Derivation of the Navier-Stokes equation

Euler's equation

The fluid velocity $\mathbf{u}$ of an inviscid (ideal) fluid of density ρ under the action of a body force $\rho\mathbf{f}$ is determined by the equation:

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{f}, \tag{1}$$

known as Euler's equation. The scalar p is the pressure. This equation is supplemented by an equation describing the conservation of mass. For an incompressible fluid this is simply

$$\nabla\cdot\mathbf{u} = 0. \tag{2}$$

Real fluids however are never truly inviscid. We must therefore see how equation (1) is changed by the inclusion of viscous forces.

The stress tensor

We first introduce the *stress tensor* σ_{ij} as follows:

σ_{ij} is the i-component of stress on a surface element δS that has a normal $\mathbf{n}$ *pointing in the j-direction.*

Then if $\mathbf{t}$ is the stress on a small surface element δS with unit normal $\mathbf{n}$ it is straightforward to demonstrate that

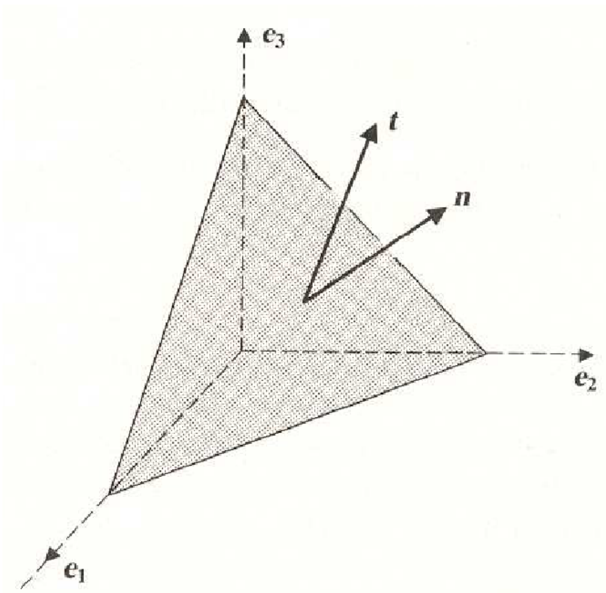
$$t_i = \sigma_{ij}n_j. \tag{3}$$

<u>Proof of (3)</u>

Take δS to be the large face of the tetrahedron shown in the figure on the following page. We shall apply Newton's 2nd law to the fluid in the tetrahedron. For definiteness consider the i-component of the force. The force of surrounding fluid on the large face is $t_i\delta S$. The i-component of the force on the 'back' face is $-\sigma_{i1}$ (since the normal is in the $-\mathbf{e}_1$ direction). Similarly for the two faces with normals $-\mathbf{e}_2$ and $-\mathbf{e}_3$. Thus the i-component of the total force exerted by surrounding fluid on the tetrahedron is

$$(t_i - \sigma_{ij} n_j)\, \delta S \tag{4}$$

(summation over j).



This force (plus any body force) will be equal, from Newton's 2nd law, to the mass of the element ($\rho\delta V$) multiplied by its acceleration. If ℓ is the linear length scale of the tetrahedron then $\delta V \sim \ell^3$ and $\delta S \sim \ell^2$. Thus if we let $\ell \to 0$ then the term (4) clearly dominates and hence $t_i = \sigma_{ij} n_j$.$\square$

The equation of motion

The ith component of force exerted by the surrounding fluid on a fluid blob with surface S and volume V is given by

$$\int_S t_i dS = \int_S \sigma_{ij} n_j dS = \int_V \frac{\partial \sigma_{ij}}{\partial x_j} dV,$$

where we have used relation (3) together with the divergence theorem. Applying Newton's 2nd law to an arbitrary fluid blob then leads to the equation of motion:

$$\rho \frac{Du_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i. \tag{5}$$

Applying an argument involving *angular* momentum to a tetrahedron, which is similar in spirit to the argument above using linear momentum, leads to the result that the tensor σ_{ij} is symmetric.

Newtonian fluids

In order actually to solve equation (5) it is necessary to relate the stress tensor σ_{ij} to the fluid velocity $\mathbf{u}$. We shall restrict our attention to fluids that are incompressible and for which the stress tensor is assumed to take the form:

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{6}$$

where μ is the viscosity and p is the pressure. Note that σ_{ij} is symmetric, as required. Fluids for which the linear relation (6) holds are said to be *Newtonian*, in recognition of the fact that Newton proposed such a relation for a simple shearing motion. For water and for most gases under non-extreme conditions the linear relation (6) holds. There are though many important fluids that are *non-Newtonian*, i.e. the simple relation (6) does not hold. These include paint, tomato ketchup, blood, toothpaste and quicksand.

Note that, since $\nabla\cdot\mathbf{u} = 0$, then

$$p = -\frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right),$$

i.e. $-p$ is the mean of the three normal stresses.

The Navier-Stokes equation

The final step in deriving the Navier-Stokes equation is to substitute expression (6) for σ_{ij} into equation (5). This leads to the equation (assuming constant viscosity μ),

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{f} + \mu\nabla^2\mathbf{u}. \tag{7}$$

Equation (7) is known as the Navier-Stokes equation for an incompressible fluid; it is to be solved in conjunction with the incompressibility condition (2). For compressible fluids — with which we shall not be concerned in this course — the viscous term is a little more complicated.

Assuming that the density ρ is constant, equation (7) can be written as

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{f} + \nu\nabla^2\mathbf{u}, \tag{8}$$

where $\nu = \mu/\rho$ is the *kinematic viscosity*.

It is often helpful to express the Navier-Stokes equation in dimensionless form. Suppose a flow has characteristic length scale L and characteristic velocity scale U. Then we may introduce dimensionless variables (denoted by tildes) as follows:

$$\mathbf{u} = U\tilde{\mathbf{u}}, \quad t = \frac{L}{U}\tilde{t}, \quad \frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial \tilde{x}}, \quad p = \rho U^2 \tilde{p}, \quad \mathbf{f} = \frac{U^2}{L}\tilde{\mathbf{f}}.$$

On substituting into (8), and dropping the tildes, we obtain the dimensionless Navier-Stokes equation

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{f} + \frac{1}{Re}\nabla^2 \mathbf{u},$$

where $Re = UL/\nu$ is the *Reynolds number*.

Further reading

The most comprehensive derivation of the Navier-Stokes equation, covering both incompressible and compressible fluids, is in *An Introduction to Fluid Dynamics* by G.K. Batchelor (Cambridge University Press), §3.3.