

Green's Theorem

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1 Overview

Green's Theorem relates the curl of a vector field to the line integral along around a simply connected region. In the right circumstances it can greatly simplify the line integral calculation. In Figure 1, I and II meet the criteria but III and IV do not. Region III has a path inside of the main path whereas IV is not connected.

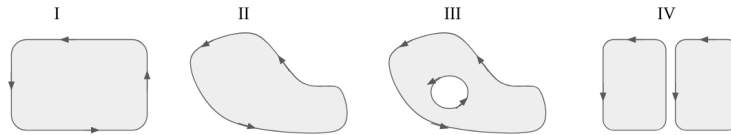


Figure 1

Given a field $F = \langle P, Q \rangle$ if R is a simply connected region with a boundary C oriented counterclockwise, and if P and Q have continuous first partial derivatives, then:

$$\int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad (1)$$

2 Alternative Line Integral Notation

Before we Expression (1) let's explain the notation used on the left, $P dx + Q dy$. We start with the formulation for line integrals along a path C :

$$\int_C F dr = \int_a^b F(r(t)) \cdot r'(t) dt$$

Let's suppose $F = \langle P, Q \rangle$, then developing the dot product would get us:

$$\int_a^b \langle P, Q \rangle \cdot \langle x'(t), y'(t) \rangle dt$$

$$\int_a^b P x'(t) + Q y'(t) dt = \int_a^b P x'(t) dt + \int_a^b Q y'(t) dt = \int_a^b P \frac{dx}{dt} dt + \int_a^b Q \frac{dy}{dt} dt$$

$$\int_C P dx + Q dy$$

The use of this notation does not affect the calculation, but it does help with organization. Line integral and Green's Theorem problems are often presented in this format.

3 Deriving Green's Theorem

Given a field $F = \langle P, Q \rangle$ and our reformulated line integral $\int_C P dx + Q dy$, we'll proceed to calculate the line integral along C . We'll show the $\int_C P dx$ on the left and $\int_C Q dy$ on the right, combining the results at the end:

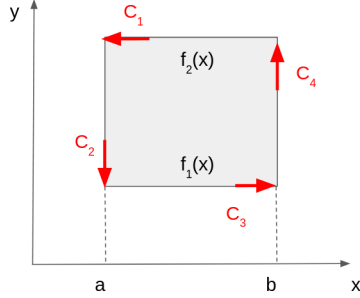


Figure 2

In Figure 2 we've decomposed C into four paths C_1, C_2, C_3, C_4 . Since we are integrating with respect to x we can say that $\int_{C_2} P dx = \int_{C_4} P dx = 0$. For paths C_1 and C_3 the net line integral is:

$$-\int_a^b P(x, f_2(x)) dx + \int_a^b P(x, f_1(x)) dx$$

The negative value refers to C_1 based on its leftward direction. We can do some additional work with the result above:

$$\begin{aligned} & -\int_a^b P(x, f_2(x)) dx + \int_a^b P(x, f_1(x)) dx \\ &= -\int_a^b P(x, f_2(x)) - P(x, f_1(x)) dx \end{aligned}$$

By the Second Fundamental Theorem of Calculus:

$$\begin{aligned} & -\int_a^b P(x, f_2(x)) - P(x, f_1(x)) dx \\ &= -\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dy dx \\ &= -\int \int_R \frac{\partial P}{\partial y} dA \end{aligned}$$

We bring both results together now:

$$\int_C P dx + Q dy = -\int \int_R \frac{\partial P}{\partial y} dA + \int \int_R \frac{\partial Q}{\partial x} dA = \int \int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

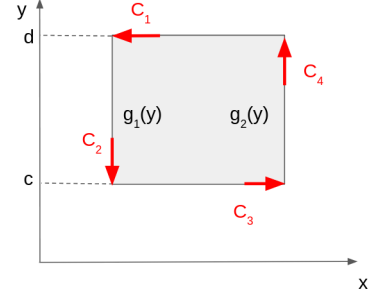


Figure 3

We can repeat this process for $\int Q dy$. Based on Figure 3's setup, we can say that $\int_{C_1} Q dy = \int_{C_3} Q dy = 0$. The net line integral we can come up with is:

$$\int_c^d Q(g_2(y), y) dy - Q(g_1(y), y) dy$$

Applying the Second Fundamental Theorem of Calculus:

$$\begin{aligned} & \int_c^d Q(g_2(y), y) dy - Q(g_1(y), y) dy \\ &= \int_c^d \int_{g_1(y)}^{g_2(y)} \frac{\partial Q}{\partial x} dx dy \\ &= \int \int_R \frac{\partial Q}{\partial x} dA \end{aligned}$$

4 Examples

Ex.1 Calculate the line integral $\int_C ye^x dx + 2e^x dy$. The path C is defined by a rectangle with points $(0, 0), (3, 0), (3, 4), (0, 4)$. Use both Green's Theorem and the traditional approach.

Source: Stewart, Calculus 8th Edition pg 1182

We will first use Green's Theorem:

$$\frac{\partial Q}{\partial x} = 2e^x \quad \frac{\partial P}{\partial y} = e^x$$

By the theorem we can evaluate the the following double integral:

$$\int_0^4 \int_0^3 2e^x - e^x dx dy = \int_0^4 (2e^x - e^x) \Big|_0^3 dy = \int_0^4 e^3 - 1 dy = (e^3 - 1)y \Big|_0^4 = 4(e^3 - 1)$$

We arrived at the answer with relative ease using Green's Theorem. We will now attempt the long way. Let's define the path going along the bottom edge as C_1 , the right edge C_2 , the top edge C_3 and the left edge C_4 .

C_1 :

We can start by defining $r(t) = \langle t, 0 \rangle$, which makes $r'(t) = \langle 1, 0 \rangle$. We can now say that $F(r(t)) = \langle 0, 2e^t \rangle$. Since $F(r(t)) \cdot r'(t) = 0$, then the line integral along C_1 is 0.

C_2 :

Here we can say that $r(t) = \langle 3, t \rangle$ so $r'(t) = \langle 0, 1 \rangle$. So then $F(r(t)) = \langle te^3, 2e^3 \rangle$, and $F(r(t)) \cdot r'(t) = 2e^3$. We can now compute the following integral:

$$\int_0^4 2e^3 dt = 8e^3$$

C_3 :

Starting the same way we have $r(t) = \langle -t, 4 \rangle$ and $r'(t) = \langle -1, 0 \rangle$. So then $F(r(t)) = \langle 4e^t, 2e^t \rangle$, and $F(r(t)) \cdot r'(t) = -4e^t$. We are left with the following integral:

$$\int_0^3 -4e^t dt = 4e^t \Big|_0^3 = 4e^3 - 4$$

C_4 :

For the last segment of the path, we have $r(t) = \langle 0, t \rangle$, $r'(t) = \langle 0, 1 \rangle$, $F(r(t)) = \langle t, 2 \rangle$, and $F(r(t)) \cdot r'(t) = 2$. We just have to complete the following integral:

$$\int_0^4 2 dt = 8$$

We can conclude the process by adding the line integral for each of the four paths. We will give the result for C_3 and C_4 as negatives since they move left and down respectively:

$$8e^3 - 8 - (4e^3 - 4) = 4(e^3 - 1).$$

Ex.2 Evaluate $\int_C \cos y \, dx + (xy - x \sin y) \, dy$ using Green's Theorem. C is the boundary of the area formed between $y = \sqrt{x}$ and $y = x$.

Source: Larson, Calculus 6th Edition, pg 1061

$$\frac{\partial Q}{\partial x} = y - \sin y \quad \frac{\partial P}{\partial y} = -\sin y$$

Here is a graph showing the bounds of integration:

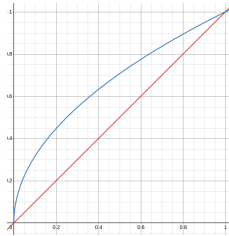


Figure 4

$$\begin{aligned} \int_0^1 \int_x^{\sqrt{x}} y - \sin y - (-\sin y) \, dy \, dx &= \int_0^1 \frac{y^2}{2} \Big|_x^{\sqrt{x}} \, dx \\ &= \frac{1}{2} \int_0^1 x - x^2 \, dx = \frac{1}{2} \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 \\ &= \frac{1}{12} \end{aligned}$$

Ex. 3 Evaluate $\int_C y^3 \, dx - x^3 \, dy$ where C is the edge of a circle with radius 2.

Source: Larson, Calculus 6th Edition, pg 1061

$$\frac{\partial Q}{\partial x} = -3x^2 \quad \frac{\partial P}{\partial y} = 3y^2$$

Applying the theorem, we get:

$$\int_a^b \int_c^d -3x^2 - 3y^2 \, dy \, dx$$

The best way to proceed is to change over to polar coordinates, so we have:

$$\begin{aligned} -3 \int_0^{2\pi} \int_0^2 (r^2)r \, dr \, d\theta &= -3 \int_0^{2\pi} \frac{r^4}{4} \Big|_0^2 \, d\theta \\ &= -3 \int_0^{2\pi} 4 \, d\theta = -3(4\theta) \Big|_0^{2\pi} \\ &= -24\pi \end{aligned}$$