Continuity for Computability

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Introduction

Consider a Denotational Semantics.

- Continuity needs infinite structures.
- Computability deals with finite objects and sets only.
- Thus, infinity has to be a potential infinite.

We will explore the following question:

Which concepts are related solely to the potential infinite?

- We do not consider constructivity, decidability, complexity, *knowledge* about existence (e.g. realizer)...
- The question is independent of the inference system. It can use classical logic, intuitionistic logic, etc.

Introduction

The Potential Infinite...

... is an indefinitely extensible finite (i.e., it is a form of finitism). This includes Dummett's understanding — a reference to all objects creates a new object. It is a *dynamic* concept.

- No use of completed sets, e.g. $\mathbb{N}$, no naive reference to $\mathbb{N}$ or universal quantification on $\mathbb{N}$.
- No naive application, e.g., to objects $a \in [\mathbb{N} \to \mathbb{N}]$.

Conceptualizing the Dynamic.

Instead of a static set $\mathcal{M}$ use a *system* $\mathcal{M}_\mathcal{I} := (\mathcal{M}_i)_{i \in \mathcal{I}}$. For instance, $\mathbb{N}_\mathbb{N} = (\mathbb{N}_i)_{i \in \mathbb{N}}$ with $\mathbb{N}_i = \{0, \ldots, i-1\}$.

Note: Index set $\mathbb{N}$ is on a naive notion on meta-level and does not necessarily refer to an actual infinite set (see later in this talk).

Systems

What are Systems?

$(\mathcal{M}_{\mathcal{I}}, \overset{p}{\mapsto})$, with $(\mathcal{I}, \leq)$ a directed index set of states or approximation levels, and a predecessor relation $\mathcal{M}_{i'} \ni a' \overset{p}{\mapsto} a \in \mathcal{M}_i$ for $i' \geq i$.

Examples.

- Direct system with $a' \overset{p}{\mapsto} a \iff a' = emb_i^{i'}(a)$. E.g. $(\mathbb{N}_i)_{i \in \mathbb{N}}$.
- Inverse system with $a' \overset{p}{\mapsto} a \iff a = proj_i^{i'}(a')$. E.g. $[\mathbb{N}_i \to \mathbb{B}]_{i \in \mathbb{N}}$ with $\mathbb{B} = \{true, false\}$ and $a' \overset{p}{\mapsto} a \iff a = a' \upharpoonright \mathbb{N}_i$.
- $[\mathbb{N}_i \to \mathbb{N}_j]_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ with product order on index set and $a' \overset{p}{\mapsto} a \iff a = a' \upharpoonright \mathbb{N}_i$. Here $\overset{p}{\mapsto}$ is a partial surjection.

Systems

More general than direct and inverse systems.

- No transitivity of $\overset{p}{\mapsto}$ due to use of logical relations.
- Roughly: In between direct and inverse system.
- In general, $\overset{p}{\mapsto}$ is a relation. If $\overset{p}{\mapsto}$ is a partial surjection, this corresponds to a standard model.

Basic Definition.

- Consistency: For $a \in \mathcal{M}_i$ and $b \in \mathcal{M}_j$ define $a \asymp b :\Longleftrightarrow \exists a' \in \mathcal{M}_{i'}$ with $a' \overset{p}{\mapsto} a$ and $a' \overset{p}{\mapsto} b$.
- Basic property is $\mathcal{M}_{i'} \ni a' \asymp a \in \mathcal{M}_i$ and $i' \geq i$ implies $a' \overset{p}{\mapsto} a$.

Systems

Finite Types.

Function space construction can be done for these systems — this allows models for STT. Needs an additional embedding-projection pair. Embedding corresponds to a choice.

Functions.

Functions are themselves indefinitely extensible sets of assignments, e.g. $0 \mapsto f(0), 1 \mapsto f(1), \ldots$.

Local Application Only.

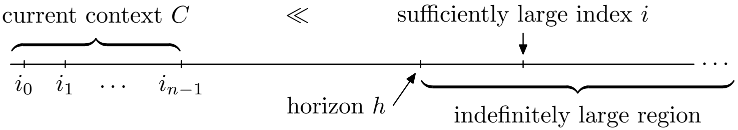
But: Application exists only locally, e.g., $[\mathbb{N}_i \to \mathbb{N}_j] \times \mathbb{N}_i \to \mathbb{N}_j$. No global application as $\bigcup_{(i,j)\in\mathbb{N}\times\mathbb{N}}[\mathbb{N}_i \to \mathbb{N}_j] \times \bigcup_{i\in\mathbb{N}} \mathbb{N}_i \to \bigcup_{j\in\mathbb{N}} \mathbb{N}_j$.

The Infinitely Large Finite

An indefinitely large finite set substitutes an actual infinite set.

A Relative Infinite.

- The infinite is not a single state $i \in \mathcal{I}$, but a region, e.g. $\{i' \in \mathcal{I} \mid i' \geq i\}$, the "indefinitely large region".
- The region depends on a context $C = (i_0, \ldots, i_{n-1}) \rightsquigarrow$ notion $C \ll i$ (or $i \gg C$), "i is indefinitely large relative to C".
- To a system we define a limit relative to C as a state with $i \gg C$.



Limits

Systems alone are not enough, we need limits of the system. But:

- Limits are not absolute and outside the system, but inside the system in the region of "indefinitely large stages". This region depends on a current context C and a relation $\ll$.
- This limit can be used to extend the system (added to the current context), i.e., it is then only an intermediate state.
- Systems formalize indefinitely extensibility, limits formalize indefinitely large finite states.
- An actual infinite set is seen as an indefinitely large finite set for which the context is irrelevant.

Limits

The Structure of Limits.

A limit $\mathcal{M}$ gets its structure from relation $\overset{p}{\mapsto}$ of the system $(\mathcal{M}_i)_{i\in\mathcal{I}}$. It consists of increasingly better PERs $\approx_i$. This family of PERs are a substitute for equality. $(\mathcal{M}, (\approx_i)_{i\in\mathcal{I}})$ is called *PER-set*.

Requirement for Limits.

There must be enough approximations, i.e., $\{i \in \mathcal{I} \mid a \in [i]\}$ with $[i] := \{a \in \mathcal{M} \mid a \approx_i a\}$ is *dense* for all $a \in \mathcal{M}$. Other formulations are: *indefinitely many*, *sufficiently many*, *almost all* states.
Formally we need $\mathfrak{D} \subseteq \mathfrak{P}(\mathcal{I})$ with $\{i \in \mathcal{I} \mid a \in [i]\} \in \mathfrak{D}$.
Properties:

- Each $\mathcal{J} \in \mathfrak{D}$ is cofinal.
- $\mathfrak{D}$ contains all (non-empty) upward-closed sets in $\mathcal{I}$.
- $\mathfrak{D}$ is a filter (necessary to do logic).
- $\mathfrak{D}$ is the cardinal aspect of infinity, $\ll$ is the ordinal aspect.

Limits

Function Space.

Given PER-sets $(\mathcal{M}, \approx_{\mathcal{I}})$ and $(\mathcal{N}, \approx_{\mathcal{J}})$, then the equivalence relation $\approx_{\mathcal{I}\times\mathcal{J}}$ of the function space $[\mathcal{M} \to \mathcal{N}]$ is defined as:

$$f \approx_{i\to j} g :\Longleftrightarrow a \approx_i b \text{ implies } f(a) \approx_j g(b) \text{ for all } a, b \in \mathcal{M}.$$

Continuity.

A function $f : \mathcal{M} \to \mathcal{N}$ is *i-j-continuous* iff $f \in [i \to j]$, that is, $a \approx_i b$ implies $f(a) \approx_j f(b)$. A function $f : \mathcal{M} \to \mathcal{N}$ is *$\mathfrak{D}$-continuous* iff there are $\mathfrak{D}$-many indices $i \to j$ such that f is i-j-continuous.
It is not a topological notion and it depends on $\mathfrak{D}$.

Totality.

No assumption of a general, global totality for type 2 and higher.

Dynamic Type Theory

Type Theory as a Basis.

Use type theory with types $\varrho, \sigma, \ldots$ and type constructors as $\rightarrow$, e.g. STT (for classical logic, but also for a realizability interpretation of intuitionistic logic). Assign to each type ϱ a set of approximation states $\mathcal{I}_\varrho$.

Approximation Declarations.

Similar as a type declaration $\Gamma \mid \mathrm{r} : \varrho$ — term r has type ϱ in type context $\Gamma = (\varrho_0, \ldots, \varrho_{n-1})$ we need an approximation declaration: $C \mid \mathrm{r} : i$, meaning, term r has an approximation in i for an approximation context $C = (i_0, \ldots, i_{n-1})$.

- C corresponds to "input bounds" and i to an "output bound".
- $C \mid \mathrm{r} : i$ states that a term r satisfies the "principle of finite support".

Dynamic Type Theory

Interpretation.

For $i \in \mathcal{I}_\varrho$ the interpretation $[\![i]\!]$ is a state in the system (a finite set), and $[\![\varrho]\!]$ is the limit of this system (an indefinitely large finite set).

Logic.

Classical logic: STT with type *bool* and further base types ι. But it needs a restriction of variables to positive and negative types:

$$\varrho^+ ::= \iota \,|\, (\varrho^- \to \varrho^+) \quad \text{and} \quad \varrho^- ::= \mathit{bool} \,|\, (\varrho^+ \to \varrho^-).$$

Positive types correspond to *objects* and to a direct limit construction, whereas negative types correspond to *properties* and an inverse limit construction.

Dynamic Type Theory

Range of Variables.

In a formula as $\forall \mathrm{x}_0 \exists \mathrm{x}_1 \forall \mathrm{x}_2 \Phi$, the variables x_0, x_1 and x_2 typically refer to different states, e.g., $(i_0, i_1) \ll i_2$.

Universal quantifier.

As a rule, not as a constant:

$$\frac{C \ll i \qquad C \mid \mathrm{r} : i \rightarrow bool}{C \mid \forall_{\varrho} \mathrm{r} : bool}$$

Interpretation with implicit reflection principle:

$$[\![\forall_{\varrho} \mathrm{r}]\!]_{\boldsymbol{a}:C} := \bigwedge_{b \in [\![i]\!]} [\![\mathrm{r}]\!]^{i \rightarrow bool}_{\boldsymbol{a}:C}(b).$$

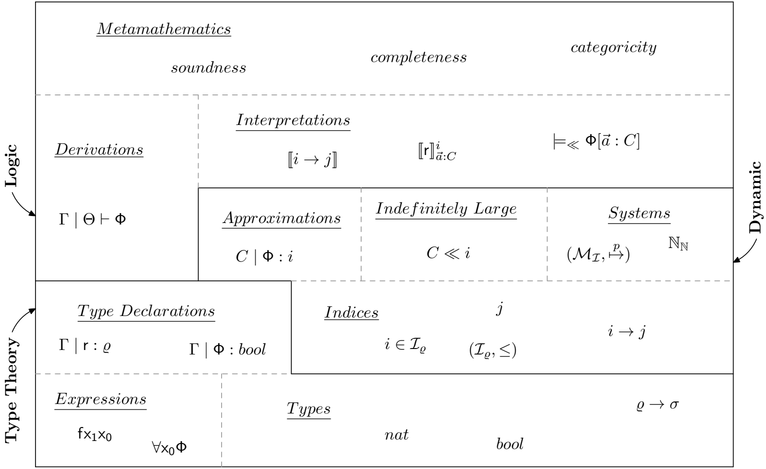
Dynamic Type Theory

Functions vs. Relations

To use a function symbol requires totality, e.g. for $\mathrm{f} : (nat \to nat) \to nat$ there must be indefinitely many indices $(i \to j) \to k$ such that $\mathrm{f} : (i \to j) \to k$. For a relation totality must be proven, e.g. $\mathrm{R} : (nat \to nat) \to nat \to bool$ holds for all $(i \to j) \to k \to bool$ (by restriction).

Correctness.

In order to show that the interpretation is correct, do a Löwenheim-Skolem constructions for elements of positive type: If such an element exists, it exists within a finite state. Restrict the elements of negative type to these finite states.
This argument uses a naive (non-constructive) notion of existence on meta-level — a universal quantified formula is valid, if there is no counter-example.


Metamathematics
soundness
completeness
categoricity
Derivations
$\Gamma \mid \Theta \vdash \Phi$
Interpretations
$[\![i \to j]\!]$
$[\![\mathsf{r}]\!]^{i}_{\vec{a}:C}$
$\models_{\ll} \Phi[\vec{a} : C]$
Approximations
$C \mid \Phi : i$
Indefinitely Large
$C \ll i$
Systems
$(\mathcal{M}_{\mathcal{I}}, \overset{p}{\mapsto})$
$\mathbb{N}_{\mathbb{N}}$
Type Declarations
$\Gamma \mid \mathsf{r} : \varrho$
$\Gamma \mid \Phi : bool$
Indices
j
$i \in \mathcal{I}_{\varrho}$
$(\mathcal{I}_{\varrho}, \leq)$
$i \to j$
Expressions
$\mathsf{f}\mathsf{x}_1\mathsf{x}_0$
$\forall \mathsf{x}_0 \Phi$
Types
$\varrho \to \sigma$
nat
bool
Logic
Dynamic
Type Theory

Real Numbers

Abstract Objects.

As first-order, base type objects of a complete ordered field: $r : real$ with base type $real$.

Concrete Approximations.

As higher-order approximation process. Different ways to approximate/represent a real number. For instance a Cauchy sequence, roughly $r : nat \to rat$ with base type rat of rational numbers.

Real Numbers as Limits.

A real number cannot identified with its approximation process, but needs an explicit limit operation. $\lim : (nat \to rat) \to real$.

Avantages

- Model theoretic approach which contains elements of computability.
- Avoids paradoxes of infinity.
- Distinguishes epistemological from ontological aspects of finitism.
- Allows a uniform model for (classical) first- and higher-order logic.
- Has a reflection principle from the beginning.
- Concepts are applicable to background model (meta level): Model of model theory uses indefinitely extensible sets, in particular, the index $\mathcal{I}$ is not actual infinite.
- Other concepts often require a notion of infinity: Domain theory due to *direct completeness*. Hyperfinte type structure due to a *Fréchet product*.
- No notion of finiteness required. Applicable to any definite vs. indefinite distinction (e.g. set vs. proper class).

Remarks

Using indefinitely large finite sets has its origin in Shaughan Lavine's book *Understanding the Infinite*, based on Jan Mycielski's *Locally finite theories*.

Done.

- FOL, classical as well as intuitionistic (with Kripke model).
- Model for HOL.
- Almost done: Interpretation for HOL.

Open questions.

- Relation to hereditary total functionals.
- Understand more the role of set $\mathfrak{D}$. E.g., a global totality for functionals of type ≥ 2 is equivalent to the fact that the direct limit is the same as the inverse limit (as in domain theory).
- Application to other logics, e.g. intuitionistic HOL.

Thank you for your attention.