Quasi-Polish spaces as spaces of ideals[1]

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Introduction

- We present recent a characterization of quasi-Polish spaces as spaces of ideals of a transitive relation on a countable set.
  - The characterization was proved in a recent paper "Overt choice" by M. d., A. Pauly and M. Schröder, and further investigated in the paper "Some notes on spaces of ideals and computable topology" by M. d.
- We then use domain theoretic techniques to demonstrate the computability of the lower, upper, double, and valuations powerspace functors on the category of quasi-Polish spaces.

- Some motivation:
  - (Domain theory) It highlights connections between domain theory and quasi-Polish spaces, and creates new possibilities for applying domain theory to their study.
  - (Computable topology) It provides a natural way to investigate computability aspects of quasi-Polish spaces, in a way that is compatible with Weihrauch's Type Two Theory of Effectivity.
  - (Logic and foundations) Makes it possible to develop the theory of quasi-Polish spaces within weak foundations, such as subsystems of second order arithmetic.
    - Many results in C. Mummert's PhD thesis on the reverse mathematics of general topology can be used in our setting.

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A topological space is quasi-Polish iff it satisfies any of the following equivalent properties:

- It is a countably based space with a topology generated by a (Smyth-) complete quasi-metric.
- It is homeomorphic to a $\mathbf{\Pi}^0_2$-subspace of $\mathcal{P}(\mathbb{N})$, the powerset of the natural numbers with the Scott-topology.
  - A subset S is $\mathbf{\Pi}^0_2$ iff there are sequences $(U_i)_{i\in\mathbb{N}}$ and $(V_i)_{i\in\mathbb{N}}$ of opens such that $x \in S \iff (\forall i \in \mathbb{N})\,[x \in U_i \Rightarrow x \in V_i]$.
- It is homeomorphic to the subspace of non-compact elements of an ω-algebraic domain.
- and more...

Polish	$\iff$	countably based & completely metrizable.
Quasi-Polish	$\iff$	countably based & completely quasi-metrizable.

(Fact: Polish = quasi-Polish + metrizable)

Examples

The following are quasi-Polish:

- Polish spaces
  - $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}^{\mathbb{N}}$, etc.
- Countably based spaces that are locally homeomorphic to some Polish space
  - the line with two origins
  - countably based non-Hausdorff topological manifolds
  - etc.
- ω-continuous domains
  - $\mathbb{S}$ (Sierpinski space), $\mathbb{N}_\perp$, $\mathcal{P}(\mathbb{N})$, etc.
- Countably based spectral spaces
  - Spec$(\mathbb{Z})$, Spec$(\mathbb{Q}[x_1, \ldots, x_n])$, etc.
- Countably based locally compact sober spaces
  - (Contains the last two categories)

(There are quasi-Polish spaces which do not fit into any of the above categories).

Counter-examples

The following are not quasi-Polish:

- Non-Polish metric spaces
    - $\mathbb{Q}$ with the subspace topology inherited from $\mathbb{R}$
    - etc.
- Non-sober spaces
    - $\mathbb{N}$ with the cofinite topology
    - $(\mathbb{N}, <)$ with the Scott-topology
    - etc.
- And some others
    - $(\mathbb{N}^{<\infty}, \preceq_{\text{prefix}})$ with the lower topology
    - the Gandy-Harrington space
    - etc.

Theorem (d., 2018) - Generalized Hurewicz Theorem

Any $\mathbf{\Pi}^1_1$-subspace of a quasi-Polish space which is not quasi-Polish will contain a $\mathbf{\Pi}^0_2$-subset homeomorphic to one of the four spaces highlighted above.

Some basic results

- Every countably based T_0-space embeds into a quasi-Polish space.
- A space is Polish if and only if it is a metrizable quasi-Polish space.
- If X is quasi-Polish, then $A \subseteq X$ is quasi-Polish iff $A \in \mathbf{\Pi}^0_2(X)$.
- Quasi-Polish spaces form the smallest (up to equivalence) full subcategory of Top that contains $\mathbb{S}$ (Sierpinski space) and is closed under countable limits.
- (R. Heckmann) The category of quasi-Polish spaces is equivalent to the category of countably presented locales.
  - Quasi-Polish spaces correspond to countably axiomatized propositional geometric theories.
  - Recent work by R. Chen extends R. Heckmann's results and further develops connections between descriptive set theory and locale theory.

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Spaces of Ideals

Definition

Let $\prec$ be a transitive relation on $\mathbb{N}$. A subset $I \subseteq \mathbb{N}$ is an ideal (with respect to $\prec$) if and only if:

1. $I \neq \emptyset$, *(I is non-empty)*
2. $(\forall a \in I)(\forall b \in \mathbb{N})\,(b \prec a \Rightarrow b \in I)$, *($I$ is a lower set)*
3. $(\forall a, b \in I)(\exists c \in I)\,(a \prec c \,\&\, b \prec c)$. *($I$ is directed)*

The collection $\mathbf{I}(\prec)$ of all ideals has the topology generated by basic open sets of the form $[n]_\prec = \{I \in \mathbf{I}(\prec) \mid n \in I\}$ for $n \in \mathbb{N}$.

- Think of the elements of $\mathbb{N}$ as encoding pieces of information about points in some space.
- The relation $a \prec b$ means that the token b contains more information than the token a.
- A point (i.e., an ideal $I \in \mathbf{I}(\prec)$) is any consistent collection of arbitrarily precise information

Spaces of Ideals

Theorem (M. d, A. Pauly, & M. Schröder, 2019)

A space is quasi-Polish if and only if it is homeomorphic to a space of the form $\mathbf{I}(\prec)$ for some transitive relation $\prec$ on $\mathbb{N}$.

- Spaces of the form $\mathbf{I}(\prec)$ for some c.e. relation $\prec$ on $\mathbb{N}$ provide an effective interpretation of quasi-Polish spaces.
- If the set $E_\prec = \{n \in \mathbb{N} \mid [n]_\prec \neq \emptyset\}$ is also c.e., then it is an effective interpretation of an *overt* quasi-Polish space.
- Effective aspects of quasi-Polish spaces has been investigated by M. Korovina, O. Kudinov, V. Selivanov, V. Becher, S. Grigorieff, A. Pauly, M. Schröder, M. Hoyrup, C. Rojas, D. Stull, and T. Kihara.

Examples

Example

If $=$ is the equality relation on $\mathbb{N}$, then $\mathbf{I}(=)$ is homeomorphic to $\mathbb{N}$ with the discrete topology.

We also consider relations on other countable sets (encoded by $\mathbb{N}$)

Example

If $\subseteq$ is the usual subset relation on the set $\mathcal{P}_{\mathrm{fin}}(\mathbb{N})$ of finite subsets of $\mathbb{N}$, then $\mathbf{I}(\subseteq)$ is homeomorphic to $\mathcal{P}(\mathbb{N})$, the powerset of the natural numbers with the Scott-topology.

Examples

- ω-algebraic domains are precisely the spaces of the form $\mathbf{I}(\prec)$, where $\prec$ is a partial order (i.e., reflexive, transitive, and anti-symmetric)
  - This yields the same definition of ideal as from order theory.
- ω-continuous domains are precisely the spaces of the form $\mathbf{I}(\prec)$, where $\prec$ is a transitive relation satisfying the following *finite interpolation property:*
  - For every finite $F \subseteq \mathbb{N}$ and $z \in \mathbb{N}$,

$$F \prec z \text{ implies } (\exists y \in \mathbb{N})\, F \prec y \prec z$$

    where $F \prec z$ is shorthand for $(\forall x \in F)\, x \prec z$.

Examples

- Removing the interpolation requirement allows us to construct important spaces other than domains:

Example

If $\prec$ is the strict prefix relation on the set $\mathbb{N}^{<\infty}$ of finite sequences of natural numbers, then $\mathbf{I}(\prec)$ is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$.

Examples: Completion of separable metric spaces

- Let (X, d) be a separable metric space. Fix a countable dense subset $D \subseteq X$, and define a transitive relation $\prec$ on $P = D \times \mathbb{N}$ as

$$\langle x, n \rangle \prec \langle y, m \rangle \iff d(x, y) < 2^{-n} - 2^{-m}.$$

  - This definition guarantees that the open ball with center x and radius 2^{-n} contains the closed ball with center y and radius 2^{-m}.
- $\mathbf{I}(\prec)$ is homeomorphic to the completion of (X, d).
  - This is related to the *formal ball* models in domain theory.

Generalization to transitive relations on arbitrary sets

- One can also consider spaces of the form $\mathbf{I}(\prec)$ for transitive relations $\prec$ on arbitrary (possibly uncountable) sets.
- However, if $\mathbf{I}(\prec)$ happens to be a countably based space, then there is a countable $B \subseteq S$ such that $\mathbf{I}(\prec)$ is homeomorphic to $\mathbf{I}(\prec|_B)$, where $\prec|_B$ is the restriction of $\prec$ to B.
- Therefore, even if we generalize to relations on arbitrary sets, the countably based spaces that can be represented are exactly the quasi-Polish spaces.
  - In particular, the rationals $\mathbb{Q}$ cannot be realized as a space of ideals of a transitive relation on some arbitrary set.
  - Removing the countability restriction from the locale theoretic characterization of quasi-Polish spaces allows you to construct all locales, which includes $\mathbb{Q}$ (but its not a group anymore).
  - This is another example of how the multiple (classically) equivalent characterizations of quasi-Polish spaces diverge when you attempt to generalize to a larger category of spaces.

Continuous (computable) functions

Definition

Let $\prec_1$ and $\prec_2$ be transitive relations on $\mathbb{N}$.

- A **code** for a partial function is any subset $R \subseteq \mathbb{N} \times \mathbb{N}$.
- Each code R represents the partial function $\ulcorner R \urcorner :\subseteq \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ defined as

$$\begin{aligned} \ulcorner R \urcorner(I) &= \{n \in \mathbb{N} \mid (\exists m \in I)\, \langle m, n \rangle \in R\}, \\ dom(\ulcorner R \urcorner) &= \{I \in \mathbf{I}(\prec_1) \mid \ulcorner R \urcorner(I) \in \mathbf{I}(\prec_2)\}. \end{aligned}$$

Theorem

A total function $f\colon \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ is continuous (computable) if and only if there is a (c.e.) code $R \subseteq \mathbb{N} \times \mathbb{N}$ such that $f = \ulcorner R \urcorner$.

Intuitively, a function $f\colon \mathbf{I}(\prec_1) \to \mathbf{I}(\prec_2)$ is computable if and only if there is an algorithm that, given an enumeration of some $I \in \mathbf{I}(\prec_1)$ produces an enumeration of $f(I) \in \mathbf{I}(\prec_2)$.

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Examples: Upper and lower powerspaces

The upper and lower powerspaces are used for

- (Topology) Constructing multi-valued functions
- (Computer science) Modeling non-deterministic programs
- (Logic) Providing semantics for modal logics

Definition

Given a topological space X with topology $\mathbf{O}(X)$, define the topological spaces $\mathbf{A}(X)$ and $\mathbf{K}(X)$ as follows:

- $\mathbf{A}(X)$ (Lower powerspace):
  - Set of closed subsets of X with lower Vietoris topology, which has subbasis $\Diamond U := \{A \in \mathbf{A}(X) \mid A \cap U \neq \emptyset\}$ for $U \in \mathbf{O}(X)$
- $\mathbf{K}(X)$ (Upper powerspace):
  - Set of saturated compact subsets of X with upper Vietoris topology, which has subbasis $\Box U := \{K \in \mathbf{K}(X) \mid K \subseteq U\}$ for $U \in \mathbf{O}(X)$

Note: $S \subseteq X$ is saturated iff $S = \bigcap\{W \in \mathbf{O}(X) \mid S \subseteq W\}$. (Every subset of a T_1-space is saturated).

Example: Lower powerspace functor

Definition (Lower powerspace endofunctor $\mathbf{A}(X)$)

- $\mathbf{A}(X)$ is the set of closed subsets of X with the lower Vietoris topology. (This is the hyperspace of (closed) overt subspaces.)
- $f\colon X \to Y$ maps to $\mathbf{A}(f)\colon \mathbf{A}(X) \to \mathbf{A}(Y)$ defined as $\mathbf{A}(f)(A) = Cl_Y(\{f(x) \mid x \in A\})$.

- This is realized by defining $\mathbf{A} = (\mathbf{A}_{\mathsf{Obj}}, \mathbf{A}_{\mathsf{Mor}})$ as

$$\begin{aligned} \mathbf{A}_{\mathsf{Obj}}(\prec) &= \prec_L \\ \mathbf{A}_{\mathsf{Mor}}(R) &= R_L, \end{aligned}$$

  where $\prec$ is a transitive relation on $\mathbb{N}$, R is a code for a total continuous function, and
  - $A \prec_L B \iff (\forall a \in A)(\exists b \in B)\, a \prec b$ for $A, B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$,
  - $R_L = \{\langle F, G\rangle \mid (\forall n \in G)(\exists m \in F)\, \langle m, n\rangle \in R\}$.
- $\prec_L$ is based on the construction by M. Smyth for ω-algebraic domains (but I am unaware of work on the morphisms).

Example: Upper powerspace functor

Definition (Upper powerspace endofunctor $\mathbf{K}(X)$)

- $\mathbf{K}(X)$ is the set of saturated compact subsets of X with upper Vietoris topology.
- $f\colon X \to Y$ maps to $\mathbf{K}(f)\colon \mathbf{K}(X) \to \mathbf{K}(Y)$ defined as $\mathbf{K}(f)(K) = Sat_Y(\{f(x) \mid x \in K\})$.

- This is realized by defining $\mathbf{K} = (\mathbf{K}_{\mathsf{Obj}}, \mathbf{K}_{\mathsf{Mor}})$ as

$$\begin{aligned} \mathbf{K}_{\mathsf{Obj}}(\prec) &= \prec_U \\ \mathbf{K}_{\mathsf{Mor}}(R) &= R_U\,, \end{aligned}$$

  where $\prec$ is a transitive relation on $\mathbb{N}$, R is a code for a total continuous function, and
  - $A \prec_U B \iff (\forall b \in B)(\exists a \in A)\, a \prec b$ for $A, B \in \mathcal{P}_{\mathrm{fin}}(\mathbb{N})$,
  - $R_U = \{\langle F, G\rangle \mid (\forall m \in F)(\exists n \in G)\, \langle m, n\rangle \in R\}$.
- $\prec_U$ is based on the construction by M. Smyth for ω-algebraic domains (but I am unaware of work on the morphisms).

Examples: Upper and lower powerspaces

$\mathbf{I}(\prec) \xrightarrow{\ulcorner R \urcorner} \mathbf{I}(\prec')$ is a total continuous function.

- Lower powerspace:

$$\begin{tikzcd} \mathbf{A}(\mathbf{I}(\prec)) \arrow[r, "\mathbf{A}(\ulcorner R \urcorner)"] \arrow[d, equal] & \mathbf{A}(\mathbf{I}(\prec')) \arrow[d, equal] \\ \mathbf{I}(\prec_L) \arrow[r, "\ulcorner R_L \urcorner"] & \mathbf{I}(\prec'_L) \end{tikzcd}$$

  - $A \prec_L B \iff (\forall a \in A)(\exists b \in B)\, a \prec b$ for $A, B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$,
  - $R_L = \{\langle F, G \rangle \mid (\forall n \in G)(\exists m \in F)\, \langle m, n \rangle \in R\}$.

- Upper powerspace:

$$\begin{tikzcd} \mathbf{K}(\mathbf{I}(\prec)) \arrow[r, "\mathbf{K}(\ulcorner R \urcorner)"] \arrow[d, equal] & \mathbf{K}(\mathbf{I}(\prec')) \arrow[d, equal] \\ \mathbf{I}(\prec_U) \arrow[r, "\ulcorner R_U \urcorner"] & \mathbf{I}(\prec'_U) \end{tikzcd}$$

  - $A \prec_U B \iff (\forall b \in B)(\exists a \in A)\, a \prec b$ for $A, B \in \mathcal{P}_{\text{fin}}(\mathbb{N})$,
  - $R_U = \{\langle F, G \rangle \mid (\forall m \in F)(\exists n \in G)\, \langle m, n \rangle \in R\}$.

Example: Double powerspace functor

Definition (Double powerspace endofunctor)

- $\mathbb{S}^{\mathbb{S}^X}$ is the space of continuous functions from $\mathbb{S}^X$ to $\mathbb{S}$
- $f\colon X \to Y$ maps to $\mathbb{S}^{\mathbb{S}^f} : \mathbb{S}^{\mathbb{S}^X} \to \mathbb{S}^{\mathbb{S}^Y}$, which is defined as $\mathbb{S}^{\mathbb{S}^f} = \lambda\mathcal{H}.\lambda\varphi.\mathcal{H}(\lambda x.\varphi(f(x)))$.
  (λ-calculus notation can be justified by embedding QPol into the cartesian closed category QCB_0.)
  - The exponentials $\mathbb{S}^X$ and $\mathbb{S}^{\mathbb{S}^X}$ in QCB_0 both have the Scott-topology, which is equivalent to the comact-open topology when X is quasi-Polish. If X is quasi-Polish then $\mathbb{S}^X$ is quasi-Polish if and only if X is locally compact.

- This is realized by composing $\mathbf{A}$ and $\mathbf{K}$, because $\mathbb{S}^{\mathbb{S}^X} \cong \mathbf{A}(\mathbf{K}(X)) \cong \mathbf{K}(\mathbf{A}(X))$ when X is quasi-Polish (d. & T. Kawai 2019), and similarly for morphisms.
  - This is closely related to work by S. Vickers on the double powerlocale and work by P. Taylor on Abstract Stone Duality.
  - See also recent work by E. Neumann investigating applications of the upper, lower, and double powerspace functors on effective represented spaces.

Example: Valuations powerspace functor

Definition (Valuations)

- A *valuation* on X is a continuous function $\nu\colon \mathbf{O}(X) \to \overline{\mathbb{R}}_+$ satisfying:
  1. $\nu(\emptyset) = 0$, and (*strictness*)
  2. $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$. (*modularity*)

  The *space of valuations* on X is the set $\mathbf{V}(X)$ of all valuations on X with the topology induced by subbasic opens of the form $\langle U, q\rangle := \{\nu \in \mathbf{V}(X) \mid \nu(U) > q\}$ with $U \in \mathbf{O}(X)$ and $q \in \overline{\mathbb{R}}_+ \setminus \{\infty\}$.
- $f\colon X \to Y$ maps to $\mathbf{V}(f)\colon \mathbf{V}(X) \to \mathbf{V}(Y)$ defined as $\mathbf{V}(f)(\nu) = \lambda U.\nu(f^{-1}(U))$.

- $\mathbf{O}(X)$ and $\overline{\mathbb{R}}_+ = [0, \infty]$ are assumed to have the Scott-topology.
- Every (locally finite) valuation on a quasi-Polish space extends (uniquely) to a Borel measure. Conversely, restricting any Borel measure to the open subsets results in a valuation.

Example: Valuations powerspace functor

- This is realized by defining $\mathbf{V} = (\mathbf{V}_{\mathsf{Obj}}, \mathbf{V}_{\mathsf{Mor}})$ as

$$\begin{aligned}\mathbf{V}_{\mathsf{Obj}}(\prec) &= \prec_V \\ \mathbf{V}_{\mathsf{Mor}}(R,) &= R_V,\end{aligned}$$

  where $\prec$ is a transitive relation on $\mathbb{N}$, R is a code for a total continuous function, and

  - $\prec_V$ is the computable relation on the (countable) set $\{r :\subseteq \mathbb{N} \to \mathbb{Q}_{>0} \mid dom(r) \text{ is finite } \}$ defined as $r \prec_V s$ iff $\sum_{b \in F} r(b) < \sum\{s(c) \mid c \in dom(s) \,\&\, (\exists b \in F)\, b \prec c\}$ for every non-empty $F \subseteq dom(r)$.
  - $R_V = \{\langle r, s\rangle \mid (\forall G \subseteq dom(s))\ [G \neq \emptyset \Rightarrow \sum_{a \in A_G} r(a) > \sum_{b \in G} s(b)]\}$. where $A_G = \{a \in dom(r) \mid (\exists a_0 \in \mathbb{N})(\exists b \in G)\, [a_0 \prec a \,\&\, \langle a_0, b\rangle \in R]\}$.

- This is related to work by C. Jones on the probabilistic powerdomain in domain theory, which is used to model probabilistic computations (but I am unaware of work on the morphisms).

Conclusion

- We introduced the recent characterization of quasi-Polish spaces as spaces of ideals of a transitive relation on $\mathbb{N}$.
- Using ideas from domain theory, we showed how to (computably) construct the lower, upper, double, and valuations powerspace functors on the category of quasi-Polish spaces.
- Open: Find maximal cartesian closed subcategories of QPol.
    - (I am only interested in full sub-CCCs of QCB_0 that are contained in QPol, so exponentials will have the compact-open topology).
    - $X \in$ QPol is exponentiable (i.e., $Y^X \in$ QPol for all $Y \in$ QPol) if and only if X is locally compact. However, the locally compact spaces do not form a cartesian closed category.
    - ωFS-domains (the largest cartesian closed full subcategory of ω-continuous domains) is a full sub-CCC of QCB_0 contained in QPol, but it is unknown if it is maximal in QPol.