Determinant

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1 Geometric Intuition of Determinants

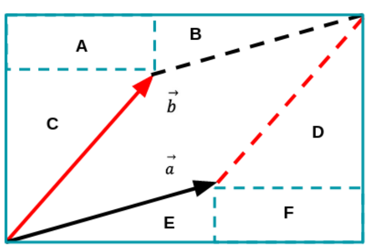
This document offers a geometric intuition of the determinant for a 2x2 and a 3x3 matrix. We start with the idea of a **square matrix** which is one that has an equal number of rows and columns. A 2x2 square matrix is therefore capable of holding two vectors each containing 2 components, and a 3x3 matrix can hold three vectors each containing 3 components.

We will show that the determinant of a 2x2 matrix is the area of the parallelogram formed by two vectors. It can also be shown that the determinant of a 3x3 matrix is the volume of a parallelepiped formed by three vectors.

2 A 2x2 matrix

To show that the determinant of a 2x2 matrix is the area of the inscribed parallelogram from figure 1 we can start by defining matrix A comprised of two generic vectors. This vector along with its determinant is shown below:

$$A = \begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix} \therefore det(A) = a_x b_y - b_x a_y$$



(Total Area) $= (a_x + b_x)(a_y + b_y) = a_x a_y + a_x b_y + b_x a_y + b_x b_y$
(A & F) $= b_x a_y$, **(B & E)**$=\frac{1}{2} a_x a_y$, **(C & D)**$=\frac{1}{2} b_x b_y$

Figure 1

The area of the inscribed parallelogram will be the total rectangle area minus the sum of the named components. This result is the determinant of matrix A:

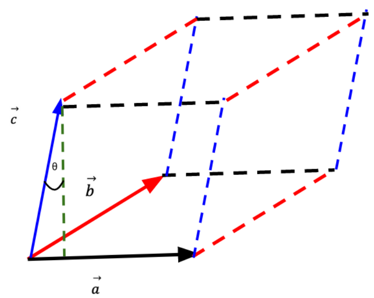
$$\begin{aligned} a_x a_y + a_x b_y + b_x a_y + b_x b_y - 2b_x a_y - a_x a_y - b_x b_y \\ = a_x b_y + b_x a_y - 2b_x a_y \\ = a_x b_y - b_x a_y \end{aligned}$$

In the case of a 2x2 matrix, the determinant is equivalent to the cross product of the two vectors. Since it's a scalar we can refer to this calculation as: $||\vec{a} \times \vec{b}||$.

3 A 3x3 matrix

The determinant of a 3x3 matrix is the volume of a parallelepiped formed by three vectors. This can be conveyed through the following matrix:

$$B = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} \therefore det(B) = (a_x b_y c_z + a_z b_x c_y + a_y b_z c_x) - (a_z b_y c_x + a_x b_z c_y + a_y b_x c_z)$$



The volume of such a figure will be its base multiplied by its height.

Base: $||\vec{a} \times \vec{b}||$

Height: $||\vec{c}||cos\theta$

Volume: $||\vec{a} \times \vec{b}||\ ||c||cos\theta$

Using the law of cosines, the volume can be rewritten to: $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

Figure 2

We can now carry out the triple scalar operation which will the determinant of matrix B:

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{pmatrix} a_y b_z - a_z b_y & -a_x b_z + a_z b_x & a_x b_y - a_y b_x \end{pmatrix} \cdot \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}$$

$$\begin{aligned} &c_x(a_y b_z - a_z b_y) + c_y(-a_x b_z + a_z b_x) + c_z(a_x b_y - a_y b_x) \\ &= a_y b_z c_x - a_z b_y c_x - a_x b_z c_y + a_z b_x c_y + a_x b_y c_z - a_y b_x c_z \\ &= (a_x b_y c_z + a_z b_x c_y + a_y b_z c_x) - (a_z b_y c_x + a_x b_z c_y + a_y b_x c_z) \end{aligned}$$

4 Determinant and Independence

There are many useful properties for the determinant, but in this document we'll highlight the relationship between linear independence and the determinant. **If a set of vectors is linearly dependent, then its determinant is zero.**

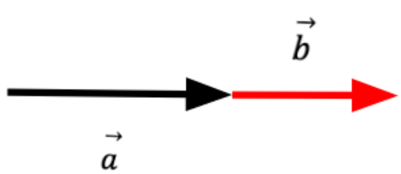


Figure 3

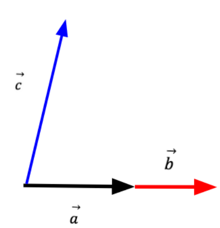


Figure 4

In figure 3, we see two generic vectors that are dependent ($\vec{b} = k\vec{a}$). There is no way a parallelogram can be formed, and thus the area (or determinant) will always be zero. Likewise, in figure 4, $\vec{a}$ and $\vec{b}$ are still dependent. A parallelepiped cannot be formed under these circumstances and as such its volume (or determinant) will be zero.