Compounding Interest and Euler's Number (e)

Andy Chong Sam

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1 Compounding Interest

Describing compounding interest for a rate of return r across time periods t for a fixed payment a takes the form:

$$a(1+r)^t \tag{1}$$

We can derive expression 1 as follows:

$$t = 0 \rightarrow a(1+r)^0 = a$$

At t=1, we need to add the original investment plus any interest earned (ar) during the period.

$$t = 1 \rightarrow a + ar$$

Now when t=2, we add the amount we started with(a+ar), to the interest earned in this meriod (a+ar)r.

$$\begin{aligned} t = 2 \rightarrow (a+ar) + (a+ar)r \\ = (a+ar)(1+r) \\ = a(1+r)(1+r) \\ = a(1+r)^2 \end{aligned}$$

Now for t=3, the initial amount is (a + ar)(1+ r), or the amount at t=2. The interest earned at t=3 is therefore (a+ar)(1+r)r.

$$\begin{aligned} t = 3 \rightarrow (a+ar)(1+r) + (a+ar)(1+r)r \\ = (a+ar)(1+r)(1+r) \\ = a(1+r)(1+r)(1+r) \\ = a(1+r)^3 \end{aligned}$$

We can see how that the total value of an investment can be described as by: $V = a(1+r)^t$, where t is the number of elapsed periods, r is the interest, and a is the initial amount.

2 Deriving Euler's Number (e)

Suppose that a $1 investment makes 100 % interest. If interest is credited once at the end of the year, then the value at the end of the year is just 1 + 1 = 2. Now suppose that the interest will be credited twice a year, the end value will be 2.25.

$$1(1+\frac{1}{2})^2 = 2.25$$

As you increase the number of times the interest is credited, we will eventually converge at e:

$$\begin{aligned} (1+\frac{1}{2})^2 &= 2.25 \\ (1+\frac{1}{100})^{100} &\approx 2.7048 \\ (1+\frac{1}{1000})^{1000} &\approx 2.7169 \\ (1+\frac{1}{10000})^{10000} &\approx 2.7181 \\ (1+\frac{1}{10000})^{100000} &\approx 2.7182 \end{aligned}$$

The value of e is 2.71828... The formal definition of e is therefore:

$$e = \lim_{x \to \infty} (1+\frac{1}{n})^n \tag{2}$$