Dot and Cross Product

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1 Norm

The magnitude of a vector is known as the **norm** in linear algebra. A commonly used notation to describe the norm of a vector $\vec{v}$ is $||\vec{v}||$.

Calculating the norm for a vector is simply an application of the Pythagorean Theorem. For a vector of two components like say, $\vec{v} =< x, y >$, we have $||\vec{v}|| = \sqrt{x^2 + y^2}$. For a vector of three components, like $\vec{v} =< x, y, z >$, we have $||\vec{v}|| = \sqrt{x^2 + y^2 + z^2}$, etc.

We can **normalize** any vector by dividing each component by its norm. Given $\vec{v}$, we describe the normalized vector with the notation $\hat{v}$. For example, the normalized vector for a three component vector can be described like so:

$$\hat{v} = \frac{1}{||\vec{v}||} < x, y, z >$$

An important aspect of the normalized vector is that its norm is always one. We can take our three component vector and show this algebraically:

$$\hat{v} =< \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} >$$

$$||\hat{v}|| = \sqrt{(\frac{x}{\sqrt{x^2 + y^2 + z^2}})^2 + (\frac{y}{\sqrt{x^2 + y^2 + z^2}})^2 + (\frac{z}{\sqrt{x^2 + y^2 + z^2}})^2}$$

$$||\hat{v}|| = \sqrt{(\frac{x^2}{x^2 + y^2 + z^2}) + (\frac{y^2}{x^2 + y^2 + z^2}) + (\frac{z^2}{x^2 + y^2 + z^2})}$$

$$||\hat{v}|| = \sqrt{\frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2}} = 1$$

Ex. 1 Normalize the vector $\vec{v} =< 4, 2, 4 >$.

$$||\vec{v}|| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{36} = 6$$

$$\hat{v} = \frac{1}{6} < 4, 2, 4 >$$

$$\hat{v} =< \frac{2}{3}, \frac{1}{3}, \frac{2}{3} >$$

2 Dot Product

The dot product between two vectors is a scalar. Given $\vec{a} =< a_1, a_2, a_3, ..., a_n >$ and $\vec{b} =< b_1, b_2, b_3, ..., b_n >$, the dot product $\vec{a} \cdot \vec{b}$ is defined as:

$$\vec{a} \cdot \vec{b} \ = \sum_{i=n}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 + ... + a_n b_n$$

A dot product is an example of an **inner product**. When discussed in this context, the notation $< a, b >$ is sometimes used.

Intuitively, the dot product is a measurement of how much in common, in terms of direction, two vectors have. Suppose we have two vectors $\vec{a} =< 3, 2 >$ and $\vec{b} =< 2, 2 >$ Let's suppose that you have a way of scaling these vector down, we can see that a_x can take on the values of $\{1, 2, 3\}$. Looking at b_x we have a little less to work with, we have $\{1, 2\}$. We can repeat this analysis for the y component of both vectors to get $\{1, 2\}$ for both a_y and b_y. In summary:

$$\begin{aligned} a_x &\in A, \text{where A} = \{1, 2, 3\} \\ a_y &\in B, \text{where B} = \{1, 2\} \\ b_x &\in C, \text{where C} = \{1, 2\} \\ b_y &\in D, \text{where D} = \{1, 2\} \end{aligned}$$

Focusing on sets A and B we can count the number of number of possible arrangements between these two sets. We get $\{1, 1\}$, $\{1, 2\}$, $\{2, 1\}$, $\{2, 2\}$, $\{3, 1\}$, $\{3, 2\}$ or a total of 6 possible arrangements.

This is an application of the **multiplication rule of counting**, where the total number of combinations is the size set A, $|A|$ multiplied by the size of set B, $|B|$. In this case, we have $|A||B| = (3)(2) = 6$ which matches the number of arrangements we came up with before. If we apply this to the y-components, we have $|C||D| = (2)(2) = 4$.

At this point we have treated the scaling of the x and y components as independent events. The **addition rule of counting** would predict that the total number of combinations is the sum of the ways we can combine the x-components and the way we can combine the y-components. We are left with:

$$|A||B| + |C||D| = a_x b_x + a_y b_y$$

Ex. 2 Determine $\vec{a} \cdot \vec{b}$, where $\vec{a} =< 3, 2, 1 >$ and $\vec{b} =< 5, 3, 5 >$

$$\vec{a} \cdot \vec{b} = (3)(5) + (2)(3) + (1)(5) = 26$$

Ex. 3 Determine $\vec{a} \cdot \vec{b}$, where $\vec{a} =< 2, 2 >$ and $\vec{b} =< -1, 1 >$

$$\vec{a} \cdot \vec{b} = (2)(-1) + (2)(1) = 0$$

Example 3 illustrates an important property of the dot product. **The dot product of two orthogonal vectors is zero.** Suppose again that we have two vectors $\vec{a}$ and $\vec{b}$, and they are perpendicular to each other, then the angle θ between them is $\frac{\pi}{2}$. This results in the dot product equaling zero:

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \; ||\vec{b}|| \cos(\frac{\pi}{2}) = 0$$

3 The Law of Cosines

This section concerns itself with the vector law of cosines, which states:

$$\vec{a} \cdot \vec{b} = ||\vec{a}||\ ||\vec{b}|| \cos\theta \tag{1}$$

The above definition of the dot product is related to the formula:

$$c^2 = a^2 + b^2 - 2ab\cos(\theta) \tag{2}$$

We will first derive expression (2):

$$\begin{aligned}
\overline{CD} &= a\cos\theta \\
\overline{DA} &= b - a\cos\theta \\
\overline{BD} &= a\sin\theta \\
\\
c^2 &= (\overline{BD})^2 + (\overline{DA})^2 \\
&= (a\sin\theta)^2 + (b - a\cos\theta)^2 \\
&= a^2\sin^2\theta + a^2 cos^2\theta + b^2 - 2ab\cos(\theta) \\
&= a^2(\sin^2\theta + \cos^2\theta) + b^2 - 2ab\cos(\theta) \\
&= a^2 + b^2 - 2ab\cos(\theta)
\end{aligned}$$

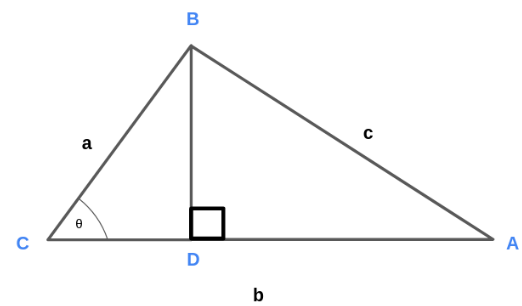

Figure 1

We can use expression (2) to derive expression (1). First, we restate expression (1) in terms of vectors.

$$||\vec{a} - \vec{b}||^2 = ||\vec{a}||^2 + ||\vec{b}||^2 - 2||\vec{a}||\ ||\vec{b}|| \cos\theta \tag{3}$$

We can restate the left hand side using the fact that the dot product of a vector with itself is its squared magnitude:

$$\begin{aligned}
||\vec{a} - \vec{b}||^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\
&= \vec{a} \cdot \vec{a} - 2(\vec{a} \cdot \vec{b}) + \vec{b} \cdot \vec{b} \\
&= ||\vec{a}||^2 + ||\vec{b}||^2 - 2(\vec{a} \cdot \vec{b})
\end{aligned}$$

If we compare the result we obtained on the last stem with expression 3, we see that $2(\vec{a} \cdot \vec{b})$ is equivalent to $2||\vec{a}||\ ||\vec{b}|| cos\theta$:

$$\begin{aligned}
||\vec{a}||^2 + ||\vec{b}||^2 - 2(\vec{a} \cdot \vec{b}) &= ||\vec{a}||^2 + ||\vec{b}||^2 - 2||\vec{a}||\ ||\vec{b}|| \cos\theta \\
\vec{a} \cdot \vec{b} &= ||\vec{a}||\ ||\vec{b}|| \cos\theta
\end{aligned}$$

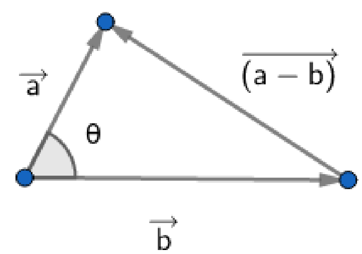

Figure 2

Ex. 4 Determine the angle between $\vec{a} =< 5, 0 >$ and $\vec{b} =< -3, 4 >$

$$\begin{aligned}
\text{Since } \vec{a} \cdot \vec{b} = -15 \text{ and } ||\vec{a}||\ ||\vec{b}|| &= (5)(5) = 25 \\
\theta &= \arccos(\frac{-15}{25}) = 2.21
\end{aligned}$$

The angle between $\vec{a}$ and $\vec{b}$ is 2.21 radians, which is approximately $126.62°$

4 Projection

A useful feature of the dot product is that it assists with calculating vector projection. In Figure 2 we have the length w, which is the projection of $\vec{a}$ onto $\vec{b}$. We can figure out w if we know θ and the norm of $\vec{a}$:

$$w = \cos\theta ||\vec{a}||$$

Solving for θ in expression (1) gets us $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||\,||\vec{b}||}$. Substituting this into the equation for w, we get:

$$w = \frac{\vec{a}\cdot\vec{b}\,||\vec{a}||}{||\vec{a}||\,||\vec{b}||}$$

$$w = \frac{\vec{a}\cdot\vec{b}}{||\vec{b}||} \tag{4}$$

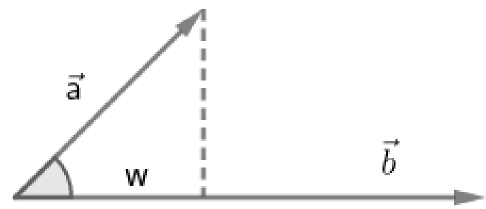

Figure 3

Expression (4) shows us how to calculate the length of a projection, but we can also derive a vector whose magnitude is equivalent to that length. Continuing with the same scenario posed by Figure 3. We first derive the unit vector for $\vec{b}$, which will have a magnitude of one:

$$\hat{b} = \frac{\vec{b}}{||\vec{b}||}$$

To obtain the projection vector we need to scale up $\hat{b}$ by the projection length w, so we have:

$$\text{proj}_b(a) = \frac{\vec{a}\cdot\vec{b}}{||\vec{b}||}\ \frac{\vec{b}}{||\vec{b}||} = \frac{\vec{a}\cdot\vec{b}}{||\vec{b}||^2}\vec{b}$$

The square of a vector's norm is the dot product of the vector with itself, $||\vec{b}||^2 = \vec{b}\cdot\vec{b}$, so we arrive at the final form of the projection vector:

$$\text{proj}_b(a) = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b} \tag{5}$$

Ex. 5 Find the projection length w of $\vec{a} =< 3, 1 >$ onto $\vec{b} =< 4, -2 >$. Determine $\text{proj}_b(a)$:

$$w = \frac{(3)(4) + (1)(-2)}{\sqrt{(4)^2 + (-2)^2}}$$

$$w = \frac{10}{\sqrt{20}}$$

$$\text{proj}_b(a) = \frac{(3)(4) + (1)(-2)}{(4)(4) + (-2)(-2)}\,\vec{b}$$

$$\text{proj}_b(a) = \frac{1}{2}\ < 4, -2 >$$

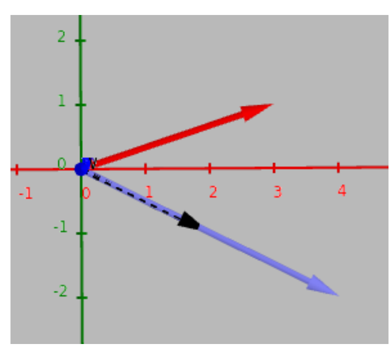

Figure 4

In Figure 4, the dotted vector is proj$b(a)$.

5 The Cross Product

The cross product between two vectors $\vec{a}$ and $\vec{b}$ is denoted $\vec{a} \times \vec{b}$. The most important aspect of the cross product is that it produces a third vector that is orthogonal to the two input vectors.

The x-component of the cross product is the determinant calculated from the y and z components of the input vectors. The y-component is the determinant calculated from the x and z components. Lastly, the z-component is the determinant from the x and y components. So given $< a_x, a_y, a_z$ and $< b_x, b_y, b_z >$, the cross product is:

$$\vec{a} \times \vec{b} =< (a_y)(b_z) - (a_z)(b_y), \;\; -(\ (a_x)(b_z) - (a_z)(b_x)\ ), \;\; (a_x)(b_y) - (a_y) - (a_y)(b_x) > \tag{6}$$

We can also take the cross product of a two 2 dimensional vectors. This will simply be the determinant of a matrix comprised of the two vectors $\vec{a} =< a_x, a_y >$ and $\vec{b} =< b_x, b_y >$, the cross product is just: $(a_x)(b_y) - (a_y)(b_x)$. In contrast to the three dimensions case, this calculation produces a scalar.

We can now show that the cross product is orthogonal to both of the input vectors. If $\vec{a} \times \vec{b}$ is orthogonal to both vectors a and b, then $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ and $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$. We start by demonstrating this for $\vec{a}$:

$$\begin{aligned}
&\vec{a} \cdot (\vec{a} \times \vec{b}) \\
&=< a_x, a_y, a_z > \cdot < a_y b_z - a_z b_y, a_x b_z - a_z b_x, a_x b_y - a_y b_x > \\
&= a_x(a_y b_z - a_z b_y) - a_y(a_x b_z - a_z b_x) + a_z(a_x b_y - a_y b_x) \\
&= a_x a_y b_z + a_x a_z b_y - a_y a_x b_z + a_y a_z b_x + a_z a_x b_y - a_z a_y b_x \\
&= 0
\end{aligned}$$

The same exercise can be done with $\vec{b}$:

$$\begin{aligned}
&\vec{b} \cdot (\vec{a} \times \vec{b}) \\
&=< b_x, b_y, b_z > \cdot < a_y b_z - a_z b_y, a_x b_z - a_z b_x, a_x b_y - a_y b_x > \\
&= b_x(a_y b_z - a_z b_y) - b_y(a_x b_z - b_z b_x) + b_z(a_x b_y - b_y b_x) \\
&= b_x a_y b_z + b_x a_z b_y - b_y a_x b_z + b_y a_z b_x + b_z a_x b_y - b_z a_y b_x \\
&= 0
\end{aligned}$$

Lastly, we want to derive the following relationship:

$$||\vec{a} \times \vec{b}|| = ||\vec{a}||\ ||\vec{b}|| sin\theta \tag{7}$$

Two vectors will always be coplanar. From the perspective of the plane both vectors are located on, the z-components can be ignored for the purposes of this derivation. One way to visualize this is to suppose we are staring straight down at that plane, the same way we would look down on the xy plane. We can therefore treat $< a_x, a_y, a_z >$ for example to be the same as $< a_x, a_y >$.

On the plane, we can now draw the parallelogram in Figure 5. From this, we can see that the parallelogram's area is $||\vec{a}||\ ||\vec{b}||\ sin\theta$ or the right half of Expression (7).

Since we are treating $\vec{a}$ and $\vec{b}$ as two dimensional vectors, its cross product is to $(a_x)(b_y) - (a_y)(b_x)$, in other words $||\vec{a} \times \vec{b}||$. In the determinants chapter we saw how this is also the area of the parallelogram.

Therefore, the area of the parallelogram is the norm of $\vec{a} \times \vec{b}$.

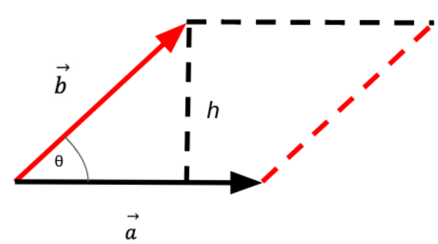

Figure 5

Ex. 6 Determine the cross product between $\vec{a} = < 7, 2, -2 >$ and $\vec{b} = < -5, 0, 3 >$.

x-component: $(2)(3) - (-2)(0) = 6$
y-component: $-(\ (7)(3) - (-2)(-5)\ ) = -11$
z-component: $(7)(0) - (2)(-5) = 10$

$$\vec{a} \times \vec{b} = < 6, -11, 10 >$$

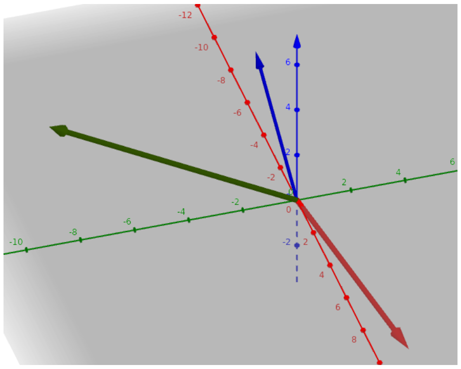

Figure 6

In Figure 6, we have plotted $\vec{a}$ in red, $\vec{b}$ in blue, and the cross product in green.