Gaussian Information Bottleneck and the Non-Perturbative Renormalization Group

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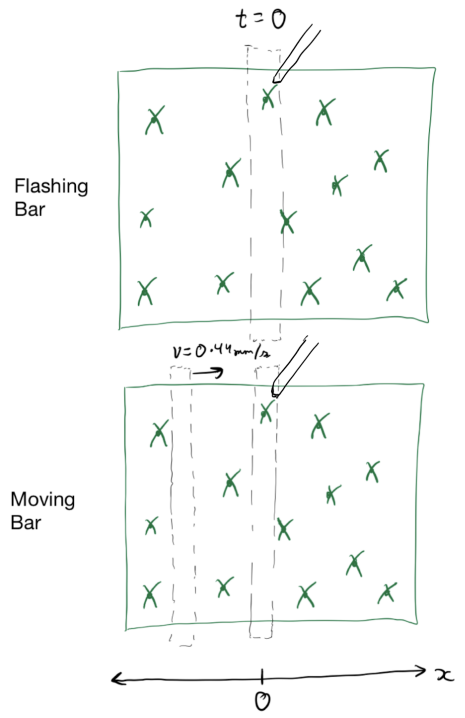
Desiderata

1. Information Bottleneck
   - Retinal Computation
   - Theoretical Framework
   - Solution
   - Gaussian Information Bottleneck
2. Renormalization Group
   - Simplest Example
3. Connection

Desiderata

Motion is predicted by retinal ganglion cells

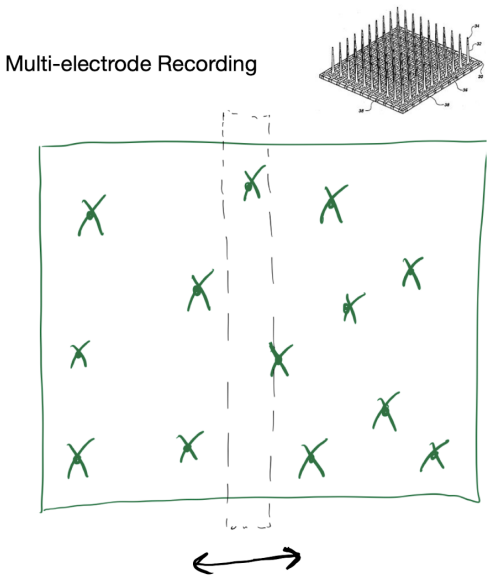
- A flashing bar and moving bar was presented to salamander retina and response of a ganglion cell was recorded.
- Retinal ganglion cells are able to predict motion of bar upto 1mm/s speed

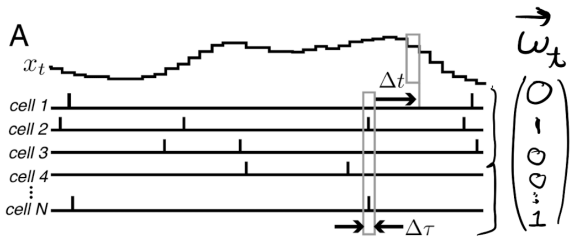


Credit: *Berry and Meister 1999.*

Recording from population of ganglion cells

- The bar performs a stochastic motion. We record from many($N \sim 50$) ganglion cells and create a binary vector $\vec{w}(t)$ representing the spike activity of neurons
- Question: How much information about bar's position is encoded in ganglion cells?





$$\frac{d^2x(t)}{dt^2} = -\omega^2 x(t) + \gamma v(t) + \eta(t)$$

Credit: *Palmer et.al.*, 2015

How much information about bar's position is encoded in ganglion cells?

Answer: We need to find mutual information between position at time t' and spike activity vector at time t.

$$I(w_t, x_{t'}) = \sum_{w_t, x_{t'}} P(w_t, x_{t'}) \log \left(\frac{P(w_t, x_{t'})}{P(w_t) P(x_{t'})} \right)$$

$$= \sum_{w_t, x_{t'}} P(w_t) P(x_{t'}|w_t) \log \left(\frac{P(x_{t'}|w_t)}{P(x_{t'})} \right)$$

To estimate $I(w_t, x_{t'})$ we do the following:

1. $P(x_{t'})$ is univariate distribution and can calculated analytically by solving Fokker-Planck equation or estimated by frequency analysis
2. $P(w_t)$ is a N dimensional distribution. To prevent undersampling we use $N \leq 7$ and then again use frequency analysis to estimate it
3. $P(x_{t'}|w_t)$ is univariate distribution and is estimated by frequency analysis

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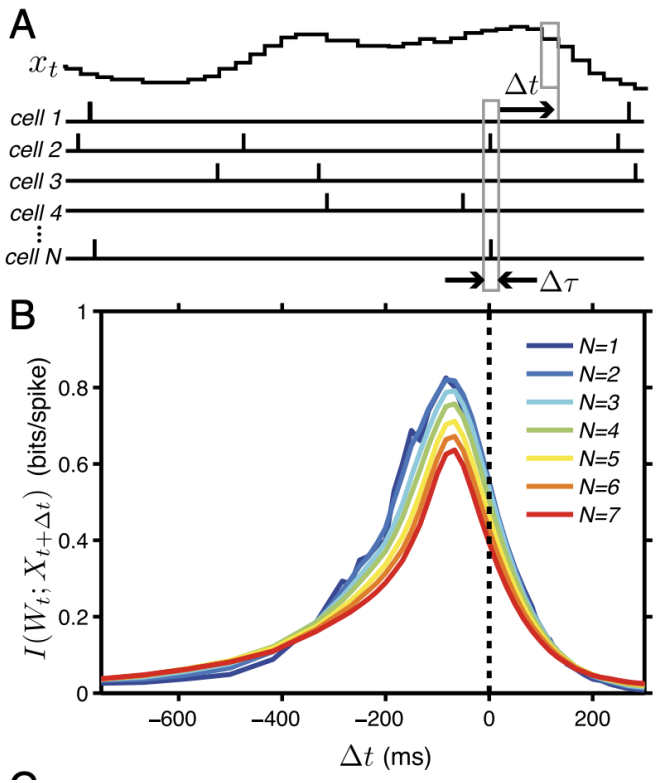
Mutual information result

1. Retina is most informative about the position of the object at $\Delta t = -80$ms because of latency in response.
2. Neural responses carry information about the position that extends far into the past and into the future.
3. Notice bits/spike goes down slightly with increasing N. This indicates redundant coding.



Credit: *Palmer et.al.*, 2015

Let's focus on future prediction: Information Bottleneck

In Information Bottleneck framework, we assume brain performs a trade-off between maximally predicting the future while minimally representing the the past.

$$X_{\text{past}} \xrightarrow{p(z|x_{past})} Z \xrightarrow{p(x_{future}|z)} X_{\text{future}}$$

The following objective function is minimized,

$$\min_{p(z|x_{past})} \mathcal{L} = I(X_{\text{past}}, Z) - \beta I(Z, X_{\text{future}})$$

The parameter β sets the trade-off between compression (reducing the information that we keep about the past, $I(X_{\text{past}}, Z)$ and prediction [increasing the information that we keep about the future, $I(Z, X_{\text{future}})$]

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Solving the Information Bottleneck problem

Objective function is:

$$\min_{p(z|x_{past})} \mathcal{L} = I(X_{\text{past}}, Z) - \beta I(Z, X_{\text{future}})$$

Since, $p(z|x_{past})$ must be normalized, we instead consider the add a Lagrange multiplier to thee objective function.

$$\min_{p(z|x_{past})} \mathcal{L} = I(X_{\text{past}}, Z) - \beta I(Z, X_{\text{future}}) - \sum_{x_{past}, z} \lambda(x_{past})(p(z|x_{past}) - 1)$$

We perform, $\frac{\delta \mathcal{L}}{\delta p(z|x_{past})} = 0$ to get,

$$p(z|x_{past}) = \frac{1}{\mathcal{Z}[\beta, \lambda(x_{past})]} \exp\left[-\beta D_{KL}(p(x_{future}|x_{past})||p(x_{future}|z))\right]$$

Note: We don't know $p(x_{future}|x_{past})$ or $p(x_{future}|z)$

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Solving the Information Bottleneck problem: Blahut-Arimoto algorithm

We have the following set of equations:

$$p(z|x_{past}) = \frac{p(z)}{\mathcal{Z}[\beta, \lambda(x_{past})]} \exp\left[-\beta D_{KL}(p(x_{future}|x_{past})||p(x_{future}|z))\right] \quad (1)$$

$$p(z) = \sum_{x_{past}} p(z|x_{past})p(x_{past}) \quad (2)$$

$$p(z|x_{future}) = \sum_{x_{past}} p(z|x_{past})p(x_{past}|x_{future}) \quad (3)$$

These can be solved iteratively using Blahut-Arimoto algorithm. In gaussian information bottleneck framework(coming soon!) these can be solved analytically.

Retinal population saturate the predictive bound

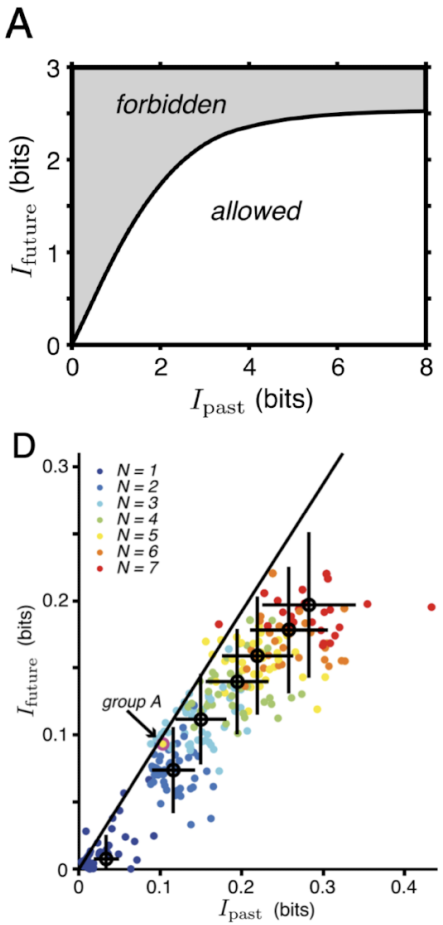
Recall, IB objective function is:

$$\mathcal{L} = I_{past} - \beta I_{future}$$

where,

$$I_{past} \triangleq I(X_{\text{past}}, Z)$$
$$I_{future} \triangleq I(Z, X_{\text{future}})$$

1. The ganglion cells maximally encode information about the future



Credit: *Palmer et.al.*, 2015

Gaussian Information Bottleneck(GIB)

$$X_{\text{past}} \xrightarrow{p(z|x_{past})} Z \xrightarrow{p(x_{future}|z)} X_{\text{future}}$$

From now on, in accordance with the paper I will use $X = X_{\text{past}}$,$\tilde{X} = Z$ and $Y = X_{\text{future}}$ In GIB framework, we assume that p(x,y) is jointly Gaussian. I will assume mean=0 while the covariance matrix looks like:

$$\Sigma = \begin{pmatrix} \Sigma_X & \Sigma_{XY} \\ \Sigma_{XY} & \Sigma_Y \end{pmatrix}$$

We must have,

$$\tilde{X} = AX + \xi$$

where ξ is a gaussian white noise($\sim \mathcal{N}(0, \Sigma_\xi)$)

GIB solution

We have,

$$X \xrightarrow{p(\tilde{x}|x)} \tilde{X} \xrightarrow{p(y|\tilde{x})} Y$$

The objective function is,

$$\mathcal{L} = I(X, \tilde{X}) - \beta I(\tilde{X}, Y)$$

The compression transformation is,

$$\tilde{X} = AX + \xi$$

For any β, the exact solution turns out to be,

$$\Sigma_\xi = \mathrm{I}$$

$$A(\beta) = \mathrm{diag}(\alpha_i(\beta))V^T$$

The matrix V represents the set of eigenvectors of $\Sigma_X^{-1}\Sigma_{X|Y}$, and $\alpha_i(\beta)$ is a complicated function of beta, eigenvalue of $\Sigma_X^{-1}\Sigma_{X|Y}$, etc.

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Reparameterization of GIB

The solution to GIB is not unique. Let's say for some β we have IB optimal solutions (A, Σ_ξ). Then,

$$X \xrightarrow[\beta]{(A,\Sigma_\xi)} \tilde{X} \to Y$$

Let's imagine having two latent variables instead.

$$X \xrightarrow[\beta_1]{(A_1,\Sigma_{\xi_1})} \tilde{X}_1 \xrightarrow[\beta_2]{(A_2,\Sigma_{\xi_2})} \tilde{X}_2 \to Y$$

It turns out,

$$\beta = \frac{\beta_2 \beta_1}{\beta_2 + \beta_1 - 1}$$

$$A = A_2 A_1$$

$$\Sigma_\xi = A_2 A_2^T + \mathrm{I}$$

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Semi-group structure of GIB

We have the following composition law:

$$\beta = \beta_2 \circ \beta_1 = \frac{\beta_2 \beta_1}{\beta_2 + \beta_1 - 1}$$

Direct computations show that the composition operator satisfies closure and associativity, and thus furnishes the space in which β values live, that is $\mathbb{R} > 1$.

$\beta = \infty$ is the identity element.

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Desiderata

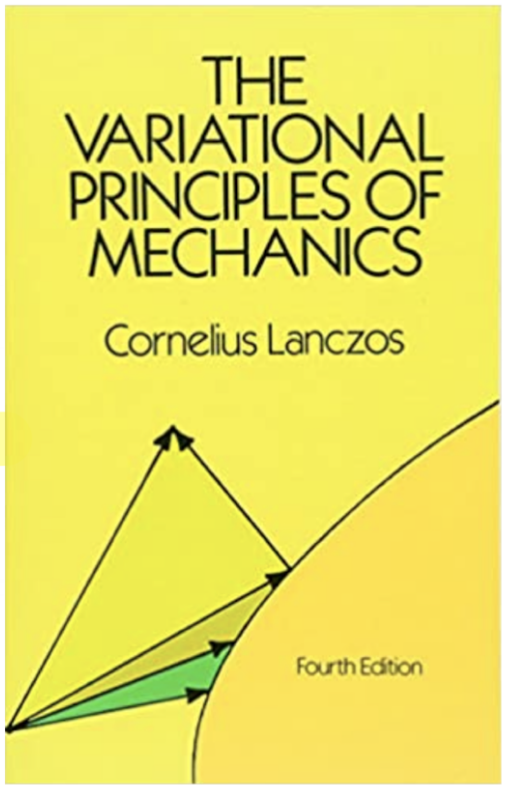
Renormalization Group: Introduction

CHAPTER VIII

THE PARTIAL DIFFERENTIAL EQUATION OF HAMILTON-JACOBI

Put off thy shoes from off thy feet, for the place whereon thou standest is holy ground. EXODUS III, 5

Introduction. We have done considerable mountain climbing. Now we are in the rarefied atmosphere of theories of excessive beauty and we are nearing a high plateau on which geometry, optics, mechanics, and wave mechanics meet on common ground. Only concentrated thinking, and a considerable amount of re-creation, will reveal the full beauty of our subject in which the last word has not yet been spoken. We start with the integration theory of Jacobi and continue with Hamilton's own investigations in the realm of geometrical optics and mechanics. The combination of these two approaches leads to de Broglie's and Schroedinger's great discoveries, and we come to the end of our journey.



Some mysteries

1. Why fluids containing a gazillion molecules are describable by only a handful of parameters like density, viscosity and temperature?
2. Why all unimodal chaotic maps show the same geometric pattern in their period doubling?

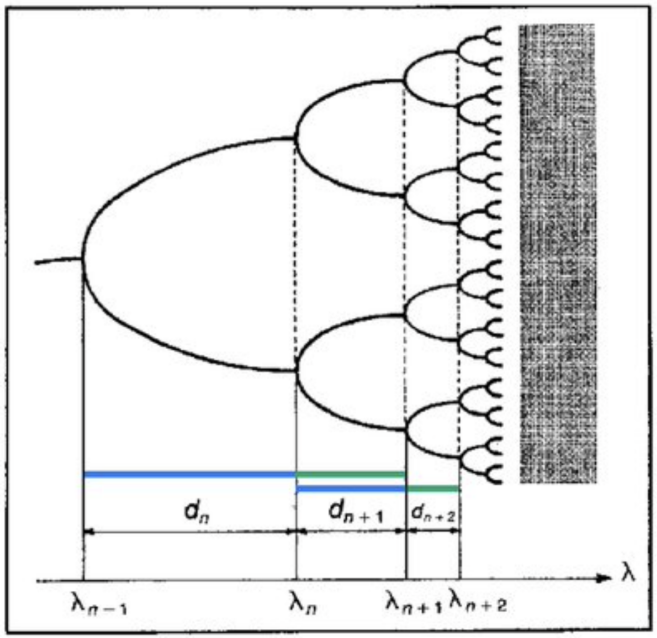


Image credit: Sights and sounds of chaos

$$\delta = \lim_{n\to\infty} \frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+2} - \lambda_{n+1}}$$

If you repeatedly take the distance between consecutive bifurcations and divide them, you arrive at Feigenbaum's number.

Fixed ratio approached on dividing

by

Approximately equals:

4.6692

Simplest Model

Let's say we have the following partition function,

$$Z(a,b) = \int dx \int dy e^{-a(x^2+y^2)} e^{-b(x+y)^4} \triangleq \int dx \int dy e^{-S(a,b;x,y)}$$

Assume we are only interested x and never in y. Then can integrate y to get,

$$\int dy e^{-a(x^2+y^2)} e^{-b(x+y)^4} = e^{a'x^2 - b'x^4 + \dots}$$

where a',b',. . . are defined by above equation. We now write the partition function as,

$$Z(a', b', \dots) = \int dx e^{-S'(a', b', \dots; x)}$$

The primed coefficients a',b',. . . are the parameters of *effective theory* for x.
$S'(a', b', \dots; x)$ defines the *effective action*. Notice the effective action modifies the *coupling of x to itself*
The evolution of parameters (a,b, c, . . .) $\rightarrow$ (a',b', c',. . .) upon the elimination of uninteresting degrees of freedom is what we mean these days by renormalization

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Simplest Model

$$Z(a,b) = \int dx \int dy e^{-S(a,b,\ldots;x,y)} = \int dx e^{-S'(a',b',\ldots;x)} = Z(a',b'\ldots)$$

x could stand for future variables and y could stand for past variables

Upon integrating out the past variables, we obtain an effective Hamiltonian for the future variable. A coupling that used to be numerically small in the full theory can grow in size and dominate (or an initially impressive one diminish into oblivion). By focusing our attention on the dominant couplings we can often get a simplified model of the problem

Desiderata