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Embeddability of graphs and Weihrauch degrees

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joint work with Arno Pauly (Swansea University)

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Introduction

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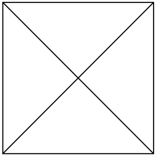
In this talk we study graph theoretic problems using the framework of Weihrauch reducibility.
By graph here we always mean *undirected* ones, without multiple edges and self-loops. In particular, we will consider the complexity of *subgraph* and *induced subgraph* relations.

(Induced) subgraphs

Definition

An undirected graph (V_0, E_0) is a subgraph of (V, E) if and only if $V_0 \subseteq V$ and $E_0 \subseteq E$. We say it is an induced subgraph if furthermore $E_0 = E \cap (V_0 \times V_0)$.



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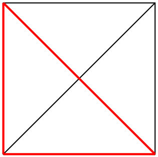

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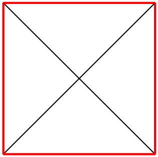

This is a subgraph, but not an induced one.

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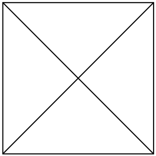

Notation

A vertex v_i belong to G is denoted by $v_i \in V^G$.
Two vertices v_i, v_j are connected in G is denoted by $(v_i, v_j) \in E^G$.

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A represented space is a pair (X, δ_X) where $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \twoheadrightarrow X$.

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Now we can think of a computational problem as a (possibly partial) *multivalued functions* $f :\subseteq X \rightrightarrows Y$, where X, Y are represented spaces.
In the space **Gr** of countable undirected graphs we have that p is a name for a graph G iff

- $p(\langle i, i \rangle) = 1$ iff $v_i \in V^G$ and
- for $i \neq j$, $p(\langle i, j \rangle) = 1$ iff $(v_i, v_j) \in E^G$.

Let f, g be (partial multivalued) functions on represented spaces.

Weihrauch Reducibility

Let f, g be (partial multivalued) functions on represented spaces.
f is Weihrauch reducible to g ($f \leq_W g$) if there are computable $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

- Given a name p for $x \in dom(f)$, $\Phi(p)$ is a name for $z \in dom(g)$;
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If the Ψ has no access to p, the reduction is called *strong* ($\leq_{\mathsf{sW}}$).

BeMent, Hirst and Wallace [BHW21] introduced the (induced) subgraph isomorphism problem as follows:

- S: Given inputs of graphs G and H, output 1 if G is isomorphic to an induced subgraph of H and 0 if it is not;
- S_G: Given a graph H as input, output 1 if G is isomorphic to an induced subgraph of H and 0 otherwise.

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- SE_G: Given a graph H as input, output 1 if G is isomorphic to a subgraph of H and 0 otherwise.

The problems in the previous slides are directly related to the subsystem of second-order arithmetic Π^1_1-CA_0. Indeed, the problem below, also known in the literature as $\chi_{\Pi^1_1}$, directly follows from Lemma VI.I.I. of Simpson [Sim09].

- WF: Given a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ as input, output 1 if T is well-founded and 0 otherwise.

Known results

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- WF: Given a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ as input, output 1 if T is well-founded and 0 otherwise.

Let L be the infinite line graph with vertex set $V = \{v_i : i \in \mathbb{N}\}$ and edges $E = \{(v_i, v_i + 1) : i \in \mathbb{N}\}$.

Theorem ([BHW21])

$\mathsf{WF} \equiv_{\mathsf{sW}} \mathsf{S}_L \equiv_{\mathsf{sW}} \mathsf{S}$.

Proof Sketch:
($\mathsf{S}_L \leq_{\mathsf{sW}} \mathsf{S}$): Trivial.

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($\mathsf{WF} \leq_{\mathsf{sW}} \mathsf{S}_L$): Just consider the input $T \subseteq \mathbb{N}^{<\mathbb{N}}$ as a graph G. Then T is ill-founded iff L is isomorphic to an induced subgraph of G.
($\mathsf{S} \leq_{\mathsf{sW}} \mathsf{WF}$): From G compute T, i.e. the tree of initial segments of isomorphisms between G and H. Then there exists an isomorphism between H and a subgraph of G iff T is ill-founded.

When G is finite

[BeHiWa] also considered the case in which G is a finite graph. The problem S_G in this case is equivalent to a well-studied principle, namely LPO.

- LPO: Given in input a sequence $p \in 2^{\mathbb{N}}$, output 1 iff $p = 0^\omega$.

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Anyway, there is a huge gap between LPO and WF (i.e. Σ^0_1 vs Π^1_1).
The authors left open the following question: What about the problem S_G for other graphs G? That is,

$$\text{is there a graph } G \text{ s.t. } \mathsf{LPO} <_{\mathsf{sW}} \mathsf{S}_G <_{\mathsf{sW}} \mathrm{WF}?$$

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Lemma (Pauly)

Let G be the graph composed by infinitely many triangles (i.e.cycles of length 3). Then $\mathrm{SE}_G \equiv_{\mathsf{W}} \mathsf{LPO}'$.

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Proof Sketch.
($\mathrm{SE}_G \leq_{\mathrm{W}} \mathsf{LPO}'$): LPO' counts how many disjoint triangles are in G.
($\mathsf{LPO}' \leq_{\mathrm{W}} \mathrm{SE}_G$): given a set build a graph having a triangle for each member of the set.

First lemma: Having L as a subgraph

Lemma

For any graph G such that L is a subgraph of G, we have that $\mathsf{WF} \equiv_{\mathsf{sW}} \mathsf{S}_G \equiv_{\mathsf{sW}} \mathsf{SE}_G$

Proof sketch.
$(\mathsf{S}_G, \mathsf{SE}_G \leq_{\mathsf{sW}} \mathsf{WF})$: Follows from [BeHiWa].

Lemma

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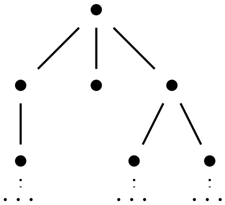
Proof sketch.
($\mathrm{S}_G, \mathrm{SE}_G \leq_{\mathsf{sW}} \mathrm{WF}$): Follows from [BeHiWa].
For the opposite direction, let $\{v_0, v_1, \dots\}$ be an enumeration of the vertices in G and let T be the tree in input for WF. Then we (computably) build the input H for SE_G, as follows.

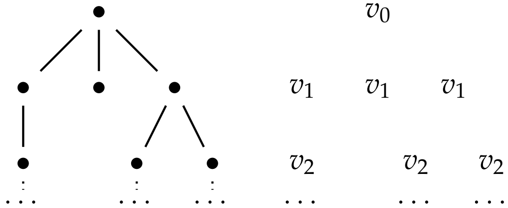
Notation

u is an initial segment of w in a tree T is denoted by $u \sqsubset_T w$.
The incomparability between u and w is denote by $u \perp_T w$.

Input $T \subseteq \mathbb{N}^{<\mathbb{N}}$

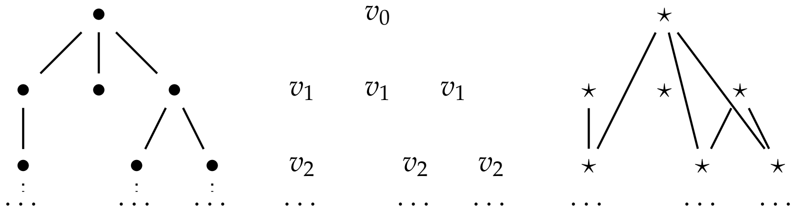


Input $T \subseteq \mathbb{N}^{<\mathbb{N}}$ Graph H, step 1.



We add copies of each v_k into H, indexed by $\mathbb{N}^k \cap T$.

Input $T \subseteq \mathbb{N}^{<\mathbb{N}}$ | Graph H, step 1. | Graph H step 2



We add copies of each v_k into H, indexed by $\mathbb{N}^k \cap T$. Then, two vertices of index u and w are connected in H iff

- $(u \sqsubset_T w \vee w \sqsubset_T u)$ and
- the vertices of index $|u|$ and $|w|$ are connected in G.

$(\mathsf{WF} \le_{\mathrm{W}} \mathsf{S}_G, \mathsf{SE}_G)$

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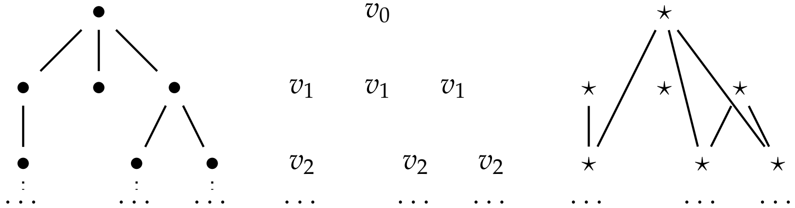

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Here for example $(v_1, v_2), (v_0, v_2) \in E^G$ and $(v_0, v_2) \notin E^G$.

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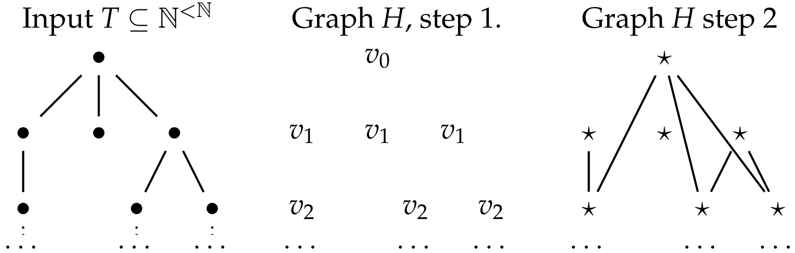

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Here for example $(v_1, v_2), (v_0, v_2) \in E^G$ and $(v_0, v_2) \notin E^G$.
If T is ill-founded, H has an induced subgraph isomorphic to G (just select the vertices of the infinite path). Otherwise, H does not even have L as subgraph. $\square$

Recall that every infinite partial order contains either an infinite chain or an infinite antichain (CAC). From it we can easily derive the following.

Corollary

Let T be a well-founded tree and S an infinite subset of T. Then S contains an infinite antichain.

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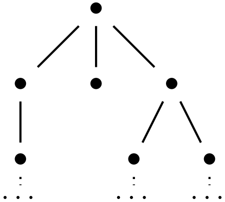
Let K_ω be the complete graph on ω vertices (where all vertices are connected each other).

Theorem (C., Pauly)

Let G be an infinite graph not having a subgraph isomorphic to K_ω. Then $\mathsf{S}_G \equiv_{\mathrm{W}} \mathsf{WF}$.

Again $\mathsf{S}_G \leq_{\mathrm{W}} \mathsf{WF}$ from [BHW21].

Input $T \subseteq \mathbb{N}^{<\mathbb{N}}$



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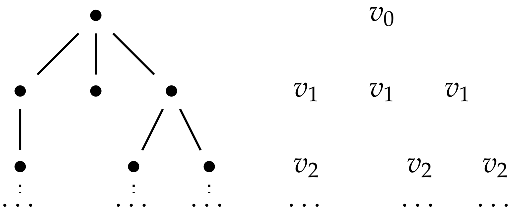

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WF $\leq_{sW}$ S_G

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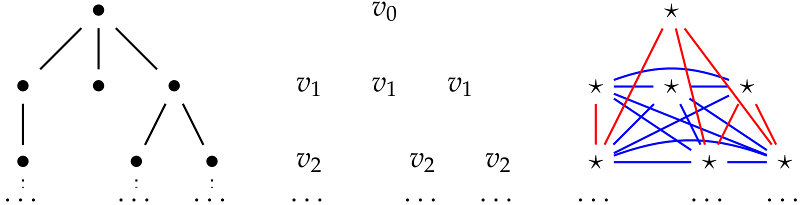

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- $u \perp_T w$ or
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For example here $(v_1, v_2), (v_0, v_2) \in E^G$ and $(v_0, v_2) \notin E^G$.

If T is ill-founded, H has an induced subgraph isomorphic to G (just select the vertices belonging to an infinite path).

Otherwise, if T is well-founded, by CAC for trees, vertices of any infinite subset of H contain an antichain in T and, by construction, in H they are isomorphic to a copy K_ω. Hence, G is not a subgraph of H. This concludes the (sketch of) the proof.

Recall the initial question

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As L is a subgraph of K_ω, applying the previous two lemmas, we get the following:

Theorem (C., Pauly)

Let G be a graph with at least two vertices. Then

- *either* $\mathsf{S}_G \equiv_{\mathsf{W}} \mathsf{LPO}$*;*
- *or* $\mathsf{S}_G \equiv_{\mathsf{W}} \mathsf{WF}$*;*

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Theorem (C., Pauly)

Let G be a graph with at least two vertices. Then

- *either* $\mathsf{S}_G \equiv_{\mathsf{W}} \mathsf{LPO}$; *if G is finite;*
- *or* $\mathsf{S}_G \equiv_{\mathsf{W}} \mathsf{WF}$; *if G is infinite;*

Thanks for your attention!

Bibliography

[BHW21] Zack BeMent, Jeffry Hirst, and Asuka Wallace. *Reverse mathematics and Weihrauch analysis motivated by finite complexity theory*. 2021. arXiv: 2105.01719 [math.LO].

[Sim09] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. 2nd ed. Perspectives in Logic. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511581007.