Binary Arithmetic (part 2) [Andy Chong Sam]

1 Binary Multiplication

(I) First, some foundational operations, with their decimal equivalents on the right:

Binary	**Decimal**
$0 \times 0 = 0$	$0 \times 0 = 0$
$1 \times 0 = 0$	$1 \times 0 = 0$
$1 \times 1 = 1$	$1 \times 1 = 1$

(II) We carry out binary multiplication the same way we would with decimal numbers, by arranging our results in columns. Let's consider the case of $2 \times 3 = 6$. In binary 2 is 10 and 3 is 11:

$$\begin{array}{cccc}
 & & \color{red}{\mathbf{1}} & \color{red}{\mathbf{0}} \\
 & \mathrm{x} & 1 & \color{red}{\mathbf{1}} \\
\hline
 & & \color{red}{\mathbf{1}} & \color{red}{\mathbf{0}} \\
 & 1 & 0 & \\
\hline
 & 1 & 1 & 0
\end{array}$$

We populate the first result column by carrying out $1 \times 0 = 0$ followed by $1 \times 1 = 1$. This is show in red above.

$$\begin{array}{cccc}
 & & \color{blue}{\mathbf{1}} & \color{blue}{\mathbf{0}} \\
 & \mathrm{x} & \color{blue}{\mathbf{1}} & 1 \\
\hline
 & & 1 & 0 \\
 & \color{blue}{\mathbf{1}} & \color{blue}{\mathbf{0}} & \\
\hline
 & 1 & 1 & 0
\end{array}$$

We now multiply the digits in blue $1 \times 0 = 0$ and $1 \times 1 = 1$ and populate the second result column. We conclude by adding the columns to get 110 which is indeed the binary representation of 6.

(III) Now let's try a trickier operation, suppose we wanted to evaluate $7 \times 7 = 49$ which in binary is 111×111. When we finish the multiplication phase we are left with the following three rows:

$$\begin{array}{ccccc}
 & & 1 & 1 & 1 \\
 & \mathrm{x} & 1 & 1 & 1 \\
\hline
 & & 1 & 1 & 1 \\
 & 1 & 1 & 1 & \\
1 & 1 & 1 & &
\end{array}$$

After adding the first two columns we have:

$$\begin{array}{ccccc}
 & & 1 & 1 & 1 \\
 & \mathrm{x} & 1 & 1 & 1 \\
\hline
 & & {}^{\color{red}{1}}1 & 1 & 1 \\
 & 1 & 1 & 1 & \\
1 & 1 & 1 & & \\
\hline
 & & & 0 & 1
\end{array}$$

On the third column, we are evaluating $1 + 1 + 1 + 1 = 4$ which in binary is 100. We will record the right most zero, send the center zero to the next column's carry, and finally the one to the column after that. This result is shown below in blue:

$$\begin{array}{cccccc}
 & & & 1 & 1 & 1 \\
 & & \mathrm{x} & 1 & 1 & 1 \\
\hline
 & & & {}^{\color{red}{1}}1 & 1 & 1 \\
 & & {}^{\color{blue}{0}}1 & 1 & 1 & \\
 & {}^{\color{magenta}{1}}_{\color{blue}{1}}1 & 1 & 1 & & \\
\hline
1 & 1 & \color{magenta}{0} & \color{blue}{0} & \color{red}{0} & 1
\end{array}$$

Adding all the columns gets us 110001, the binary representation of 49.

2 Binary Division

(I) As you might have guessed we can recycle another technique from decimal math. Binary division is best handled through long division. We will use the following terms: the dividend (the number we are dividing), divisor (what we are dividing by), the quotient, and the remainder.

(II) Let's consider the case of $4 \div 2 = 2$. The number 4 in binary is 100 and 2 is 10, so we have:

```
      1
   ______
10 | 100
     10
   ----
      0
```

On the calculation above, the first step is shown, the 10 in the divisor goes evenly into the first two digits of the dividend producing a remainder of 0. We pull down the next digit in the dividend:

```
      10
   ______
10 | 100
     10
   ----
      00
```

Since 00 is less than 10 we place a 0 on the quotient next to the one. Lastly 10 can go 0 times into 00 so documenting this gets us a quotient of 10, the binary representation of 2:

```
      10
   ______
10 | 100
     10
   ----
      00
      00
     ---
       0
```

(III) Now let's try the operation $15 \div 2$, which will result in a quotient of 7 and a remainder of 1. In binary, 15 is 1111. Here is the first step:

```
      1
   _______
10 | 1111
     10
   ----
     011
```

In the next step, after the 1 is pulled down, we can fit 10 into 11 but with a remainder of 1 (this operation is the subtraction $011 - 010$):

```
      11
   _______
10 | 1111
     10
   ----
     011
     010
     ----
       11
```

After again pulling down the next digit, which happens to be a 1, we can fit the 10 into 11 with a remainder of 1:

```
      111
   _______
10 | 1111
     10
   ----
     011
     010
     ----
       11
       10
       --
        1
```

There are no more digits to pull down so we are done. Upon completing the final subtraction we find that the result is a quotient of 111 (binary for 7) and a remainder of 1.