Determinant

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1 Geometric Intuition of Determinants

This introduction offers a geometric intuition of the determinant in two and three dimensions. We start with the idea of a **square matrix** which is one that has an equal number of rows and columns. A 2x2 square matrix is therefore capable of holding two vectors each containing 2 components, and a 3x3 matrix can hold three vectors each containing 3 components.

Let's consider a generic 2x2 matrix containing two linearly independent vectors: $\{\vec{a}, \vec{b}\}$.

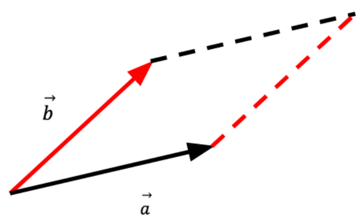


Figure 1

The two vectors form a parallelogram. The determinant of a matrix containing $\{\vec{a}, \vec{b}\}$ will be the area of this parallelogram.

Given a matrix A comprised of vectors $\vec{a}$ and $\vec{b}$:

$$A = \begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}$$

The determinant is:

$$det(A) = a_x b_y - b_x a_y$$

How the formula is derived will be the subject of section 2.

Let's consider a generic 3x3 matrix containing three linearly independent vectors: $\{\vec{a}, \vec{b}, \vec{c}\}$.

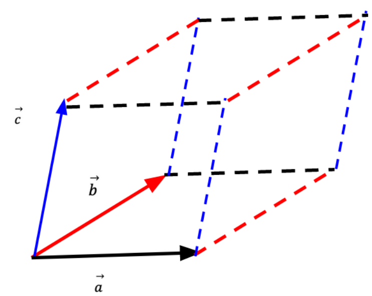


Figure 2

The three vectors form a parallelepiped. The determinant of a matrix containing $\{\vec{a}, \vec{b}, \vec{c}\}$ when calculated will be the volume of the parallelepiped.

Given a matrix B comprised of $\vec{a}$, $\vec{b}$ $\vec{c}$:

$$B = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$$

The determinant is:

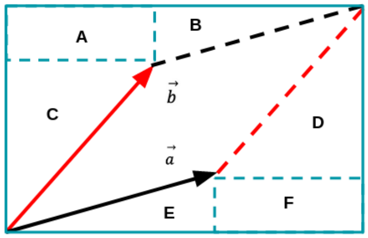
$$det(B) = (a_x b_y c_z + a_z b_x c_y + a_y b_z c_x) - (a_z b_y c_x + a_x b_z c_y + a_y b_x c_z)$$

The above formula was derived using the **triple scalar product**, and will be expanded upon in section 3.

2 A 2x2 matrix

To show that the determinant of a 2x2 matrix is the area of the parallelogram from figure 1 we can start by defining matrix A made comprised of two generic vectors. This vector along with its determinant is shown below:

$$A = \begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix} \therefore det(A) = a_x b_y - b_x a_y$$



(Total Area) $= (a_x + b_x)(a_y + b_y) = a_x a_y + a_x b_y + b_x a_y + b_x b_y$
(A & F) $= b_x a_y$, **(B & E)**$=\frac{1}{2} a_x a_y$, **(C & D)**$=\frac{1}{2} b_x b_y$

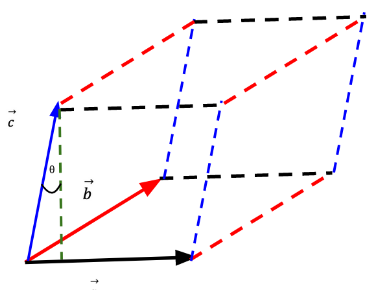
The area of the inscribed parallelogram can will be the Total Rectangle Area minus the sum of the named components. This result is the determinant of matrix A:

$$\begin{aligned} a_x a_y + a_x b_y + b_x a_y + b_x b_y - 2b_x a_y - a_x a_y - b_x b_y \\ = a_x b_y + b_x a_y - 2b_x a_y \\ = a_x b_y - b_x a_y \end{aligned}$$

3 A 3x3 matrix

The determinant of a 3x3 matrix is the volume of a parallelepiped. This can be conveyed through the following matrix:

$$B = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} \therefore det(B) = (a_x b_y c_z + a_z b_x c_y + a_y b_z c_x) - (a_z b_y c_x + a_x b_z c_y + a_y b_x c_z)$$



The volume of such a figure will be its base multiplied by its height.

The base will be the parallelogram $||\vec{a} \times \vec{b}||$, and the height will be $||\vec{c}||cos\theta$. We can simplify the volume $||\vec{a} \times \vec{b}||\ ||c||cos\theta$ using the law of cosines to: $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

We can now carry out the triple scalar operation which will the determinant of matrix B:

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} \cdot \begin{pmatrix} a_y b_z - a_z b_y & -a_x b_z + a_z b_x & a_x b_y - a_y b_x \end{pmatrix}$$

$$\begin{aligned} c_x(a_y b_z - a_z b_y) + c_y(-a_x b_z + a_z b_x) + c_z(a_x b_y - a_y b_x) \\ = a_y b_z c_x - a_z b_y c_x - a_x b_z c_y + a_z b_x c_y + a_x b_y c_z - a_y b_x c_z \\ = (a_x b_y c_z + a_z b_x c_y + a_y b_z c_x) - (a_z b_y c_x + a_x b_z c_y + a_y b_x c_z) \end{aligned}$$