT.
- Univalence and its consequences to interpret extensional judgemental equalities
- HIT to interpret quotients
- Universe levels and h-levels to faithfully interpret MF-sorts

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