Toward verified real computation

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Continuity, Computability, Constructivity From Logic to Algorithms 2021
September 23, 2021

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- how do we write *correct* programs in the languages?
- imperative: Hoare-style verification based on

Park, Sewon, Franz Brauße, Pieter Collins, SunYoung Kim, Michal Konečný, Gyesik Lee, Norbert Müller, Eike Neumann, Norbert Preining, and Martin Ziegler "Foundation of Computer (Algebra) ANALYSIS Systems: Semantics, Logic, Programming, Verification." arXiv preprint arXiv:1608.05787 (2021).

- functional: program extraction based on

Michal Knenčný, Sewon Park, and Holger Thies "Axiomatic Reals and Certified Efficient Exact Real Computation." 27th Workshop on Logic, Language, Information and Computation. Springer, 2021 (accepted)

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REAL Rump(REAL x, REAL y)

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is this specification correct?

iRRAM Example 2

For any $x \in \mathbb{R}$, compute $n \in \mathbb{Z}$ such that $x < 2^n$

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int magnitude(REAL x)

    REAL y = 1;
    int n = 0;
    while (y < x){
        y = y * 2;
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int magnitude(REAL x)
    {x ∈ ℝ}
    REAL y = 1;
    int n = 0;
    while (choose(y < x + 1, x < y) = 1){
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```

- y goes $1, 2, 2^2, 2^3, \cdots$. Throughout the loop, $y = 2^n$
- for any y, at least one of two tests $y < x + 1$ and $x < y$ hold
- at some point, $y < x + 1$ must evaluate to `false`. Hence, the loop gets escaped
- when the loop is escaped, $x < y = 2^n$ holds.

Our Goal

A systematic way to verify the correctness of program specifications.

- Design a simple imperative language that can model core fragments of real number computation languages including nondeterministic choose
- Computable semantics in the sense of computable analysis (exact arithmetical operations and partial comparison)
- Convenient-to-use precondition-postcondition-style program verification
- Some thoughts on extending it with further continuous objects

Language Design

Syntax

While-language based on Peano Arithmetic and Boolean logic:

data types	τ	$::=$	B	Boolean
		$\mid$	Z	integer
expressions	e	$::=$	$\texttt{true} \mid \texttt{false} \mid 0 \mid 1 \mid \cdots$	constant
		$\mid$	$e_1 + e_2 \mid e_1 - e_2 \mid e_1 \times e_2$	integer arithmetic
		$\mid$	$e_1 = e_2 \mid e_1 \leq e_2$	integer comparison
commands	c	$::=$	$\texttt{skip}$	do nothing
		$\mid$	$c_1 ; c_2$	composition
		$\mid$	$x := e$	assignment
		$\mid$	$\texttt{if}\ e\ \texttt{then}\ c_1\ \texttt{else}\ c_2\ \texttt{end}$	conditional
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\text{data types} & \tau & ::= & \mathsf{B} & \text{Boolean} \\
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 & & \mid & \mathsf{R} & \text{real number} \\
\text{expressions} & e & ::= & \mathtt{true} \mid \mathtt{false} \mid 0 \mid 1 \mid \cdots & \text{constant} \\
 & & \mid & e_1 + e_2 \mid e_1 - e_2 \mid e_1 \times e_2 & \text{integer arithmetic} \\
 & & \mid & e_1 = e_2 \mid e_1 \leq e_2 & \text{integer comparison} \\
 & & \mid & e_1 + e_2 \mid e_1 - e_2 \mid e_1 \times e_2 \mid e^{-1} & \text{real arithmetic} \\
 & & \mid & e_1 \lesssim e_2 & \text{partial real comparison} \\
 & & \mid & \mathsf{choose}(e_1, \cdots, e_n) & \text{nondeterminism} \\
 & & \mid & 2^e \mid \iota(e) & \text{coercion from } \mathsf{Z} \text{ to } \mathsf{R} \\
\text{commands} & c & ::= & \mathtt{skip} & \text{do nothing} \\
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Denotational Semantics

1. Given a state γ, an *expression* evaluates to a value:

$$x + y \rightsquigarrow_\gamma 42 \quad \text{when } x \rightsquigarrow_\gamma 21 \text{ and } y \rightsquigarrow_\gamma 21$$

2. Due to *nondeterminism*, there are several possible evaluations:

$$\mathsf{choose}(\mathtt{true}, \mathtt{true}) \rightsquigarrow_\gamma 1 \text{ or } 2$$

3. There are nonterminating evaluations

$$0^{-1} \rightsquigarrow_\gamma \bot \qquad \pi \lesssim \pi \rightsquigarrow_\gamma \bot$$

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$$[\![\mathsf{choose}(e_1, \cdots, e_n)]\!]\gamma = \{i \mid \texttt{true} \in [\![e_i]\!]\gamma\} \cup \{\bot \mid \forall i.\ \bot \text{ or } \texttt{false} \in [\![e_i]\!]\gamma\}$$

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Theorem
The language is approximately complete in that to any computable (partial) real function, there is a program that rigorously approximates it.

Formal Verification

Specification

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All states satisfying ϕ make c terminate and results states satisfying ψ.

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Theorem

The first order logic over the structures of integers and reals connected via $\mathbb{Z} \ni z \mapsto z \in \mathbb{R}$ and $\mathbb{Z} \ni z \mapsto 2^z \in \mathbb{R}$ is expressive for the expression language;

for any expression e, there is a predicate $(\!|e|\!)(y)$ that defines the denotation of e

$$(\!|e|\!)(v) \textit{ iff } v \textit{ is in (but } \bot \textit{ is not in) the denotation of } e$$

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For example, (in the simplified form)

$$(\!|x \lesssim y|\!)(v) \equiv (v = \mathtt{true} \land x < y) \lor (v = \mathtt{false} \land y < x)$$
$$(\!|\mathsf{choose}_n(e_1, \cdots, e_n)|\!)(v) \equiv (v = 1 \land (\!|e_1|\!)(\mathtt{true})) \lor \cdots \lor (v = n \land (\!|e_n|\!)(\mathtt{true}))$$

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$$\begin{aligned} \exists v.\ (\!|e|\!)(v) \quad &\text{iff} \quad \bot \notin [\![e]\!]\gamma \\ \forall v.\ (\!|e|\!)(v) \Rightarrow \psi(v) \quad &\text{iff} \quad \forall v \in [\![e]\!]\gamma \text{ satisfy } \psi \end{aligned}$$

Proof Rule: assignment

$$\frac{}{\left\{\exists v.\ (\!|e|\!)(v) \land \forall v.\ (\!|e|\!)(v) \Rightarrow \psi[v/x]\right\}\ x := e\ \left\{\psi\right\}}$$

For an initial state γ,

- γ satisfying $\exists v.\ (\!|e|\!)(v)$ ensures $\bot \notin [\![e]\!]\gamma$
- Resulting state is $\gamma[x \mapsto v]$ for each $v \in [\![e]\!]\gamma$
- If γ satisfies $\forall v.\ (\!|s|\!)(v) \Rightarrow \psi[v/x]$, any $v \in [\![e]\!]\gamma$, γ satisfies $\psi[v/x]$
- Hence, resulting states satisfy ψ

Proof Rule: assignment

$$\frac{}{\left\{\exists v.\ (\!|e|\!)(v) \land \forall v.\ (\!|e|\!)(v) \Rightarrow \psi[v/x]\right\}\ x := e\ \left\{\psi\right\}}$$

For an initial state γ,

- γ satisfying $\exists v.\ (\!|e|\!)(v)$ ensures $\bot \notin [\![e]\!]\gamma$
- Resulting state is $\gamma[x \mapsto v]$ for each $v \in [\![e]\!]\gamma$
- If γ satisfies $\forall v.\ (\!|s|\!)(v) \Rightarrow \psi[v/x]$, any $v \in [\![e]\!]\gamma$, γ satisfies $\psi[v/x]$
- Hence, resulting states satisfy ψ

Example:

$$b := x \lesssim y\ \left\{b = \mathtt{true}\right\}$$

Proof Rule: assignment

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Example:

$$\left\{(\exists v.\ (\!|x \lesssim y|\!)(v)) \wedge \forall v.\ (\!|x \lesssim y|\!)(v) \Rightarrow (b = \texttt{true})[v/b]\right\}\ b := x \lesssim y\ \left\{b = \texttt{true}\right\}$$

Proof Rule: assignment

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$$\begin{aligned}&\left\{(\exists v.\ (v = \texttt{true} \land x < y) \lor (v = \texttt{false} \land y < x))\right.\\ &\quad \left.\land \forall v.\ ((v = \texttt{true} \land x < y) \lor (v = \texttt{false} \land y < x)) \Rightarrow v = \texttt{true}\right\}\ b := x \lesssim y\ \left\{b = \texttt{true}\right\}\end{aligned}$$

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$$\frac{}{\left\{\exists v.\ (\!|e|\!)(v) \land \forall v.\ (\!|e|\!)(v) \Rightarrow \psi[v/x]\right\}\ x := e\ \left\{\psi\right\}}$$

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$$\left\{(\exists v.\ (v = \mathtt{true} \land x < y) \lor (v = \mathtt{false} \land y < x))\right.$$
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$$\left\{(x < y \lor y < x) \land x \leq y\right\}\ b := x \lesssim y\ \left\{b = \mathtt{true}\right\}$$

Proof Rule: assignment

$$\frac{}{\left\{\exists v.\ (\!|e|\!)(v) \wedge \forall v.\ (\!|e|\!)(v) \Rightarrow \psi[v/x]\right\}\ x := e\ \left\{\psi\right\}}$$

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Example:

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$$\left\{(x < y \vee y < x) \wedge x \leq y\right\}\ b := x \lesssim y\ \left\{b = \texttt{true}\right\}$$

$$\left\{x < y\right\}\ b := x \lesssim y\ \left\{b = \texttt{true}\right\}$$

Proof Rule: while loop

$$\frac{\left\{ (\!|e|\!)(\texttt{true}) \wedge I \wedge V = \xi \wedge L = \xi' \right\}\ c\ \left\{ I \wedge V \leq \xi - \xi' \wedge L = \xi' \right\}}{\left\{ I \right\}\ \texttt{while}\ e\ \texttt{do}\ c\ \texttt{end}\ \left\{ I \wedge (\!|e|\!)(\texttt{false}) \right\}}$$

- $I \wedge (\!|e|\!)(\texttt{true}) \Rightarrow L > 0$
- $I \Rightarrow ((\!|e|\!)(\texttt{true}) \vee (\!|e|\!)(\texttt{false}))$
- $I \wedge V \leq 0 \Rightarrow \forall k.\ (\!|e|\!)(k) \Rightarrow k = \texttt{false}$

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- I is a loop-invariant formula, L is a loop-invariant quantity, and V is a loop-variant quantity

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- I is a loop-invariant formula, L is a loop-invariant quantity, and V is a loop-variant quantity
- L bounds the decrement of V in each iteration
- Side-conditions ensure
    1. When I holds and $e \rightsquigarrow \texttt{true}$, L is positive
    2. When I holds, $e \rightsquigarrow \bot$ does not happen
    3. When I holds and V is negative, e evaluates only to `false`

Proof Rule: while loop

$$\frac{\left\{ (\!|e|\!)(\texttt{true}) \wedge I \wedge V = \xi \wedge L = \xi' \right\} \; c \; \left\{ I \wedge V \leq \xi - \xi' \wedge L = \xi' \right\}}{\left\{ I \right\} \; \texttt{while } e \texttt{ do } c \texttt{ end} \; \left\{ I \wedge (\!|e|\!)(\texttt{false}) \right\}}$$

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Theorem
The proof rules are sound w.r.t. the denotational semantics.

Verification Condition

annotated commands	c	$::=$	`skip`	do nothing
		$\mid$	$c_1 ; c_2$	composition
		$\mid$	$x := e$	assignment
		$\mid$	`if` e `then` c_1 `else` c_2 `end`	conditional
		$\mid$	$\{I, V, L\}$`while` e `do` c `end`	loop

$$
\begin{array}{llcll}
\text{annotated commands} & c & ::= & \texttt{skip} & \text{do nothing} \\
 & & | & c_1 ; c_2 & \text{composition} \\
 & & | & x := e & \text{assignment} \\
 & & | & \texttt{if } e \texttt{ then } c_1 \texttt{ else } c_2 \texttt{ end} & \text{conditional} \\
 & & | & \{I, V, L\}\texttt{while } e \texttt{ do } c \texttt{ end} & \text{loop}
\end{array}
$$

Theorem
For c and a postcondition ψ, there is a precondition $\texttt{vc}(c, \psi)$

$$\left\{\texttt{vc}(c, \psi)\right\} \ c \ \left\{\psi\right\}$$

In order to show

$$\left\{\phi\right\} \ c \ \left\{\psi\right\}$$

it suffices to show

$$\phi \Rightarrow \texttt{vc}(c, \psi)$$

Verification Condition Example

For any $x \in \mathbb{R}$, compute $n \in \mathbb{Z}$ such that $x < 2^n$

```
int magnitude(REAL x){
    {x ∈ ℝ}
    let y := 1;
    let n := 0;

    while choose(y < x + 1, x < y) = 1
    do
        y = y × 2;
        n = n + 1
    end
    {x < 2^n}
    return n
}
```

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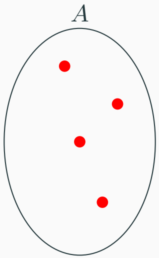
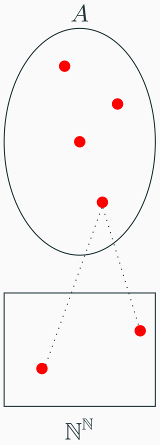
```
int magnitude(REAL x){
    {x ∈ ℝ}
    let y := 1;
    let n := 0;
    {I ≡ y = 2^n ∧ n ≥ 0     V ≡ x − y + 1     L ≡ 1}
    while choose(y < x + 1, x < y) = 1
    do
        y = y × 2;
        n = n + 1
    end
    {x < 2^n}
    return n
}
```

Extension



Definition
An assembly is a pair $(A, \Vdash_A)$ of a set A and a relation $\Vdash_A \subseteq \mathbb{N}^{\mathbb{N}} \times A$ such that

$$\forall x \in A \,.\, \exists \varphi \in \mathbb{N}^{\mathbb{N}} \,.\, \varphi \Vdash_A x$$



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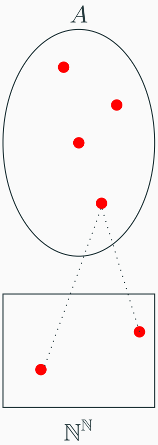
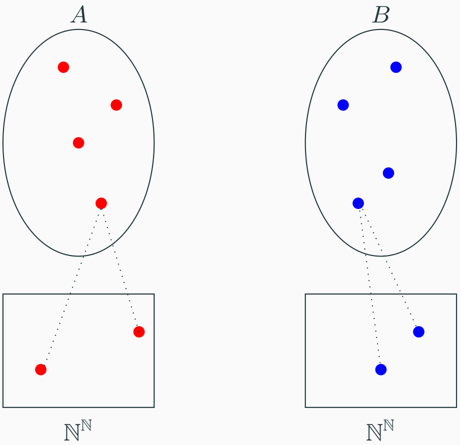


Definition
A function $f : A \to B$ is computable if there is computable $\tau : \mathbb{N}^{\mathbb{N}} \rightharpoonup \mathbb{N}^{\mathbb{N}}$ *tracks* f:

$$\forall x \in A \,.\, \forall \varphi \in \mathbb{N}^{\mathbb{N}} \,.\, \varphi \Vdash_A x \Rightarrow \tau(\varphi) \Vdash_B f(x)$$



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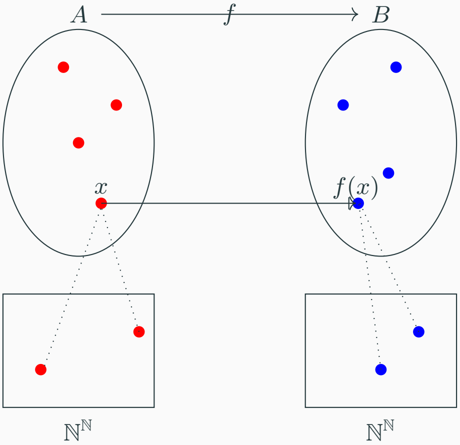

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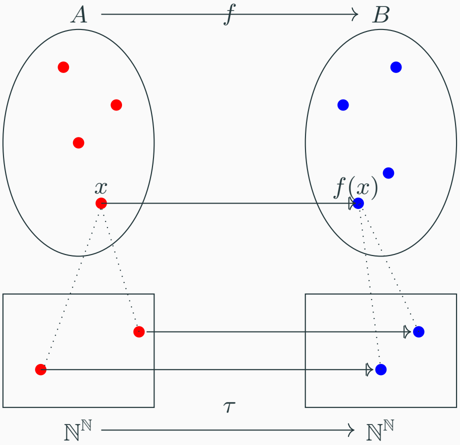

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- Assmeblies and computable functions form $\mathsf{Asm}(\mathbb{N}^{\mathbb{N}})$

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- $\flat$ classifies partial functions which diverge outside of their domains

$$\lesssim : \mathbb{R} \times \mathbb{R} \to \flat\{\mathtt{true}, \mathtt{false}\} \qquad \text{s.t.} \qquad \pi \lesssim \pi = \flat$$

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- $\sharp$ classifies partial functions which correct outside of their domains

$$\mathsf{lim} : \mathbb{R}^{\mathbb{N}} \to \sharp\mathbb{R} \qquad \text{s.t.} \qquad \mathsf{lim}(0, 1, 2, 3, \cdots) = \sharp$$

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- $\natural$ classifies general partial functions without any out-of-domain specification

$$\lesssim : \mathbb{R} \times \mathbb{R} \to \natural\{\mathtt{true}, \mathtt{false}\} \qquad \text{and} \qquad \mathsf{lim} : \mathbb{R}^{\mathbb{N}} \to \natural\mathbb{R}$$

Monads

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- M defines nondeterministic functions

$$\lesssim_n : \mathbb{R} \times \mathbb{R} \to \mathsf{M}\{\mathtt{true}, \mathtt{false}\} \qquad \text{s.t.} \qquad \pi \lesssim_n \pi = \{\mathtt{true}, \mathtt{false}\}$$

Interpretation

Denotational Semantics

In Set:

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$$\bullet \; [\![\tau]\!]$$

- Denotation of data type $[\![\tau]\!]$

Denotational Semantics

In Set:

$[\![\tau]\!]$

$[\![\mathsf{R}]\!] = \mathbb{R}$

- Denotation of data type $[\![\tau]\!]$

Denotational Semantics

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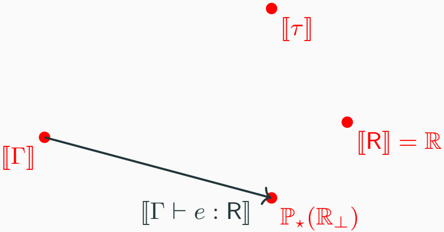

- Denotation of data type $[\![\tau]\!]$
- Denotation of a context $[\![\Gamma]\!]$ of all states
- Denotation of a expression

$$[\![\Gamma \vdash e : \tau]\!] : [\![\Gamma]\!] \to \mathbb{P}_\star([\![\tau]\!]_\perp)$$

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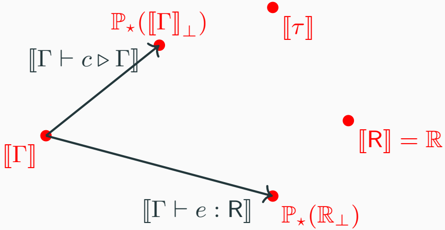

- Denotation of data type $\llbracket\tau\rrbracket$
- Denotation of a context $\llbracket\Gamma\rrbracket$ of all states
- Denotation of a expression
$$\llbracket\Gamma \vdash e : \tau\rrbracket : \llbracket\Gamma\rrbracket \to \mathbb{P}_\star(\llbracket\tau\rrbracket_\perp)$$
- Denotation of a command
$$\llbracket\Gamma \vdash c \triangleright \Gamma\rrbracket : \llbracket\Gamma\rrbracket \to \mathbb{P}_\star(\llbracket\Gamma\rrbracket_\perp)$$

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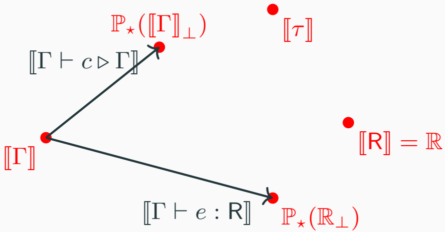

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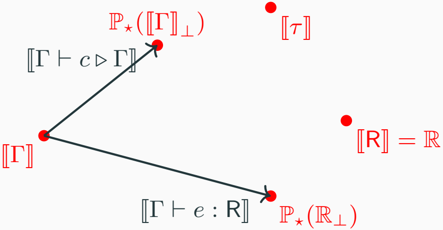
Interpretation

In $\mathsf{Asm}(\mathbb{N}^\mathbb{N})$:

Interpretation

Denotational Semantics

In Set:



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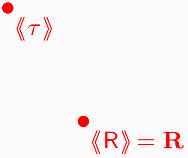
Interpretation

In $\mathsf{Asm}(\mathbb{N}^\mathbb{N})$:

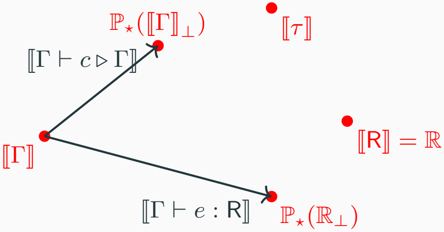


- Interpretation of data type $\langle\!\langle\tau\rangle\!\rangle$

Interpretation

Denotational Semantics

In Set:



- Denotation of data type $\llbracket\tau\rrbracket$
- Denotation of a context $\llbracket\Gamma\rrbracket$ of all states
- Denotation of a expression
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Interpretation

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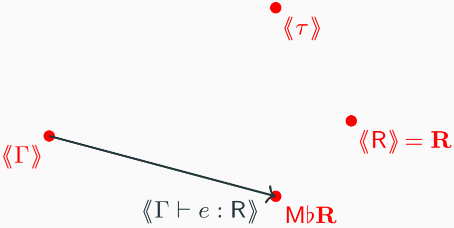

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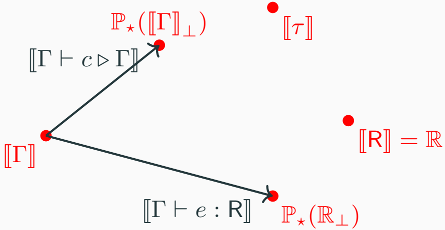
Denotational Semantics

In Set:



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- Denotation of a command
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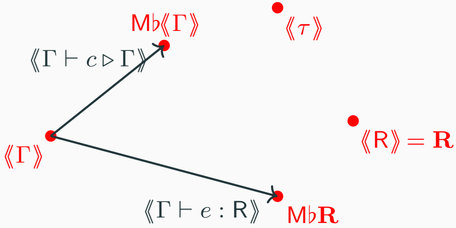

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Denotational Semantics

In Set:

Interpretation

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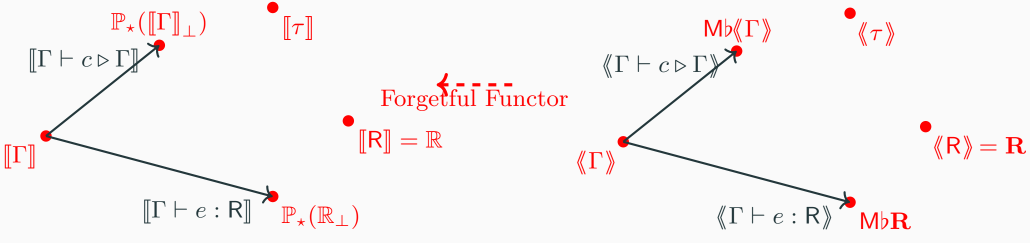

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Denotational Semantics

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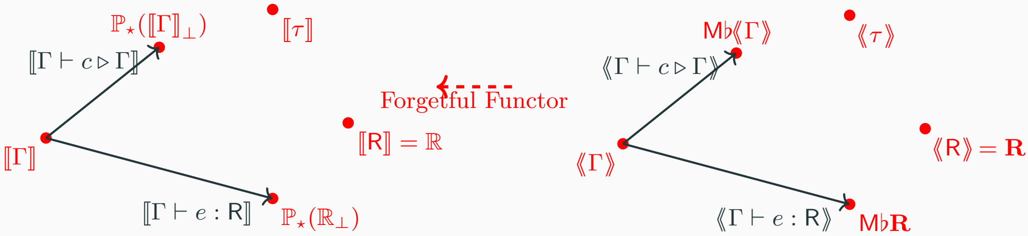

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Theorem
The denotational semantics is computable.

Extension

Denotational Semantics

In Set:

$$\begin{tikzcd} & \mathbb{P}_\star(\llbracket\Gamma\rrbracket_\perp) & \llbracket\tau\rrbracket \\ \llbracket\Gamma\rrbracket \arrow[ur, "\llbracket\Gamma \vdash c \triangleright \Gamma\rrbracket"] \arrow[rr, "\llbracket\Gamma \vdash e : \mathsf{R}\rrbracket"'] & & \mathbb{P}_\star(\mathbb{R}_\perp) \end{tikzcd}$$

$\llbracket\mathsf{R}\rrbracket = \mathbb{R}$

$\xleftarrow{\text{Forgetful Functor}}$

Interpretation

In $\mathsf{Asm}(\mathbb{N}^{\mathbb{N}})$:

$$\begin{tikzcd} & \mathsf{M}\flat\langle\!\langle\Gamma\rangle\!\rangle & \langle\!\langle\tau\rangle\!\rangle \\ \langle\!\langle\Gamma\rangle\!\rangle \arrow[ur, "\langle\!\langle\Gamma \vdash c \triangleright \Gamma\rangle\!\rangle"] \arrow[rr, "\langle\!\langle\Gamma \vdash e : \mathsf{R}\rangle\!\rangle"'] & & \mathsf{M}\flat\mathbf{R} \end{tikzcd}$$

$\langle\!\langle\mathsf{R}\rangle\!\rangle = \mathbf{R}$

Denotational Semantics

In Set:

$$\begin{tikzcd}[column sep=small]
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\llbracket \Gamma \rrbracket \arrow[ur, "\llbracket \Gamma \vdash c \triangleright \Gamma \rrbracket"] \arrow[rr, "\llbracket \Gamma \vdash e : \mathsf{R} \rrbracket"'] & & \mathbb{P}_\star(\mathbb{R}_\perp) & \llbracket \mathsf{R} \rrbracket = \mathbb{R}
\end{tikzcd}$$

Interpretation

In $\mathsf{Asm}(\mathbb{N}^{\mathbb{N}})$:

$$\begin{tikzcd}[column sep=small]
 & \mathsf{M}\flat\langle\!\langle \Gamma \rangle\!\rangle & \langle\!\langle \tau \rangle\!\rangle & \\
\langle\!\langle \Gamma \rangle\!\rangle \arrow[ur, "\langle\!\langle \Gamma \vdash c \triangleright \Gamma \rangle\!\rangle"] \arrow[rr, "\langle\!\langle \Gamma \vdash e : \mathsf{R} \rangle\!\rangle"'] & & \mathsf{M}\flat\mathbf{R} & \langle\!\langle \mathsf{R} \rangle\!\rangle = \mathbf{R}
\end{tikzcd}$$

$$\mathsf{Asm}(\mathbb{N}^{\mathbb{N}}) \xrightarrow{\text{Forgetful Functor}} \mathsf{Set}$$

- Add a new data type τ' and a new expression construct $\mathsf{f}(t_1, \cdots, t_n) : \tau'$

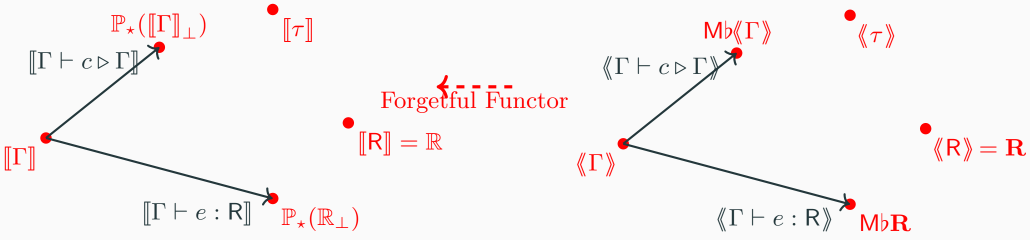
Extension

Denotational Semantics

In Set:

Interpretation

In $\mathsf{Asm}(\mathbb{N}^{\mathbb{N}})$:



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Denotational Semantics

In Set:

Interpretation

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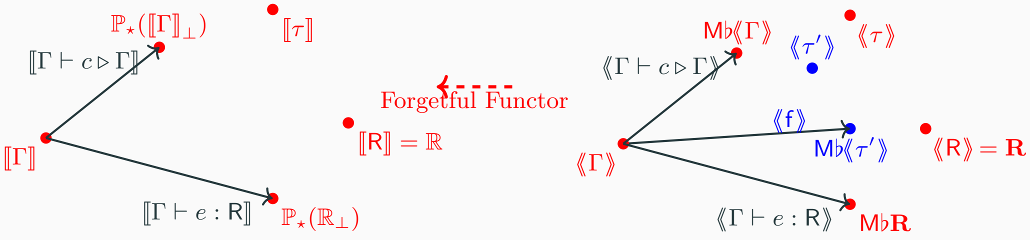

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Denotational Semantics

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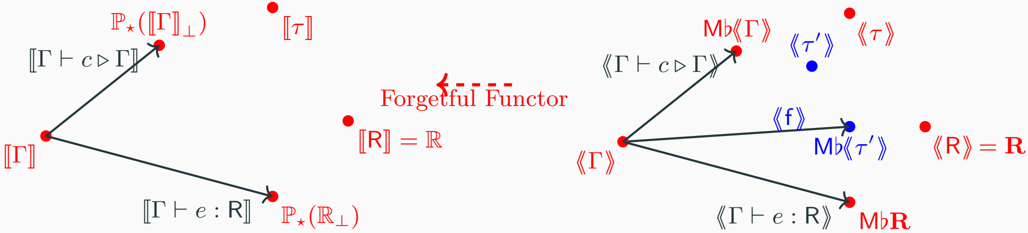

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Denotational Semantics

In Set:

Interpretation

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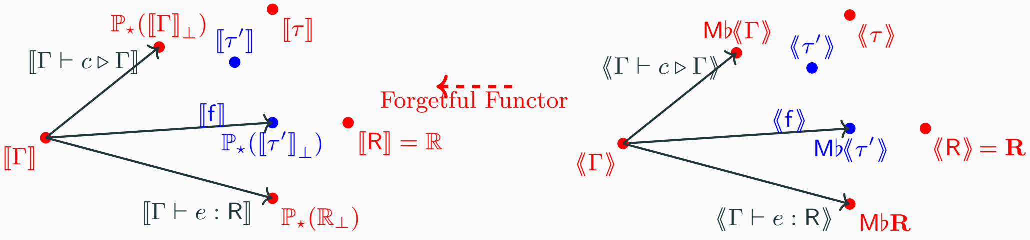

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Extension

Denotational Semantics

In Set:

Interpretation

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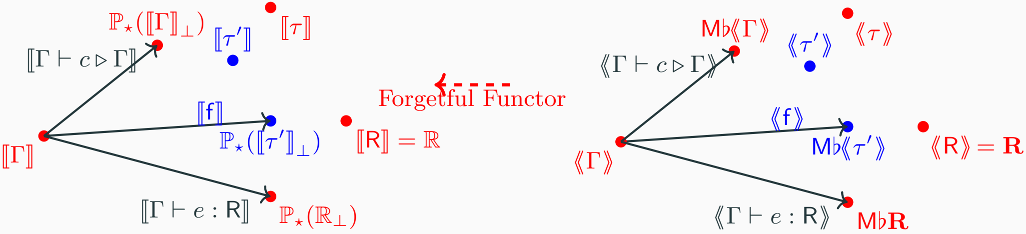

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Interpretation

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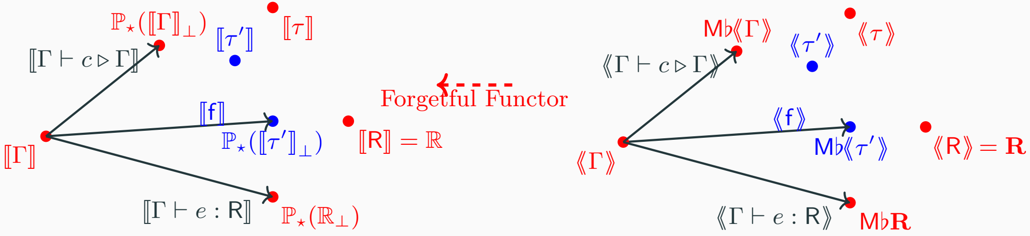

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- per collection of assemblies and morphisms $\mathcal{E}$ (with consistency property), we define a while language over $\mathcal{E}$
  - with laziness
  - with matrices
  - with continuous real functions

Extension with Continuous Real Functions

- Add a new data type C and a term construct $\mathsf{eval} : \mathsf{C} \times \mathsf{R} \to \mathsf{R}$
- Interpret $\langle\!\langle \mathsf{C} \rangle\!\rangle = \mathbf{R}^{\mathbf{R}}$ and $\langle\!\langle \mathsf{eval} \rangle\!\rangle$ the evaluation map of $\mathbf{R}^{\mathbf{R}}$ in $\mathsf{Asm}(\mathbb{N}^{\mathbb{N}})$
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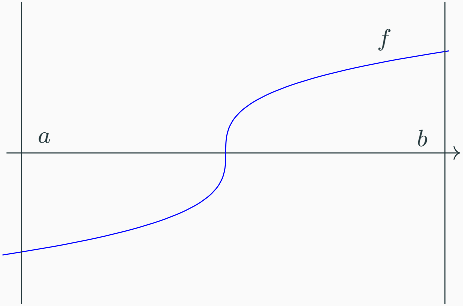
When $\quad a, b : \mathsf{R}, f : \mathsf{C}$



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When $\quad x, y, a, b : \mathsf{R}, f : \mathsf{C}$

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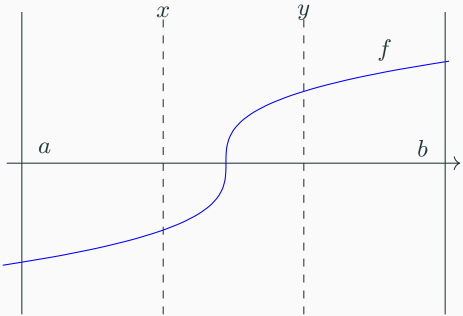

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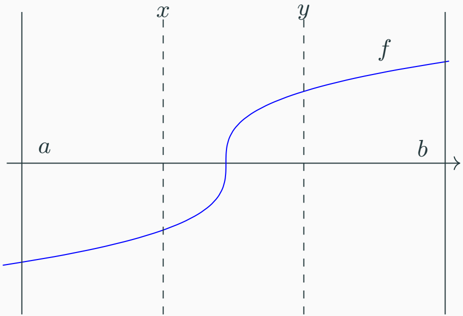

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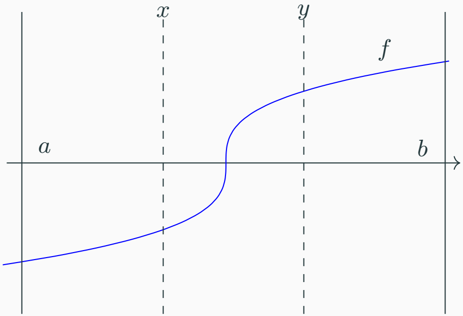

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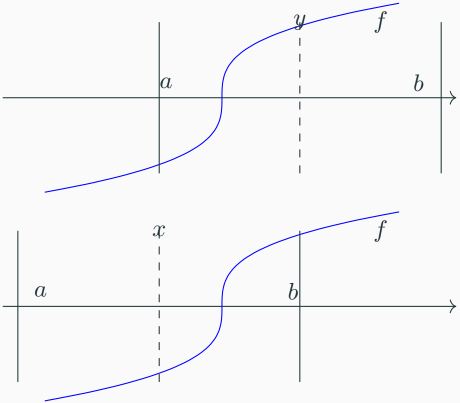

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When $p : \mathsf{Z}, x, y, a, b : \mathsf{R}, f : \mathsf{C}$

while $b - a \lesssim 2^p$ **do**
  $x := (2 \times a + b)/3;$
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  if $\mathsf{choose}(\mathsf{eval}(f, x) \lesssim 0, 0 \lesssim \mathsf{eval}(f, y)) = 1$
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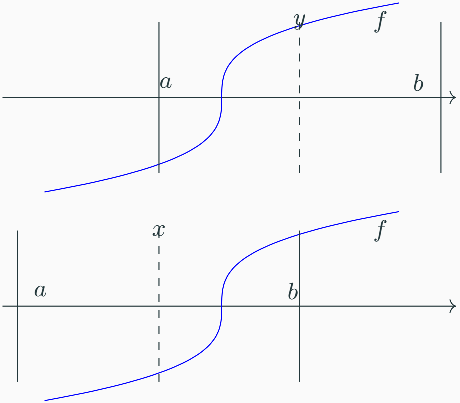
  end
end



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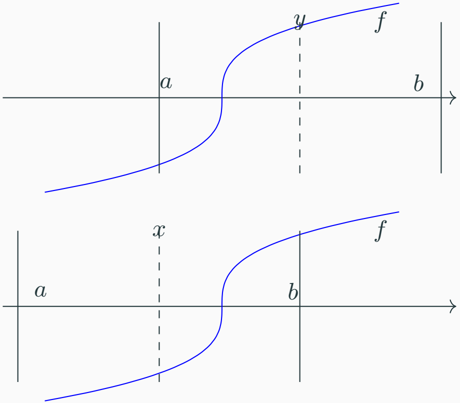
When $p : \mathsf{Z}, x, y, a, b : \mathsf{R}, f : \mathsf{C}$

$\{f$ has unique root in $(a, b) \wedge f(a) < 0 < f(b)\}$
`while` $b - a \lesssim 2^p$ `do`
  $x := (2 \times a + b)/3;$
  $y := (a + 2 \times b)/3;$
  `if` $\mathsf{choose}(\mathsf{eval}(f, x) \lesssim 0, 0 \lesssim \mathsf{eval}(f, y)) = 1$
    `then` $a := x$
    `else` $b := y$
  `end`
`end`
$\{a$ approximates the root of f by $2^p\}$



Concluding Remark (of the section)

So far,

- we defined a simple imperative language that supports exact real computation and nondeterminism
- we devised a sound verification calculus and VC generator
- we suggested a way to extend the language and the verification calculus with other continuous objects

In the future,

- strong verification automation
- practical tool for verifying iRRAM programs

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- we devised a sound verification calculus and VC generator
- we suggested a way to extend the language and the verification calculus with other continuous objects

In the future,

- strong verification automation
- practical tool for verifying iRRAM programs
- imperative language with explicit limit operator (j.w.w. Andrej Bauer and Alex Simpson, CCA 2020)

Program Extraction

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- axiomatic real in dependent type theories

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- Implement the axioms in Coq, adjust Coq's extraction mechanism: axiomatic real $\to$ AERN's primitive data type `CReal`
- Proved interesting lemmas including intermediate value theorem, boundedness of real numbers, square roots, and complex square roots.
- Experiment with extracted AERN programs.

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- axiomatic real in dependent type theories
- In a dependent type theory that admits $\mathsf{Asm}(\mathbb{N}^{\mathbb{N}})$-interpretation, axiomatize the nondeterminism monad $\mathsf{M} : \mathsf{Type} \to \mathsf{Type}$
- Using the monad, axiomatize the set of real numbers; i.e., axiomatize an assembly that is (provably) equivalent to the Cauchy assembly
- Implement the axioms in Coq, adjust Coq's extraction mechanism: axiomatic real $\to$ AERN's primitive data type `CReal`
- Proved interesting lemmas including intermediate value theorem, boundedness of real numbers, square roots, and complex square roots.
- Experiment with extracted AERN programs.
- j.w.w. Michal Konečný and Holger Thies. Talk on October 9 (Saturday) at WoLLIC (Workshop on Logic, Language, Information and Computation) workshop
- implementation available at `https://github.com/holgerthies/coq-aern`

Thank you for listening! :)