# Modular Arithmetic

Andy Chong Sam

September 15, 2022

## 1 Remainder Quotient Formula

When performing an integer division we can document a quotient (q) and a remainder (r). In this document we will use the following notation to denote each:

$$a/b = q$$
$$a \mod b = r$$

Given a dividend $a$, and a divisor $b$ that results in a quotient $q$ and a remainder $r$ , we can derive the following expression:

$$a = bq + r \tag{1}$$

Consider a simple integer division of 25 divided by 11. Since $25/11 = 2$ and $25 \mod 11 = 3$, all pieces of the operation can be encapsulated as: $25 = (11)(2) + 3$. Expression (1) can be used to demonstrate various properties in modular arithmetic.

## 2 Modular Arithmetic

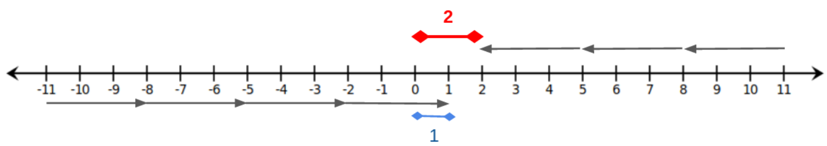
### 2.1 Overview

A straightforward interpretation of the modulo operation (mod) is that its output is the remainder of the division between two integers. A cyclical pattern is observed by varying x in $x \mod s$, when s is left constant.

For $x \mod 1$, the set of all possible outcomes is {0}.
For $x \mod 2$, the set of all possible outcomes is {0,1}.
For $x \mod 5$, the set of all possible outcomes is {0,1,2,3,4}.

### 2.2 Negative Dividends

When the dividend is negative, the results are less intuitive. For example: $-11 \mod 3 = 1$. One way to visualize this outcome is by imagining a number line with an emphasis on the range of mod 3: $\{0, 1, 2\}$




In the case of $-11 \mod 3$ we can see that there are 4 skips needed to enter the modulo range. The distance between where the last skip lands and 0 is the modulo. In this case, this distance is 1.

In the case of $11 \mod 3$ we can see that there are 3 skips needed to enter the modulo range. The distance between where the last skip lands and 0 is the modulo. In this case, this distance is 2.

For the operation $a \mod b$ and if a is negative, then we can calculate the modulo using the following expression:

$$a + |\lfloor \frac{a}{b} \rfloor|(b) + b \tag{2}$$

## 2.3 Modular Addition

The property of modular addition is as follows:

$$(a+b) \mod c = ((a \mod c) + (b \mod c)) \mod c \tag{3}$$

This relationship can be derived by using expression (1). Recall that if a is divisible by c, and b is divisible by c, then a+b is divisible by c. We first start by restating a + b:

$$\begin{aligned} a = cq_1 + r_1 \therefore a \mod c = r_1 \\ b = cq_2 + r_2 \therefore b \mod c = r_2 \\ a + b = (cq_1 + r_1 + cq_2 + r_2) \\ a + b = c(q_1 + q_2) + r_1 + r_2 \end{aligned}$$

Plugging the above result back into $(a+b) \mod c$ we get:

$$(c(q_1 + q_2) + r_1 + r_2) \mod c$$

We can apply the following rule to simplify the above expression. If we have an operation $a \mod b$, then we know that adding a multiple of b (say kb), will result in the same modulo value: $(a + kb) \mod b = a \mod b$. We can now simplify further:

$$\begin{aligned} (c(q_1 + q_2) + r_1 + r_2) \mod c \\ = (r_1 + r_2) \mod c \end{aligned}$$

On the right hand side of expression (3) we can simplify further, using:

$$\begin{aligned} a = cq_1 + r_1 \therefore a \mod c = r_1 \\ b = cq_2 + r_2 \therefore b \mod c = r_2 \end{aligned}$$

So the right hand side becomes $(r_1 + r_2) \mod c$ as well.

## 2.4 Modular Multiplication

The property of modular multiplication is as follows:

$$(ab) \mod c = ((a \mod c)\ (b \mod c)) \mod c \tag{4}$$

This can be derived in a way similar to modular addition. The left hand side can be rewritten like so:

$$\begin{aligned} &((cq_1 + r_1)(cq_2 + r_2)) \mod c \\ = &(c^2 q_1 q_2 + cq_1 r_2 + cq_2 r_1 + r_1 r_2) \mod c \\ = &(c(cq_1 q_2 + q_1 r_2 + q_2 r_1) + r_1 r_2) \mod c \end{aligned}$$

Since $c(cq_1q_2 + q_1r_2 + q_2r_1)$ is a multiple of c, we are left with:

$$(r_1 r_2) \mod c$$

On to the right hand side of expression (4), using:

$$\begin{aligned} a = cq_1 + r_1 \therefore a \mod c = r_1 \\ b = cq_2 + r_2 \therefore b \mod c = r_2 \end{aligned}$$

We see that the right hand side of expression (4) becomes:

$$(r_1 r_2) \mod c$$

## 2.5 Modular Exponentiation

Modular exponentiation takes the form of evaluation a problem like $a^x \mod b$. The challenge here is that $a^x$ could easily become a very large number, causing errors on calculators. A commonly used technique to overcome this problem is known as "fast modular exponentiation" and it involves restating x using base-2. Suppose we are trying to evaluate $7^{15} \mod 17$. The first step is to translate the exponent into base-2. The number 15 thus becomes $(1111)_2$. We can expand this base-2 number with each term representing the binary symbol {0,1} times 2 raised to the power of the place value it appears in:

$$15 = (1)(2)^3 + (1)(2)^2 + (1)(2)^1 + (1)(2)^0$$

With this expansion in mind, we can restate the original problem like so:

$$\begin{aligned} &7^{15} \mod 17 \\ = &(7^{(1)2^3+(1)2^2+(1)2^1+(1)2^0}) \mod 17 \\ = &(7^{8+4+2+1}) \mod 17 \end{aligned}$$

Finally, we can apply the algebraic rule of exponents and the property of modular multiplication:

$$\begin{aligned} &(7^{8+4+2+1}) \mod 17 \\ = &((7^8 \mod 17)\ (7^4 \mod 17)\ (7^2 \mod 17)\ (7^1 \mod 17)) \mod 17 \end{aligned}$$

We have now broken the problem into individual components, thus making calculations easier and reducing the likelikhood of overflow errors.

# 3 Euclidean Algorithm

## 3.1 GCD and Algorithm Steps

The Greatest Common Divisor (GCD) is the largest common divisor for a set of numbers.

The Euclidean Algorithm outlines a series of steps that can be followed to arrive at the GCD for two integers (say a and b). We start by taking the larger of the two numbers, let's say a, and rewrite it using the remainder quotient formula: $a = bq_1 + r_1$. We will then evaluate $b = r_1q_2 + r_2$. We could potentially evaluate $r_1 = r_2q_3 + r_3$.

We could encapsulate this recursive operation as $E(s,t) \to s = tq_n + r_n$. We stop when s or t becomes 0, and the non-zero value will be the GCD.

**Ex. 1** Evaluate $gcd(93, 42)$

$$\begin{aligned} E(93, 42) \to 93 &= (2)(42) + 9 \\ E(42, 9) \to 42 &= (4)(9) + 6 \\ E(9, 6) \to 9 &= (6)(1) + 3 \\ E(6, 3) \to 6 &= (3)(2) + 0 \\ E(3, 0) & \end{aligned}$$

**Solution:** $gcd(93, 42) = 3$

**Ex. 2** Evaluate $gcd(4278, 8602)$

$$\begin{aligned} E(8602, 4278) \to 8602 &= (4278)(2) + 96 \\ E(4278, 96) \to 4278 &= (46)(93) + 0 \\ E(46, 0) & \end{aligned}$$

So $gcd(4278, 8602) = 46$

### 3.2 Bézout's Lemma

There are integers x and y such that $gcd(a,b) = xa + yb$. Matrices with elementary row operations can be used to find $x$ and $y$. We will demonstrate the algorithm using the two examples above.

**Ex. 3** Our claim is that there are integers x and y such that $gcd(93,42) = 93x + 42y$.

| **State Matrix** | **Division** | **Elementary Row Operation** |
|---|---|---|
| $\begin{bmatrix}1 & 0 & 93\\0 & 1 & 42\end{bmatrix}$ | $93/42 = 2$<br>$93 \mod 42 = 9$ | $R_1 - 2R_2 \to R_1$ |
| $\begin{bmatrix}1 & -2 & 9\\0 & 1 & 42\end{bmatrix}$ | $42/9 = 4$<br>$42 \mod 92 = 6$ | $R_2 - 4R_{21} \to R_2$ |
| $\begin{bmatrix}1 & -2 & 9\\-4 & 9 & 6\end{bmatrix}$ | $9/6 = 1$<br>$9 \mod 6 = 3$ | $R_1 - R_2 \to R_1$ |
| $\begin{bmatrix}5 & -11 & 3\\-4 & 9 & 6\end{bmatrix}$ | $6/3 = 2$<br>$6 \mod 3 = 0$ | The stopping condition, that 6 mod $3 = 0$, has been reached. |

**Solution:** So $x = 5$ and $y = -11$. We can verify this: $(93)(5) + (42)(-11) = 3$.

**Ex. 4** Our claim is that there are integers x and y such that $gcd(4278, 8602) = 8602x + 4278y$.

| **State Matrix** | **Division** | **Elementary Row Operation** |
|---|---|---|
| $\begin{bmatrix}1 & 0 & 8602\\0 & 1 & 4278\end{bmatrix}$ | $8602/4278 = 2$<br>$8602 \mod 4278 = 46$ | $R_1 - 2R_2 \to R_1$ |
| $\begin{bmatrix}1 & -2 & 46\\0 & 1 & 4278\end{bmatrix}$ | $4278/46 = 93$<br>$4278 \mod 46 = 0$ | The stopping condition has been reached. |

**Solution:** So $x = 1$ and $y = -2$. We can verify this: $(8602)(1) + (4278)(-2) = 46$

# 4 Congruence

### 4.1 Overview

The statement $a \equiv b \mod n$, is another way of stating $a \mod n = b \mod n$. For example, $7 \equiv 12 \mod 5$ is a true statement, since $7 \mod 5 = 2$ and $12 \mod 5 = 2$. We also find that $17 \mod 5 = 2$. These 3 integers belong to the same Congruence Class for mod 5.

Referring back to section 2.1, we know that the only possible outcomes of any integer x mod 5 is {0,1,2,3,4}. Therefore there are 5 different congruence classes for mod 5.

### 4.2 Solving Congruence Problems

Several strategies can be applied to solve the problems of the form $ax \equiv b(\mod n)$. For the next few examples, we will use the following five propositions:

**(p1)** If d divides n and we have $ad \equiv bd(\mod n)$, we can simplify this to $a \equiv b(\mod \frac{n}{d})$

**(p2)** If $gcd(a, n) = 1$ and we have $ad \equiv bd(\mod n)$, we can simplify this to $a \equiv b(\mod n)$

**(p3)** There is a solution to $ax \equiv b(\mod n)$ if $gcd(a, b)|n$

**(p4)** For $a\bar{a} = 1(\mod n)$, the inverse $\bar{a}$ exists if $gcd(a, n) = 1$

**(p5)** For $ax \equiv b(\mod n)$ where multiple solutions exist, the spread between solutions is $\frac{n}{gcd(a,n)}$

Sometimes the solution to a problem is trivial and involves only algebra, consider the following two problems:

**Ex. 5** Solve $8x \equiv 16(\mod 5)$ :

First, we observe that $gcd(8, 5) = 1$. We can divide both sides of the congruence by 8 (applying **p2**).

$$\begin{aligned} x &\equiv 2(\mod 5) \\ &\text{To verify our solution:} \\ (8)(2) &\equiv 16(\mod 5) \\ 16 &\equiv 16(\mod 5) \end{aligned}$$

**Solution:** $x \equiv 2(\mod 5)$

**Ex. 6** Solve $2x \equiv 8(\mod 3)$ :

First, we observe that $gcd(2, 3) = 1$. We can divide both sides of the congruence by 2 (applying **p2**).

$$\begin{aligned} x &\equiv 4(\mod 3) \\ &\text{To verify our solution:} \\ (2)(4) &\equiv 8(\mod 3) \\ 8 &\equiv 8(\mod 3) \end{aligned}$$

**Solution:**$4x \equiv (\mod 3)$

In some cases, algebra by itself will not work as the division on both sides of the congruence would not produce an integer. For these cases, we can first verify if a solution exists, and determine if a and n are coprime. If a and n are coprime, we we can use the modulo multiplicative inverse.

For a problem $ax \equiv b(\mod n)$, we first check if $gcd(a, n) = 1$. If so, we we can evaulate $a\bar{a} \equiv 1(\mod n)$. Once we determine the value of $\bar{a}$. We can multiply both sides of the original congruence by $\bar{a}$:

$$\begin{aligned} a\bar{a}x &\equiv \bar{a}b(\mod n) \\ x &\equiv \bar{a}b(\mod n) \end{aligned}$$

Let's clarify what happens to the dropped coefficient on the left. The expression $a\bar{a}x \equiv \bar{a}b(\mod n)$ can be restated as $a\bar{a}x \mod n = \bar{a}b \mod n$.

On the left hand side, $\bar{a}$ is an integer, that when multiplied by $ax$ and divided by n, produces a remainder of 1: $a\bar{a}x \mod n = 1$. This is functionally equivalent to saying $(1)(x) \mod n = 1$. We can now rewrite the left hand side:

$$\begin{aligned} a\bar{a}x \mod n &= \bar{a}b \mod n \\ (1)x \mod n &= \bar{a}b \mod n \\ x &\equiv \bar{a}b(\mod n) \end{aligned}$$

Let's consider a few examples:

**Ex. 7** Solve $3x \equiv 7(\mod 11)$

We first determine that gcd(3,11)=1, so the method can be applied. Since 1|11, an inverse $\bar{a}$ exists. After some tests, we determine that $\bar{a}$ is 4.

$$\begin{aligned} (3)(4) \mod 11 &= 1 \\ \therefore \bar{a} &= 4, \end{aligned}$$

We can now use $\bar{a}$ to solve the original problem:

$$\begin{aligned} 4x &\equiv (4)(7)(\mod 11) \\ x &\equiv 28(\mod 11) \end{aligned}$$

We can improve the answer by reporting the smallest positive value. The spread is 11, so $x \equiv 6(\mod 11)$ is the best answer. Finally, we can do a quick verification:

$$\begin{aligned} (3)(6) &\mod 11 \\ = 18 &\mod 11 \\ &= 7 \end{aligned}$$

Solution: $x \equiv 6(\mod 11)$

**Ex. 8** Solve $21x \equiv 14(\mod 91)$

We notice that $gcd(21, 91) = 7$, so at first glance the inverse technique might not work. However, we can simplify the congruence by dividing both sides by 7, and also divide 91 by 7 (Applying **p1**):

$$\begin{aligned} 21x &\equiv 14(\mod 91) \\ 3x &\equiv 2(\mod 13) \end{aligned}$$

We can now apply the technique. We find $gcd(3, 13) = 1$, and since $1|13$ an inverse exists. After testing some numbers we find:

$$\begin{aligned} (3)(9) \mod 13 &= 1 \\ \therefore \bar{a} &= 9 \end{aligned}$$

We can now use $\bar{a}$ to solve the original problem:

$$\begin{aligned} (9)(3)x &\equiv (9)(2)(\mod 13) \\ x &\equiv 18(\mod 13) \end{aligned}$$

Like the previous problem, we can report a slightly better answer by using the spread, which is 13, leaving us with $x \equiv 5(\mod 13)$. We can verify the answer:

$$\begin{aligned} (21)(5) &\mod 91 \\ = 105 &\mod 91 \\ &= 14 \end{aligned}$$

Solution: $x \equiv 5(\mod 13)$

**Ex. 9** Solve $19x \equiv 4(\mod 141)$

Since $gcd(19, 141) = 1$ and $1|141$, the inverse technique can be applied. First we find $\bar{a}$:

$$\begin{aligned} (19)(52) \mod 141 = 1 \\ \therefore \bar{a} = 52 \end{aligned}$$

We can now use $\bar{a}$ to solve the original problem:

$$\begin{aligned} (52)(19)x &\equiv (52)(4)(\mod 141) \\ x &\equiv 208(\mod 141) \end{aligned}$$

Since the spread is 141, a better answer to report would be $x \equiv 67(\mod 141)$. Let's verify the result:

$$\begin{aligned} (19)(67) &\mod 141 \\ = 1273 &\mod 141 \\ &= 4 \end{aligned}$$

Solution: $x \equiv 67(\mod 141)$

# 5 The Chinese Remainder Theorem

The Chinese remainder Theorem outlines an algorithm that can be used to solve a system of congruences. The steps are illustrated below using a system of three congruences. Suppose we want to solve the following:

$$\begin{aligned} x &\equiv b_1(\mod c_1) \\ x &\equiv b_2(\mod c_2) \\ x &\equiv b_3(\mod c_3) \end{aligned}$$

The theorem can be used if $c_1, c_2, c_3$ are coprime. We first calculate $N = (c_1)(c_2)(c_3)$. We then setup the following table:

| $b_i$ | $N_i$ | $\bar{a}$ | $b_i N_i a_i$ |
|---|---|---|---|
| $b_1$ | $N_1 = n_2 n_3$ | $\bar{a}_1$ | $b_1 N_1 a_1$ |
| $b_2$ | $N_2 = n_1 n_3$ | $\bar{a}_2$ | $b_2 N_2 a_2$ |
| $b_3$ | $N_3 = n_1 n_2$ | $\bar{a}_3$ | $b_3 N_3 a_3$ |

The column $N_i$ represents the calculation $N_i = \frac{N}{n_i}$. The column $\bar{a}$ is the solution to $\bar{a}$ in $N_i \bar{a}_i \equiv 1(\mod c_i)$. The solution to the system is the sum of the last column mod N:

$$(\sum_{i=1}^{3} b_i N_i a_i) \mod N$$

Here is an example of the algorithm in action.

**Ex. 10** Suppose that there is a group of students, and the instructor has a choice to group everyone into teams of 3, 4, or 5. If groups of 3 are created, there will be 2 students unassigned students left over. If groups of 4 are made, then there will be 3 students left over, and in the case of groups of 5, then there will be 1 student left over. We want to find a class size that would result in the above outcomes.

The problem can be summarized into the following system of congruences:

$$x \equiv 3(\mod 4)$$
$$x \equiv 1(\mod 5)$$
$$x \equiv 2(\mod 3)$$

From the initial setup, we can see that $N = (4)(5)(3) = 60$. We can start to fill in some of the table details:

| $b_i$ | $N_i$ | $\bar{a}$ | $b_i N_i a_i$ |
|---|---|---|---|
| 3 | 15 | $\bar{a}_1$ | $b_1 N_1 a_1$ |
| 1 | 12 | $\bar{a}_2$ | $b_2 N_2 a_2$ |
| 2 | 20 | $\bar{a}_3$ | $b_3 N_3 a_3$ |

The next step is to figure out $\bar{a}_i$. After some trial and error, we determine the following:

$$(15)(3) \equiv 1(\mod 15) \therefore \bar{a}_1 = 3$$
$$(12)(3) \equiv 1(\mod 12) \therefore \bar{a}_2 = 3$$
$$(20)(2) \equiv 1(\mod 20) \therefore \bar{a}_3 = 2$$

We can now complete the table:

| $b_i$ | $N_i$ | $\bar{a}$ | $b_i N_i a_i$ |
|---|---|---|---|
| 3 | 15 | 3 | 135 |
| 1 | 12 | 3 | 36 |
| 2 | 20 | 2 | 80 |

The sum of the last column is 251. At this point, we have $x \equiv 251(\mod 60)$. Since the spread is 60, the best reportable answer is $x \equiv 11(\mod 60)$. We can verify this result:

$$11 \mod 4 = 3 \qquad 11 \mod 5 = 1 \qquad 11 \mod 3 = 2$$

So a class size of 15 meets the criteria, with 3 students left if groups of 4 are made, 1 student left if groups of 5 are made, and 2 left with groups of 3.

**Ex. 11** Here is an additional example. solve the following system:

$$x \equiv 3(\mod 5)$$
$$x \equiv 1(\mod 7)$$
$$x \equiv 6(\mod 8)$$

First, we determine that $N = (5)(7)(8) = 280$. We can partially fill our table:

| $b_i$ | $N_i$ | $\bar{a}$ | $b_i N_i a_i$ |
|---|---|---|---|
| 3 | 56 | $\bar{a}_1$ | $b_1 N_1 a_1$ |
| 1 | 40 | $\bar{a}_2$ | $b_2 N_2 a_2$ |
| 6 | 35 | $\bar{a}_3$ | $b_3 N_3 a_3$ |

The next step is to figure out $\bar{a}_i$. After some trial and error, we determine the following:

$$(56)(1) \equiv 1(\mod 8) \therefore \bar{a}_1 = 1$$
$$(40)(5) \equiv 1(\mod 7) \therefore \bar{a}_2 = 3$$
$$(40)(5) \equiv 1(\mod 7) \therefore \bar{a}_3 = 3$$

We complete the table:

| $b_i$ | $N_i$ | $\bar{a}$ | $b_i N_i a_i$ |
|---|---|---|---|
| 3 | 56 | 1 | 168 |
| 1 | 40 | 3 | 120 |
| 6 | 35 | 3 | 630 |

Given the sum of the last column, we have $x \equiv 918(\mod 280)$. Since the spread is 280, the best reportable answer is $x \equiv 78(\mod 280)$