

Curl

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1 Overview

The curl operator describes the rotation tendency of a vector field. Both its input and output are vectors. In the cases when we consider only two dimensions, the result of curl is a scalar.

If a vector field is defined by $F = \langle P, Q \rangle$ then its two dimensional curl is defined as:

$$\nabla \times F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \quad (1)$$

If $F = \langle P, Q, R \rangle$ we can calculate a three dimensional curl with:

$$\nabla \times F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k} \quad (2)$$

From Expressions (1) and (2) we again note that the latter produces a vector, and that its third component is just the two dimensional curl.

2 Geometric Intuition

To start we will note that the orthodox direction of rotation is counter clockwise. Fields that induce a counter clockwise rotation will have a positive curl. We can start out by imagining a particle on the xy plane.

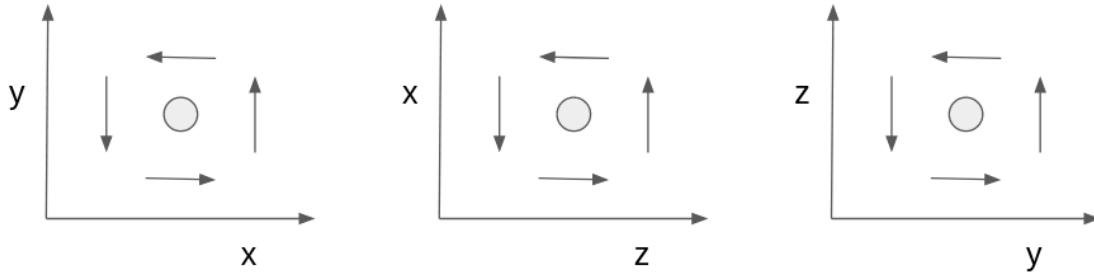


Figure 1

In Figure 1's left graph, we've generalized what happens to a particle when a field induces positive curl. In these diagrams the arrows represent the field. We note that as x increases, the Q component of the field has a tendency to shift from negative to positive. Likewise as y increases, the P component becomes more negative. In order to guarantee a positive value if the field is inducing a counter clockwise rotation we can take the change of R with respect to y (which will be positive) and subtract from it the change of P with respect to y which will be negative. From this intuition, we get $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ as the two dimensional curl.

In our analysis so far we've imagined that the field can only induce rotation on the xy plane. Because z is fixed in this scenario, the two dimensional curl is the \vec{k} component of the three dimensional curl. We can think of this as the observed rotation viewed from z 's perspective.

Looking again at Figure 1, the central diagram describes a hypothetical counter clockwise rotation from the perspective of y . As z increases, the P component of the field becomes more positive and as x increases, the R component becomes more negative. So to guarantee a positive value we can write $\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}$. The same analysis can be performed with Figure 1's right graph.

3 Derivation of Two Dimension Curl

Suppose we have a patch of xy like the one described in Figure 2. We have here a simple connected region where the path C can be divided into $C1 + C2 + C3 + C4$. We have also drawn here some vectors that approximate what the field $F = \langle P, Q, R \rangle$ is doing near these points along the four sub paths.

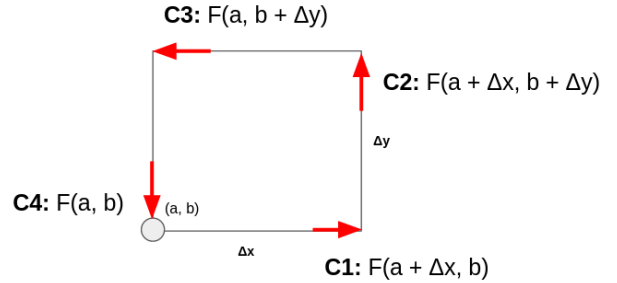


Figure 2

So the line integral for the complete path is:

$$\oint_C F dr = \sum_{i=1}^4 \int_{C_i} F dr$$

For very small values of Δx and Δy we can state the following:

$$\begin{aligned} \int_{C1} F dr &\approx P(a + \Delta x, b) \Delta x \\ \int_{C2} F dr &\approx Q(a + \Delta x, b + \Delta y) \Delta y \\ \int_{C3} F dr &\approx P(a, b + \Delta y) \Delta x \\ \int_{C4} F dr &\approx Q(a, b) \Delta y \end{aligned}$$

Our goal is ultimately to describe the rotational tendency of a field per area of $\Delta x \Delta y$, albeit an infinitesimally small area. Our strategy here is to look at the horizontal (C1, C3) and vertical paths (C2, C4). Let's start with the horizontal:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{P(a + \Delta x, b) \Delta x - P(a, b + \Delta x) \Delta x}{\Delta x \Delta y}$$

Note how we treat C3 as a negative because it goes in the opposite direction of C1. Upon canceling out the Δx and evaluating the limit as x approaches zero we are left with:

$$\lim_{y \rightarrow 0} \frac{P(a, b) - P(a, b + \Delta y)}{\Delta y}$$

If we modify this result by making it negative, we arrive at one of the two components of the two dimension curve:

$$-\lim_{y \rightarrow 0} \frac{P(a, b + \Delta y) - P(a, b)}{\Delta y} = -\frac{\partial P}{\partial y}$$

We can repeat this analysis for the vertical components:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{Q(a + \Delta x, b + \Delta y)\Delta y - Q(a, b)\Delta y}{\Delta x \Delta y}$$

We take the limit as y approaches zero first and are left with the definition of a partial derivative with respect to x , the remaining component of two dimension curl:

$$\lim_{x \rightarrow 0} \frac{Q(a + \Delta x, b) - Q(a, b)}{\Delta x} = \frac{\partial Q}{\partial x}$$

4 Examples

Ex. 1 Calculate $\nabla \times F$ given $F(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x \vec{i} + y \vec{j} + z \vec{k})$

Source:

$$\begin{aligned} \frac{\partial R}{\partial y} &= \frac{zy}{2\sqrt{x^2 + y^2 + z^2}} & \frac{\partial Q}{\partial z} &= \frac{zy}{2\sqrt{x^2 + y^2 + z^2}} & \frac{\partial P}{\partial z} &= \frac{xz}{2\sqrt{x^2 + y^2 + z^2}} \\ \frac{\partial R}{\partial x} &= \frac{xz}{2\sqrt{x^2 + y^2 + z^2}} & \frac{\partial Q}{\partial x} &= \frac{xy}{2\sqrt{x^2 + y^2 + z^2}} & \frac{\partial P}{\partial y} &= \frac{xy}{2\sqrt{x^2 + y^2 + z^2}} \end{aligned}$$

$$\nabla \times F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\nabla \times F = 0 \vec{i} + 0 \vec{j} + 0 \vec{k}$$

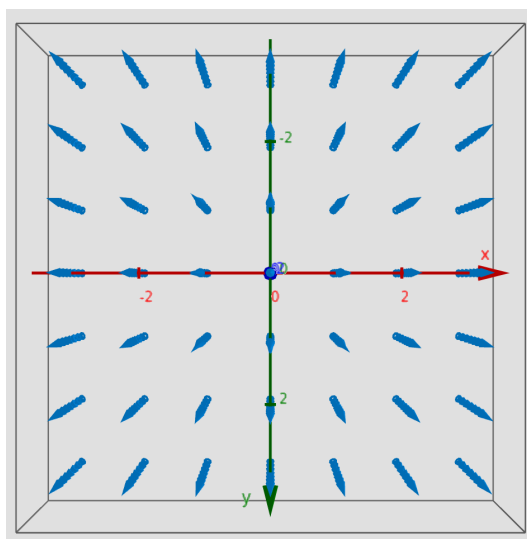


Figure 2

No curl is present. A graph of the field (top down) is shown in Figure 2. We can intuitively see that this field is incapable of inducing rotation.

Ex. 2 Calculate $\nabla \times F$ where $F = xye^z \vec{i} + yze^x \vec{k}$

Source:

$$\begin{aligned} \frac{\partial R}{\partial y} &= ze^x & \frac{\partial Q}{\partial z} &= 0 & \frac{\partial P}{\partial z} &= xye^z \\ \frac{\partial R}{\partial x} &= yze^x & \frac{\partial Q}{\partial x} &= 0 & \frac{\partial P}{\partial y} &= xe^z \end{aligned}$$

$$\nabla \times F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

$$\nabla \times F = ze^x \vec{i} + xye^z - yze^x \vec{j} - xe^z \vec{k}$$