Kuratowski Mrowka Theorem for internal preneighbourhood spaces

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Compactness in classical topology

Theorem 1.1 (Classical Kuratowski-Mrowka Theorem)

The following are equivalent for a topological space X:

(a) *X is compact (i.e., every open cover has a finite subcover).*

(b) *Every filter $\mathcal{F}$ on X cluster (i.e., $\bigcap_{T\in\mathcal{F}} \mathrm{cl}_X T \neq \emptyset$)*

(c) *The projection map $X \times Y \xrightarrow{p_2} Y$ is closed for any topological space Y.*

Spaces internalised

The notion of a *space* has been discussed in categorical setup, e.g., by *categorical closure operators* (see Dikranjan and Tholen, *Categorical structure of closure operators*; Dikranjan, Giuli, and Tholen, "Closure operators. II"; Dikranjan and Giuli, "Compactness, minimality and closedness with respect to a closure operator"; Dikranjan, Giuli, and Tozzi, "Topological categories and closure operators"; Dikranjan and Giuli, "Closure operators. I", for instance),

or by *categorical interior operators* (see Vorster, "Interior operators in general categories"; Castellini, "Some remarks on interior operators and the functional property"; Castellini, "Interior operators, open morphisms and the preservation property"; Castellini and Murcia, "Interior operators and topological separation"; Castellini, "Interior operators in a category: idempotency and heredity"; Castellini and Ramos, "Interior operators and topological connectedness"; Castellini, "Discrete objects, splitting closure and connectedness"; Castellini and Holgate, "A link between two connectedness notions"; Razafindrakoto and Holgate, "A lax approach to neighbourhood operators"; Razafindrakoto and Holgate, "Interior and neighbourhood"; Holgate and Šlapal, "Categorical neighborhood operators", for instance)

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by axiomatisation of *proper morphisms* (see Hofmann and Tholen, "Lax algebra meets topology")...

Spaces internalised

The present approach is by *internalising* the notion of a neighbourhood, which can be done in a wide variety of categories, e.g., in all small complete and small cocomplete and well powered categories, see Ghosh, “Internal neighbourhood structures”.

Context: setting the stage

Definition 2.1 (see Ghosh, "Internal neighbourhood structures", §2, Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", §1)

A *context* is a triple $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ such that:

(a) $\mathbb{A}$ is finitely complete

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(d) For each object X of $\mathbb{A}$, the (possibly large) set $\mathtt{Sub}_{\mathsf{M}}(X)$ of admissible subobjects of X is a complete lattice.

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(d) $(\mathtt{Meas}, \mathtt{Epi}, \mathtt{ExtMon})$
(e) $(\mathtt{Grp}, \mathtt{RegEpi}, \mathtt{Mon})$
(f) $((\mathit{\Omega}, \mathit{\Xi})\text{-}\mathtt{Alg}, \mathtt{RegEpi}, \mathtt{Mon})$
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(l) $(\mathbb{A}, \mathtt{Epi}(\mathbb{A}), \mathtt{ExtMon}(\mathbb{A}))$, where $\mathbb{A}$ is a small complete and small cocomplete well powered category

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(g) $(\mathtt{Loc}, \mathtt{Epi}, \mathtt{RegMon})$
(h) $(\mathtt{CRing}^{\mathrm{op}}, \mathtt{Epi}, \mathtt{RegMon})$ or $(\mathbb{A}^{\mathrm{op}}, \mathtt{Epi}, \mathtt{RegMon})$, where $\mathbb{A}$ is a Zariski category (see Diers, *Categories of commutative algebras*, Definition I.2)
(i) any topos
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(k) if $(\mathbb{A}, \mathsf{E}, \mathsf{M})$ is a context then for any object B, then $((\mathbb{A} \downarrow B), (\mathsf{E} \downarrow B), (\mathsf{M} \downarrow B))$ is also a context (see Clementino, Giuli, and Tholen, "A functional approach to general topology")

(l) $(\mathbb{A}, \mathtt{Epi}(\mathbb{A}), \mathtt{ExtMon}(\mathbb{A}))$, where $\mathbb{A}$ is a small complete and small cocomplete well powered category

Adjunction between filters

Definition 2.2 (Filters)

Given any object X, a *filter* F on X is a subset of $\mathtt{Sub}_{\mathrm{M}}(X)$ such that

(a) $x \geq y \in F \Rightarrow x \in F$, and

(b) $x, y \in F \Rightarrow x \wedge y \in F$

The set of all filters on X is $\mathtt{Fil}X$.

Adjunction between filters

$\mathtt{Fil}X$ is a complete algebraic lattice, with compact elements being

$$\uparrow x = \{p \in \mathtt{Sub}_{\mathrm{M}}(X) : x \leq p\}.$$

$\mathtt{Fil}X$ is distributive if and only if $\mathtt{Sub}_{\mathrm{M}}(X)$ is distributive (see Iberkleid and McGovern, "A natural equivalence for the category of coherent frames", Theorem 1.2).

Internal neighbourhoods of three kinds

Definition 2.2 (Neighbourhoods, see Ghosh, "Internal neighbourhood structures", Definition 3.1)

Let X be an object of $\mathbb{A}$.

(a) A *preneighbourhood system* on X is an order preserving function $\mathtt{Sub}_{\mathsf{M}}(X)^{\mathrm{op}} \xrightarrow{\mu} \mathtt{Fil}X$ such that for each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$

$$p \in \mu(x) \Rightarrow x \leq p.$$

The pair (X, μ) is called an *internal preneighbourhood space* of $\mathbb{A}$.

(b) A preneighbourhood system μ on X is a *weak neighbourhood system* if

$$p \in \mu(x) \Rightarrow (\exists q \in \mu(x))\big(p \in \mu(q)\big).$$

The pair (X, μ) is called an *internal weak neighbourhood space* of $\mathbb{A}$.

(c) A weak neighbourhood system μ on X is a *neighbourhood system* on X if

$$\mu\left(\bigvee_{i\in I} p_i\right) = \bigcap_{i\in I} \mu(p_i).$$

The pair (X, μ) is called an *internal neighbourhood space* of $\mathbb{A}$.

Morphisms of neighbourhoods

Definition 2.3 (Morphisms of Neighbourhoods, see Ghosh, "Internal neighbourhood structures", Definition 3.39)

Let (X,μ), (Y,ϕ) be internal preneighbourhood spaces of $\mathbb{A}$ and $X \xrightarrow{f} Y$ be a morphism of $\mathbb{A}$.

(a) The morphism f is a *preneighbourhood morphism*, written $(X,\mu) \xrightarrow{f} (Y,\phi)$, if for each $y \in \mathrm{Sub}_{\mathrm{M}}(Y)$

$$p \in \phi(y) \Rightarrow f^{-1}p \in \mu(f^{-1}y).$$

(b) If (X,μ) and (Y,ϕ) are internal neighbourhoods of $\mathbb{A}$ then a preneighbourhood morphism $(X,\mu) \xrightarrow{f} (Y,\phi)$ is a *neighbourhood morphism* if for any family $\langle y_i : i \in I \rangle$ of admissible subobjects of Y

$$f^{-1}(\textstyle\bigvee_{i\in I} y_i) = \bigvee_{i\in I} f^{-1}y_i.$$

Categories of neighbourhoods

Definition 2.4 (Categories of Neighbourhoods, see Ghosh, "Internal neighbourhood structures", Definition 4.1)

(a) $\mathtt{pNbd}[\mathbb{A}]$ is the category of all internal preneighbourhood spaces of $\mathbb{A}$ and preneighbourhood morphisms.

(b) $\mathtt{wNbd}[\mathbb{A}]$ is the category of all internal weak neighbourhood spaces of $\mathbb{A}$ and preneighbourhood morphisms.

(c) $\mathtt{Nbd}[\mathbb{A}]$ is the category of all internal neighbourhood spaces of $\mathbb{A}$ and neighbourhood morphisms.

Generality of present approach

Theorem 2.5 (Prenieghbourhoods more general, see Ghosh, “Internal neighbourhood structures II: Closure and Closed Morphisms”, Proposition 2.1)

Let $\mathcal{G}(X)$ *be the complete lattice of all grounded, monotone and extensional endomaps on* $\mathtt{Sub}_{\mathsf{M}}(X)$, $\mathtt{pnbd}[X]$ *the complete lattice of all preneighbourhood systems on* X,

$$\begin{tikzcd} \mathcal{G}(X) \arrow[r, shift left, "\Phi"] & \mathtt{pnbd}[X]^{\mathrm{op}} \arrow[l, shift left, "\Psi"] \end{tikzcd}$$ *are order preserving functions defined by*

$\Phi(c)(x) = \{p \in \mathtt{Sub}_{\mathsf{M}}(X) : p \geq c(x)\}$ *and* $\Psi(\mu)(x) = \bigwedge_{u\in\mu(x)} u$.

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Then $\Phi \dashv \Psi$ *with* $\Psi(\Phi(c)) = c$ *for all* $c \in \mathcal{G}(X)$.

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Then $\Phi \dashv \Psi$ *with* $\Psi(\Phi(c)) = c$ *for all* $c \in \mathcal{G}(X)$*.*

In the context $(\mathtt{FinSet}, \mathsf{Surjections}, \mathsf{Injections})$ Φ is an isomorphism; in $(\mathtt{Set}, \mathsf{Surjections}, \mathsf{Injections})$ Φ is an embedding.

Thus grounded monotone extensional operators are dually coreflective inside preneighbourhood systems.

Topologicity results

Theorem 2.6 (Topologicity, see Ghosh, “Internal neighbourhood structures”, Theorem 4.8)

The categories $\mathtt{pNbd}[\mathbb{A}]$ *and* $\mathtt{wNbd}[\mathbb{A}]$ *are topological over* $\mathbb{A}$*.*

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The category $\mathtt{Nbd}[\mathbb{A}]$ *is topological over* $\mathbb{A}$ *provided preimage for every morphism preserve joins.*

Closure et al...

Definition 3.1 (Definition of closure, closed subobject and closed morphism, see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", definitions in §3)

Let (X, μ) be an internal preneighbourhood space and $p \in \mathtt{Sub}_{\mathsf{M}}(X)$. The admissible subobject:

$$\mathrm{cl}_{\mu}p = \bigvee\left\{u \in \mathtt{Sub}_{\mathsf{M}}(X)_{\neq 1} : x \in \mu(u) \Rightarrow x \wedge p \neq \sigma_X\right\} \tag{1}$$

is called the *μ-closure of p*.

The subobject p is *μ-closed* if $\mathrm{cl}_{\mu}p = p$.

For any internal preneighbourhood space (X, μ), $\mathfrak{C}_{\mu} = \{p \in \mathtt{Sub}_{\mathsf{M}}(X) : p = \mathrm{cl}_{\mu}p\}$.

Given the internal preneighbourhood spaces (X, μ) and (Y, ϕ), a morphism $X \xrightarrow{f} Y$ is said to be *μ-ϕ closed* or simply *closed* if it *preserves* closed subobjects, i.e., $p \in \mathfrak{C}_{\mu} \Rightarrow \exists_f p \in \mathfrak{C}_{\phi}$.

$\mathbb{A}_{\mathsf{cl}}$ is the (possibly large) set of all closed morphisms of $\mathbb{A}$.

Properties of the closure operator

Theorem 3.2 (Properties of Closure, see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 3.1)

- *Given any internal preneighbourhood space* (X, μ), *the function* $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{\mathrm{cl}_\mu} \mathtt{Sub}_{\mathsf{M}}(X)$ *defines a closure operation on* $\mathtt{Sub}_{\mathsf{M}}(X)$ *such that* $\mathrm{cl}_\mu \sigma_X = \sigma_Y$.

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- *If* $(X,\mu) \xrightarrow{f} (Y,\phi)$ *is a preneighbourhood morphism reflecting zero then it is* μ-ϕ continuous, *i.e., for any* $p \in \mathtt{Sub}_{\mathsf{M}}(X)$*:*

$$\exists_f \mathrm{cl}_\mu p \leq \mathrm{cl}_\phi \exists_f p \tag{2}$$

Properties of the closure operator

Definition 3.2 (Reflecting Zero)

A morphism $X \xrightarrow{f} Y$ is said to *reflect zero* if $f^{-\mathbf{1}}\sigma_Y = \sigma_X$.

Properties of the closure operator

The following three statements are equivalent for any morphism $X \xrightarrow{f} Y$:

(a) f reflects zero

(b) For each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\exists_f x = \sigma_Y \Rightarrow x = \sigma_X$

(c) For each $x \in \mathtt{Sub}_{\mathsf{M}}(X)$ and $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$, $y \wedge \exists_f x = \sigma_Y \Rightarrow x \wedge f^{-1} y = \sigma_X$

see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 9.2

Properties of the closure operator

In any context if every morphism reflects zero then the initial object $\emptyset$ is strict.
Conversely, if the initial object $\emptyset$ is strict and the unique morphism $\emptyset \to 1$ is an admissible monomorphism then every morphism reflects zero.
see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 9.2

Properties of the closure operator

A category was called *quasi-pointed* (see Bourn, “3×3 lemma and protomodularity”, §1, and see Goswami and Janelidze, “On the structure of zero morphisms in a quasi-pointed category”) if the unique morphism $\emptyset \to 1$ is a monomorphism.

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A context shall be called *admissibly quasi-pointed* if the unique morphism $\emptyset \to 1$ is an admissible monomorphism.

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Several contexts are admissibly quasi-pointed — e.g., sets and functions, topological spaces and continuous maps, locales and localic maps, where this unique morphism is a regular monomorphism and hence admissible; however the context of rings and their homomorphisms (rings with identity and homomorphisms preserving identity) is **not** quasi-pointed even, although the context for $\mathtt{CRing}^{\mathrm{op}}$, and likewise $\mathbb{A}^{\mathrm{op}}$ (for $\mathbb{A}$ a Zariski category), is still quasi-pointed.

Properties of the closure operator

The following three statements are equivalent for any morphism $X \xrightarrow{f} Y$:

(a) f reflects zero

(b) For each $x \in \mathtt{Sub}_{\mathrm{M}}(X)$, $\exists_f x = \sigma_Y \Rightarrow x = \sigma_X$

(c) For each $x \in \mathtt{Sub}_{\mathrm{M}}(X)$ and $y \in \mathtt{Sub}_{\mathrm{M}}(Y)$, $y \wedge \exists_f x = \sigma_Y \Rightarrow x \wedge f^{-1} y = \sigma_X$

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Thus: in admissibly quasi-pointed contexts, the initial object is strict if and only if every morphism reflects zero.

Properties of the closure operator

Theorem 3.2 (Properties of Closure, see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 3.1)

- *Given any internal preneighbourhood space* (X, μ), *the function* $\mathtt{Sub}_{\mathsf{M}}(X) \xrightarrow{\mathrm{cl}_\mu} \mathtt{Sub}_{\mathsf{M}}(X)$ *defines a closure operation on* $\mathtt{Sub}_{\mathsf{M}}(X)$ *such that* $\mathrm{cl}_\mu \sigma_X = \sigma_Y$.
- *If* $(X, \mu) \xrightarrow{f} (Y, \phi)$ *is a preneighbourhood morphism reflecting zero then it is* μ-ϕ continuous, *i.e., for any* $p \in \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\exists_f \mathrm{cl}_\mu p \leq \mathrm{cl}_\phi \exists_f p \tag{2}$$

- *If every filter of* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is contained in a prime filter then* cl_μ *is additive, i.e., for each* $x, y \in \mathtt{Sub}_{\mathsf{M}}(X)$:

$$\mathrm{cl}_\mu(x \vee y) = \mathrm{cl}_\mu x \vee \mathrm{cl}_\mu y. \tag{3}$$

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The existence of prime filters in non-distributive lattices is not automatic, even under Axiom of Choice.

In such cases, a maximal filter need not be prime.

See Erné, "Prime and maximal ideals of partially ordered sets" for details.

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- *The closure operation is hereditary, i.e., given* $A \overset{a}{\rightarrowtail} M \overset{m}{\rightarrowtail} X$ *,*

$$\mathrm{cl}_{(\mu|_m)} a = m^{-1}(\mathrm{cl}_\mu(m \circ a));$$

hence a and m closed imply $m \circ a$ *closed.*

Proper Morphisms

Definition 4.1 (see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Definition 6.1)

A preneighbourhood morphism $(X,\mu) \xrightarrow{f} (Y,\phi)$ is *proper* if for every preneighbourhood morphism $(Z,\psi) \xrightarrow{g} (Y,\phi)$ and pullback

$$\begin{tikzcd} X \times_Y Z \arrow[r, "f_g"] \arrow[d, "g_f"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

, the morphism $(X \times_Y Z, \mu \times_\phi \psi) \xrightarrow{f_g} (Z,\psi)$ is a closed morphism.

The set $\mathbb{A}_{\text{pr}}$ is the (possibly large) set of all proper morphisms.

Proper Morphisms

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A preneighbourhood morphism $(X,\mu)\xrightarrow{f}(Y,\phi)$ is *proper* if for every preneighbourhood morphism $(Z,\psi)\xrightarrow{g}(Y,\phi)$ and pullback

$$\begin{tikzcd} X\times_Y Z \arrow[r, "f_g"] \arrow[d, "g_f"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

, the morphism

$(X\times_Y Z,\mu\times_\phi\psi)\xrightarrow{f_g}(Z,\psi)$ is a closed morphism.

The set $\mathbb{A}_{\mathsf{pr}}$ is the (possibly large) set of all proper morphisms.

Examples
$(\mathtt{Set},\mathtt{Sur},\mathtt{Inj})$
$(\mathtt{Top},\mathtt{Epi},\mathtt{ExtMon})$
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, the morphism $(X \times_Y Z, \mu \times_\phi \psi) \xrightarrow{f_g} (Z, \psi)$ is a closed morphism.

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Examples	
$(\mathtt{Set}, \mathtt{Sur}, \mathtt{Inj})$	for internal neighbourhood spaces, usual proper maps of topological spaces
$(\mathtt{Top}, \mathtt{Epi}, \mathtt{ExtMon})$	for internal neighbourhood spaces, usual proper maps between the second topology
$(\mathtt{Loc}, \mathtt{Epi}, \mathtt{RegMon})$	for locales with T-neighbourhood systems, usual proper maps of locales

Proper Morphisms

The *T-neighbourhood system* was investigated in the papers Dube and Ighedo, “More on locales in which every open sublocale is z-embedded”; Dube and Ighedo, “Characterising points which make P-frames”; for a locale X, it is the order preserving map $\mathtt{Sub}_{\mathtt{RegMon}}(X)^{\mathrm{op}} \xrightarrow{\mathfrak{o}_X} \mathtt{Fil}X$ defined by:

$$\mathfrak{o}_X(S) = \left\{ T \in \mathtt{Sub}_{\mathtt{RegMon}}(X) : (\exists a \in X)\big(S \subseteq \mathfrak{O}[a] \subseteq T\big) \right\}.$$

It is a neighbourhood system on X, and the functor with object function $X \mapsto (X, \mathfrak{o}_X)$ is right inverse to the forgetful functor $\mathtt{pNbd}[\mathtt{Loc}] \xrightarrow{U} \mathtt{Loc}$ (see Ghosh, “Internal neighbourhood structures”, Theorem 3.38).
This ensures the theory of locales a special case of the present theory.

Proper Morphisms

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$(\mathtt{Set}, \mathtt{Sur}, \mathtt{Inj})$	for internal neighbourhood spaces, usual proper maps of topological spaces
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Proper Morphisms

Theorem 4.1 (Alternative characaterisation of proper morphisms, see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", Theorem 6.1(a))

A preneighbourhood morphism $(X, \mu) \xrightarrow{f} (Y, \phi)$ *is proper if and only if for every preneighbourhood space* (Z, ψ)*, every corestriction of* $(X \times Z, \mu \times \psi) \xrightarrow{f \times \mathbf{1}_Z} (Y \times Z, \phi \times \psi)$ *is a closed morphism.*

Properties of closed and proper morphisms

Summary of properties of closed/proper morphisms, given the preneighbourhood morphisms $(X,\mu) \xrightarrow{f} (Y,\phi) \xrightarrow{g} (Z,\psi)$

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$\mathbb{A}_{\mathsf{cl}}$	
contain isomorphisms	
closed under compositions	
$g \circ f$ is closed, f is a continuous formal surjection imply g is closed	
if $m \in \mathfrak{C}_\phi$, f is continuous then $f^{-1}m \in \mathfrak{C}_\mu$; f is closed and continuous imply f_m is closed	

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$\mathbb{A}_{\mathrm{cl}}$	$\mathbb{A}_{\mathrm{pr}}$
contain isomorphisms	contain closed embeddings in a reflecting zero context
closed under compositions	closed under compositions
$g\circ f$ is closed, f is a continuous formal surjection imply g is closed	$g\circ f$ is proper, f is continuously stably in E imply g is proper
	$g\circ f$ is proper, g is a monomorphism imply f is proper
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A morphism $X \xrightarrow{f} Y$ is a *formal surjection* if $y \in \mathtt{Sub}_{\mathsf{M}}(Y) \Rightarrow (\exists x \in \mathtt{Sub}_{\mathsf{M}}(X))(y = \exists_f x)$, or equivalently, for each $y \in \mathtt{Sub}_{\mathsf{M}}(Y)$, $f_y \in \mathsf{E}$.

Properties of closed and proper morphisms

Summary of properties of closed/proper morphisms, given the preneighbourhood morphisms $(X,\mu)\xrightarrow{f}(Y,\phi)\xrightarrow{g}(Z,\psi)$

	$g \circ f$ is proper, f is continuously stably in E imply g is proper

A preneighbourhood morphism $(X,\mu)\xrightarrow{f}(Y,\phi)$ is *continuously stably in E* if for every preneighbourhood morphism $(Z,\psi)\xrightarrow{g}(Y,\phi)$, the pullback f_g of f along g is $((\mu\times_{\phi}\psi),\psi)$-continuous and is in E.

Properties of closed and proper morphisms

Summary of properties of closed/proper morphisms, given the preneighbourhood morphisms $(X, \mu) \xrightarrow{f} (Y, \phi) \xrightarrow{g} (Z, \psi)$

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Compare the properties for similar morphisms in Clementino, Giuli, and Tholen, "A functional approach to general topology", where *continuous* condition is automatic.

Compact preneighbourhood spaces

The smallest preneighbourhood system on an object X is $\mathtt{Sub}_{\mathsf{M}}(X)^{\mathrm{op}} \xrightarrow{\nabla_X} \mathtt{Fil}X$, where:

$$\nabla_X(p) = \begin{cases} \mathtt{Sub}_{\mathsf{M}}(X), & \text{if } p = \sigma_X \\ \{\mathbf{1}_X\}, & \text{if } p \neq \sigma_X \end{cases}.$$

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The terminal object 1 being the empty product is considered as an internal preneighbourhood space with its smallest preneighbourhood system ∇_1.

Compact preneighbourhood spaces

Definition 5.1 (see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", §7.2)

An internal preneighbourhood space (X, μ) is said to be *compact* if the unique morphism $(X, \mu) \xrightarrow{\mathsf{t}_X} (1, \nabla_1)$ is proper.

$\mathbb{K}[\mathbb{A}]$ is the full subcategory of compact preneighbourhood spaces.

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In $(\mathtt{Set}, \mathtt{Surjection}, \mathtt{Injection})$ the terminal object is singleton, $\nabla_1 = \uparrow_1$.

However, in $(\mathtt{CRing}^{\mathrm{op}}, \mathtt{Epi}, \mathtt{RegMon})$ the terminal object is $\mathbb{Z}$ and $\nabla_1 < \uparrow_1$.

The category $\mathbb{K}[\mathbb{A}]$

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- *An internal preneighbourhood space* (X, μ) *is compact if and only if for all preneighbourhood spaces* (Y, ϕ)*, the projection* $(X \times Y, \mu \times \phi) \xrightarrow{p_2} (Y, \phi)$ *is a closed morphism.*

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- *If* $(X, \mu) \xrightarrow{f} (Y, \phi)$ *is proper and* (Y, ϕ) *is compact then* (X, μ) *is compact.*

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- *If* $(X, \mu) \xrightarrow{f} (Y, \phi)$ *is stably in* E *and* (X, μ) *is compact then* (Y, ϕ) *is compact.*

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- *If* $(X, \mu) \xrightarrow{f} (Y, \phi)$ *is stably in* E *and* (X, μ) *is compact then* (Y, ϕ) *is compact.*
- *The category* $\mathtt{K}[\mathbb{A}]$ *is finitely productive and closed hereditary.*

Alternative descriptions of compactness

A set $\mathfrak{a} \subseteq \mathtt{Sub}_{\mathsf{M}}(X)$ is said to have finite intersection property if $\mathbf{1}_X \in \mathfrak{a}$ and $x, y \in \mathfrak{a} \Rightarrow x \wedge y \neq \sigma_X$.

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A filter $A \in \mathtt{Fil}X$ of a preneighbourhood space (X, μ) is said to cluster if $\bigwedge\{\mathrm{cl}_\mu p : p \in A\} \neq \sigma_X$

Alternative descriptions of compactness

Theorem 6.1 (Filters cluster imply compactness)

In a reflecting zero context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ *with finite product projections in E, if every filter cluster then* (X, μ) *is compact.*

Alternative descriptions of compactness

- A category with finite sums is *extensive* if the sum functor $(\mathbb{A} \downarrow A) \times (\mathbb{A} \downarrow B) \xrightarrow{+} (\mathbb{A} \downarrow A + B)$ is an equivalence of categories (see Carboni, Lack, and Walters, "Introduction to extensive and distributive categories", for details...); lextensive is a finitely complete extensive category.
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In an extensive context:

- the initial object is strict
- the coproduct injections are admissible monomorphisms
- the unique morphism $\emptyset \to 1$ is an admissible monomorphism, hence is an admissibly quasi-pointed context
- $\sigma_X = \mathtt{i}_X$, every morphism reflect zero, every preneighbourhood morphism is continuous, see Ghosh, "Internal neighbourhood structures II: Closure and Closed Morphisms", §9

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Theorem 6.2 (Compactness imply filters cluster)

In a lextensive context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ *with finite product projections in E, if* (X, μ) *is compact then every filter cluster.*

Alternative descriptions of compactness

A lattice L is *pseudocomplemented* if for each $a \in L$ the set $\{x \in L : x \wedge a = 0\}$ has a maximum element, denoted a^*, called *pseudocomplement of a* (see Blyth, *Lattices and ordered algebraic structures*, §7.1).

Alternative descriptions of compactness

Definition 6.1 (see Ghosh, "Internal neighbourhood structures", §3.1.3)

Let (X, μ) be a preneighbourhood space, $p \in \mathtt{Sub}_{\mathsf{M}}(X)$; p is *μ-open* if $p \in \mu(p)$, $\mathfrak{O}_\mu$ is the set of all μ-open sets and $\mathrm{int}_\mu p = \bigvee \left\{ x \in \mathfrak{O}_\mu : x \leq p \right\}$ is the *μ-interior of p*.

Alternative descriptions of compactness

The assignment $p \mapsto \mathrm{int}_\mu p$ is monotone, idempotent and intensional, fixing every μ-open set; $\sigma_X \in \mathfrak{O}_\mu$, $\mathfrak{O}_\mu$ closed under finite meets; $\mathfrak{O}_\mu$ is closed under arbitrary joins if and only if for each $p \in \mathtt{Sub}_{\mathsf{M}}(X)$, $\mathrm{int}_\mu p \in \mathfrak{O}_\mu$.

Furthermore, in such a case:

$$\mu(m) = \{p \in \mathtt{Sub}_{\mathsf{M}}(X) : m \leq \mathrm{int}_\mu p\} \Leftrightarrow \mu(m) = \bigcup\{\uparrow q : m \leq q \in \mathfrak{O}_\mu\}.$$

(see Ghosh, "Internal neighbourhood structures", Theorem 3.20).
In particular, neighbourhood systems have this property.

Alternative descriptions of compactness

Fact 6.1

If (X,μ) *is a preneighbourhood space with* μ*-interiors open and* $\mathtt{Sub}_{\mathsf{M}}(X)$ *pseudocomplemented, then pseudocomplementation produce an adjunction* $\begin{tikzcd}\mathfrak{C}_\mu \arrow[r, shift left, "*"] \arrow[r, phantom, "\perp"] & {\mathfrak{O}_\mu}^{\mathrm{op}} \arrow[l, shift left, "*"]\end{tikzcd}$ *between the complete* $\wedge$*-semilattices.*

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The Galois connection restricts to a dual isomorphism between the complete $\wedge$-semilattices ${\mathfrak{C}_\mu}^* = \left\{p \in \mathfrak{C}_\mu : p = p^{**}\right\}$ of $*$-closed subobjects and ${\mathfrak{O}_\mu}^* = \left\{p \in \mathfrak{O}_\mu : p = p^{**}\right\}$ of $\star$-open subobjects.

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Corollary 6.2

Every set of $*$*-closed subsets with finite intersection property has nonempty intersection if and only if every* $*$*-open cover has a finite subcover.*

Alternative descriptions of compactness

Theorem 6.1 (Filters cluster imply compactness)

In a reflecting zero context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ *with finite product projections in E, if every filter cluster then* (X, μ) *is compact.*

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In a lextensive context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ *with finite product projections in E, if* (X, μ) *is compact then every filter cluster.*

Theorem 6.3 (The Kuratowski-Mrowka Theorem)

In any lextensive context with finite product projections in E, if (X, μ) *is a preneighbourhood space with* μ*-interiors* μ*-open and* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is pseudocomplemented then the following are equivalent:*

(a) (X, μ) *is compact.*

(b) Every filter cluster.

(c) Every set of $*$*-closed subobjects of* X *with finite intersection property has a nonempty intersection.*

(d) Every $*$*-open cover has a finite subcover.*

Alternative descriptions of compactness

Proof of *filter clustering* imply *compactness*.

□

Alternative descriptions of compactness

Proof of *filter clustering* imply *compactness*.

Assume every filter in (X,μ) cluster, (Y,ϕ) a preneighbourhood space, $(X\times Y,\mu\times\phi)\xrightarrow{p_2}(Y,\phi)$ is the product projection, $a\in\mathfrak{C}_{\mu\times\phi}$ and $\sigma_Y\neq y\leq\mathrm{cl}_\phi\exists_{p_2}a$.
To show $y\leq\exists_{p_2}a$.

$\square$

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Assume every filter in (X, μ) cluster, (Y, ϕ) a preneighbourhood space,
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To show $y \leq \exists_{p_2} a$.
$A = \left\{\mathrm{cl}_\mu \exists_{p_1} (a \wedge {p_2}^{-1} u) : u \in \phi(y)\right\}$ has finite intersection property, hence $x = \bigwedge A \neq \sigma_X$.

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Proof of *filter clustering* imply *compactness*.

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Here $(X \times Y, \mu \times \phi) \xrightarrow{p_1} (X, \mu)$ is the other product projection.

□

Alternative descriptions of compactness

Proof of *filter clustering* imply *compactness*.

Assume every filter in (X,μ) cluster, (Y,ϕ) a preneighbourhood space,
$(X\times Y,\mu\times\phi)\xrightarrow{p_2}(Y,\phi)$ is the product projection, $a\in\mathfrak{C}_{\mu\times\phi}$ and $\sigma_Y\neq y\leq \mathrm{cl}_\phi\exists_{p_2}a$.
To show $y\leq\exists_{p_2}a$.
$A=\left\{\mathrm{cl}_\mu\exists_{p_1}(a\wedge {p_2}^{-1}u) : u\in\phi(y)\right\}$ has finite intersection property, hence $x=\bigwedge A\neq\sigma_X$.
Hence:

$$\begin{aligned} a\wedge {p_1}^{-1}x &\geq a\wedge \mathrm{cl}_{\mu\times\phi}{p_1}^{-1}\left(\bigwedge_{u\in\phi(y)}\exists_{p_1}(a\wedge {p_2}^{-1}u)\right) && \text{(continuity of } p_1\text{)}\\ &\geq a\wedge {p_2}^{-1}y\neq\sigma_{X\times Y}, \end{aligned}$$

□

Alternative descriptions of compactness

Proof of *filter clustering* imply *compactness*.

Assume every filter in (X,μ) cluster, (Y,ϕ) a preneighbourhood space, $(X\times Y,\mu\times\phi)\xrightarrow{p_2}(Y,\phi)$ is the product projection, $a\in\mathfrak{C}_{\mu\times\phi}$ and $\sigma_Y\neq y\leq \mathrm{cl}_\phi\exists_{p_2}a$.
To show $y\leq\exists_{p_2}a$.
$A=\left\{\mathrm{cl}_\mu\exists_{p_1}(a\wedge {p_2}^{-1}u) : u\in\phi(y)\right\}$ has finite intersection property, hence $x=\bigwedge A\neq\sigma_X$.
Hence:

$$\begin{aligned} a\wedge {p_1}^{-1}x &\geq a\wedge \mathrm{cl}_{\mu\times\phi}{p_1}^{-1}\left(\bigwedge\nolimits_{u\in\phi(y)}\exists_{p_1}(a\wedge {p_2}^{-1}u)\right) && \text{(continuity of } p_1\text{)}\\ &\geq a\wedge {p_2}^{-1}y\neq\sigma_{X\times Y}, \end{aligned}$$

since: If (X,μ) and (Y,ϕ) are preneighbourhood spaces in a context with finite product projections in E, $X\times Y\xrightarrow{p_2}Y$ the product projection, $a\in\mathfrak{C}_{\mu\times\phi}$ then for any $y\in\mathrm{Sub}_{\mathrm{M}}(Y)$:
$a\wedge {p_2}^{-1}y\neq\sigma_{X\times Y}\Leftrightarrow y\wedge\exists_{p_2}a\neq\sigma_Y\Leftrightarrow y\leq\mathrm{cl}_\phi\exists_{p_2}a$.

□

Alternative descriptions of compactness

Proof of *filter clustering* imply *compactness*.

Assume every filter in (X, μ) cluster, (Y, ϕ) a preneighbourhood space, $(X \times Y, \mu \times \phi) \xrightarrow{p_2} (Y, \phi)$ is the product projection, $a \in \mathfrak{C}_{\mu \times \phi}$ and $\sigma_Y \neq y \leq \mathrm{cl}_\phi \exists_{p_2} a$.
To show $y \leq \exists_{p_2} a$.
$A = \left\{ \mathrm{cl}_\mu \exists_{p_1} (a \wedge {p_2}^{-1} u) : u \in \phi(y) \right\}$ has finite intersection property, hence $x = \bigwedge A \neq \sigma_X$.
Hence:

$$\begin{aligned} a \wedge {p_1}^{-1} x &\geq a \wedge \mathrm{cl}_{\mu \times \phi} {p_1}^{-1} \left(\bigwedge_{u \in \phi(y)} \exists_{p_1} (a \wedge {p_2}^{-1} u) \right) && \text{(continuity of } p_1\text{)} \\ &\geq a \wedge {p_2}^{-1} y \neq \sigma_{X \times Y}, \end{aligned}$$

Hence, for $v \in \mu(x)$, $u \in \phi(y)$:

$$(v \times u) \wedge a \geq a \circ a^{-1} (a \wedge {p_2}^{-1} y) \neq \sigma_{X \times Y},$$

implying $x \times y \leq \mathrm{cl}_{\mu \times \phi} a = a$.

□

Alternative descriptions of compactness

Proof of *filter clustering* imply *compactness*.

Assume every filter in (X,μ) cluster, (Y,ϕ) a preneighbourhood space,
$(X\times Y,\mu\times\phi)\xrightarrow{p_2}(Y,\phi)$ is the product projection, $a\in\mathfrak{C}_{\mu\times\phi}$ and $\sigma_Y\neq y\leq\mathrm{cl}_\phi\exists_{p_2}a$.
To show $y\leq\exists_{p_2}a$.
$A=\left\{\mathrm{cl}_\mu\exists_{p_1}(a\wedge {p_2}^{-1}u):u\in\phi(y)\right\}$ has finite intersection property, hence $x=\bigwedge A\neq\sigma_X$.
Hence:

$$\begin{aligned} a\wedge {p_1}^{-1}x &\geq a\wedge\mathrm{cl}_{\mu\times\phi}{p_1}^{-1}\left(\bigwedge\nolimits_{u\in\phi(y)}\exists_{p_1}(a\wedge {p_2}^{-1}u)\right) && \text{(continuity of } p_1\text{)}\\ &\geq a\wedge {p_2}^{-1}y\neq\sigma_{X\times Y}, \end{aligned}$$

Hence, for $v\in\mu(x)$, $u\in\phi(y)$:

$$(v\times u)\wedge a\geq a\circ a^{-1}(a\wedge {p_2}^{-1}y)\neq\sigma_{X\times Y},$$

implying $x\times y\leq\mathrm{cl}_{\mu\times\phi}a=a$.
Since product projections are in E, $y=\exists_{p_2}(x\times y)\leq\exists_{p_2}a$. □

Alternative descriptions of compactness

Proof of *compactness* implying *filter clustering*.

□

Alternative descriptions of compactness

Proof of *compactness* implying *filter clustering*.

In a lextensive context:

- $\sigma_X = \mathtt{i}_X$, every morphism reflect zero, every preneighbourhood morphism is continuous

$\square$

Alternative descriptions of compactness

Proof of *compactness* implying *filter clustering*.

In a lextensive context:

- $\sigma_X = \mathtt{i}_X$, every morphism reflect zero, every preneighbourhood morphism is continuous
- If finite sum of admissible subobjects is an admissible subobject, then

$\mathtt{Sub}_{\mathsf{M}}(X+Z) = \left\{u_X + u_Z : u_X \in \mathtt{Sub}_{\mathsf{M}}(X), u_Z \in \mathtt{Sub}_{\mathsf{M}}(Z)\right\}$, where

$$\begin{tikzcd}
U_X \arrow[r, tail, "\iota_{U_X}"] \arrow[d, tail, "u_X"'] & U \arrow[d, tail, "u_X+u_Z"'] & U_Z \arrow[l, tail, "\iota_{U_Z}"'] \arrow[d, tail, "u_Z"] \\
X \arrow[r, tail, "\iota_X"'] & X+Z & Z \arrow[l, tail, "\iota_Z"]
\end{tikzcd} .$$

In fact, under these conditions:

$$\begin{tikzcd}
\mathtt{Sub}_{\mathsf{M}}(X) \arrow[r, shift left, "\exists_{\iota_X}"] \arrow[r, phantom, "\perp"] & \mathtt{Sub}_{\mathsf{M}}(X+Z) \arrow[l, shift left, "{\iota_X}^{-1}"] \arrow[r, shift right, "{\iota_Z}^{-1}"'] \arrow[r, phantom, "\perp"] & \mathtt{Sub}_{\mathsf{M}}(Z) \arrow[l, shift right, "\exists_{\iota_Z}"']
\end{tikzcd}$$

is a biproduct in $\vee$-$\mathtt{SemLat}$, (see Ghosh, "Internal neighbourhood structures III: Finite sum of subobjects", Theorem 8.1); in general: $\mathtt{Sub}_{\mathsf{M}}(X+Z) \subseteq \mathtt{Sub}_{\mathsf{M}}(X) + \mathtt{Sub}_{\mathsf{M}}(Z)$.

In particular, $u_X + u_Z \in \mathtt{Sub}_{\mathsf{M}}(X) \Leftrightarrow u_Z = \mathtt{i}_Z$.

Alternative descriptions of compactness

Proof of *compactness* implying *filter clustering*.

Assume (X, μ) is compact and A is a filter of closed subobjects of X.

$\square$

Alternative descriptions of compactness

Proof of *compactness* implying *filter clustering*.

Assume (X, μ) is compact and A is a filter of closed subobjects of X.
Choose and fix an object Z and define a preneighbourhood system on $X + Z$ by:
$\phi(x + z) = \left\{x' + z' \in \mathtt{Sub}_{\mathsf{M}}(X + Z) : (\exists k \in A)(x' \geq x \vee k) \text{ and } z' \geq z\right\}$. where

$$\begin{tikzcd} X \arrow[r, tail, "\iota_X"] & X + Z & Z \arrow[l, tail, "\iota_Z"'] \end{tikzcd}$$

is the coproduct. Rename: $X + Z$ as Y.

$\square$

Alternative descriptions of compactness

Proof of *compactness* implying *filter clustering*.

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Choose and fix an object Z and define a preneighbourhood system on $X + Z$ by:
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$X \overset{\iota_X}{\rightarrowtail} X + Z \overset{\iota_Z}{\leftarrowtail} Z$ is the coproduct. Rename: $X + Z$ as Y.
Evidently, $\mathrm{cl}_\phi \iota_X = \mathbf{1}_Y$.

□

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$X \overset{\iota_X}{\rightarrowtail} X + Z \overset{\iota_Z}{\leftarrowtail} Z$ is the coproduct. Rename: $X + Z$ as Y.
Evidently, $\mathrm{cl}_\phi \iota_X = \mathbf{1}_Y$.
Take $a = (a_X, a_Y) = \mathrm{cl}_{\mu \times \phi}(\mathbf{1}_X, \iota_X)$. Since $(X \times Y, \mu \times \phi) \xrightarrow{p_2} (Y, \phi)$ is closed and continuous: $\exists_{p_2} a \in \mathfrak{C}_\phi$ and $\exists_{p_2} a = \exists_{p_2} \mathrm{cl}_{\mu \times \phi}(\mathbf{1}_X, \iota_X) = \mathrm{cl}_\phi \exists_{p_2}(\mathbf{1}_X, \iota_X) = \mathrm{cl}_\phi \iota_X = \mathbf{1}_Y$.

$\square$

Alternative descriptions of compactness

Proof of *compactness* implying *filter clustering*.

Assertion:

$$
\begin{aligned}
&\forall z \in \mathtt{Sub}_{\mathsf{M}}(Z)_{\neq 0},\\
&\qquad \forall (x, y) \in \left\{(u, v) \in \mathtt{Sub}_{\mathsf{M}}(X \times (Y)) : (u, v) \leq a, u \neq \mathtt{i}_X, \exists_{p_2}(u, v) = \mathtt{i}_X + z\right\},\\
&\qquad\qquad \forall p \in \mu(\exists_{p_1}(x, y)), \forall k \in A, p \wedge k \neq \mathtt{i}_X\\
&\qquad\qquad\qquad \Rightarrow \bigwedge A \neq \mathtt{i}_X
\end{aligned}
$$

$\square$

Alternative descriptions of compactness

Proof of *compactness* implying *filter clustering*.

Assertion:

$$\forall z \in \mathtt{Sub}_{\mathrm{M}}(Z)_{\neq 0}, \\ \forall (x, y) \in \{(u, v) \in \mathtt{Sub}_{\mathrm{M}}(X \times (Y)) : (u, v) \leq a, u \neq \mathtt{i}_X, \exists_{p_2}(u, v) = \mathtt{i}_X + z\}, \\ \forall p \in \mu(\exists_{p_1}(x, y)), \forall k \in A, p \wedge k \neq \mathtt{i}_X \\ \Rightarrow \bigwedge A \neq \mathtt{i}_X$$

If $\bigwedge A = \mathtt{i}_X$ then: there exists one $z \in \mathtt{Sub}_{\mathrm{M}}(Z)_{\neq 0}$, one $(x, y) \leq a$ such that $x \neq \mathtt{i}_X$, $\exists_{p_2}(x, y) = \mathtt{i}_X + z$, one $p \in \mu(\exists_{p_1}(x, y)$ and one $k \in A$ such that $p \wedge k = \mathtt{i}_X$.

But $p \times (k + \mathtt{i}_Z) \in (\mu \times \phi)(x, y)$ and $(p \times (k + \mathtt{i}_Z)) \wedge (\mathbf{1}_X, \iota_X) = (\mathbf{1}_X, \iota_X) \circ (p \wedge k) = \mathtt{i}_{X \times Y}!$

Hence $\bigwedge A \neq \mathtt{i}_X$

$\square$

Alternative descriptions of compactness

Proof of *compactness* implying *filter clustering*.

Assertion:

$$\begin{aligned}
&\forall z \in \mathtt{Sub}_{\mathsf{M}}(Z)_{\neq 0},\\
&\qquad \forall (x,y) \in \left\{(u,v) \in \mathtt{Sub}_{\mathsf{M}}(X \times (Y)) : (u,v) \le a, u \neq \mathtt{i}_X, \exists_{p_2}(u,v) = \mathtt{i}_X + z\right\},\\
&\qquad\qquad \forall p \in \mu(\exists_{p_1}(x,y)), \forall k \in A, p \wedge k \neq \mathtt{i}_X\\
&\qquad\qquad\qquad \Rightarrow \bigwedge A \neq \mathtt{i}_X
\end{aligned}$$

Choose a $z \in \mathtt{Sub}_{\mathsf{M}}(Z)_{\neq 0}$, $(x,y) \le a$ such that $x \neq \mathtt{i}_X$, $\exists_{p_2}(x,y) = \mathtt{i}_X + z$. Replace A by $\hat{A} = A \vee \left\{\mathrm{cl}_\mu p : p \in \mu(\exists_{p_1}(x,y))\right\}$. With such a choice, $\exists_{p_2} a = \mathrm{cl}_{\phi_{\hat{A}}} \iota_X = \mathbf{1}_Y$, $\phi_{\hat{A}}$ is defined like ϕ from $\hat{A}$. Consequently,
$\mathtt{i}_X \neq \exists_{p_1}(x,y) \le \mathrm{cl}_\mu k = k \in \hat{A} \Rightarrow \mathtt{i}_X \neq \exists_{p_1}(x,y) \le \bigwedge \hat{A} \le \bigwedge A$.

□

Alternative descriptions of compactness

Proof of *compactness* implying *filter clustering*.

Assume (X, μ) is compact and A is a filter of closed subobjects of X.
Choose and fix an object Z and define a preneighbourhood system on $X + Z$ by:
$\phi(x + z) = \left\{x' + z' \in \mathtt{Sub}_{\mathsf{M}}(X + Z) : (\exists k \in A)(x' \geq x \vee k) \text{ and } z' \geq z\right\}$. where
$X \overset{\iota_X}{\rightarrowtail} X + Z \overset{\iota_Z}{\leftarrowtail} Z$ is the coproduct. Rename: $X + Z$ as Y.
Evidently, $\mathrm{cl}_\phi \iota_X = \mathbf{1}_Y$.
Take $a = (a_X, a_Y) = \mathrm{cl}_{\mu\times\phi}(\mathbf{1}_X, \iota_X)$. Since $(X \times Y, \mu \times \phi) \xrightarrow{p_2} (Y, \phi)$ is closed and continuous: $\exists_{p_2} a \in \mathfrak{C}_\phi$ and $\exists_{p_2} a = \exists_{p_2} \mathrm{cl}_{\mu\times\phi}(\mathbf{1}_X, \iota_X) = \mathrm{cl}_\phi \exists_{p_2}(\mathbf{1}_X, \iota_X) = \mathrm{cl}_\phi \iota_X = \mathbf{1}_Y$.
If $\bigwedge A = \mathtt{i}_X$ then: there exists one $z \in \mathtt{Sub}_{\mathsf{M}}(Z)_{\neq 0}$, one $(x, y) \leq a$ such that $x \neq \mathtt{i}_X$, $\exists_{p_2}(x, y) = \mathtt{i}_X + z$, one $p \in \mu(\exists_{p_1}(x, y)$ and one $k \in A$ such that $p \wedge k = \mathtt{i}_X$.

But $p \times (k + \mathtt{i}_Z) \in (\mu \times \phi)(x, y)$ and $(p \times (k + \mathtt{i}_Z)) \wedge (\mathbf{1}_X, \iota_X) = (\mathbf{1}_X, \iota_X) \circ (p \wedge k) = \mathtt{i}_{X\times Y}$!

Hence $\bigwedge A \neq \mathtt{i}_X$

□

Alternative descriptions of compactness

Theorem 6.1 (Filters cluster imply compactness)

In a reflecting zero context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ *with finite product projections in E, if every filter cluster then* (X, μ) *is compact.*

Theorem 6.2 (Compactness imply filters cluster)

In a lextensive context $\mathcal{A} = (\mathbb{A}, \mathsf{E}, \mathsf{M})$ *with finite product projections in E, if* (X, μ) *is compact then every filter cluster.*

Theorem 6.3 (The Kuratowski-Mrowka Theorem)

In any lextensive context with finite product projections in E, if (X, μ) *is a preneighbourhood space with* μ*-interiors* μ*-open and* $\mathtt{Sub}_{\mathsf{M}}(X)$ *is pseudocomplemented then the following are equivalent:*

- *(a)* (X, μ) *is compact.*
- *(b) Every filter cluster.*
- *(c) Every set of* $*$*-closed subobjects of* X *with finite intersection property has a nonempty intersection.*
- *(d) Every* $*$*-open cover has a finite subcover.*

- Blyth, T. S. *Lattices and ordered algebraic structures.* Universitext. Springer-Verlag London, Ltd., London, 2005, pp. x+303. ISBN: 1-85233-905-5. MR2126425, rev. by Bernd S. W. Schröder (cit. on p. 77).
- Bourn, D. “3 × 3 lemma and protomodularity”. In: *J. Algebra* 236.2 (2001), pp. 778–795. ISSN: 0021-8693. DOI: `10.1006/jabr.2000.8526`. MR1813501, rev. by R. H. Street (cit. on pp. 33–35).
- Carboni, A., S. Lack, and R. F. C. Walters. “Introduction to extensive and distributive categories”. In: *J. Pure Appl. Algebra* 84.2 (1993), pp. 145–158. ISSN: 0022-4049. DOI: `10.1016/0022-4049(93)90035-R`. MR1201048, rev. by R. H. Street (cit. on pp. 70–75).
- Castellini, G. “Discrete objects, splitting closure and connectedness”. In: *Quaest. Math.* 31.2 (2008), pp. 107–126. ISSN: 1607-3606. DOI: `10.2989/QM.2008.31.2.1.473`. MR2529128 (cit. on pp. 3–5).
- — .“Interior operators in a category: idempotency and heredity”. In: *Topology Appl.* 158.17 (2011), pp. 2332–2339. ISSN: 0166-8641. DOI: `10.1016/j.topol.2011.06.030`. MR2838382, rev. by Esfandiar Haghverdi (cit. on pp. 3–5).
- — .“Interior operators, open morphisms and the preservation property”. In: *Appl. Categ. Structures* 23.3 (2015), pp. 311–322. ISSN: 0927-2852. DOI: `10.1007/s10485-013-9337-4`. MR3351083, rev. by E. Lowen-Colebunders (cit. on pp. 3–5).
- — .“Some remarks on interior operators and the functional property”. In: *Quaest. Math.* 39.2 (2016), pp. 275–287. ISSN: 1607-3606. DOI: `10.2989/16073606.2015.1070379`. MR3483373, rev. by Ando Razafindrakoto (cit. on pp. 3–5).

- Castellini, G. and D. Holgate. “A link between two connectedness notions”. In: *Appl. Categ. Structures* 11.5 (2003), pp. 473–486. ISSN: 0927-2852. DOI: `10.1023/A:1025732820692`. MR2006796 (cit. on pp. 3–5).
- Castellini, G. and E. Murcia. “Interior operators and topological separation”. In: *Topology Appl.* 160.12 (2013), pp. 1476–1485. ISSN: 0166-8641. DOI: `10.1016/j.topol.2013.05.023`. MR3072710, rev. by Maria Manuel Clementino (cit. on pp. 3–5).
- Castellini, G. and J. Ramos. “Interior operators and topological connectedness”. In: *Quaest. Math.* 33.3 (2010), pp. 290–304. ISSN: 1607-3606. DOI: `10.2989/16073606.2010.507322`. MR2755522 (cit. on pp. 3–5).
- Clementino, M. M., E. Giuli, and W. Tholen. “A functional approach to general topology”. In: *Categorical foundations*. Vol. 97. Encyclopedia Math. Appl. Cambridge Univ. Press, Cambridge, 2004, pp. 103–163. DOI: `https://doi.org/10.1017/cbo9781107340985.006`. MR2056582 (cit. on pp. 4, 5, 11–16, 53).
- Diers, Y. *Categories of commutative algebras*. English. Oxford: Clarendon Press, 1992, pp. ix + 271. ISBN: 0-19-853586-4 (cit. on pp. 11–16).
- Dikranjan, D. and E. Giuli. “Closure operators. I”. In: *Proceedings of the 8th international conference on categorical topology (L’Aquila, 1986)*. Vol. 27. 2. 1987, pp. 129–143. DOI: `10.1016/0166-8641(87)90100-3`. MR911687 (cit. on pp. 3–5).
- — .“Compactness, minimality and closedness with respect to a closure operator”. In: *Categorical topology and its relation to analysis, algebra and combinatorics (Prague,*

1988). World Sci. Publ., Teaneck, NJ, 1989, pp. 284–296. MR1047908, rev. by H. Herrlich (cit. on pp. 3–5).

- Dikranjan, D., E. Giuli, and W. Tholen. “Closure operators. II”. In: *Categorical topology and its relation to analysis, algebra and combinatorics (Prague, 1988)*. World Sci. Publ., Teaneck, NJ, 1989, pp. 297–335. MR1047909, rev. by H. Herrlich (cit. on pp. 3–5).
- Dikranjan, D., E. Giuli, and A. Tozzi. “Topological categories and closure operators”. In: *Quaestiones Math.* 11.3 (1988), pp. 323–337. ISSN: 0379-9468. MR953772, rev. by H. Herrlich (cit. on pp. 3–5).
- Dikranjan, D. and W. Tholen. *Categorical structure of closure operators*. Vol. 346. Mathematics and its Applications. With applications to topology, algebra and discrete mathematics. Kluwer Academic Publishers Group, Dordrecht, 1995, pp. xviii+358. ISBN: 0-7923-3772-7. DOI: `10.1007/978-94-015-8400-5`. MR1368854, rev. by D. Pumplün (cit. on pp. 3–5).
- Dube, T. and O. Ighedo. “Characterising points which make *P*-frames”. In: *Topology Appl.* 200 (2016), pp. 146–159. ISSN: 0166-8641. DOI: `10.1016/j.topol.2015.12.017`. MR3453411, rev. by Ali Akbar Estaji (cit. on p. 44).
- — .“More on locales in which every open sublocale is *z*-embedded”. In: *Topology Appl.* 201 (2016), pp. 110–123. ISSN: 0166-8641. DOI: `10.1016/j.topol.2015.12.030`. MR3461158, rev. by Mack Zakaria Matlabyana (cit. on p. 44).
- Erné, M. “Prime and maximal ideals of partially ordered sets”. In: *Math. Slovaca* 56.1 (2006), pp. 1–22. ISSN: 0139-9918. MR2217576, rev. by Viorica Sofronie-Stokkermans (cit. on p. 39).

- Ghosh, P. P. “Internal neighbourhood structures”. In: *Algebra Universalis* 81.2 (2020), Paper No. 12, 53 pages. ISSN: 0002-5240. DOI: `10.1007/s00012-020-0640-2`. MR4066491 (cit. on pp. 6–16, 19–21, 25, 26, 44, 78, 79).
- — . “Internal neighbourhood structures II: Closure and Closed Morphisms”. submitted, available from arXiv site: https://arxiv.org/abs/2004.06238. Sept. 2021 (cit. on pp. 7–10, 22–24, 27–29, 31, 32, 36, 37, 40–43, 45, 46, 57–65, 75).
- — . “Internal neighbourhood structures III: Finite sum of subobjects”. submitted, available from: arXiv site: https://arxiv.org/abs/2012.03125. Feb. 2021 (cit. on p. 94).
- Goswami, A. and Z. Janelidze. “On the structure of zero morphisms in a quasi-pointed category”. In: *Appl. Categ. Structures* 25.6 (2017), pp. 1037–1043. ISSN: 0927-2852. DOI: `10.1007/s10485-016-9462-y`. MR3720399 (cit. on pp. 33–35).
- Hofmann, D. and W. Tholen. “Lax algebra meets topology”. In: *Topology Appl.* 159.9 (2012), pp. 2434–2452. ISSN: 0166-8641. DOI: `10.1016/j.topol.2011.09.049`. MR2921832, rev. by Sergey A. Solovyov (cit. on p. 5).
- Holgate, D. and J. Šlapal. “Categorical neighborhood operators”. In: *Topology Appl.* 158.17 (2011), pp. 2356–2365. ISSN: 0166-8641. DOI: `10.1016/j.topol.2011.06.031`. MR2838385, rev. by A. Pultr (cit. on pp. 3–5).
- Iberkleid, W. and W. W. McGovern. “A natural equivalence for the category of coherent frames”. In: *Algebra Universalis* 62.2-3 (2009), pp. 247–258. ISSN: 0002-5240. DOI: `10.1007/s00012-010-0058-3`. MR2661378, rev. by Mojgan Mahmoudi (cit. on p. 18).
- Razafindrakoto, A. and D. Holgate. “Interior and neighbourhood”. In: *Topology Appl.* 168 (2014), pp. 144–152. ISSN: 0166-8641. DOI: `10.1016/j.topol.2014.02.019`. MR3196846, rev. by Zbigniew Duszyński (cit. on pp. 3–5).

- Razafindrakoto, A. and D. Holgate. “A lax approach to neighbourhood operators”. In: *Appl. Categ. Structures* 25.3 (2017), pp. 431–445. ISSN: 0927-2852. DOI: `10.1007/s10485-016-9441-3`. MR3654180, rev. by Jǐrié Rosický (cit. on pp. 3–5).
- Vorster, S. J. R. “Interior operators in general categories”. In: *Quaest. Math.* 23.4 (2000), pp. 405–416. ISSN: 1607-3606. DOI: `10.2989/16073600009485987`. MR1810290, rev. by A. Pultr (cit. on pp. 3–5).