Logical and computational aspects of Gleason's theorem in probability theory

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In this work , we explore the identification of the computational content, and implications of having constructive proofs of the important Gleason theorem [2] (1957) and how it relates to incompatibilty of measurements and the notion of contextuality in statistics. The implications are not only for quantum theory but also for understanding various stochastic noises in probability theory, for example.

A constructive proof of a suitable classical reformulation of Gleason's theorem appears in Richman and Bridges [5] (1999). We shall discuss some of the implications for probability and logic of these results. We also indicate that these results lead to interesting challenges of a combinatorial and computational nature.

The present discussion is heavily influenced by the arguments in Hrushovski and Pitowsky [3] (2004) (HP). This paper serves as a good example of dealing with the pitfalls encountered when attempting to extract the algorithmic content from a constructive, or at least, ‘effective’, proof. We shall argue that there still are some interesting problems that need to be addressed. What still is a challenge is to find effective and clearly explicit ways of expressing these results in the first-order real-closed ordered fields (I have shown that is possible effectively in principle) and yet to have explicit and effective bounds for finitary versions of Gleason’s theorem.

In the sequel, H will denote a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$ of finite dimension n.
The unit sphere in H will be denoted by S^{n-1}. For z a unit vector in H, we shall denote by P_z the projection of H onto the line spanned by z. In Dirac notation

$$P_z = |z\rangle\langle z|.$$

A frame f in H is an orthonormal basis in H. Obviously $f \subset S^{n-1}$.
Let Γ be a homogeneous subset of S^{n-1} meaning that if $x \in \Gamma$ and α is a unit scalar then $\alpha x \in \Gamma$.
A function $p : \Gamma \to [0, 1]$ is said to be a *frame function* if $p(\alpha x) = p(x)$ for all α with $|\alpha| = 1$ and $x \in \Gamma$, and, moreover, if the elements of $\{x_1, \ldots, x_n\} \subset \Gamma$ are orthonormal, then

$$\sum_j p(x_j) = 1.$$

Theorem

(Gleason 1957 in finite dimensional case)
Let $p : S^{n-1} \to [0,1]$ *be a frame function, where* $n \geq 3$. *Then there is an orthonormal basis* $(z_1, \ldots, z_n)$ *of* H *such that,*

$$p(x) = \sum_j p(z_j) |\langle x | z_j \rangle|^2, \ x \in S^{n-1}.$$

It follows that a frame function is continuous. The most difficult part of the theorem is in fact to show that any frame function is continuous. Even more difficult is to show that this statement has a constructive proof.
A constructive proof of a suitable classically equivalent reformulation of Gleason's theorem appears in Richman and Bridges [5] (1999).

Bell (1966) made the observation that there is no $\{0,1\}$-valued frame function on $S^{n-1} \subset E^3$ when $n \geq 3$ by invoking Gleason's theorem (and related it to the Kochen Specker argument in quantum theory.)
We now present a slightly different and generalised view of his argument:

Theorem

Let $r \geq 2$ be a natural number. If $\chi : S^r \to \{0, \ldots, r-1\}$, where S^r is the unit sphere in the real Hilbert space E^{r+1} of dimension $r+1$, is any `non-constant` *function such that $\chi(\alpha) = \chi(-\alpha)$ for all $\alpha \in S^r$, then, for each $0 < s \leq r$ there is a frame f in S^r such that $\sum_{\alpha \in f} \chi(\alpha) \neq s$.*
In particular, there are no frame functions p on $S^r \subset E^{r+1}$ that assume at most r values when $r \geq 2$. There are no 2-valued frame functions on S^r.

EXAMPLE: Suppose that we *arbitrarily* partition the unit sphere $S^2 \subset \mathbb{R}^3$ into two disjoint parts, such that α is in one block of the partition, then $-\alpha$ is in the same block.
We can think of it as an arbitrary mapping $\chi : S^r \to \{0,1\}$. Then there are three orthonormal elements, a "frame" $f = \{\alpha_1, \alpha_2, \alpha_3\}$ such that

$$\sum_{\alpha \in f} \chi(\alpha) \neq 1$$

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$$\chi(-\alpha) = \chi(\alpha) \tag{1}$$

for every $\alpha \in S^2$.
Thus if α is in one block of the partition so is $-\alpha$. By Gleason's result, the requirement $\chi(-\alpha) = \chi(\alpha)$ forces, as it were, χ to be continuous. This is the core of the statement, and its constructive and computational understanding is what the research is about.

Under these conditions, there is always a frame $f = \{\alpha_1, \alpha_2, \alpha_3\}$, which upon application of χ the outcome can only be one of

$$(0,0,0), (1,1,0), (1,0,1), (0,1,1), (1,1,1).$$

The colour 1 under χ cannot appear exactly once. This statement is independent from the choice of the colouring χ associated with the original partition of S^r into two parts. The outcomes (1,0,0) , (0,1,0), (0,0,1) are not allowed here for the conclusion of the statement for all pre-given partitions.
By symmetry, we have have a frame, where the outcomes can only be:

$$(1,1,1), (0,0,1), (0,1,0), (1,0,0), (0,0,0).$$

Proof: Suppose there is some non-constant function $\chi : S^r \to \{0, \ldots, r-1\}$ such that, for every frame f in S^r it is the case that $\sum_{\alpha \in f} \chi(\alpha) = s$, and where, moreover, $\chi(\alpha) = \chi(-\alpha)$ for all $\alpha \in S^r$.
Define $p : S^r \to [0,1]$ by

$$p = \frac{\chi}{s}.$$

Thus p is a frame function and hence continuous by Gleason's theorem. This also means that χ is continuous. But it is topologically impossible to have a continuous mapping χ from S^r to $\mathbb{R}$, assuming only finitely many values unless the mapping is constant, in the sense that it maps all of S^r onto a single value. This is because S^r is a *connected* topological space. This means that S^r cannot be partitioned into more than one non-empty open subspace.

Here is a finitary but concrete case of Bell's observation:

Theorem

(F). Suppose $\dim H = 3$, *where* $H = E^3$ *is the Hilbert space over the reals. Let* Γ_0 *be a finite subset of* $S^2 \subset H$ *and let* $z_0 \in S^2$ *and* $\epsilon > 0$. *Choose some* $0 < \theta < \frac{\pi}{2}$ *with* $\cos^2 \theta > \epsilon$. *Assume that* Γ_0 *contains a frame* f *with two elements in the union* $S_{z_0\theta}$ *of two semi-circles*

$$S_{z_0\theta} := \{z \in S^2 : |\langle z|z_0\rangle|^2 = \cos^2 \theta\}.$$

Let Γ *be a finite homogeneous subset of* S^2 *such that* $(\Gamma_0, \epsilon, z_0) \Rightarrow \Gamma$. *Then any frame function* p *on* Γ *with* $p(z_0) = 1$ *is not* $\{0, 1\}$*-valued on* Γ_0.

Proof. Let p be a frame function on Γ with $p(z_0) = 1$. Suppose that the restriction to Γ_0 is $\{0, 1\}$-valued. By assumption, there is a frame f in Γ_0 having two elements in $S_{z_0\theta}$ such that

$$\sum_{\alpha \in f} p(\alpha) = 1.$$

Hence, since p is $\{0, 1\}$-valued on Γ_0, there is some $\alpha \in f$ belonging to $S_{z_0\theta}$ for which $p(\alpha) = 0$. But since $(\Gamma_0, \epsilon, z_0) \Rightarrow \Gamma$,

$$|\langle \alpha | z_0 \rangle|^2 = |p(\alpha) - |\langle \alpha | z_0 \rangle|^2| < \epsilon,$$

which is a contradiction, since $\alpha \in S_{z_0\theta}$ and $|\langle \alpha | z_0 \rangle|^2 = \cos^2 \theta > \epsilon$.

It should be mentioned, that for some finite subsets Γ of S^{n-1}, we can find frame functions on Γ which are not the restrictions of frame functions on S^{n-1}. We can even find $\{0,1\}$-valued such functions.

For example, let $n = 3$ and consider the unit sphere S^2 in Euclidean space E^3. Let e_1, e_2, e_3 be a frame and set $e_4 = \frac{1}{\sqrt{2}}(e_1 + e_2)$. Set $\Gamma = \{e_1, \ldots, e_4\}$. Define $p : \Gamma \to \{0,1\}$ by

$$p(e_1) = p(e_2) = 0,\ p(e_3) = p(e_4) = 1.$$

Then p is a frame function on Γ, in which e_1, e_2, e_3 is the only frame. Suppose p is the restriction of a frame function on S^2 , then, by the indeterminacy principle, since $0 < (e_1, e_4) = \frac{1}{\sqrt{2}} < 1$, the function p is not the restriction of a frame function on S^2. This is because not both $p(e_1)$ and $p(e_4)$ are 0 while being $\{0,1\}$-valued.

Another way of seeing this: if p on Γ were the restriction of a frame function on S^2, then necessarily, since $p(\alpha_3) = 1$, we have $p(e_4) = |(e_4, e_3)|^2$, which is a contradiction, since e_4 is orthogonal to e_3 and $p(e_4) = 1$

Hrushovski and Pitowsky [3] (2004) showed the following:

Theorem

(finitary Born rule.)(HP 2004)
Fix $\epsilon > 0$, an element $z_0 \in S^{n-1}$ and a finite subset Γ_0 of S^{n-1}. Then there is a finite homogeneous set $\Gamma \subset S^{n-1}$ depending on ϵ, Γ_0 and z_0 which contains Γ_0 and z_0, such that for all frame functions $p : \Gamma \to [0, 1]$, it is the case that

$$p(z_0) = 1 \Rightarrow |p(x) - |\langle x|z_0\rangle|^2| < \epsilon,$$

for all $x \in \Gamma_0$.

We can abbreviate the theorem as follows:

$$\forall_{\Gamma_0, z_0, \epsilon} \exists_{\Gamma} \ (\Gamma_0, \epsilon, z_0) \Rightarrow \Gamma.$$

To be more explicit: The predicate

$$(\Gamma_0, \epsilon, z_0) \Rightarrow \Gamma,$$

means the following:
For a given finite subset $\Gamma_0 \subset S^{n-1}$, $z_0 \in S^{n-1}$ and $\epsilon > 0$, the finite set Γ in S^{n-1} is such that it contains both Γ_0 and z_0 such that for any frame function p on Γ it will be the case that

$$p(z_0) = 1 \Rightarrow |p(x) - |\langle x|z_0\rangle|^2| < \epsilon,$$

for all $x \in \Gamma_0$.
Thus a frame function p on Γ with $p(z_0) = 1$, will be, on Γ_0, within ϵ the restriction of the state determined by the pure state P_{z_0} on H.

The statement $(\Gamma_0, \epsilon, z_0) \Rightarrow \Gamma$, poses interesting computational problems.

For example, how can we "compute" Γ from the triple $(\Gamma_0, \epsilon, z_0)$ and what is the complexity of such a computation, both in terms of space and time?

By using the combinatorial - geometric arguments by Lovasz et al [4] (1989) much progress can be made with this problem. This work is under development.

Perhaps the model-theoretic approach to Gleason's theorem opens up many intriguing things to explore. I believe that by using the ideas of Hrushovski and Pitowsky [3] ,one can show that Gleason's theorem, is, essentially geometric. This statement to follow is my inference of their work, and it requires some further argument, which is done but not yet published, though I state it because it opens the way for looking at the problems stated so far.

Theorem

We can algorithmically and explicitly construct, for every $n \geq 3$, a first-order statement Π_n in the theory **R** *of real closed ordered fields which is* `classically` *equivalent to Gleason's theorem for E^n. An analogous result holds for the first order theory* **C** *for the field $\mathbb{C}$ of complex numbers and the version of Gleason's theorem for finite dimensional Hilbert spaces over $\mathbb{C}$.*

There is also an interesting conceptual point to be made in relation to **incompatibility of measurements.** Usually, this is taken to be a postulate of quantum mechanics, and specific to the quantum-mechanical formalism of non-commuting observables. However, in the light of general results such as those obtained in this paper, in a line of work going back to that of Fine ... , a different view emerges. The incompatibility of certain measurements can be interpreted as the impossibility — in the sense of mathematically provable non-existence — of joint distributions on all measurements which marginalize to yield the observed empirical distributions.

Thus, if we refer to the experimental scenario with which we began Section 2, this shows that there cannot be, even in principle, any such scenario in which all measurements can be performed jointly, which is consistent with the actually observed outcomes. Thus the incompatibility of certain measurements is revealed as a theory-independent structural impossibility result for certain families of empirical distributions. These families include those predicted by quantum mechanics, and confirmed by experiment; but the result itself is completely independent of quantum mechanics. Thus in this sense, we can say that incompatibility is derived rather than assumed. (Abramsky and Brandenberger [1](2011) *The Sheaf-Theoretic Structure of Non-Locality and Contextuality*, New Journal of Physics, 13.)

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