Green's Theorem

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1 Overview

Green's Theorem relates the curl of a vector field to the line integral along around a simply connected region. In the right circumstances it can greatly simplify the line integral calculation. In Figure 1, I and II meet the criteria but III and IV do not. Region III has a path inside of the main path whereas IV is not connected.

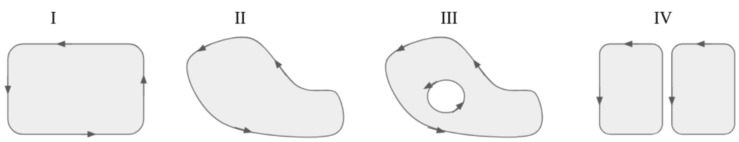

Figure 1

Given a field $F =< P, Q >$ if R is a simply connected region with a boundary C oriented counterclockwise, and if P and Q have continuous first partial derivatives, then:

$$\int_C P\ dx + Q\ dy = \int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA \tag{1}$$

2 Alternative Line Integral Notation

Before we Expression (1) let's explain the notation used on the left, $P\ dx + Q\ dy$. We start with the formulation for line integrals along a path C:

$$\int_C F\ dr = \int_a^b F(r(t)) \cdot r'(t)\ dt$$

Let's suppose $F =< P, Q >$, then developing the dot product would get us:

$$\int_a^b < P, Q > \cdot < x'(t), y'(t) >\ dt$$

$$\int_a^b Px'(t) + Qy'(t)\ dt = \int_a^b Px'(t)\ dt + \int_a^b Qy'(t)\ dt = \int_a^b P\ \frac{dx}{dt}\ dt + \int_a^b Q\ \frac{dy}{dt}\ dt$$

$$\int_C P\ dx + Q\ dy$$

The use of this notation does not affect the calculation, but it does help with organization. Line integral and Green's Theorem problems are often presented in this format.

3 Deriving Green's Theorem

Given a field $F =< P, Q >$ and our reformulated line integral $\int_C P\ dx + Q\ dy$, we'll proceed to calculate the line integral along C. We'll show the $\int_C\ P\ dx$ on the left and $\int_C\ Q\ dy$ on the right, combining the results at the end:

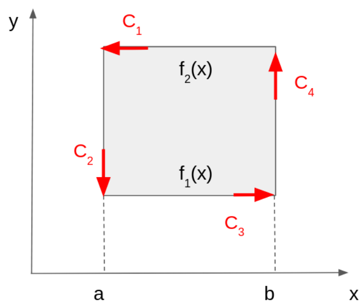

Figure 2

In Figure 2 we've decomposed C into four paths C_1, C_2, C_3, C_4. Since we are integrating with respect to x we can say that $\int_{C_2}\ P\ dx = \int_{C_4}\ P\ dx = 0$. For paths C_1 and C_3 the net line integral is:

$$-\int_a^b\ P(x, f_2(x))\ dx + \int_a^b\ P(x, f_1(x))\ dx$$

The negative value refers to C_1 based on its leftward direction. We can do some additional work with the result above:

$$\begin{aligned}
&-\int_a^b\ P(x, f_2(x))\ dx + \int_a^b\ P(x, f_1(x))\ dx \\
&= -\left(\int_a^b\ P(x, f_2(x)) -\ P(x, f_1(x))\ dx \right)
\end{aligned}$$

By the Second Fundamental Theorem of Calculus:

$$\begin{aligned}
-\left(\int_a^b\ P(x, f_2(x)) -\ P(x, f_1(x))\ dx \right) \\
= -\int_a^b\ \int_{f_1(x)}^{f_2(x)}\ \frac{\partial P}{\partial y}\ dy\ dx \\
= -\int\ \int_R \frac{\partial P}{\partial y}\ dA
\end{aligned}$$

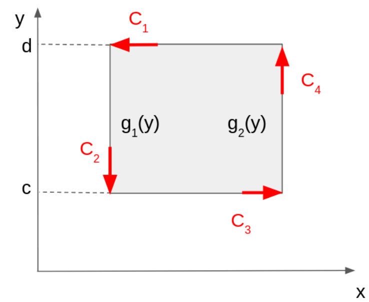

Figure 3

We can repeat this process for $\int Q\ dy$. Based on Figure 3's setup, we can say that $\int_{C_1} Q\ dy = \int_{C_3} Q\ dy = 0$. The net line integral we can come up with is:

$$\int_c^d\ Q(g_2(y), y)\ dy - Q(g_1(y), y)\ dy$$

Applying the Second Fundamental Theorem of Calculus:

$$\begin{aligned}
\int_c^d\ Q(g_2(y), y)\ dy - Q(g_1(y), y)\ dy \\
= \int_c^d\ \int_{g_1(y)}^{g_2(y)}\ \frac{\partial Q}{\partial x}\ dx\ dy \\
= \int\ \int_R \frac{\partial Q}{\partial x}\ dA
\end{aligned}$$

We bring both results together now:

$$\int_C P\ dx + Q\ dy = -\int\ \int_R \frac{\partial P}{\partial y}\ dA + \int\ \int_R \frac{\partial Q}{\partial x}\ dA = \int\ \int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA$$

4 Examples

Ex.1 Calculate the line integral $\int_C ye^x\ dx + 2e^x\ dy$. The path C is defined by a rectangle with points $(0,0), (3,0), (3,4), (0,4)$. Use both Green's Theorem and the traditional approach.

Source: Stewart, Calculus 8th Edition pg 1182

We will first use Green's Theorem:

$$\frac{\partial Q}{\partial x} = 2e^x \qquad \frac{\partial P}{\partial y} = e^x$$

By the theorem we can evaluate the the following double integral:

$$\int_0^4 \int_0^3 2e^x - e^x\ dx\ dy = \int_0^4 (2e^x - e^x)\Big|_0^3\ dy = \int_0^4 e^3 - 1\ dy = (e^3 - 1)y\Big|_0^4 = 4(e^3 - 1)$$

We arrived at the answer with relative ease using Green's Theorem. We will now attempt the long way. Let's define the path going along the bottom edge as C_1, the right edge C_2, the top edge C_3 and the left edge C_4.

C_1:
We can start by defining $r(t) =< t, 0 >$, which makes $r'(t) =< 1, 0 >$. We can now say that $F(r(t)) =< 0, 2e^t >$. Since $F(r(t)) \cdot r'(t) = 0$, then the line integral along C_1 is 0.

C_2:
Here we can say that $r(t) =< 3, t >$ so $r'(t) =< 0, 1 >$. So then $F(r(t)) =< te^3, 2e^3 >$, and $F(r(t)) \cdot r'(t) = 2e^3$. We can now compute the following integral:

$$\int_0^4 2e^3\ dt = 8e^3$$

C_3:
Starting the same way we have $r(t) =< -t, 4 >$ and $r'(t) =< 1, 0 >$. So then $F(r(t)) =< 4e^t, 2e^t >$, and $F(r(t)) \cdot r'(t) = -4e^{-t}$. We are left with the following integral:

$$\int_0^3 -4e^t\ dt = 4e^t\Big|_0^4 = 4e^3 - 4$$

C_4:
For the last segment of the path, we have $r(t) =< 0, t >$, $r'(t) =< 0, 1 >$, $F(r(t)) =< t, 2 >$, and $F(r(t)) \cdot r'(t) = 2$. We just have to complete the following integral:

$$\int_0^4 2\ dt = 8$$

We can conclude the process by adding the line integral for each of the four paths. We will give the result for C_3 and C_4 as negatives since they move left and down respectively:

$$8e^3 - 8 - (4e^3 - 4) = 4(e^3 - 1).$$

Ex.2 Evaluate $\int_C \cos y \; dx + (xy - x \sin y) \; dy$ using Green's Theorem. C is the boundary of the area formed between $y = \sqrt{x}$ and $y = x$.

Source: Larson, Calculus 6th Edition, pg 1061

$$\frac{\partial Q}{\partial x} = y - \sin y \qquad \frac{\partial P}{\partial y} = -\sin y$$

Here is a graph showing the bounds of integration:

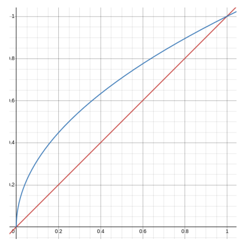

Figure 4

$$\begin{aligned} \int_0^1 \int_x^{\sqrt{x}} y - \sin y - (-\sin y) \; dy \; dx &= \int_0^1 \frac{y^2}{2}\Big|_x^{\sqrt{x}} \; dx \\ &= \frac{1}{2}\int_0^1 x - x^2 \; dx = \frac{1}{2}(\frac{x^2}{2} - \frac{x^3}{3})\Big|_0^1 \\ &= \frac{1}{12} \end{aligned}$$

Ex. 3 Evaluate $\int_C \; y^3 \; dx - x^3 \; dy$ where C is the edge of a circle with radius 2.

Source: Larson, Calculus 6th Edition, pg 1061

$$\frac{\partial Q}{\partial x} = -3x^2 \qquad \frac{\partial P}{\partial y} = 3y^2$$

Applying the theorem, we get:

$$\int_a^b \int_c^d -3x^2 - 3y^2 \; dy \; dx$$

The best way to proceed is to change over to polar coordinates, so we have:

$$\begin{aligned} -3\int_0^{2\pi} \int_0^2 (r^2) r \; dr \; d\theta = -3\int_0^{2\pi} &= -3\int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 \; d\theta \\ -3\int_0^{2\pi} 4 \; d\theta &= -3(4\theta)\Big|_0^{2\pi} \\ &= -24\pi \end{aligned}$$