Computing the Mandelbrot Set, Reliably

Jihoon Hyun, Hyunwoo Lee, Martin Ziegler

KAIST

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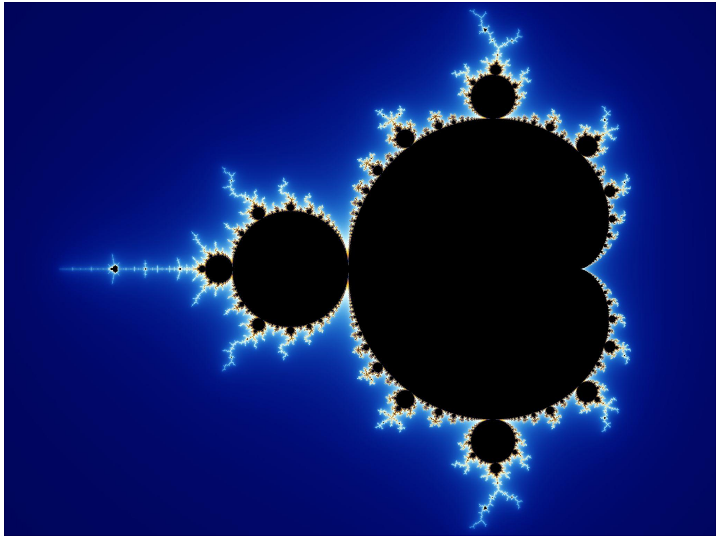
Mandelbrot set

- $M = \{ c \in \mathbb{C} \mid \forall k, | p_c^k(c) | \text{ is bounded} \}$ where $p_c(z) = z^2 + c$

- **Proposition 1**. $c \notin M \leftrightarrow \exists k, | p_c^k(c) | > 2$

- Proposition 1 is not sufficient to draw the Mandelbrot set; it only checks if the point is in the exterior.

- Floating point rounding errors propagate



Source: Wikipedia

References

Peter Hertling:
“Is the Mandelbrot Set Computable?”,
MLQ (2005)

Mark Braverman, Michael Yampolsky:
Computability of Julia Sets,
Springer (2009)

Heinz-Otto Peitgen:
The Beauty of Fractals,
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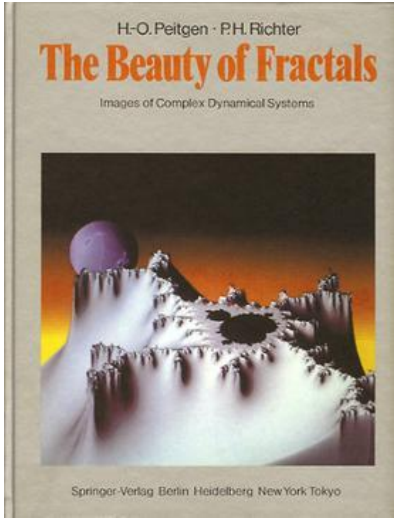


Mandelbrot set

- Definition: Period
  - z_i: fixed points of p^k
  - z_0 is periodic point for p if $\exists\ k > 0$ that $p^k(z_0) = z_0$.
    Smallest such k is called the period of z_0.
  - If z_0 is a periodic point for p with period k,
    then $\{ z_0, z_1, \dots z_{k-1} \}$ where $z_j = p^j(z_0)$ is called a cycle.
  - $(p^k)`(z_0)$ is called the multiplier of the cycle.
    Note that $(p^k)`(z_0) = (p^k)`(z_j)$ for $0 \le j \le k - 1$.
  - Attracting cycle is a cycle whose multiplier is less than 1.

Mandelbrot set

- Definition: Union of Hyperbolic Components of Mandelbrot Set
  - $H(M) = \{ c \in \mathbb{C} \mid p_c$ has an attracting cycle $\}$
- Facts about $H(M)$
  - $H(M)$ is an open subset of the Mandelbrot set.
  - ∂M is contained in the closure of $H(M)$.
- Hyperbolicity Conjecture
  - $H(M) = M^0$

Mandelbrot set

Peter Hertling gave two propositions:

- **Proposition 2**. M^c is semidecidable.

Proof. $c \notin M \leftrightarrow \exists k, |\, p_c^{\,k}(c) \,| > 2$

- **Proposition 3**. $H(M)$ is semidecidable.

Proof. For each k, check if $p_c^{\,k}$ has an attracting cycle.

But...

Drawing a set in $\mathbb{R}^2$

- Goal: Drawing a set in $\mathbb{R}^2$
  with equally spaced $r \times r$ grid on $[0, 1] \times [0, 1]$
  by coloring each pixel with some color

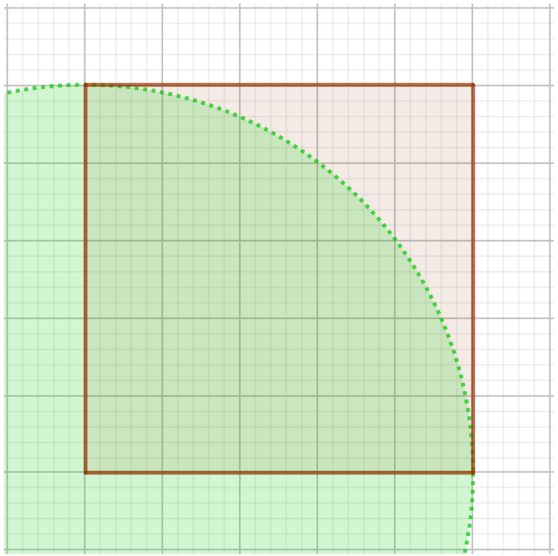
- 'Covering the set' is the most common method.

Drawing a set in $\mathbb{R}^2$

- Three types of colors are used to draw a set.
  - Pixel meets a set if the color of the pixel is positive.
  - Pixel does not meet a set if the color of the pixel is negative.
  - Pixel contains a point which is close enough ($< 1 / r$) to the boundary if the color of the pixel is zero.

Example of drawing of a set in $\mathbb{R}^2$

- $x^2 + y^2 < 1$ on 5×5 grid



0	0	0	-	-
+	+	+	0	-
+	+	+	+	0
+	+	+	+	0
+	+	+	+	0

0	-	0	-	0
0	0	0	0	-
+	+	+	0	0
+	+	+	0	-
+	+	+	0	0

Why three colors?

- $H(M)$ is an open subset of the Mandelbrot set.
- ∂M is contained in the closure of $H(M)$.

- $\partial M = \mathrm{cl}(H(M)) \cap M^c$

- Simpler approach to draw the set

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Drawing the Mandelbrot set

Theorem. If the Hyperbolicity Conjecture is true, then the algorithm halts, and returns a drawing of the Mandelbrot set.

Pixel meets a set if the color of the pixel is positive.
Pixel does not meet a set if the color of the pixel is negative.
Pixel contains a point which is close enough ($< 1 / r$) to the boundary if the color of the pixel is zero.

Drawing the Mandelbrot set

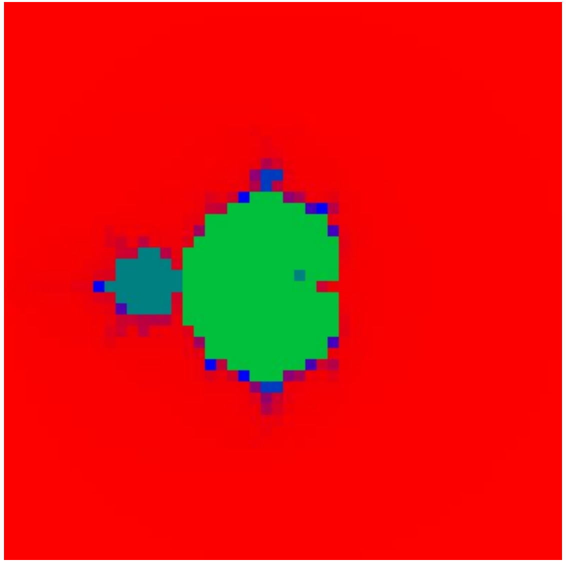
Algorithm (Brief):

Initialize.

- Choose points in pixels.
- Color negative/positive.
- Make a helper grid.
- Color points of the helper grid negative/positive.
- Color zero.
- Iterate until all pixels are colored.

Drawing the Mandelbrot set

Theorem. If the Hyperbolicity Conjecture is true, then the algorithm halts, and returns a drawing of the Mandelbrot set.



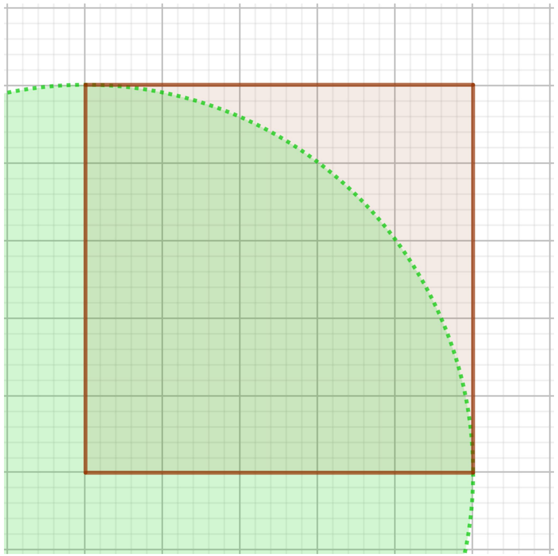
50 × 50 drawing of the Mandelbrot Set

Drawing the Mandelbrot set

Theorem. If the Hyperbolicity Conjecture is true, then the algorithm halts, and returns a drawing of the Mandelbrot set.

0. Initialize.

Let the color of every pixels as unknown.



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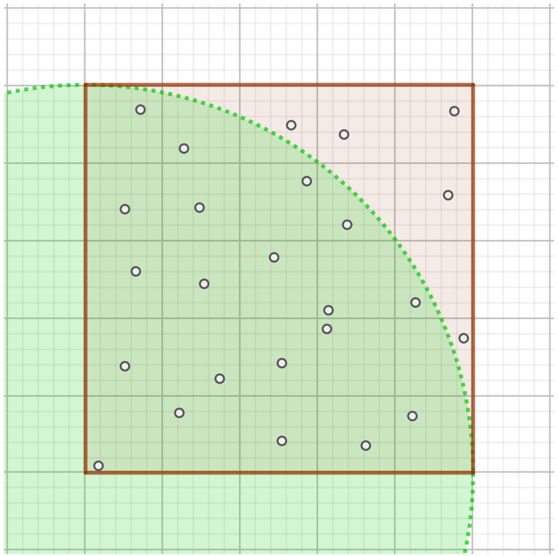
Drawing the Mandelbrot set

Theorem. If the Hyperbolicity Conjecture is true, then the algorithm halts, and returns a drawing of the Mandelbrot set.

1. Choose points.

For each pixel with unknown color, choose a point in a pixel. The closure of union of chosen points in each iteration should be the corresponding pixel.



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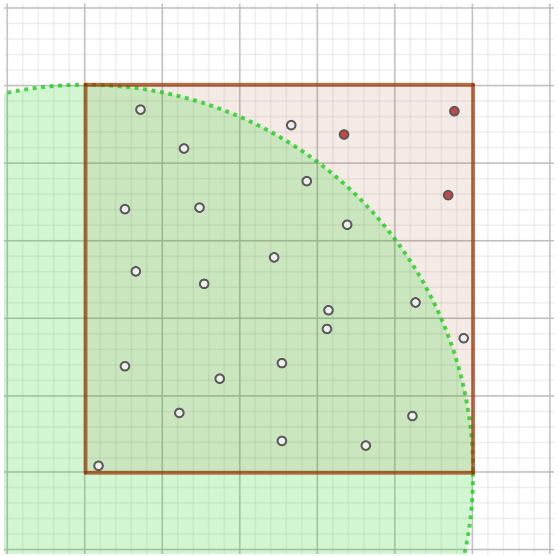
Drawing the Mandelbrot set

Theorem. If the Hyperbolicity Conjecture is true, then the algorithm halts, and returns a drawing of the Mandelbrot set.

2. Color negative.

If the selected point is known to be in the exterior, color the pixel negative.



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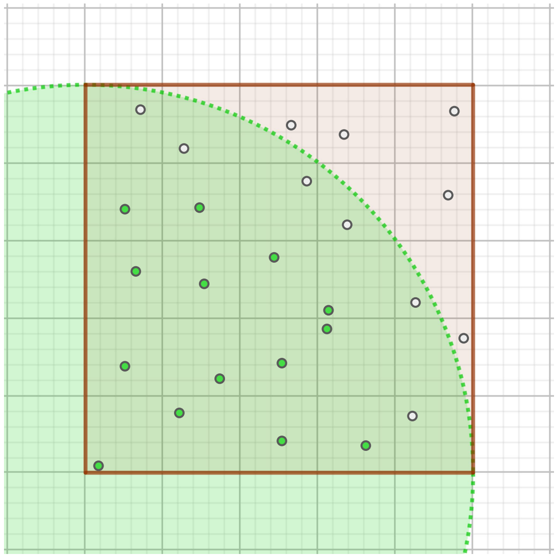
Drawing the Mandelbrot set

Theorem. If the Hyperbolicity Conjecture is true, then the algorithm halts, and returns a drawing of the Mandelbrot set.

3. Color positive.

If the selected point is known to be in the interior, color the pixel positive.



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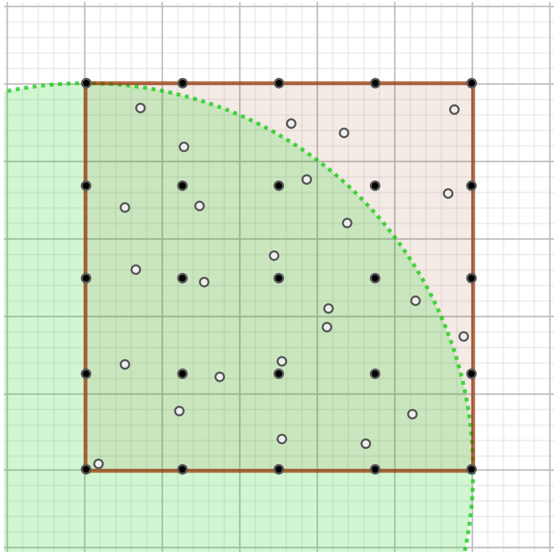
+	+	?	?	-
+	+	+	+	?
+	+	+	+	?
+	+	+	+	?

Drawing the Mandelbrot set

Theorem. If the Hyperbolicity Conjecture is true, then the algorithm halts, and returns a drawing of the Mandelbrot set.

4. Make a helper grid.

Choose a set of equally distanced points.
The points should get denser as the number of iteration increases.



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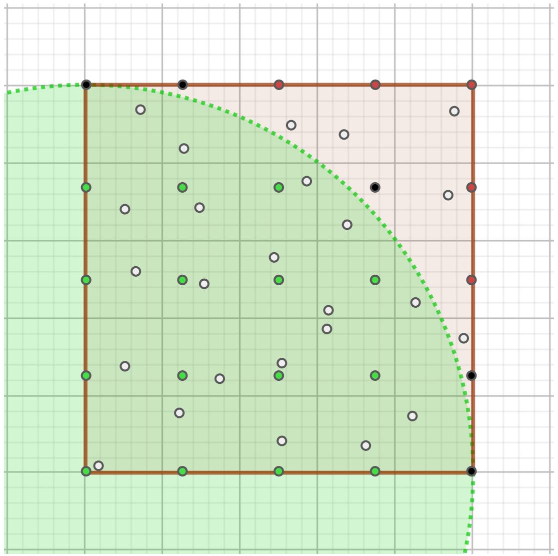
+	+	+	+	?
+	+	+	+	?
+	+	+	+	?

Drawing the Mandelbrot set

Theorem. If the Hyperbolicity Conjecture is true, then the algorithm halts, and returns a drawing of the Mandelbrot set.

5. Color the helper grid.

Color the points from 4 as done in 2 ~ 3.



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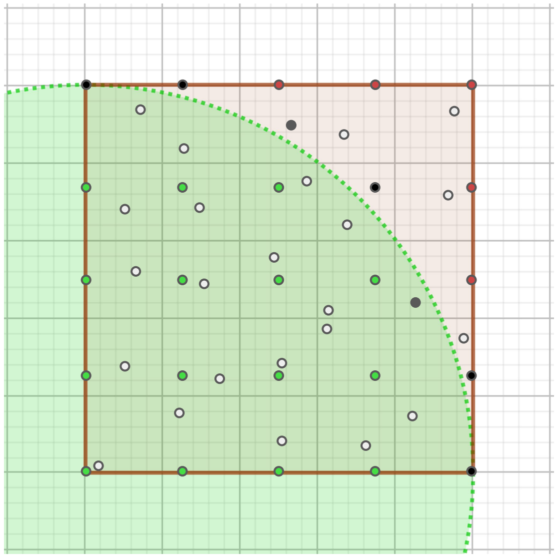
+	+	+	+	?
+	+	+	+	?

Drawing the Mandelbrot set

Theorem. If the Hyperbolicity Conjecture is true, then the algorithm halts, and returns a drawing of the Mandelbrot set.

6. Color zero.

Color the (white) point W zero if there are at least one red point R and one green point G which $d(W, R) < 1 / r$ and $d(W, G) < 1 / r$.



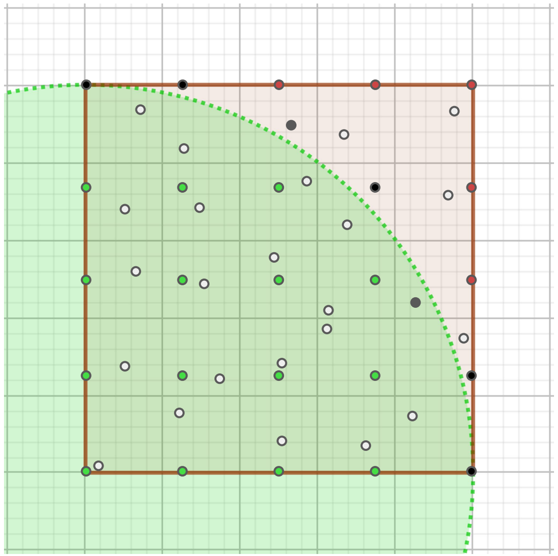
?	?	0	-	-
+	+	?	?	-
+	+	+	+	0
+	+	+	+	?
+	+	+	+	?

Drawing the Mandelbrot set

Theorem. If the Hyperbolicity Conjecture is true, then the algorithm halts, and returns a representation of the Mandelbrot set.

7. Iterate.

Do 1 ~ 6 until all pixels are colored.



?	?	0	-	-
+	+	?	?	-
+	+	+	+	0
+	+	+	+	?
+	+	+	+	?

Drawing the Mandelbrot set

2-1. In iteration n, check if a point z is in the exterior.

$k := 1$

while $k \leq n$ do:
    if $| p_z^k(z) | > 2$
    then return *true*

return *false*

Drawing the Mandelbrot set

3-1. In iteration n, check if a point z is in the interior.

Assuming that the Hyperbolicity Conjecture is true,
we can check if $z \in M^0$.

Drawing the Mandelbrot set

3-1. In iteration n, check if a point z is in the interior.

$p(x) := x$, $k := 1$

while $k \leq n$, do:

  $p := p \circ p_z$

  roots := set of fixed points of $p(x)$

  for r in *roots*:

    if $p`(r) < 1$ then return *true*

return *false*

Sketch of proof of termination

- Assume that the point is on the boundary.
  - If not, the point will be classified positive or negative when iteration count is large enough.

- $\partial M = \mathrm{cl}(H(M)) \cap M^c$

- Any circle centered at $x \in \partial M$ contains point of both $\mathrm{cl}(H(M))$ and M^c.

Drawing the Mandelbrot set

For full implementation, please check:
https://github.com/realcomputation/MANDELBROT

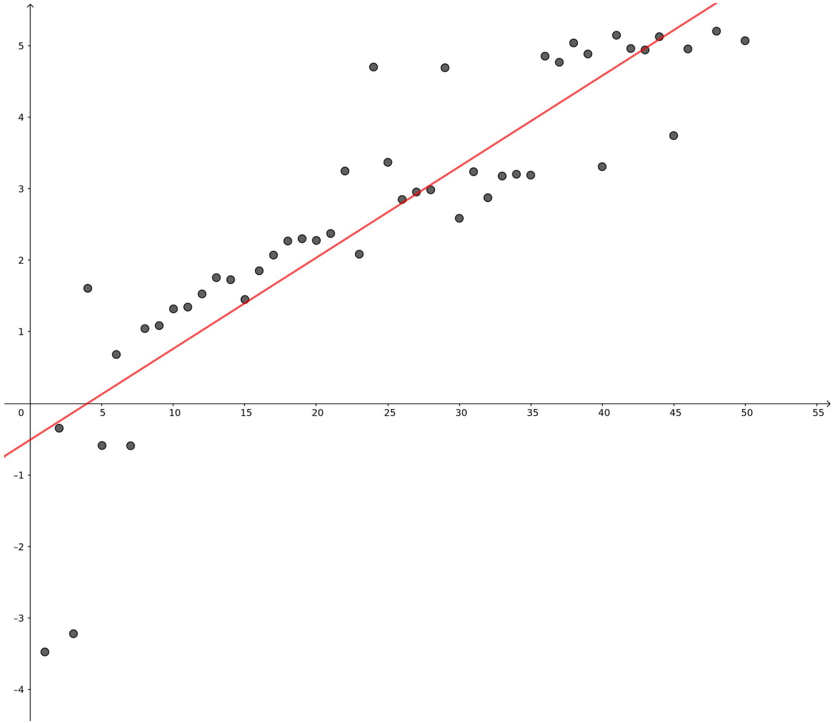
using Norbert Müller’s C++ **iRRAM** library
for Exact Real Computation

Drawing the Mandelbrot set: Evaluation time

Algorithm (Brief):

Initialize.

Choose points in pixels.
Color negative/**positive**.
Make a helper grid.
Color points of the helper grid.
Color zero.
Iterate until all pixels are colored.



resolution vs log time

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Thank you!