From Daniell spaces to the integration spaces of Bishop and Cheng

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Overview

- The Daniell approach to classical integration and measure theory
- The integrable functions in the Daniell approach
- The integrable functions in the Daniell-Mikusiński approach
- The integration spaces of Bishop and Cheng
- Why partiality
- Why strong extensionality
- The integrable functions in the Bishop-Cheng approach
- The integration spaces yesterday, today and tomorrow

The Daniell approach to classical integration and measure theory

Two approaches to measure and integration theory

- The popular approach: a "from sets to functions"-approach

$$\mu \to \int$$

- The Daniell approach: a "from functions to sets"-approach

$$\int \to \mu$$

Daniell (1918), Weil (1940), Kolmogoroff (1948), Carathéodory (1956), Segal's algebraic integration theory (1954, 1965)

Two approaches to topology

- The popular approach: from open sets to continuous functions

$$(X, \mathcal{T}) \to C(X)$$

- From continuous functions to open sets:

$$C(X) \to (X, \mathcal{T})$$

From open sets to continuous functions in classical and constructive topology

$$\mathcal{T} \to C(X)$$

- Topological spaces
- Formal spaces
- Apartness spaces
- Intuitionistic topological spaces
- Neighborhood spaces

From continuous functions to open sets in classical and constructive topology

$$C(X) \to \mathcal{T}$$

- limit spaces
- Spanier's quasi-topological spaces
- Bishop spaces: $F \subseteq \mathbb{F}(X)$ and for every $f \in F$:

$$U(f) := \{x \in X \mid f(x) > 0\}$$

Functions suit better to (classical and) constructive study rather than sets

- The theory of $C(X)$ is used in the study of X.
- To define the characteristic function of a subset we need PEM:

$$\chi_A(x) := \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}$$

- In constructive topology the function-theoretic approach of Bishop spaces is shown to be very fruitful.

From measure to integral in classical measure theory

$$\mu \to \int$$

- We start from a measure space $(X, \mathcal{A}, \mu)$
- We define simple functions:

$$\phi = \sum_{i=1}^{n} c_i \chi_{A_i}, \quad \bigcup_{i=1}^{n} A_i = X, \quad A_i \cap A_j = \varnothing$$

- We define measurable functions $f \colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ through the Borel sets in $\mathbb{R}$
- A simple function is measurable
- Every positive measurable function is the limit of an increasing sequence $(\phi_n)_{n \in \mathbb{N}}$ of positive, simple functions

From measure to integral in classical measure theory

- The integral of a simple function ϕ is well-defined:

$$\int \phi d\mu := \sum_{i=1}^{n} c_i \mu(A_i)$$

- The integral of a positive measurable function f is well-defined:

$$\int f d\mu := \lim_n \int \phi_n d\mu$$

  f is μ-integrable, if $\int f d\mu \in \mathbb{R}$.

- If f is measurable, then $f_+, f_- \geqslant 0$ are measurable and if f_+, f_- are μ-integrable, let

$$\int f d\mu := \int f_+ d\mu - \int f_- d\mu$$

$$L^1 := \left\{ f : X \to \overline{\mathbb{R}} \mid f \text{ measurable \& } \int |f| d\mu \in \mathbb{R} \right\}$$

The Daniell approach: from integral to measure

$$\int \to \mu$$

- We start from a Daniel space $\left(X, L, \int\right)$, where $L \ni f \mapsto \int f \in \mathbb{R}$
- We extend L to L^1 and $\int$ to $\int^1$, using the Bolzano-Weierstrass theorem and the completeness axiom for real numbers.
- Measurable functions are approximated by elements of L^1
- $A \subseteq X$ is measurable, if χ_A is measurable.
- A measure is defined on the measurable sets (Stone).
- The Daniell aproach is followed by Bourbaki in [6] with a slight variation.

Daniell space

$X \neq \emptyset$, L a Riesz space in $\mathbb{F}(X)$, and

$$\int : L \to \mathbb{R}$$

is a positive, linear functional, continuous under monotone limits:

($\texttt{IntLin}$) $\int(af + bg) = a\int f + b\int g$, for every $a, b \in \mathbb{R}$ and $f, g \in L$.
($\texttt{IntPos}$) $f \geqslant 0 \Rightarrow \int f \geqslant 0$, for every $f \in F$.
($\texttt{IntCont}$) For every sequence $(f_n)_{n=1}^{\infty}$ in L with $f_n \geqslant f_{n+1}$,

$$\lim_n f_n = 0 \Rightarrow \lim_n \int f_n = 0.$$

$\int$ is called an integral on L and $\mathcal{X} := \left(X, L, \int\right)$ an Daniell space.

If $\mathcal{R}[0,1]$ is the set of Riemann-integrable functions on $[0,1]$ and $\int_{\mathcal{R}}$ is the Riemann integral on $\mathcal{R}$, then $\left([0,1], \mathcal{R}[0,1], \int_{\mathcal{R}}\right)$ is a Daniell space, which is not complete;

If $(q_n)_{n=1}^{\infty}$ is a fixed enumeration of $\mathbb{Q}$, and

$$f_n(x) := \begin{cases} 1 & , x \in \{q_1, \dots, q_n\} \\ 0 & , \text{otherwise}, \end{cases}$$

then $\lim f_n$ is the Dirichlet function, which is not in $\mathcal{R}[0,1]$.

To get the completeness property for $\int_{\mathcal{R}}$ on $[0,1]$ it suffices that $(f_n)_{n=1}^{\infty}$ converges uniformly to f.

- If $C(M)$ is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ which are 0 outside $[-M, M]$, for some $M > 0$, and

$$\int_M f := \int_{\mathcal{R}} f := \int_{-\infty}^{+\infty} f(x)dx,$$

then $\left(\mathbb{R}, C(M), \int_{\mathcal{R}}\right)$ is a Daniell space.

- If $C^{\mathrm{supp}}(\mathbb{R}^n)$ is the set of continuous real-valued functions with compact support i.e., the closure of

$$[f \neq 0] := \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$$

is compact, and if K is compact with $[f \neq 0] \subseteq K$, let

$$\int f := \int_K f(x)dx := \int_{\mathcal{R}} f\chi_K.$$

Then $\mathcal{D} := \left(\mathbb{R}^n, C^{\mathrm{supp}}(\mathbb{R}^n), \int\right)$ is a Daniell space and its completion $\mathcal{D}^1$ is the Lebesgue (Daniell) space.

- If X is a locally compact Hausdorff space and $L = C^{\mathrm{supp}}(X)$, then every positive, linear functional on L is an integral.

- If $(X, \mathcal{A}, \mu)$ is a σ-finite measure space, $L(\mu)$ is the set of μ-integrable functions from X to $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and

$$\int_\mu f := \int f d\mu,$$

for every $f \in L(\mu)$, then $\left(X, L(\mu), \int_\mu\right)$ is a Daniell space.

The integrable functions in the Daniell approach

- We extend L and $\int$ as follows:

$$L^+ := \left\{ f \in \mathbb{F}(X, \overline{\mathbb{R}}) \mid \exists_{(f_n)_{n=1}^{\infty} \in \mathbb{F}^+(\mathbb{N}, L)} \left(f = \lim_n f_n \right) \right\}$$

$$a, b \geqslant 0 \;\&\; f, g \in L^+ \Rightarrow af + bg \in L^+$$

$$\int^+ : L^+ \to \overline{\mathbb{R}}$$

$$\int^+ f = \lim_n \int f_n \in \overline{\mathbb{R}}$$

- If $f : X \to \overline{\mathbb{R}}$, let

$$\overline{\int} f := \inf \left\{ \int^{+} g \in \mathbb{R} \mid g \in L^{+} \ \& \ g \geqslant f \right\}$$

$$\underline{\int} f := \sup \left\{ \int^{+} h \in \mathbb{R} \mid h \in L^{+} \ \& \ h \leqslant f \right\}$$

If $f \in L^{+}$, then clearly

$$\overline{\int} f = \underline{\int} f = \int^{+} f.$$

- A function $f : X \to \overline{\mathbb{R}}$ is called integrable, if

$$\overline{\int} f = \underline{\int} f \quad \& \quad \overline{\int} f \in \mathbb{R}.$$

- L^1 is the set of integrable functions and if $f \in L^1$,

$$\int^{1} f := \overline{\int} f$$

The proof of the following theorem is classical, as it uses PEM:

$$\lim_n \int f_n = +\infty \ \vee \ \lim_n \int f_n \in \mathbb{R}.$$

Theorem (CLASS)

If $\mathcal{D} := \big(X, L, \textstyle\int\big)$ is a Daniell space, then $\mathcal{D}^1 := \big(X, L^1, \textstyle\int^1\big)$ is a Daniell space that extends $\mathcal{D}$ i.e., $L \subseteq L_1$ and

$$\int^1 f = \int f,$$

for every $f \in L$. Moreover, if $(f_n)_{n=1}^{\infty}$ is an increasing sequence in L^1, and $f : X \to \overline{\mathbb{R}}$ such that $f = \lim_n f_n$, then

$$f \in L^1 \Leftrightarrow \lim_n \int^1 f_n \in \mathbb{R}.$$

$$\int^1 f = \lim_n \int^1 f_n.$$

Definition

Let $\mathcal{D} := \left(X, L, \int\right)$ be a Daniell space. A function $f\colon X \to [0, +\infty]$ is called measurable,

$$\forall_{g \in L^1}\left(f \wedge g \in L^1\right).$$

$A \subseteq X$ is measurable, if χ_A is measurable.

A is integrable, if $\chi_A \in L^1$.

Let $\mathcal{A}$ be the set of all measurable sets.

The formulation and the proof of the next theorem, which connects Daniell integration to standard measure integration, relies on PEM.

Theorem (Stone, CLASS)

Let a Daniel space $\mathcal{D}^1 := \left(X, L^1, \int^1\right)$. If $X \in \mathcal{A}$, then $(X, \mathcal{A}, \mu)$ is a measure space, where for every $A \in \mathcal{A}$

$$\mu(A) := \begin{cases} \int^1 \chi_A & , A \text{ is integrable} \\ +\infty & , \text{otherwise.} \end{cases}$$

Moreover, $f \in L^1$ if and only if f is μ-integrable, and then

$$\int^1 f = \int f d\mu.$$

The integrable functions in the Daniell-Mikusiński approach

Mikusiński's approach on the Daniell integral in [13], [5] is based on the observation, introduced by his father in [12], that a function $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is Lebesgue-integrable, if there is a sequence of simple functions $(f_n)_{n=1}^{\infty}$ such that the following conditions hold:

$$\texttt{Integr}_1(f) \qquad \sum_{n=1}^{\infty} \int |f_n| \in \mathbb{R}$$

$$\texttt{Integr}_2(f) \qquad \forall_{x \in X} \left(\sum_{n=1}^{\infty} |f_n(x)| \in \mathbb{R} \Rightarrow f(x) = \sum_{n=1}^{\infty} f_n(x) \right).$$

Mikusiński defined L^1 as the set of all functions in $\mathbb{F}(X)$ such that there is a a sequence $(f_n)_{n=1}^{\infty}$ in L such that $\texttt{Integr}_1(f)$ and $\texttt{Integr}_2(f)$ are satisfied. In this case we write

$$(f_n)_{n=1}^{\infty} : \texttt{Integr}(f).$$

Using this notion of integrable function “a very fast and natural way of developing the theory of the Lebesgue integral as well as the Bochner integral” is developed.

The function $\int^1 \colon L^1 \to \mathbb{R}$, defined by

$$\int^1 f = \sum_{n=1}^{\infty} \int f_n,$$

where $(f_n)_{n=1}^{\infty} \colon \mathtt{Integr}(f)$, is well-defined and $\mathcal{X}^1 := \big(X, L^1, \int^1\big)$ is a complete Daniell space i.e., if $(f_n^1)_{n=1}^{\infty} \subseteq L^1$ and $f \colon X \to \mathbb{R}$, then

$$(f_n^1)_{n=1}^{\infty} \colon \mathtt{Integr}(f) \Rightarrow f \in L^1.$$

If $\mathcal{X} := (X, L, \int)$ and $\mathcal{Y} := (Y, M, \oint)$ are Daniell spaces, we call a function $h : X \to Y$ a Daniell morphism, if $\forall_{g \in M}(g \circ h \in L)$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "L \ni g \circ h"'] & Y \arrow[d, "g \in M"] \\ & \mathbb{R}, \end{tikzcd}$$

and for every $g \in M$ we have that

$$\int g \circ h = \oint g.$$

If $h \in \mathrm{Mor}(\mathcal{X}, \mathcal{Y})$, the induced mapping $h^* : M \to L$ from h

$$h^*(g) := g \circ h; \qquad g \in M,$$

is a morphism of Riesz spaces. Let **Dan** be the category of Daniell spaces with Daniell morphisms.

If $h \in \mathrm{Mor}(\mathcal{X}, \mathcal{Y})$, then $h \in \mathrm{Mor}(\mathcal{X}^1, \mathcal{Y}^1)$

$$(g_n)_{n=1}^{\infty} \colon \texttt{Integr}(g) \Rightarrow (g_n \circ h)_{n=1}^{\infty} \colon \texttt{Integr}(g \circ h)$$

$$\int^1 g \circ h = \sum_{n=1}^{\infty} \int (g_n \circ h) = \sum_{n=1}^{\infty} \oint g_n = \oint^1 g$$

$$\mathrm{Compl} \colon \mathbf{Dan} \to \mathbf{Dan}$$

$$\mathrm{Compl}_0(\mathcal{X}) := \mathcal{X}^1$$

$$\mathrm{Compl}_1(h \colon \mathcal{X} \to \mathcal{Y}) := h \colon \mathcal{X}^1 \to \mathcal{Y}^1$$

The integration spaces of Bishop and Cheng

A Bishop-Cheng integration space:

$(X, L, \int)$, where $(X, =_X, \neq_X)$ is inhabited,

L is a subset of the set $\mathfrak{F}^{se}(X)$ of strongly extensional, real-valued partial functions

$\int : L \to \mathbb{R}$, s.t.

- If $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$, $|f|$, and $f \wedge 1$ are in L
- $\int$ is linear
- There is $u \in L$ s.t. $\int u = 1$
- $(\mathtt{IntBishCheng})$ and $(\mathtt{IntInfinity})$ and $(\mathtt{IntZero})$

Arrows between integration spaces can be defined as between Daniell spaces (composition of partial functions with a total one).

In a Daniell space:

A subset A of X is null, if there is $(f_n)_{n=1}^{\infty} \subseteq L$:

$(\mathtt{Null}_1)$ $f_n \leqslant f_{n+1}$, for every $n \in \mathbb{N}^+$.

$(\mathtt{Null}_2)$ There is $M > 0$ such that $\int f_n \leqslant M$, for every $n \in \mathbb{N}^+$.

$(\mathtt{Null}_3)$ The sequence $(f_n(x))_{n=1}^{\infty}$ is divergent, for every $x \in A$.

A property $P(x)$ on X holds almost everywhere (a.e.) in X, if there is a null $A \subseteq X$ such that $P(x)$ holds, for every $x \in A^c$.

Basic lemma for Daniell spaces in CLASS

Let $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty} \subseteq L$ and $f, g \colon X \to \mathbb{R}$.

(i) If $(f_n)_{n=1}^{\infty}$ satisfies $(\mathtt{Null}_1)$ and $(\mathtt{Null}_2)$, and if

$$A = \big\{x \in A \mid (f_n(x))_{n=1}^{\infty} \text{ is divergent}\big\},$$

and $g \colon X \to \mathbb{R}$ is defined by

$$h(x) = \begin{cases} \lim_n f_n(x) & , x \in A^c \\ 0 & , x \in A, \end{cases}$$

then A is null and $f_n \xrightarrow{\text{a.e.}} h$.

(ii) If $f_n \geqslant f_{n+1}$ and $f_n \geqslant 0$, for every $n \in \mathbb{N}^+$, then

$$f_n \xrightarrow{\text{a.e.}} 0 \Rightarrow \lim_n \int f_n = 0.$$

(iii) If $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}$ satisfy $(\mathtt{Null}_1)$ and $(\mathtt{Null}_2)$, if $f_n \xrightarrow{\text{a.e.}} f$ and $g_n \xrightarrow{\text{a.e.}} g$, and if $f \geqslant g$ a.e., then

$$\lim_n \int f_n \geqslant \lim_n \int g_n.$$

Continuity properties of $\int$ on L:

$(\mathtt{IntCont}')$ For every monotone sequence $(f_n)_{n=1}^{\infty}$ in L such that $f(x) = \lim_n f_n(x) \in \mathbb{R}$, for every $x \in X$, and $f \in L$,

$$\int f = \int \lim_n f_n = \lim_n \int f_n.$$

$(\mathtt{IntDan})$ For every sequence $(f_n)_{n=1}^{\infty}$ in L with $f_n \leqslant f_{n+1}$, for every $n \in \mathbb{N}^+$, and for every $f \in L$,

$$f \leqslant \lim_n f_n \Rightarrow \int f \leqslant \lim_n \int f_n.$$

$(\mathtt{IntDan}')$ For every sequence $(f_n)_{n=1}^{\infty}$ in L with $f_n \geqslant 0$, for every $n \in \mathbb{N}^+$, and for every $f \in L$,

$$f \leqslant \sum_{n=1}^{\infty} f_n \Rightarrow \int f \leqslant \sum_{n=1}^{\infty} \int f_n.$$

Let $\int$ be a positive, linear functional on a Riesz space L in $\mathbb{F}(X)$.

(i) (BISH) $(\mathtt{IntCont}) \Leftrightarrow (\mathtt{IntCont}')$.

(ii) (CLASS) $(\mathtt{IntDan}) \Leftrightarrow (\mathtt{IntCont})$.

(iii) (BISH) $(\mathtt{IntDan}) \Leftrightarrow (\mathtt{IntDan}')$.

$(\mathtt{IntCont}) \Rightarrow (\mathtt{IntDan})$ *is non-trivial and requires the previous lemma.*

($\mathtt{IntDan}'$) for every sequence $(f_n)_{n=1}^{\infty}$ in L with $f_n \geqslant 0$, for every $n \in \mathbb{N}^+$, and for every $f \in L$,

$$f \leqslant \sum_{n=1}^{\infty} f_n \Rightarrow \int f \leqslant \sum_{n=1}^{\infty} \int f_n.$$

Classically this implication is equivalent to

$$\neg\left(\int f \leqslant \sum_{n=1}^{\infty} \int f_n\right) \Rightarrow \neg\left(f \leqslant \sum_{n=1}^{\infty} f_n\right),$$

($\mathtt{IntBishCheng}$)

$$\left[\sum_{n=1}^{\infty} \int f_n \in \mathbb{R} \ \& \ \int f > \sum_{n=1}^{\infty} \int f_n\right] \Rightarrow$$

$$\exists_{x \in X}\left(\sum_{n=1}^{\infty} f_n(x) \in \mathbb{R} \ \& \ f(x) > \sum_{n=1}^{\infty} f_n(x)\right)$$

A Riesz space L in $\mathbb{F}(X)$ satisfies the Stone condition, $\texttt{Stone}(L)$, if

$$(\texttt{Stone}) \qquad \forall_{f \in L}\big(f \wedge 1 \in L\big).$$

Used by Stone [24] to prove the integrability of $[f > a]$, $f \in L$

If $1 \in L$, then $\texttt{Stone}(L)$

$\texttt{Stone}\big(C^{\text{supp}}(\mathbb{R}^n)\big)$, as $[f \neq 0] = [(f \wedge 1) \neq 0]$, and $1 \notin C^{\text{supp}}(\mathbb{R}^n)$.

$$f \vee (-1) = -\big[(-f) \wedge 1\big] \in L$$

If $a \neq 0$, then

$$f \wedge a = a\left[\left(\frac{1}{a}f\right) \wedge 1\right] \in L,$$

$$(\texttt{IntInfinity}) \qquad \lim_n \int (f \wedge n) = \int f,$$

$$(\texttt{IntZero}) \qquad \lim_n \int \left(|f| \wedge \frac{1}{n} \right) = 0.$$

(CLASS): $(\texttt{IntDan}) \Rightarrow (\texttt{IntInfinity})$

(BISH): $(\texttt{IntCont}) \Rightarrow (\texttt{IntZero})$

$$
\begin{tikzcd}
(\texttt{IntZero}) & (\texttt{IntCont}) \arrow[l, "\text{BISH}"'] \arrow[r, leftrightarrow, "\text{BISH}"] \arrow[d, "\text{CLASS}"'] & (\texttt{IntCont}') & \\
(\texttt{IntInfinity}) & (\texttt{IntDan}) \arrow[l, "\text{CLASS}"] \arrow[r, leftrightarrow, "\text{BISH}"'] \arrow[ur, "\text{CLASS}"] & (\texttt{IntDan}') \arrow[r, leftrightarrow, "\text{CLASS}"'] & (\texttt{IntBishCheng}) \arrow[ul, "\text{CLASS}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\mathtt{IntZero}) & (\mathtt{IntCont}) \arrow[l, "\mathrm{BISH}"'] \arrow[r, leftrightarrow, "\mathrm{BISH}"] \arrow[d, "\mathrm{CLASS}"'] & (\mathtt{IntCont}') & \\
(\mathtt{IntInfinity}) & (\mathtt{IntDan}) \arrow[l, "\mathrm{BISH}+\exists \lim_n \int f_n"] \arrow[r, leftrightarrow, "\mathrm{BISH}"'] \arrow[ur, "\mathrm{BISH}+\exists \lim_n \int f_n"'] & (\mathtt{IntDan}') \arrow[r, leftrightarrow, "\mathrm{CLASS}"'] & (\mathtt{IntBishCheng}) \arrow[ul, "\mathrm{BISH}+\exists \lim_n \int f_n"']
\end{tikzcd}
$$

Why partiality

A key feature of the Daniell approach is that the transition form functions to sets requires the use of characteristic functions:

$A \subseteq X$ is measurable, if χ_A is measurable.

A is integrable, if $\chi_A \in L^1$

To curry out this constructively one has to use partial functions

$$X \rightharpoonup 2$$

A is not a subset of X, but a complemented subset of X w.r.t. a given inequality $\neq$ on X.

If partial functions $X \rightharpoonup 2$ are in L^1, then L has to be a set of real-valued, partial functions on X.

Let X be a set. An inequality on X, or an apartness relation on X, is a relation $x \neq_X y$ s.t. the following conditions are satisfied:

(Ap_1) $\forall_{x,y \in X}\big(x =_X y \ \& \ x \neq_X y \Rightarrow \bot\big)$.
(Ap_2) $\forall_{x,y \in X}\big(x \neq_X y \Rightarrow y \neq_X x\big)$.
(Ap_3) $\forall_{x,y \in X}\big(x \neq_X y \Rightarrow \forall_{z \in X}(z \neq_X x \ \vee \ z \neq_X y)\big)$.

If $(X, =_x, \neq_X)$ and $(Y, =_Y, \neq_Y)$ a function $f \colon X \to Y$ is strongly extensional, if for every $x, x' \in X$

$$f(x) \neq_Y f(x') \Rightarrow x \neq_X x'.$$

A complemented subset of a set $(X, =_X, \neq_X)$ is a quadruple

$$\boldsymbol{A} := (A^1, i^X_{A^1}, A^0, i^X_{A^0}),$$

or simply $\boldsymbol{A} := (A^1, A^0)$, where $(A^1, i^X_{A^1})$ and $(A^0, i^X_{A^0}) \subseteq X$ s.t.

$$A^1 \rrbracket\llbracket A^0 :\Leftrightarrow \forall_{a^1 \in A^1} \forall_{a^0 \in A^0} \big(i^X_{A^1}(a^1) \neq_X i^X_{A^0}(a^0)\big).$$

$$\boldsymbol{A} \subseteq \boldsymbol{B} :\Leftrightarrow A^1 \subseteq B^1 \ \& \ B^0 \subseteq A^0,$$

Let $\mathcal{P}^{\rrbracket\llbracket}(X)$ be their totality, equipped with the equality

$$\boldsymbol{A} =_{\mathcal{P}^{\rrbracket\llbracket}(X)} \boldsymbol{B} :\Leftrightarrow \boldsymbol{A} \subseteq \boldsymbol{B} \ \& \ \boldsymbol{B} \subseteq \boldsymbol{A}.$$

$\mathcal{P}^{\rrbracket\llbracket}(X)$ is a proper class.

$$x \in \boldsymbol{A} :\Leftrightarrow x \in A^1 \quad \& \quad x \notin \boldsymbol{A} :\Leftrightarrow x \in A^0$$

If $\mathrm{dom}(\boldsymbol{A}) := A^1 \cup A^0$ is the domain of $\boldsymbol{A}$, the indicator function of a $\boldsymbol{A}$, or its characteristic function, is the assignment routine

$$\chi_{\boldsymbol{A}} : \mathrm{dom}(\boldsymbol{A}) \rightsquigarrow 2$$

$$\chi_{\boldsymbol{A}}(x) := \begin{cases} 1 & , x \in A^1 \\ 0 & , x \in A^0 \end{cases}$$

$\chi_{\boldsymbol{A}}$ is a strongly extensional partial function

- Shulman [22]: Bishop's complemented subsets correspond roughly to the Chu construction
- P. [16]: there is a full embedding of $\mathcal{P}^{\mathbb{II}}(X)$ into $\mathbf{Chu}(\mathbf{Set}, X \times X)$
- The Chu construction is a method of generating a $*$-autonomous category from a closed symmetric monoidal category $\mathcal{C}$ and some $\gamma \in C_0$
- $*$-autonomous categories provide models for classical (multiplicative) linear logic
- The abstract lattice version of $\mathcal{P}^{\mathbb{II}}(X)$ is not a Heyting algebra, it is what can be called a Bishop algebra.

Why strong extensionality

If $(X, =_X, \neq_X)$, let the proper class-assignment routines

$$\chi^X : \mathcal{P}^{\mathbb{I}\mathbb{I}}(X) \rightsquigarrow \mathfrak{F}^{\mathrm{se}}(X, 2), \qquad \boldsymbol{A} \mapsto \chi^X(\boldsymbol{A}) =: \chi_{\boldsymbol{A}}$$

$$\chi_{\boldsymbol{A}} := \big(A^1 \cup A^0, i^X_{A^1 \cup A^0}, \chi^2_{A^1 \cup A^0}\big),$$

$$\delta^X : \mathfrak{F}^{\mathrm{se}}(X, 2) \rightsquigarrow \mathcal{P}^{\mathbb{I}\mathbb{I}}(X), \qquad f_A := \big(A, i^X_A, f^2_A\big) \mapsto \boldsymbol{\delta}^X(f_A)$$

$$\boldsymbol{\delta}^X(f_A) := \left(\delta^1_0(f^2_A),\ (i^X_A)_{|\delta^1_0(f^2_A)},\ \delta^0_0(f^2_A),\ (i^X_A)_{|\delta^0_0(f^2_A)}\right),$$

where

$$\delta^1_0(f^2_A) := \big\{a \in A \mid f^2_A(a) =_2 1\big\} =: [f^2_A =_2 1],$$

$$\delta^0_0(f^2_A) := \big\{a \in A \mid f^2_A(a) =_2 0\big\} =: [f^2_A =_2 0],$$

(i) χ^X *is a well-defined, proper class-function.*

(ii) δ^X *is a well-defined, proper class-function.*

(iii) χ^X *and* δ^X *are inverse to each other.*

The integrable functions in the Bishop-Cheng approach

Exactly the Mikusiński-definition

$f \in \mathfrak{F}^{se}(X)$ is integrable, if there is $(f_n)_{n=1}^{\infty} \subseteq L$ s.t.

$$\sum_{n=1}^{\infty} \int |f_n| \in \mathbb{R}$$

and for every $x \in X$

$$\sum_{n=1}^{\infty} |f_n(x)| \in \mathbb{R} \Rightarrow f(x) = \sum_{n=1}^{\infty} f_n(x).$$

$$L^1 := \left\{ f \in \mathfrak{F}^{se}(X) \mid f \text{ integrable} \right\}$$

As far as we know, this fact is not known in the history of the Daniell approach, and it is a rare example of a theory developed first constructively and, independently, classically afterwards.

The separation scheme on a proper class does not define, in general, a set. L^1 is a set, only if $\mathfrak{F}^{se}(X)$ is considered to be a set. This is not predicatively correct, as the membership condition of $\mathfrak{F}^{se}(X)$, or of $\mathfrak{F}(X)$, requires quantification over the universe of sets $\mathbb{V}_0$.

Integration spaces yesterday, today and tomorrow

- Algebraic approach to integration theory (Spitters, Coquand, Palmgren)
- Semeria implemented the basic theory in Coq
- Chan 2021: constructive probability theory and constructive theory of stochastic processes
- Ishihara/Schwichtenberg
- Integration spaces within BST, a predicative approach, avoiding countable choice (Zeuner, Miyamoto, Wessel; work in progress).

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