Elementary Row Operations

Andy Chong Sam

October 23, 2022

1 The Three Operations

Given a matrix that represents the coefficients in a linear system, the three operations are:

- Exchange any two rows
- Multiply a row by a constant
- Add a multiple of one row to another row

2 Graphical Intuition

Elementary row operations can be applied to solve systems of linear equations. We will first consider a 2 variable example (which can be solved more easily with algebra) and a 3 variable example (where these operations become more useful):

Ex. 1 Find the point of intersection between $y = 1 + 2x$ and $y = 5 - 2x$
We will first rewrite these as $-2x + y = 1$ and $2x + y = 5$ and fill in the matrix:

$$\left(\begin{array}{cc|c} -2 & 1 & 1 \\ 2 & 1 & 5 \end{array}\right) \xrightarrow{-R_1 = R_1} \left(\begin{array}{cc|c} 2 & -1 & -1 \\ 2 & 1 & 5 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 = R_2} \left(\begin{array}{cc|c} 2 & -1 & -1 \\ 1 & \frac{1}{2} & \frac{5}{2} \end{array}\right) \xrightarrow{R_1 - R_2 = R_1} \left(\begin{array}{cc|c} 1 & -\frac{3}{2} & -\frac{7}{2} \\ 1 & \frac{1}{2} & \frac{5}{2} \end{array}\right) \xrightarrow{R_2 - R_1 = R_2}$$

$$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 2 & 6 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 = R_2} \left(\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1 - \frac{1}{2}R_2 = R_1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \end{array}\right)$$

Solution: $x = 1, y = 3$.

Ex. 2 Describe the intersection between $2x + 3y - 4z = 3$ and $x + 6y - 5z = 8$.

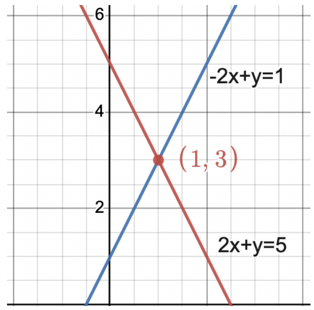
$$\left(\begin{array}{ccc|c} 2 & 3 & -4 & 3 \\ 1 & 6 & -5 & 8 \end{array}\right) \xrightarrow{R_1 - R_2 = R_1} \left(\begin{array}{ccc|c} 1 & -3 & 1 & -5 \\ 1 & 6 & -5 & 8 \end{array}\right) \xrightarrow{R_2 - R_1 = R_2} \left(\begin{array}{ccc|c} 1 & -3 & 1 & -5 \\ 0 & 9 & -6 & 13 \end{array}\right) \xrightarrow{\frac{1}{9}R_2 = R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & -5 \\ 0 & 1 & -\frac{2}{3} & \frac{13}{9} \end{array}\right) \xrightarrow{R_1 + 3R_2 = R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{2}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{13}{9} \end{array}\right)$$

We are left with the equations $x - z = -\frac{2}{3}$ abd $y - \frac{2}{3}z = \frac{13}{9}$. If we parameterize z with s we obtain the parametric equation of a line:

Solution: $x = -\frac{2}{3} + s$, $y = \frac{13}{9} + \frac{2}{3}s$, $z = s$, where $s \in \mathbb{R}$

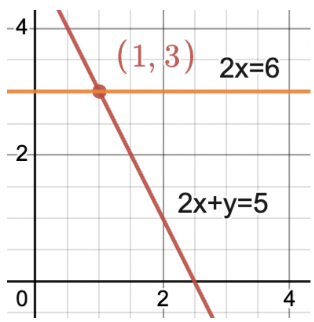
Shown below are the two lines from example 1.



Let's suppose we applied the operation $R_1 + R_2 = R_1$:

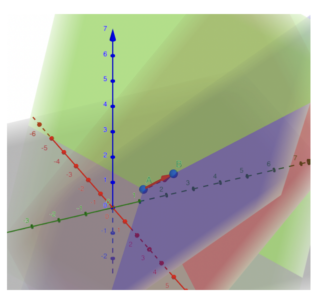
$$\left(\begin{array}{cc|c} -2 & 1 & 1 \\ 2 & 1 & 5 \end{array}\right) \xrightarrow{R_1+R_2=R_1} \left(\begin{array}{cc|c} 0 & 2 & 6 \\ 2 & 1 & 5 \end{array}\right)$$

Here is now the graph containing with the new R_1, $2y = 6$.



Notice that the solution has not changed.

Repeating this experiment with Example 2 leads us to the same conclusion. Suppose we applied the operation $R_1 - R_2 = R_1$ to the matrix in its starting state. On the graph below, the original R_1 is the purple plane, and R_2 is the red plane. The transformed R_1 is the mint plane. The formula for the transformed R_1 is $x - 3y + z = -5$. The small red arrow is a vector representative of the solution line obtained in Example 2. We note that the solution line is shared by all three planes.



In fact, none of the elementary row operations will change the solution to a linear system. On section 4, we'll show the intuition for why this is the case.

3 Elementary Matrices

An elementary matrix is the result of applying an elementary row operation to an identity matrix. Our claim is that multiplying an elementary matrix by the original matrix is the same as applying the row operation directly to an original matrix.

Let's suppose we applied the operation $R_1 - R_2 = R_1$ to the matrix we started with in Example 2:

$$\left(\begin{array}{ccc|c} 2 & 3 & -4 & 3 \\ 1 & 6 & -5 & 8 \end{array}\right) \xrightarrow{R_1 - R_2 = R_1} \left(\begin{array}{ccc|c} 1 & -3 & 1 & -5 \\ 1 & 6 & -5 & 8 \end{array}\right)$$

Let's see what happens when we apply the same operation to an identity matrix, then multiply the resulting elementary matrix by the original matrix. The identity matrix I_2 is suitable in this case given the dimensions of the original array.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2 = R_1} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 2 & 3 & -4 & 3 \\ 1 & 6 & -5 & 8 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -3 & 1 & -5 \\ 1 & 6 & -5 & 8 \end{array}\right)$$

We arrive at the same matrix regardless of the approach taken. We can show that this will always be the case for these dimensions regardless of the type of row operation.

Exchanging two rows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \end{array}\right) = \left(\begin{array}{ccc|c} e & f & g & h \\ a & b & c & d \end{array}\right)$$

Notice that this is equivalent to applying $R_1 \Leftrightarrow R_2$ directly to the original matrix:

$$\left(\begin{array}{ccc|c} e & f & g & h \\ a & b & c & d \end{array}\right) \xrightarrow{R_1 \Leftrightarrow R_2} \left(\begin{array}{ccc|c} e & f & g & h \\ a & b & c & d \end{array}\right)$$

Multiply a row by a constant:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{kR_1 = R_1} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \end{array}\right) = \left(\begin{array}{ccc|c} ka & kb & kc & kd \\ e & f & g & h \end{array}\right)$$

This is equivalent to applying $kR_1 \Leftrightarrow R_1$ directly to the original matrix:

$$\left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \end{array}\right) \xrightarrow{kR_1 = R_1} \left(\begin{array}{ccc|c} ka & kb & kc & kd \\ e & f & g & h \end{array}\right)$$

Add a multiple of one row to another:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 + kR_2 = R_1} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \end{array}\right) = \left(\begin{array}{ccc|c} a + ke & b + kf & c + kg & d + kh \\ e & f & g & h \end{array}\right)$$

This is equivalent to applying $R_1 + kR_2 = R_1$ directly to the original matrix:

$$\left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \end{array}\right) \xrightarrow{R_1 + kR_2 = R_1} \left(\begin{array}{ccc|c} a + ke & b + kf & c + kg & d + kh \\ e & f & g & h \end{array}\right)$$

4 Why Solutions Don't Change

We previously saw how elementary operations preserve the solution to a system, if it exists. In other words applying a row operation will not change the solution. Here is an insight into how this works, suppose we have a generic system of three equations and three variables with the equations on the left and the matrix on the right.

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \\ ix + jy + kz &= l \end{aligned} \qquad \left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{array}\right)$$

Exchanging two rows: A silly proof for this is as follows - changing the order in which you plot the equations onto a graph does not affect the solution ☺

Multiply the row by a constant: Suppose that we performed the operation $kR_1 = R_1$. So we get the following: $k(ax + by + cz) = kd$. Since d is just $ax + by + cz$ we can do the following substitution:

$$\begin{aligned} k(ax + by + cz) &= kd \\ k(ax + by + cz) &= k(ax + by + cz) \\ \cancel{k}(ax + by + cz) &= \cancel{k}(ax + by + cz) \end{aligned}$$

We see that we ended up in a situation where if we cancel the k term on both sides we are left with the original, and therefore the solution to the system cannot change.

Adding a multiple of one row to another: Suppose we wanted to perform the operation $R_1 + kR_2 = R_1$:

$$\begin{aligned} ax + by + cz + k(ex + fy + gz) &= d + kh \\ ax + by + cz + k(ex + fy + gz) &= d + k(ex + fy + gz) \\ ax + by + cz + \cancel{k(ex + fy + gz)} &= d + \cancel{k(ex + fy + gz)} \\ ax + by + cz &= ax + by + cz \end{aligned}$$

We have once again arrived at a situation where there are cancellable terms that allow us to revert back to the original equation, and therefore the solution cannot change.

The key word here is **invertability**. There is some mechanism that allows us to undo the work performed by the row operation. This process can be visualized through matrices as well. We already saw how we could apply elementary row operations by using elementary matrices.

Suppose our target matrix is A and we have an identity matrix of suitable dimensions I_n which has been transformed using a row operation to J. There exists an inverse matrix J^{-1} such that $J^{-1} \cdot J = I_n$. We therefore observe the following when applying an elementary row operation:

$$\begin{aligned} &J^{-1} \cdot J \cdot A \\ &= I_n \cdot A \\ &= A \end{aligned}$$