Eigenvectors

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April 5, 2023

1 Introduction

An **eigenvector** is a vector that preserves its direction in a transformation. A more formal definition is as follows - it is a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. In this formula, A is the transformation matrix, $\vec{v}$ represents the vector input to the transformation and λ is a scalar called an **eigenvalue**. The intuition behind the more formal definition is that if indeed a vector's direction remains unchanged, then it should only differ from the transformation output in scaling.

The process to find an eigenvalue relies on a rearrangement of the definition we discussed:

$$A\vec{v} = \lambda\vec{v}$$

Since $\lambda\vec{v}$ is the same as $\lambda\vec{v}I$, where I is a suitable identity matrix:

$$A\vec{v} - \lambda\vec{v}I = 0 \therefore \vec{v}(A - \lambda I) = 0$$

Of particular interest is $det(A - \lambda I) = 0$ as it produces a **characteristic polynomial**. The λ values that solve for this equation are the eigenvalues.

2 Calculation

Eigenvector and Eigenvalue problems typically involve the following steps:

1. Determine the vector $A - \lambda I$
2. Derive the characteristic polynomial by taking the determinant of $A - \lambda I$
3. Calculate the root(s) of the characteristic polynomial
4. Substitute the root into λ for the matrix obtained in step 1
5. Simplify step 4 by using an augmented matrix
6. Collect the eigenvector

3 Sheer Example

We saw in the linear transformation section an example of sheering:

$$\begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{pmatrix}$$

Evaluate $A - \lambda I$:

$$\begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & \frac{3}{2} \\ 0 & 1-\lambda \end{pmatrix}$$

Evaluate $det(A - \lambda I)$:

$$det(A - \lambda I) = (1-\lambda)^2$$

Determine the eigenvalue(s) λ:

$$(1-\lambda)^2 = 0 \therefore \lambda = 1$$

Insert the eigenvalue(s) into $A - \lambda I$:

$$\begin{pmatrix} 1-1 & \frac{3}{2} \\ 0 & 1-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solve for $\vec{v}$ in the system $\vec{v}(A - \lambda I) = \vec{0}$:

$$\begin{pmatrix} 0 & \frac{3}{2} \\ 0 & 0 \end{pmatrix} \xrightarrow{\frac{2}{3}R_1 = R_1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_1 <-> R_2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

If $\vec{v} =< x, y >$, then $y = 0$ and $x \in \mathbb{R}$. In other words, any vector that has $y = 0$ and any value of x is a candidate to be an eigenvector. We typically report the simplest vector that meets this criteria, so $< 1, 0 >$ is an eigenvector.

We can verify this answer by plugging in a vector that conforms to the description above (any real number for x, and 0 for y):

$$\begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

The resulting vector $< 5, 0 >$ differs from the eigenvector only by a factor, and thus it has the same direction as $< 1, 0 >$.

The transformation above is an example of a sheer. In the context of an image transformation, we can imagine that every vector shifts to the right or left only. On the image transformation in Figure 1, the blue arrow will have the same direction as the transformed vector, differing only by scale.

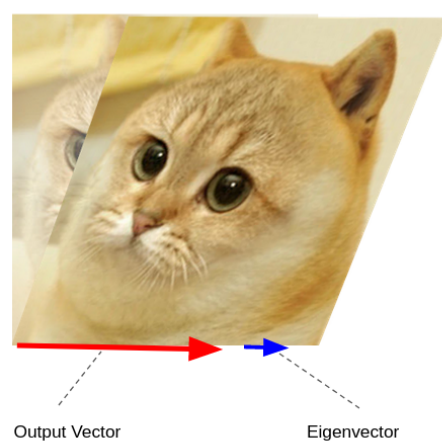

Figure 1

Suppose $\vec{v} =< -1, 0 >$. The transformation produces $< -1, 0 >$. The eigenvector is still valid, as we can get $< -1, 0 >$ by multiplying the eigenvector by -1. Although $< -1, 0 >$ points in the opposite direction, in the mathematical sense, this is still considered scaling and not a change in direction.

4 Vertical Scaling Example

Previously we examined this transformation:

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

Let's determine its eigenvalue and eigenvectors:
Evaluate $A - \lambda I$:

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 0 \\ 0 & \frac{3}{2}-\lambda \end{pmatrix}$$

Evaluate $det(A - \lambda I)$:

$$det(A - \lambda I) = (1-\lambda)(\frac{3}{2} - \lambda)$$

Determine the eigenvalue(s) λ:

$$\lambda = 1, \frac{3}{2}$$

Determine the eigenvector for $\lambda = 1$:

$$\begin{pmatrix} 1-1 & 0 \\ 0 & \frac{3}{2}-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \xrightarrow{2R_2 = R_2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Given $\vec{v} =< x, y >$, x can be any real number, but y has to equal 0. The eigenvector is $< 1, 0 >$.

Determine the eigenvector for $\lambda = \frac{3}{2}$:

$$\begin{pmatrix} 1-\frac{3}{2} & 0 \\ 0 & \frac{3}{2}-\frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{-2R_2 = R_2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Given $\vec{v} =< x, y >$, x is zero, but y can be any real number. The eigenvector is $< 0, 1 >$.

We can spot check with an input vector that meets the above criteria, in this case $< 0, 2 >$

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

The resulting vector $< 0, 3 >$ differs from the eigenvector $< 0, 1 >$ by only scaling. If we had plugged in $< 2, 0 >$ we would have obtained $< 2, 0 >$ which differs from the eigenvector $< 1, 0 >$ only by scale.

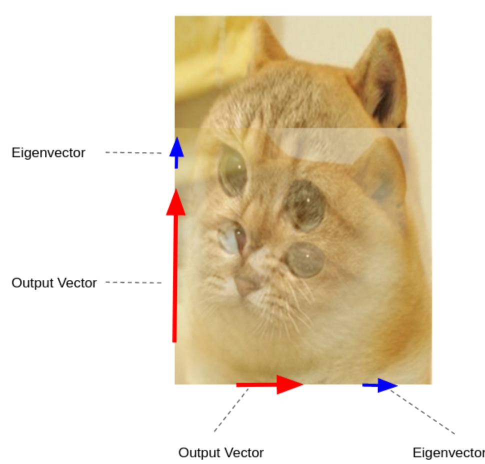

Figure 2

5 Additional Problems

Ex. 1 Determine the eigenvalues and eigenvectors for matrix A:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

First we determine the eigenvalues:

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} \therefore det(A - \lambda I) = \lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1)$$

We have two eigenvalues: -2 and -1.

We can start deriving the eigenvector starting with $\lambda = -1$:

$$A - (-1)I = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_2 = R_2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2 = R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Given $\vec{v} = (x, y)$ then the solution to the system is $x + y = 0$ or $x = -y$, where $x, y \in \mathbb{R}$. Any vector that meets this criteria will work, so we pick the simplest. **The eigenvector is** $< 1, -1 >$**.**

We now derive the eigenvector for $\lambda = -2$:

$$A - (-2)\lambda = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$$

$$A - (-2)\lambda = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \xrightarrow{R_2 + R_1 = R_2} \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$$

Given $\vec{v} = (x, y)$, the solution to they system is $2x + y = 0$, or $y = -2x$, where $x, y \in \mathbb{R}$. **The eigenvector is** $< 1, -2 >$.

Ex. 2 Determine the eigenvalues and eigenvectors for matrix A:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}$$

$$A - I\vec{\lambda} = \begin{pmatrix} 1-\lambda & 0 & -1 \\ 1 & -2-\lambda & 1 \\ 1 & 2 & -3-\lambda \end{pmatrix}$$

To determine $det(A - I\lambda)$ we will use the triple scalar product:

$$\begin{aligned} (1) &: (1-\lambda)(-2-\lambda)(-3-\lambda) = -\lambda^3 - 4\lambda^2 - \lambda + 6 \\ (2) &: (0)(1)(1) = 0 \\ (3) &: (-1)(1)(2) = -2 \\ (4) &: (1)(-2-\lambda)(-1) = 2 + \lambda \\ (5) &: (2)(1)(1-\lambda) = 2 - 2\lambda \\ (6) &: (-3-\lambda)(1)(0) = 0 \end{aligned}$$

We take the add operations (1), (2) and (3):

$$\begin{aligned} &(-\lambda^3 - 4\lambda^2 - \lambda + 6) + 0 - 2 \\ &= -\lambda^3 - 4\lambda^2 - \lambda + 4 \end{aligned}$$

We take the add operations (4), (5) and (6):

$$\begin{aligned} &(2 + \lambda) + (2 - 2\lambda) + 0 \\ &= 4 - \lambda \end{aligned}$$

To get the determinant, we take the sum of operations (1), (2), and (3) and subtract from it the sum of (4), (5), and (6):

$$\begin{aligned} det(A - I\lambda) &= (-\lambda^3 - 4\lambda^2 - \lambda + 4) - (4 - \lambda) \\ det(A - I\lambda) &= -\lambda^3 - 4\lambda^2 \end{aligned}$$

We now want to solve for $-\lambda^3 - 4\lambda^2 = 0$. After a minor rewrite we get $-\lambda^2(\lambda + 4)$. This reveals two roots which are the eigenvalues. **The eigenvalues are 0 and -4**.

Let's determine the eigenvector for $\lambda = 0$:

$$A - (0)I = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \xrightarrow{R_2 - R_1 = R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 1 & 2 & -3 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_2 = R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 2 & -3 \end{pmatrix} \xrightarrow{R_3 - R_1 = R_3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{R_3 - 2R_2 = R_3}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_3 = R_3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + R_3 = R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The eigenvector is $< 1, 1, 1 >$

Ex. 2 continued...

Let's determine the eigenvector for $\lambda = -4$:

$$A - (-4)\lambda = \begin{pmatrix} 5 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2 = R_3} \begin{pmatrix} 5 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 <-> R_2} \begin{pmatrix} 1 & 2 & 1 \\ 5 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 - 5R_1 = R_2}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -10 & -6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-\frac{1}{10}R_2 = R_2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - 2R_2 = R_1} \begin{pmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & 0 \end{pmatrix}$$

We are left with a system consisting of $x - \frac{1}{5}s = 0$, $y + \frac{3}{5}s = 0$, $z = s$, where $s \in \mathbb{R}$.

A suitable eigenvector that matches the above criteria is $< 1, -3, 5 >$