Conservative Vector Fields

Andy Chong Sam

1 Fundamental Theorem of Line Integrals

If C is a smooth curve defined parametrically by $r(t)$ where $a \leq t \leq b$, and f is a function whose gradient ∇f is continuous along C then:

$$\int_C \nabla f \; dr = f(\ r(a)\ ) - f(\ r(b)\ ) \tag{1}$$

Expression (1) is the **Fundamental Theorem of Line Integrals**. The main implication of this theorem is that of **path independence**. The line integral for the gradient of a function is independent of the path C, provided we start at point a and end at point b. If a vector field exhibits path independence then we say that the field is **conservative**.

We will discuss the steps to derive Expression (1). Let's suppose f is a function of x, y, and z. The path C is described parametrically by a vector function $r(t) =< x(t), y(t), z(t) >$ We know the following from the definition of line integrals for a vector field:

$$\int_C \nabla f \; dr = \int_a^b \nabla f(\ r(t)\ ) \cdot r'(t) \; dt$$

Let's expand the dot product on the right hand side:

$$\begin{aligned} & \int_a^b \nabla f(\ r(t)\ ) \cdot r'(t) \; dt \\ = \int_a^b < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} > \cdot < \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} > \; dt \\ = \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \; dt \end{aligned}$$

This result is an example of a **type I partial derivative**, and is what we would have obtained if we evaluated: $\frac{d}{dt}[f(\ r(t)\ )]$.The integral can now be rewritten:

$$\begin{aligned} & \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \; dt \\ & = \int_a^b \frac{d}{dt}[\ f(r(t))\ ] \; dt \end{aligned}$$

By the First Fundamental Theorem of Calculus:

$$\int_a^b \frac{d}{dt}[\ f(r(t))\ ] \; dt = f(\ r(b)\ ) - f(\ r(a)\ )$$

Ex. 1 Evaluate $\int_C \nabla f\ d\vec{r}$ where $f(x,y) = ye^{x^2-1} + 4x\sqrt{y}$. The path C is defined by $r(t) = < 1-t\ \ ,\ \ 2t^2 - 2t >$, where $0 \leq t \leq 2$:

Source: Paul's Online Notes
https://tutorial.math.lamar.edu/Problems/CalcIII/FundThmLineIntegrals.aspx

We can first calculate ∇f:

$$\begin{aligned}\nabla f &= < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > \\ &= < 2xye^{x^2-1} + 4\sqrt{y}\ \ ,\ \ e^{x^2-1} + \frac{2x}{\sqrt{y}} >\end{aligned}$$

We proceed to calculate $r(0)$ and $r(2)$:

$$\begin{aligned}r(2) &= < -1, 4 > \\ r(0) &= < 1, 0 >\end{aligned}$$

We can now evaluate $f(-1,4)$ and $f(1,0)$:

$$\begin{aligned}f(-1,4) = 4e^0 + -4\sqrt{4} &= -4 \\ f(1,0) &= 0\end{aligned}$$

We can finally use the fundamental theorem:

$$\begin{aligned}\int_C \nabla f\ d\vec{r} = f(\ r(2)\ ) - f(\ r(1)\ ) = f(-1,4) - f(1,0) &= -4 - 0 \\ &= -4\end{aligned}$$

2 Test for Conservative Fields

In the previous section we started with a function f and showed that its gradient, ∇f, is conservative. We could start out with a vector field F and determine if it's conservative. Such a test is useful as we can only apply the fundamental theorem of line integrals if the field is conservative.

We will deal with the case of a vector field containing an x and y component. Given $F =< P, Q >$, F is conservative if:

$$\frac{\partial}{\partial x}[Q] = \frac{\partial}{\partial y}[P] \tag{2}$$

Deriving this requires using curl which we'll discuss in detail on the next chapter. If $\vec{F} =< P, Q >$, $P = x(t)$, and $Q = y(t)$, then the curl is defined as:

$$\text{curl } F = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \tag{3}$$

For a function f, its gradient is a vector field $\nabla f =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$. So if, $\nabla f = F$ from the definition of curl, then:

$$\text{curl } F = \frac{\partial}{\partial x}[\frac{\partial f}{\partial y}] - \frac{\partial}{\partial y}[\frac{\partial f}{\partial x}] = 0$$

The curl above evaluates to zero because of **Clairaut's theorem** which states that when taking second partial derivatives the order of differentiation is not important. We are left with the following equality:

$$\frac{\partial}{\partial x}[\frac{\partial f}{\partial y}] = \frac{\partial}{\partial y}[\frac{\partial f}{\partial x}]$$
$$\frac{\partial}{\partial x}[Q] = \frac{\partial}{\partial y}[P]$$

So if we have a vector field we can test to see if it's conservative by checking its components.

Ex. 2 is the vector field $(xy + y^2)\vec{i} + (x^2 + 2xy)\vec{j}$ conservative?

Source: Stewart, Calculus 8th Edition pg 1094

$$\frac{\partial}{\partial x}[x^2 + 2xy] = 2x + 2y \qquad \frac{\partial}{\partial y}[xy + y^2] = x + 2y$$

Since $\frac{\partial}{\partial x}[Q] \neq \frac{\partial}{\partial y}[P]$, the field is not conservative.

Ex. 3 is the vector field $(y^2 e^{xy})\vec{i} + ((1 + xy)e^{xy})\vec{j}$ conservative?

Source: Stewart, Calculus 8th Edition pg 1094

$$\frac{\partial}{\partial x}[(1 + xy)e^{xy}] = ye^{xy} + (1 + xy)e^{xy}y = 2e^{xy}y + e^{xy}xy^2 \qquad \frac{\partial}{\partial y}[y^2 e^{xy}] = 2e^{xy}y + e^{xy}xy^2$$

Since $\frac{\partial}{\partial x}[Q] = \frac{\partial}{\partial y}[P]$, the field is conservative.

3 Potential Functions

If a field is conservative it's possible to recover f from ∇f. The recovered function is known as a **potential function**. Applying the algorithm to extract f is shown on the following examples:

Ex. 4 Find a potential function for $(x^2y^3)\vec{i} + (x^3y^2)\vec{j}$.

Source: Stewart, Calculus 8th Edition pg 1094

Step 1: We first check to see if the field is conservative.

$$\frac{\partial}{\partial x}[x^3y^2] = \frac{\partial}{\partial y}[x^2y^3] = 3x^2y^2$$

The field is conservative, we can proceed.

Step 2: Derive a candidate function with respect to x:

If x^2y^3 is the x component of ∇f, then we can integrate it with respect to x to find f_c:

$$f_c = \int \frac{\partial f}{\partial x}\, dx = \int x^2y^3\, dx = \frac{x^3y^3}{3} + h(y)$$

Step 3: Derive a candidate function with respect to y:

We start out f_c from the previous step and differentiate with respect to y:

$$\frac{\partial}{\partial y}[\frac{x^3y^3}{3}] = x^3y^2 + h'(y)$$

Since $\frac{\partial f_c}{\partial y}$ is equivalent to the y component of the field, $h'(y)$ must be zero. We can conclude that the original candidate is a potential function:

$$f = \frac{x^3y^3}{3}$$

Ex. 5 Find a potential function for $(y^2e^{xy})\vec{i} + (\ (1+xy)e^xy\ )\vec{j}$

Source: Stewart, Calculus 8th Edition pg 1095

The field is conservative:

$$\frac{\partial}{\partial y}[y^2e^xy] = \frac{\partial}{\partial x}[(1+xy)e^{xy}] = 2ye^{xy} + xy^2e^{xy}$$

Next, we try and find a candidate f_c by integrating the field's x component with respect to x:

$$f_c = \int y^2e^{xy}\ dx = ye^{xy} + g(y)$$

We can now differentiate the candidate function with respect to y and proceed to compare the result with the field's y component:

$$\frac{\partial}{\partial y}[ye^{xy}] = e^{xy} + xye^{xy} + g'(y)$$

Since the result above matches the y component in the field, we can conclude that $g'(y)$ is zero. The candidate happens to be a potential function:

$$f = ye^{xy}$$

Ex. 6 Find a potential function for $(yz)\vec{i} + (xz)\vec{j} + (xy + 2z)\vec{k}$. Assume the field is conservative.

Source: Stewart, Calculus 8th Edition pg 1095

$$f_c = \int yz\ dx = xyz + g(y, z)$$

Next, differentiate the candidate function with respect to y and compare it against the y component of the field:

$$\frac{\partial}{\partial y}[xyz] = xz + g'(y, z)$$

Our result is equivalent to the field's y component so we're done with y. Next we differentiate the candidate function with respect to z:

$$\frac{\partial}{\partial z}[xyz] = xy + h'(z)$$

Since the z component of the field is $xy + 2z$, $h'(z)$ must be $2z$. We now integrate $h'(z)$ to obtain $h(z)$:

$$\int 2z\ dz = z^2 + C$$

We collect our findings starting with the candidate function:

$$f = xyz + z^2$$

4 Final Comprehensive Example

The Fundamental Theorem of Line Integrals can simplify the calculation of a line integral. Consider the following problem:

Ex. 7 Given a vector field $F = (yze^{xz})\vec{i} + (e^{xz}\vec{j}) + (xye^{xz})\vec{k}$ calculate $\int_C F\ dr$, where C is defined by $r(t) =< t^2+1, t^2-1, t^2-2t >$.

Source: Stewart, Calculus 8th Edition pg 1095

Our first inclination might be to calculate $\int_a^b F(r(t)) \cdot r'(t)\ dt$. We would find out that this results in a rather lengthy integral.

However, if F is conservative we can extract a potential function f and apply the First Fundamental Theorem. In this problem, the field is conservative. If P, Q, and R are the components of F, we find that:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = ze^{xz} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = y(e^{xz} + xze^{xz}) \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = xe^{xz}$$

We can now recover the potential function.

$$\begin{aligned} f_c = \int yze^{xz}\ dx &= ye^{xz} + g(y,z) \\ \frac{\partial f_c}{\partial y} &= e^{xz} + g'(y,z) \end{aligned}$$

Since partial f_c with respect to y is equivalent to the y component of the field, $g'(y,z)$ must be zero. Let's check partial f_c with respect to z:

$$\frac{\partial f_c}{\partial z} = xye^{xz} + h'(z)$$

Since the result is equivalent to the z component of the field, we are done. We can conclude that $f = ye^{xz}$. We are now free to apply the First Fundamental Theorem.

First, let's calculate $r(0)$ and $r(2)$:

$$r(2) =< 5, 3, 0 > \qquad r(0) =< 1, -1, 0 >$$

We can now calculate the integral:

$$\begin{aligned} \int_a^b \vec{F}\ d\vec{r} &= f(\ r(2)\ ) - f(\ r(1)\ ) \\ &= 3 - -1 \\ &= 4 \end{aligned}$$

Ex. 1