Green's Theorem

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1 Overview

Green's Theorem relates the curl of a vector field to the line integral along around a simply connected region. In the right circumstances it can greatly simplify the line integral calculation. In Figure 1, I and II meet the criteria but III and IV do not. Region III has a path inside of the main path whereas IV is not connected.

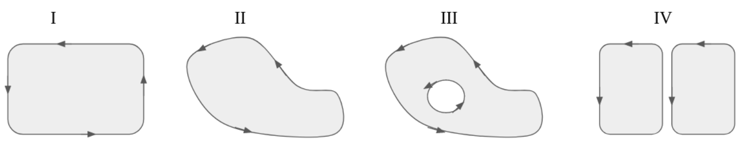

Figure 1

Given a field $F =< P, Q >$ if R is a simply connected region with a boundary C oriented counterclockwise, and if P and Q have continuous first partial derivatives, then:

$$\int_C P\ dx + Q\ dy = \int \int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\ dA \tag{1}$$

2 Alternative Line Integral Notation

Before we Expression (1) let's explain the notation used on the left, $P\ dx + Q\ dy$. We start with the formulation for line integrals along a path C:

$$\int_C F\ dr = \int_a^b F(r(t)) \cdot r'(t)\ dt$$

Let's suppose $F =< P, Q >$, then developing the dot product would get us:

$$\int_a^b < P, Q > \cdot < x'(t), y'(t) >\ dt$$

$$\int_a^b Px'(t) + Qy'(t)\ dt = \int_a^b Px'(t)\ dt + \int_a^b Qy'(t)\ dt = \int_a^b P\ \frac{dx}{dt}\ dt + \int_a^b Q\ \frac{dy}{dt}\ dt$$

$$\int_C P\ dx + Q\ dy$$

The use of this notation does not affect the calculation, but it does help with organization. Line integral and Green's Theorem problems are often presented in this format.