Change of Variables

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1 Overview

If there is a transformation that translates from a (u, v) coordinate system to the standard (x, y) system then we can use the Jacobian to integrate in the (u, v) system instead. We would want to do this if integrating on (u, v) is less calculation intensive than on (x, y).

2 Deriving the Jacobian

In Figure 1 we have a transformation T which maps from (u, v) to (x, y). Using linear algebra terms, the graph on the right is an image of the transformation. In this transformation, we have two hypothetical functions $x = g(u, v)$ and $y = h(u, v)$. We can summarize the transformation as a vector function r.

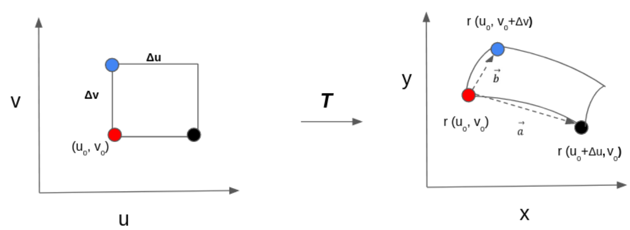

Figure 1

From this same example we can derive the following two secant vectors:

$$\vec{a} = r(u_0 + \Delta v, v_0) - r(u_0, v_0)$$
$$\vec{b} = r(u_0, v_0 + \Delta v) - r(u_0, v_0)$$

If r describes the image on (x, y) using (u, v), then r_u and r_v are partial derivatives that measure the change in r with respect to u and v.

$$r_u = g_u(u_0, v_0)\vec{i} + h_u(u_0, v_0)\vec{j}$$
$$r_v = g_v(u_0, v_0)\vec{i} + h_v(u_0, v_0)\vec{j}$$

We know that the formal definition of r_u is:

$$r_u = \lim_{\Delta \to 0} \frac{r(u_0 + \Delta u, v_0) - r(u_0, v_0)}{\Delta u}$$

For small values Δu we can say that:

$$r_u \Delta u = r(u_0 + \Delta u, v_0) - r(u_0, v_0)$$

The vector $r_u \Delta u$ serves as an approximation of $\vec{a}$, and $r_v \Delta v$ serves as an approximation of $\vec{b}$. This is illustrated in Figure 2.

The cross product $\Delta u r_u \times \Delta v r_v$ forms a parallelogram that approximates the area covered as a result of extending from (u_0, v_0) by the amounts of Δu and Δv. If we simplify the cross product further:

$$\mid \Delta u r_u \times \Delta v r_v \mid = \Delta u \Delta v |r_u \times r_v|$$

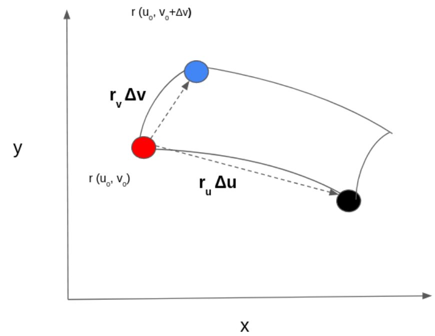

Figure 2

The cross product $r_u \times r_v$ can be obtained by calculating the following determinant:

$$det \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} = \frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}$$

Because g and h are just descriptions of the x and y coordinates in the transformation, we typically write this result as:

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{1}$$

Expression (1) is the Jacobian.

In Figure 3 we've squared off several regions of length Δu and Δv. Let's suppose that the transformation maps these to the regions on the right on the xy plane. Let's label the surface on the right f. The volume under a section of f is the double integral:

$$\int\int f(x,y)\, dA \approx \sum_{i=1}^{m}\sum_{i=1}^{n} f(x,y)\Delta A$$

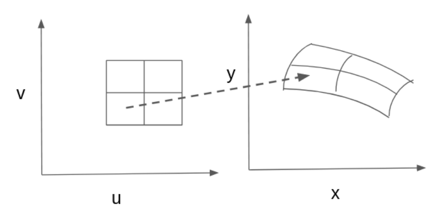

Figure 3

In the previous section we saw how to approximate a region of f with the rectangle $\Delta u \Delta v |r_u \times r_v|$. If we take infinitesimal measurements of Δu and Δv, we get the following:

$$\sum_{i=1}^{m}\sum_{i=1}^{n} f(x,y)\Delta A = \sum_{i=1}^{m}\sum_{i=1}^{n} f(g(u,v),h(u,v)) \mid \frac{\partial(x,y)}{\partial(u,v)} \mid \Delta u\ \Delta v$$

With infinitesimal measurements for Δu and Δv we obtain the following double integral:

$$\int\int f(g(u,v),h(u,v)) \mid \frac{\partial(x,y)}{\partial(u,v)} \mid du\ dv \tag{2}$$

Expression (2) represents the general methodology for how to apply change of variables to a double integral.

Ex. 1 Evaluate $\int\int_R (x-3y)\ dA$. R is defined as the parallelogram with points $(0,0)$, $(3,3)$, $(7,3)$, $(4,0)$. The following transformation can be used $x = u + v$ and $y = u$.

Source:

The parallelogram can be defined with four lines. We can apply the transformation to each of the lines:

$$\begin{aligned} \text{Line A:} \quad & y = 0 & \rightarrow \quad & u = 0 \\ \text{Line B:} \quad & y = x - 4 & \rightarrow \quad & v = 4 \\ \text{Line C:} \quad & y = 3 & \rightarrow \quad & u = 3 \\ \text{Line D:} \quad & y = x & \rightarrow \quad & v = 0 \end{aligned}$$

The new area of intetegration will be a rectangle with the bounds $0 \leq u \leq 3$ and $0 \leq v \leq 4$.

We calculate the Jacobian and evaluate the integral:

$$J = det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = det \begin{pmatrix} 1 & 1 \\ \\ 1 & 0 \end{pmatrix} = -1$$

$$\begin{aligned} \int\int_R (x-3y)\ dA = \int_0^3 \int_0^4 uv\ |-1|\ dvdu = \int_0^3 \int_0^4 uv\ \ dvdu \\ \int_0^3 \frac{uv^2}{2}\Big|_0^4 du = 8(\frac{u^2}{2}\Big|_0^3) = 36 \end{aligned}$$

Ex. 2 Evaluate $\int\int_R x - 3y\ dA$. R is a region described by a triangle with the following points $(0,0)$, $(1,2)$, $(2,1)$. The following transformation is available: $x = 2u + v$ and $y = u + 2v$.

Source:

The triangle can be described with three lines. We apply the transformation to each of these:

$$\begin{aligned} \text{Line A:} \quad y = \frac{1}{2}x \quad &\rightarrow \quad v = 0 \\ \text{Line B:} \quad y = 3 - x \quad &\rightarrow \quad v = 1 - u \\ \text{Line C:} \quad y = 2x \quad &\rightarrow \quad u = 0 \end{aligned}$$

For Line B the u-intercept is 1. This combined with the transformations above give us the following bounds, $0 \leq u \leq 1$, and $0 \leq v \leq 1 - u$.

We calculate the Jacobian and evaluate the integral:

$$J = det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = det \begin{pmatrix} 2 & 1 \\ \\ 1 & 2 \end{pmatrix} = 3$$

$$\int\int_R x - 3y\ dA = 3\int_0^1 \int_0^{1-u} -u - 5v\ dvdu$$
$$3\int_0^1 -uv - \frac{5v^2}{2}\Big|_0^{1-u} du = \int_0^1 = 3\int_0^1 -u(1-u) - \frac{5}{2}(1-u)^2 = -3$$

3 Polar Coordinates

The evaluation of double integrals using polar coordinates is an application of these techniques. We know that $x = r\cos\theta$ and $y = r\sin\theta$. If we compute the Jacobian:

$$J = det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \\ \sin\theta & r\cos\theta \end{pmatrix} = r(cos^2\theta + sin^2\theta) = r$$

If we had a function $f(x, y)$ and apply a change of variables:

$$\int\int f(x, y)\ dydx = \int_{r=b}^{r=a}\int_{\theta=d}^{\theta=c} f(x(\theta, r), y(\theta, r))r\ drd\theta$$

Ex. 3 Evaluate $\int\int_R x^2\ dA$. R is an ellipse, $9x^2 + 4y^2 = 36$. The following transformation can be used $x = 2u$ and $y = 3v$.

Source:

Let's just try and plug in the transformation into R, we get:

$$9(2u)^2 + 4(3v)^2 = 36$$
$$u^2 + v^2 = 1$$

The transformation has given us a circle, which will be easier to integrate. Next, we figure out the Jacobian:

$$J = det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = det \begin{pmatrix} 2 & 0 \\ \\ 0 & 3 \end{pmatrix} = 6$$

We are left with the following integral:

$$24\int\int u^2\ dudv$$

We can now apply another change of variables to polar coordinates to complete the integration:

$$24\int_0^{2\pi}\int_0^1 r^2\cos^2\theta\ r\ drd\theta$$

$$24\int_0^{2\pi}\frac{r^4}{4}\cos^2\theta\Big|_0^1\ d\theta = 6\int_0^{2\pi}\cos^2\theta\ d\theta = 6(\frac{\theta}{2} + \frac{\sin(2\theta)}{4})\Big|_0^{2\pi} = 6\pi$$