Conservative Vector Fields

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1 Fundamental Theorem of Line Integrals

If C is a smooth curve defined by $\vec{r}(t)$ where $a \leq t \leq b$, and we have a function f whose gradient ∇f is continuous along C then:

$$\int_C \nabla f \; d\vec{r} = f(\,\vec{r}(a)\,) - f(\,\vec{r}(b)\,) \tag{1}$$

Expression (1) is the **Fundamental Theorem of Line Integrals**. The implication of this theorem is that of **path independence**. The line integral for the gradient of a function is independent of the path C, provided we start at point a and end at point b. If a vector field exhibits path independence then we say that the field is **conservative**.

We will discuss the steps to derive Expression (1). Let's suppose f is a function of x, y, and z. The path C is described parametrically by a vector function $r(t) = <x(t), y(t), z(t)>$ We know the following from the definition of line integrals for a vector field:

$$\int_C \nabla f \; d\vec{r} = \int_a^b \nabla f(\,r(t)\,) \cdot r'(t) \; dt$$

Let's expand the dot product of the right hand side:

$$\int_a^b \nabla f(\,r(t)\,) \cdot r'(t) \; dt$$

$$= \int_a^b < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} > \cdot < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dy} > \; dt$$

$$= \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \; dt$$

The above result is an example of a **type I partial derivative**, and is what we would have obtained if we evaluated: $\frac{d}{dt}[f(\,r(t)\,)]$. The integral can now be rewritten like so:

$$\int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \; dt$$

$$= \int_a^b \frac{d}{dt}[\,f(r(t))\,] \; dt$$

By the First Fundamental Theorem of Calculus:

$$\int_a^b \frac{d}{dt}[\,f(r(t))\,] \; dt = f(\,r(b)\,) - f(\,r(a)\,)$$

Ex. 1 Evaluate $\int_C \nabla f \ d\vec{r}$ where $f(x,y) = ye^{x^2-1}+4x\sqrt{y}$. The path C is defined by $r(t) =< 1-t \ , \ 2t^2-2t >$, where $0 \le t \le 2$:

Source: Paul's Online Notes
https://tutorial.math.lamar.edu/Problems/CalcIII/FundThmLineIntegrals.aspx

We can first calculate ∇f:

$$\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$$

$$=< 2xye^{x^2-1} + 4\sqrt{y} \ , \ e^{x^2-1} + \frac{2x}{\sqrt{y}} >$$

We proceed to calculate $r(0)$ and $r(2)$:

$$r(2) =< -1, 4 >$$
$$r(0) =< 1, 0 >$$

We can now evaluate $f(-1,4)$ and $f(1,0)$:

$$f(-1,4) = 4e^0 + -4\sqrt{4} = -4$$
$$f(1,0) = 0$$

We can finally use the fundamental theorem:

$$\int_C \nabla f \ d\vec{r} = f(\ r(2)\ ) - f(\ r(1)\ ) = f(-1,4) - f(1,0)$$

$$= -4$$

2 Test for Conservative Fields

In the previous section we started our analysis from a function f and showed that its gradient, ∇f, is conservative. We could start out with a vector field $\vec{F}$ and determine if it's conservative. Such a test is useful as we can only apply the fundamental theorem of line integrals if the field is conservative.

We will deal with the case of a vector field containing an x and a y component. Given $\vec{F} = <P, Q>$, $\vec{F}$ is conservative if:

$$\frac{\partial}{\partial x}[Q] = \frac{\partial}{\partial y}[P] \tag{2}$$

Deriving this requires understanding curl which we'll discuss in detail on the next chapter. If $\vec{F} = <P, Q>$, $P = x(t)$, and $Q = y(t)$, then the curl is defined as:

$$\text{curl } \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \tag{3}$$

For a function f, its gradient is a vector field $\nabla f = <\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}>$. So if, $\nabla f = \vec{F}$ from the definition of curl, then:

$$\text{curl } \vec{F} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] - \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = 0$$

The curl above evaluates to zero because of **Clairaut's theorem** which states that when taking second partial derivatives the order of differentiation is not important, in other words:

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

$$\frac{\partial}{\partial x}[Q] = \frac{\partial}{\partial y}[P]$$

So if we have a vector field we can test if it's conservative or not by simply knowing the vector's components.

Ex. 2 is the vector field $(xy + y^2)\vec{i} + (x^2 + 2xy)\vec{j}$ conservative?

Source: Stewart, Calculus 8th Edition pg 1094

$$\frac{\partial}{\partial x}[x^2 + 2xy] = 2x + 2y \qquad \frac{\partial}{\partial y}[xy + y^2] = x + 2y$$

Since $\frac{\partial}{\partial x}[Q] \neq \frac{\partial}{\partial y}[P]$, the field is not conservative.

Ex. 3 is the vector field $(y^2 e^{xy})\vec{i} + ((1 + xy)e^{xy})\vec{j}$ conservative?

Source: Stewart, Calculus 8th Edition pg 1094

$$\frac{\partial}{\partial x}[(1 + xy)e^{xy}] = ye^{xy} + (1 + xy)e^{xy}y = 2e^{xy}y + e^{xy}xy^2 \qquad \frac{\partial}{\partial y}[y^2 e^{xy}] = 2e^{xy}y + e^{xy}xy^2$$

Since $\frac{\partial}{\partial x}[Q] = \frac{\partial}{\partial y}[P]$, the field is conservative.

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3 Potential Functions

If a field is conservative, it's possible to recover f from ∇f. The recovered function is known as a **potential function**. Applying algorithm to extract f is shown on the two examples below:

Ex. 4 Find a potential function for $(x^2 y^3)\vec{i} + (x^3 y^2)\vec{j}$.

Source: Stewart, Calculus 8th Edition pg 1094

Step 1: We first check to see if the field is conservative.

$$\frac{\partial}{\partial x}[x^3 y^2] = \frac{\partial}{\partial y}[x^2 y^3] = 3x^2 y^2$$

The field is conservative, we can proceed.

Step 2: Derive a candidate function with respect to x:

If $x^2 y^3$ is the x component of ∇f, then we can integrate it with respect to x to derive f_c:

$$f_c = \int \frac{\partial f}{\partial x}\, dx = \int x^2 y^3\, dx = \frac{x^3 y^3}{3} + h(y)$$

Step 3: Derive a candidate function with respect to y:

We start out f_c from the previous step and differentiate with respect to y:

$$\frac{\partial}{\partial y}\left[\frac{x^3 y^3}{3}\right] = x^3 y^2 + h'(y)$$

Since $\frac{\partial f_c}{\partial y}$ is equivalent to the y component of the field, $h'(y)$ must be zero. We can conclude that the original candidate is a potential function:

$$f = \frac{x^3 y^3}{3}$$

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Ex. 5 Find a potential function for $(y^2 e^{xy})\vec{i} + (\ (1+xy)e^x y\ )\vec{j}$

Source: Stewart, Calculus 8th Edition pg 1095

The field is conservative:

$$\frac{\partial}{\partial y}[y^2 e^x y] = \frac{\partial}{\partial x}[(1+xy)e^{xy}] = 2ye^{xy} + xy^2 e^{xy}$$

Next, we try and find a candidate f_c by integrating the field's x component with respect to x:

$$f_c = \int y^2 e^{xy}\ dx = ye^{xy} + g(y)$$

We can now differentiate the candidate function with respect to y and proceed to compare the result with the field's y component:

$$\frac{\partial}{\partial y}[ye^{xy}] = e^{xy} + xye^{xy} + g'(y)$$

Since the result above matches the y component in the field, we can conclude that $g'(y)$ is zero. The candidate happens to be a potential function:

$$f = ye^{xy}$$

Ex. 6 Find a potential function for $(yz)\vec{i} + (xz)\vec{j} + (xy + 2z)\vec{k}$. Assume the field is conservative.

Source: Stewart, Calculus 8th Edition pg 1095

$$f_c = \int yz \, dx = xyz + g(y, z)$$

Next, differentiate the candidate function with respect to y and compare it against the y component of the field:

$$\frac{\partial}{\partial y}[xyz] = xz$$

Our result is equivalent to the field's y component so we're done with y. Next we differentiate the candidate function with respect to z:

$$\frac{\partial}{\partial z}[xyz] = xy + h'(z)$$

Since the z component of the field is $xy + 2z$, $h'(z)$ must be $2z$. We can conclude the process by integrating $h'(z)$ to derive $h(z)$:

$$\int 2z \, dz = z^2 + C$$

We collect our findings starting with the candidate function:

$$f = xyz + z^2$$

4   Final Comprehensive Example

The Fundamental Theorem of Line Integrals can simplify the calculation of a line integral. Consider the following problem:

Ex.   7 Given a vector field $\vec{F} = (yze^{xz})\vec{i} + (e^{xz}\vec{j}) + (xye^{xz})\vec{k}$ calculate $\int_C \vec{F}\ d\vec{r}$, where C is defined by $r(t) = <t^2 + 1, t^2 - 1, t^2 - 2t>$.

Source: Stewart, Calculus 8th Edition pg 1095

Our first inclination might be to evaluate $\int_a^b F(r(t)) \cdot r'(t)\ dt$. We would quickly find out that this results in a rather lengthy integral.

However, if we can determine that $\vec{F}$ is conservative then we can extract a potential function f and apply the First Fundamental Theorem. In this case, the field is conservative. If P, Q, and R are the components of $\vec{F}$, we find that:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = ze^{xz} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = y(e^{xz} + xze^{xz}) \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = xe^{xz}$$

We can now recover the potential function.

$$f_c = \int yze^{xz}\ dx = ye^{xz} + g(y, z)$$

$$\frac{\partial f_c}{\partial y} = e^{xz} + g'(y, z)$$

Since partial f_c with respect to y is equivalent to the y component of the field, $g'(y, z)$ must be zero. Let's check partial f_c with respect to z:

$$\frac{\partial f_c}{\partial z} = xye^{xz}$$

Since the result is equivalent to the z component of the field, we are done. We can conclude that $f = ye^{xz}$. We are now free to apply the First Fundamental Theorem.

First, let's calculate $r(0)$ and $r(2)$:

$$r(2) = <5, 3, 0> \quad r(0) = <1, -1, 0>$$

We can now calculate the integral:

$$\int_a^b \vec{F}\ d\vec{r} = f(\ r(2)\ ) - f(\ r(1)\ )$$

$$= 3 - -1$$

$$= 4$$