An introduction to Edalat's theory of R-integration

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October 18, 1997

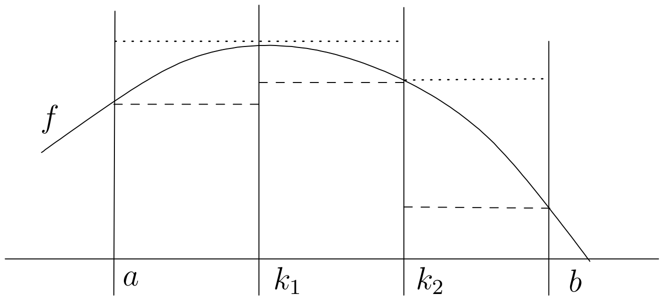
1 Introduction and motivation

1.1 The Riemann integral on the real line

Let $[a, b]$ be a compact interval of the real line and $f\colon [a, b] \to \mathbb{R}$ be a (bounded) function. For the *Riemann-integral* of f over $[a, b]$ one defines lower and upper (Darboux) sums with respect to a subdivision p of the form $a = k_0 < k_1 < \ldots < k_n = b$:

$$\begin{array}{rcl} S^l(f,p) & := & \sum_{i=1}^n (k_i - k_{i-1}) \inf\{f(x) \mid k_{i-1} \leq x \leq k_i\} \\ S^u(f,p) & := & \sum_{i=1}^n (k_i - k_{i-1}) \sup\{f(x) \mid k_{i-1} \leq x \leq k_i\} \end{array}$$

These have a geometric interpretation as areas below, resp. around the given function



and in some intuitive sense we expect the integral to lie somewhere between lower and upper sum.

$$S^l(f,p) \leq \int_a^b f\,dx \leq S^u(f,p)$$

This is made precise through a series of lemmas as follows.

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Lemma 1 *For any subdivision $S^l(f,p) \leq S^u(f,p)$ holds.*

Lemma 2 *If q refines p then*

$$S^l(f,p) \leq S^l(f,q)$$
$$S^u(f,q) \leq S^u(f,p)$$

hold.

Lemma 3 *For any two subdivisions p, q, $S^l(f,p) \leq S^u(f,q)$ holds.*

We say that a function satisfies the *Riemann condition* if for every $\epsilon > 0$ there is a subdivision p of $[a,b]$ such that $S^u(f,p) - S^l(f,p) < \epsilon$ holds. In such a situation one defines the *Riemann integral* of f over $[a,b]$ as

$$\int_a^b f dx = \sup \{S^l(f,p) \mid p \text{ a subdivision of } [a,b]\}$$

Observe the style of limit we are dealing with here: an increasing net of values S^l and a decreasing net of values S^u such that every S^l is below every S^u and such that their distance tends to zero. This is a particular kind of limiting process, different from the standard limit definition in which the limit is known beforehand

$$x_n \longrightarrow x \qquad \text{if} \qquad |x - x_n| \longrightarrow 0$$

and also different from Cauchy sequences

$$(x_n)_{n \in \mathbb{N}} \text{ convergent} \qquad \text{if} \qquad \forall \epsilon > 0 \exists N \in \mathbb{N} \forall n, m \geq N.\ |x_n - x_m| < \epsilon$$

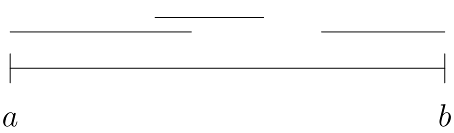
The R-integral, which we are going to introduce, operates through the same principle.

Finally, we note the following.

Theorem 4 *Every continuous function $f \colon [a,b] \to \mathbb{R}$ is Riemann integrable.*

1.2 From subdivisions to coverings

Is it possible to extend the range of subdivisions to more general coverings or even partial coverings?



If such a covering is sufficiently "fine grained", the corresponding lower and upper sums ought to be close to the integral of the function.

Why would we want to do this?

1. The measure may be unevenly distributed and we may want to follow the measure more closely this way. (We will see an example of this later.)

2. In general, a special covering may ease the computation.

3. In a more general space it may be quite difficult to do a proper partition.

A few trials will convince the reader that it is actually quite difficult to make sense of this suggestion. What is needed is a proper theory of coverings. It is the purpose of these notes to show that a very satisfying theory can be obtained from the Theory of Domains.

1.3 Linear functionals and integration

We fix a σ-algebra $\mathcal{M}$ on a compact second countable space X, the *Borel sets* of X. The set $\mathbf{M}X$ of all measures on $(X, \mathcal{M})$ forms a cone with respect to addition and multiplication by positive numbers.

Similarly, the set $\mathbf{C}X$ of continuous real-valued functions on X forms a vector space under the pointwise operations.

Integration is a *bilinear* mapping $\int : \mathbf{M}X \times \mathbf{C}X \to \mathbb{R}$. In the case of the Riemann integral, one operates on the side of measures (through subdivisions) and forms a limit. In the case of the Lebesgue integral one operates on the side of functions through the introduction of so-called *simple measurable functions*. These are mappings with finite image such that every pre-image is measurable. Note that they do not belong to $\mathbf{C}X$, so in effect we work with a larger class of functions.

Edalat's theory of the R-integral is a proper extension of the Riemann approach. Rather than extending the class of functions, he extends the class of measures to a space of so-called *valuations*. Every Borel measure will be a supremum of simple valuations, just as every measurable function is a supremum of simple measurable functions.

1.4 Construction of measures

There is the obvious question how one arrives at a measure on a space in the first place. One standard way is to employ the *Riesz Representation Theorem*:

Theorem 5 *For every positive linear operator $F : \mathbf{C}X \to \mathbb{R}$ there is a unique measure μ on the Borel sets of X such that $Ff = \int f d\mu$.*

This is a very indirect definition, of course, and it is not clear how one would use it for a concrete calculation.

As a (running) example, we consider *Iterated Function Systems with probabilities.* These consist of a finite list of pairs (f_i, p_i) where each f_i is a continuous endofunction on the compact space X and $0 \leq p_i \leq 1$ with $\sum p_i = 1$. An IFS is called *hyperbolic*, if the maps f_i are contracting.

With such an IFS we have the following *Markov operator* $\mathbf{T}$

$$\mathbf{T}\mu B = \sum_{i=1}^{n} p_i \mu(f_i^{-1}(B))$$

which maps Borel measures to Borel measures. The following is a classical theorem in the theory of IFS's:

Theorem 6 (Hutchinson) *The operator* $\mathbf{T}\colon \mathbf{M}^1 X \to \mathbf{M}^1 X$ *has a unique fixpoint, called the* invariant measure *of the IFS.*

(The superscript 1 indicates that the measures are normalised to $\mu(X) = 1$, that is, they are *probability measures.*) This is an instance where a measure is defined indirectly and any integration with respect to such an invariant measure must take this mode of definition into account.

One of the contributions of Edalat's theory of integration is that the invariant measure in Hutchinson's theorem can be approximated order-theoretically in such a way that integrals can be evaluated using the approximations.

1.5 Further information

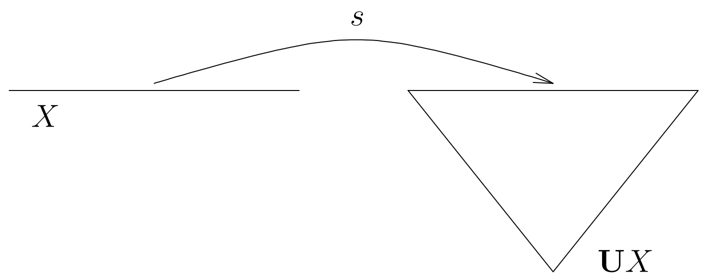
These notes are based on the publications [Eda95b, Eda95a, Edaar] by Abbas Edalat, all of which are available on the World Wide Web from `http://theory.doc.ic.ac.uk/people/Edalat`. General reference for Domain Theory is [AJ94], for Measure Theory consult [Rud87]. [Eda97] gives a wealth of further references.

2 The embedding theorem

Throughout these lecture notes let X be a second-countable compact Hausdorff space. Underlying the whole theory is the continuous domain $\mathbf{U}X$, consisting of all compact non-empty subsets of X, ordered by reversed inclusion. It is called the *upper space* of X.

Proposition 7 $\mathbf{U}X$ *is a continuous Scott-domain. A basis for the Scott-topology is given by the sets* $\Box O = \{c \in \mathbf{U}X \mid c \subseteq O\}$ *for* O *open in* X.

The space X is embedded into $\mathbf{U}X$ through the *singleton map* s: $s(x) = \{x\}$.



Proposition 8 *The singleton map is a homeomorphism from X onto the set of maximal elements of $\mathbf{U}X$, equipped with the relative Scott-topology.*

We will make extensive use of the *probabilistic powerdomain* $\mathbf{P}^1\mathbf{U}X$ of the upper space $\mathbf{U}X$. It was introduced in [SD80] and studied extensively in [Jon90, Kir93]. For any topological space X the elements of the probabilistic powerdomain are so-called *valuations*, mappings v from the topology into $[0,\infty]$ satisfying

- $v(\emptyset) = 0$;
- $v(O \cap U) + v(O \cup U) = v(O) + v(U)$;
- v Scott-continuous.

We extend the singleton map s to probability measures by setting:

$$\sigma\colon \mathbf{M}^1X \to \mathbf{P}^1\mathbf{U}X, \mu \mapsto \mu \circ s^{-1}$$

Theorem 9 (Edalat, Lawson) *The function σ is a bijection between $\mathbf{M}^1X$ and the maximal elements of $\mathbf{P}^1\mathbf{U}X$. If $\mathbf{M}^1X$ is equipped with the weak topology then σ is indeed a homeomorphism.*

A proof of this theorem is beyond the scope of these notes. But we can certainly appreciate its significance. For example, we deduce that every ascending chain of simple valuations which approximates a maximal element in $\mathbf{P}^1\mathbf{U}X$ is in fact an approximation to a probability measure on X.

Also, the convergence of an ascending chain means convergence with respect to the Scott-topology. This can sometimes be translated into convergence of an associated sequence of measures with respect to the weak topology on $\mathbf{M}^1X$. As an example, take the Markov operator associated with an IFS with probabilities from above:

$$\mathbf{T}\mu B = \sum_{i=1}^{n} p_i \mu(f_i^{-1}(B))$$

and also consider the operator

$$\mathbf{H}vO = \sum_{i=1}^{n} p_i v(\hat{f}_i^{-1}(O))$$

on $\mathbf{P}^1\mathbf{U}X$ where $\hat{f}_i(K) = \{f_i(x) \mid x \in K\}$ is the direct image function. Now, the point evaluation η_X is the least element of $\mathbf{P}^1\mathbf{U}X$ and therefore we have $\eta_X \leq \sigma(\mu) = \mu \circ s^{-1}$ for *any* probability measure μ. Then

$$\mathbf{H}\eta_X \leq \mathbf{H}(\sigma(\mu)) = \mathbf{H}(\mu \circ s^{-1}) = (\mathbf{T}\mu) \circ s^{-1} = \sigma(\mathbf{T}\mu)$$

using $s^{-1} \circ \hat{f}^{-1}(O) = f^{-1} \circ s^{-1}(O)$. Generally, we have

$$\mathbf{H}^m\eta_X \leq \mathbf{H}^m(\sigma(\mu)) = \mathbf{H}^m(\mu \circ s^{-1}) = (\mathbf{T}^m\mu) \circ s^{-1} = \sigma(\mathbf{T}^m\mu)$$

and so $\mathbf{T}^m(\mu)$ converges weakly to the invariant measure. It follows that in order to evaluate an integral with respect to the invariant measure, one can use the sequence $\mathbf{T}^m\mu$ for approximate results. In particular, μ can be a Dirac measure and then all $\mathbf{T}^m\mu$ will be linear combinations of Dirac measures.

We also note the following:

Corollary 10 *The following are equivalent for a normalised valuation v on $\mathbf{U}X$:*

1. *v is maximal in $\mathbf{P}^1\mathbf{U}X$;*
2. *the support of v is a subset of* $\max \mathbf{U}X = s(X)$*;*
3. *v belongs to the image of σ.*

3 The R-integral

Suppose now that μ is a probability measure on a compact second countable space X. We know that $\sigma(\mu) = \mu \circ s^{-1}$ is a valuation on $\mathbf{U}X$ and is in fact a maximal element of $\mathbf{P}^1\mathbf{U}X$. From the theory of the probabilistic powerdomain we know that $\mathbf{P}^1\mathbf{U}X$ is a continuous domain, hence $\sigma(\mu) = \bigsqcup^{\uparrow} v_i$ where each v_i is a simple valuation. The overall goal is to define the integral $\int_X f d\mu$ for (suitably well-behaved) functions $f\colon X \to \mathbb{R}$ through the approximations v_i. The construction is performed in a number of stages.

3.1 Notation

A simple valuation v on $\mathbf{U}X$ is of the form $\sum_{i=1}^n r_i\eta_i$, where each η_i is a point valuation. The points of $\mathbf{U}X$ are compact subsets of X. Instead of $\sum_{i=1}^n r_i\eta_i$ we therefore write $\sum_{c\in M} r_c\delta_c$ where M is a finite set of compact subsets of X and δ_c is the point valuation η_c corresponding to the *element* c of $\mathbf{U}X$. This notation is inspired by the use of δ for Dirac measures. Indeed, simple valuations δ_c can be seen as generalised Dirac measures where

$$\delta_c(O) = \begin{cases} 1 & \text{if } c \subseteq O; \\ 0 & \text{otherwise.} \end{cases}$$

3.2 Darboux sums

For $v = \sum_{c\in M} r_c\delta_c$ a simple valuation and $f\colon X \to \mathbb{R}$ a *bounded* function define

$$S^l(f, v) = \sum_{c\in M} \inf f(c) \cdot r_c$$

the *lower Darboux sum*, and

$$S^u(f, v) = \sum_{c\in M} \sup f(c) \cdot r_c$$

the *upper Darboux sum*. These values exist because f is assumed to be bounded. The following is immediate

Lemma 11 *$S^l(f, v) \le S^u(f, v)$.*

3.3 Refinement

This is the crucial step on which the whole construction hinges.

Lemma 12 *Assume $v \le w$ are simple normalised measures and $f\colon X \to \mathbb{R}$ is bounded. Then*

$$\begin{array}{rcl} S^l(f, v) & \le & S^l(f, w) \\ S^u(f, w) & \le & S^u(f, v) \end{array}$$

Proof. Let $v = \sum_{c\in M} r_c\delta_c$, $w = \sum_{d\in N} s_d\delta_d$. The Splitting Lemma for normalised valuations provides us with numbers $t_{c,d}$ such that

$$r_c = \sum_{d\in N} t_{c,d} \qquad s_d = \sum_{c\in M} t_{c,d}$$

and, furthermore, $t_{c,d} \ne 0 \Longrightarrow c \supseteq d$. The rest is a simple calculation:

$$\begin{array}{rcl} S^l(f, v) & = & \sum_{c\in M} r_c \inf f(c) \\ & = & \sum_{c\in M} \sum_{d\in N} t_{c,d} \inf f(c) \\ & = & \sum_{c\in M, d\in N} t_{c,d} \inf f(c) \\ & \le & \sum_{c\in M, d\in N} t_{c,d} \inf f(d) \\ & = & \sum_{d\in N} \sum_{c\in M} t_{c,d} \inf f(d) \\ & = & \sum_{d\in N} s_d \inf f(d) \\ & = & S^l(f, w) \end{array}$$

The calculation for the upper sum is exactly the same. ■

Note that the Splitting Lemma for general valuations only yields $s_d \ge \sum_{c\in M} t_{c,d}$. This is not a problem for the lower sums but goes in the wrong direction for upper sums.

Taking up the discussion about more general "subdivisions" from the Introduction, we can now say that it is precisely the order on finite valuations and the Splitting Lemma which give us the necessary framework.

3.4 The bounding sequences

Lemma 13 *let $(v_i)_{i\in I}$ be a directed set of simple normalised valuations on* $\mathbf{U}X$*. Then $S^l(f, v_i) \leq S^u(f, v_j)$ holds for every $i, j \in I$ (and every bounded function $f\colon X \to \mathbb{R}$).*

Proof. Let $k \geq i, j$ by directedness. Then by the previous two lemmas $S^l(f, v_i) \leq S^l(f, v_k) \leq S^u(f, v_k) \leq S^u(f, v_j)$. ■

3.5 The R-condition

If $(v_i)_{i\in I}$ is a directed set of simple normalised valuations and $f\colon X \to \mathbb{R}$ then we say that f satisfies the *R-condition* with respect to $(v_i)_{i\in I}$ if $\forall \epsilon > 0 \exists i \in I.\ S^u(f, v_i) - S^l(f, v_i) < \epsilon$.

3.6 The R-integral (tentative)

Let $v = \bigsqcup^{\uparrow} v_i$ be a normalised valuation and all v_i simple. Then for a bounded function $f\colon X \to \mathbb{R}$, which satisfies the R-condition, we define the *R-integral* $\int_X f dv$ with respect to $(v_i)_{i\in I}$ by $\int_X f dv = \sup S^l(f, v_i)$.

I have called this definition of the R-integral "tentative" because it depends on the approximating sequence $(v_i)_{i\in I}$. In the remainder of this section we shall try to find conditions under which the integral is the same, no matter which approximating sequence is used.

3.7 Surpassing approximations

Let $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ be directed sets in a poset D. We say that $(y_j)_{j\in J}$ *surpasses* $(x_i)_{i\in I}$ if for every x_i there is y_j with $x_i \leq y_j$.

Lemma 14 *If f satisfies the R-condition for a directed set $(v_i)_{i\in I}$ of simple valuations and $(w_j)_{j\in J}$ surpasses $(v_i)_{i\in I}$, then f satisfies the R-condition for $(w_j)_{j\in J}$ as well and the two R-integrals coincide.*

Proof. For every v_i there is w_j with $v_i \leq w_j$. Hence by Lemma 12 we have $S^l(f, v_i) \leq S^l(f, w_j) \leq S^u(f, w_j) \leq S^u(f, v_i)$. If the v_i-interval is smaller than ϵ then so is the w_j-interval. Therefore f satisfies the R-condition with respect to $(w_j)_{j\in J}$.

The inequality also shows that $\sup S^l(f, v_i) \leq \sup S^l(f, w_j) \leq \inf S^u(f, v_i)$ and since the outer two values are both equal to the R-integral of f with respect to $(v_i)_{i\in I}$, the inner value equals the integral as well. ■

3.8 Strongly directed sets

A subset $(x_i)_{i\in I}$ of a dcpo is called *strongly directed* if $I \neq \emptyset$ and $\forall i,j \in I\ \exists k \in I.\ i,j \ll k$.

Example 15 *$\Downarrow\! x$ is strongly directed in any continuous domain.*

Lemma 16 *If $(x_i)_{i\in I}$ is a strongly directed subset of a dcpo and $\bigsqcup^{\uparrow}_{i\in I} x_i \leq \bigsqcup^{\uparrow}_{j\in J} y_j$, then $(y_j)_{j\in J}$ surpasses $(x_i)_{i\in I}$.*

3.9 R-integrability

We say that $f\colon X \to \mathbb{R}$ is *R-integrable* with respect to a valuation v, if is satisfies the R-condition for a *strongly* directed set of simple valuations with limit v.

Theorem 17 *If $f\colon X \to \mathbb{R}$ is R-integrable with respect to a valuation v then the value of the R-integral $\int_X f dv$ does not depend on the directed set of simple valuations (approximating v) through which it is calculated.*

3.10 Concluding remarks

We end up with a definition of integrability and integral that applies to all valuations on $\mathbf{U}X$. It may be appropriate to point out that we are somewhat at a loss when we are asked to interpret a general such valuation. For example, a simple valuation δ_c as such could be said to be a generalised *Dirac measure*. However, in order to be integrable with respect to δ_c, a function has to be constant on c.

Typically, therefore, we are interested in valuations which have their support on $\max \mathbf{U}X$, i.e. those which derive from measures on X. The question then arises how the new integral compares to the Lebesgue integral. This is the topic of the next two sections.

3.11 Exercises

1. For $v = \sum_{c\in M} r_c\delta_c$ let ξ be a mapping which selects an element from every compact set c. A generalised Riemann sum with respect to such a selection function is given by $S^\xi(f,v) = \sum_{v\in M} r_c f(\xi)$. Show that if f is R-integrable then every sequence of generalised Riemann sums derived from a sequence of simple valuations $v = \bigsqcup^{\uparrow} v_i$ converges to the R-integral of f.

2. If f is continuous then f is R-integrable with respect to v if and only if it satisfies the R-condition for *some* (not necessarily strongly) directed set of simple valuations approximating v.

3. If f, g are R-integrable with respect to v then so are $f+g$ and λf $(\lambda \in \mathbb{R})$. Furthermore
$$\begin{array}{rcl} \int_X (f+g) dv &=& \int_X f dv + \int_X g dv \\ \int_X \lambda f dv &=& \lambda \int_X f dv \\ |\int_X f dv| &\leq& \int_X |f| dv \end{array}$$
4. Let $(v_i)_{i\in I}$ be a sequence of simple (*not necessarily normalised*) valuations approximating v. Suppose that f is R-integrable with respect to v. Show that the sequences $S^l(f, v_i)$ and $S^u(f, v_i)$ of lower and upper sums converge to the R-integral.

4 R-integration of continuous functions

Lemma 18 *Let X be compact metric and μ a probability measure on X. Then for all $\epsilon, \kappa > 0$ there exists a simple valuation $v = \sum_{c\in M} r_c \delta_c \ll \sigma(\mu)$ such that $\sum_{|c|>\kappa} r_c < \epsilon$.*

Proof. Consider $U = \bigcup \{\Box O \mid \mathrm{diam}(O) < \delta\}$. This is an open neighbourhood of $s(X) = \max \mathbf{U}X$. Since $\lim_{v \ll \sigma(\mu)} v(U) = \sigma(\mu)(U) = \mu(s^{-1}(U)) = \mu(X) = 1$, there exists v such that $v(U) > 1 - \epsilon$. As $v = \sum_{c\in M} r_c \delta_c$ and $v(U) = \sum_{c\in M\cap U} r_c$ and also, since $c \in U \iff c \in \Box O$ for some open set O with $\mathrm{diam}(O) < \delta$, we have $\sum_{c\in M, |c|>\delta} r_c = \sum_{c\in M\setminus U} r_c = 1 - \sum_{c\in M\cap U} r_c < \epsilon$. ■

Theorem 19 *Let $f\colon X \to \mathbb{R}$ be continuous and μ a probability measure on the compact metric space X. The f is R-integrable.*

Proof. Exercise. ■

5 R-integration vs Lebesgue integration

The aim is to prove the following:

Theorem 20 *X compact second-countable, μ a probability measure on X and $f\colon X \to \mathbb{R}$ R-integrable with respect to $\sigma\mu$. Then f is Lebesgue integrable and the two integrals coincide.*

The proof proceeds in a number of stages.

5.1 Lebesgue integral

Recall the definition of Lebesgue-integrability: If f is (Borel) measurable then f is the pointwise supremum of simple measurable functions f_i. The integral for a simple measurable function is defined as $\sum_{r\in \mathsf{im}(f)} \mu(f_i^{-1}(r)) \cdot r$. The integral for f is then defined as the supremum of all such integrals.

5.2 The idea

We shall construct "explicitly" simple measurable functions which both are close to f and also have a connection to lower and upper R-sums.

To this end let X be covered with finitely many open sets O_i. In such a situation, X is actually partitioned into *crescents*, that is, sets of the form $O \setminus U$, O, U open; namely, every x belongs to a unique crescent of the form $D_x = (\bigcap_{x \in O_i} O_i) \setminus (\bigcup_{x \notin O_i} O_i)$. Every crescent is Borel-measurable.

If $\mathcal{C} = (O_i)_{i \in I}$ is a finite cover of X with open sets we define simple measurable functions

$$\begin{array}{rcl} f^-_{\mathcal{C}}(x) & = & \inf f(D_x) \\ f^+_{\mathcal{C}}(x) & = & \sup f(D_x) \end{array}$$

The connection with finite valuations is made by choosing for every crescent D_x a compact subset $B_x \supseteq D_x$. Let M be the finite collection of these compact subsets. The simple valuation of interest is then given by $\sum_{B_x \in M} \mu(D_x)\delta_{B_x}$.

Again, we end up with a fairly complicated situation of subsets and partitions and again, we can clean up the situation considerably by using results from Domain Theory.

5.3 Deflations

A *deflation* on a dcpo D is a Scott-continuous function $d\colon D \to D$ such that $d \leq \mathsf{id}_D$ and $\mathsf{im}(d)$ finite.

Proposition 21 *A continuous Scott-domain allows a directed set of deflations $(d_i)_{i \in I}$ such that $\mathsf{id}_D = \bigsqcup^{\uparrow} d_i$.*

Proposition 22 *If X is compact then $\mathbf{U}X$ is a Scott-domain.*

5.4 Deflations and crescents

If $d\colon \mathbf{U}X \to \mathbf{U}X$ is a deflation with image M then we have the following:

1. $d^{-1}(\uparrow x)$ is Scott-open in $\mathbf{U}X$.
2. d^{-1} as a mapping from upper sets in the image of d to Scott-open subsets has finite range.
3. $s^{-1} \circ d^{-1}$ as a mapping from upper sets in the image of d to open subsets of X has finite range.
4. For $c \in \mathsf{im}(d)$ we have $d^{-1}(c) = d^{-1}(\uparrow c) \setminus \bigcup_{e > c} d^{-1}(\uparrow e)$.
5. $s^{-1}(d^{-1}(c))$ is a crescent in X.
6. $s^{-1}(d^{-1}(c)) \subseteq c$.

The sets $s^{-1}(d^{-1}(c))$ and c will be our crescents and compact subsets as discussed at the beginning of this section.

5.5 Deflations and measures

If d is a deflations on $\mathbf{U}X$ then

$$v_d = \sigma(\mu) \circ d^{-1} = \mu \circ s^{-1} \circ d^{-1}$$

is a finite normalised valuation on $\mathbf{U}X$. It can be written as $\sum_{c \in M} r_c \delta_c$ where $r_c = \mu(s^{-1}(d^{-1}(c)))$ and $M = \mathsf{im}(d)$.

Lemma 23 *If $(d_i)_{i \in I}$ is a directed set of deflations with $\bigsqcup^{\uparrow}_{i \in I} d_i = \mathsf{id}_{\mathbf{U}X}$ and μ a probability measure on X then $\sigma(\mu) = \bigsqcup^{\uparrow}_{i \in I} v_{d_i}$.*

5.6 Deflations and functions

For $f\colon X \to \mathbb{R}$ bounded and d a deflation on $\mathbf{U}X$ with image M, define $B_x = d(s(x))$, $D_x = s^{-1}(d^{-1}(B_x))$, and $f_d^-, f_d^+\colon X \to \mathbb{R}$ by

$$\begin{array}{rcl} f_d^-(x) & = & \inf f(D_x) \\ f_d^+(x) & = & \sup f(D_x) \end{array}$$

as above. These are simple measurable functions with $f^- \leq f \leq f^+$ pointwise.

For clarity we denote the Lebesgue integral by $\mathbf{L}\int$. We have

$$\begin{array}{rcl} \mathbf{L}\int f_d^- d\mu & = & \sum_{B_x \in M} \mu(D_x) f_d^-(D_x) \\ & = & \sum_{B_x \in M} \mu(D_x) \inf f(D_x) \\ & \geq & \sum_{B_x \in M} \mu(D_x) \inf f(B_x) \\ & = & S^l(f, v_d) \end{array}$$

Similarly, $\mathbf{L}\int f_d^+ d\mu \leq S^u(f, v_d)$.

5.7 The limit argument

Let $(d_i)_{i \in I}$ be a directed set of deflations with $\bigsqcup^{\uparrow}_{i \in I} d_i = \mathsf{id}_{\mathbf{U}X}$. If $f\colon X \to \mathbb{R}$ is R-integrable then we have

$$S^u(f, v_{d_i}) - S^l(f, v_{d_i}) \longrightarrow 0$$

By the previous subsection we also have

$$\mathbf{L}\int f_{d_i}^+ d\mu - \mathbf{L}\int f_{d_i}^- d\mu \longrightarrow 0$$

and

$$\mathbf{L}\int f_{d_i}^- d\mu \longrightarrow \int f d\sigma(\mu)$$

Let f^- be the pointwise limit of the $f_{d_i}^-$ and similarly f^+ the pointwise limit of the $f_{d_i}^+$. Then

$$\begin{array}{c} \mathbf{L}\int f^- d\mu = \lim \mathbf{L}\int f_{d_i}^- d\mu = \int f d\sigma(\mu) \\ \mathbf{L}\int f^+ d\mu = \lim \mathbf{L}\int f_{d_i}^+ d\mu = \int f d\sigma(\mu) \end{array}$$

Finally, $\mathbf{L}\int |f^+ - f^-| d\mu = \mathbf{L}\int (f^+ - f^-) d\mu = \mathbf{L}\int (f^+ - \mathbf{L}\int f^- d\mu = 0$. This implies that f^+, f^- and f agree almost everywhere with respect to μ and therefore the Lebesgue integral of f exists and is the same as the R-integral.

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