Dot and Cross Products

Andy Chong Sam

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1 The Law of Cosines

We start with the derivation of the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos(C) \tag{1}$$

$$\begin{aligned} \overline{CD} &= a\cos C \\ \overline{DA} &= b - a\cos C \\ \overline{BD} &= a\sin C \end{aligned}$$

$$\begin{aligned} c^2 &= (\overline{BD})^2 + (\overline{DA})^2 \\ &= (a\sin C)^2 + (b - a\cos C)^2 \\ &= a^2\sin^2 C + a^2 cos^2 C + b^2 - 2ab\cos(C) \\ &= a^2(\sin^2 C + \cos^2 C) + b^2 - 2ab\cos(C) \\ &= a^2 + b^2 - 2ab\cos(C) \end{aligned}$$

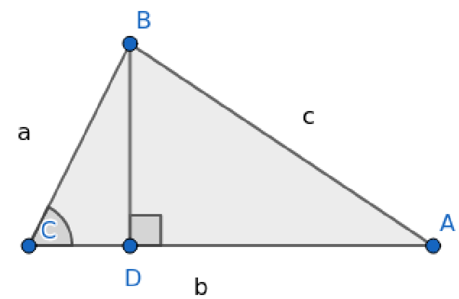

Figure 1

We can restate the law of cosines in terms of vectors and derive this expression:

$$\vec{a}\cdot\vec{b} = ||\vec{a}||\;||\vec{b}||\cos\theta \tag{2}$$

First, we restate expression (1) in terms of vectors.

$$||\vec{a} - \vec{b}||^2 = ||\vec{a}||^2 + ||\vec{b}||^2 - 2||\vec{a}||\;||\vec{b}||\cos\theta$$

We can restate the left hand side using the fact that the dot product of a vector with itself is its squared magnitude:

$$\begin{aligned} ||\vec{a} - \vec{b}||^2 &= (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b}) \\ = \vec{a}\cdot\vec{a} - 2(\vec{a}\cdot\vec{b}) + \vec{b}\cdot\vec{b} &= ||\vec{a}||^2 - 2(\vec{a}\cdot\vec{b}) + ||\vec{b}||^2 \end{aligned}$$

Now we substitute this result back into the left hand side of the original expression:

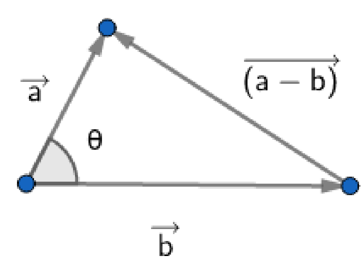

Figure 2

$$\begin{aligned} ||\vec{a}||^2 - 2(\vec{a}\cdot\vec{b}) + ||\vec{b}||^2 &= ||\vec{a}||^2 + ||\vec{b}||^2 - 2||\vec{a}||\;||\vec{b}||\cos\theta \\ \vec{a}\cdot\vec{b} &= |\vec{a}||\;||\vec{b}||\cos\theta \end{aligned}$$

2 Projection

A useful feature of the dot product is that it assists with calculating vector projection. In Figure 3 we have the length w, which is the projection of $\vec{a}$ onto $\vec{b}$. We can start with the definition of w with respect to $cos\theta$ and the magnitude of $\vec{a}$:

$$w = \cos\theta ||\vec{a}||$$

We know from expression (2) that $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{||\vec{a}||\,||\vec{b}||}$. Substituting this into the equation for w, we get:

$$w = \frac{\vec{a}\cdot\vec{b}\,||\vec{a}||}{||\vec{a}||\,||\vec{b}||}$$

$$w = \frac{\vec{a}\cdot\vec{b}}{||\vec{b}||}$$

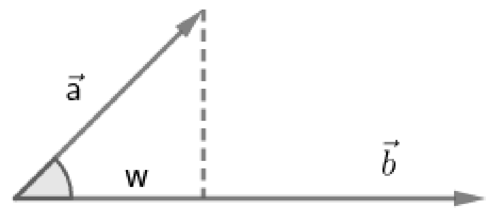

Figure 2

3 The Dot Product of Orthogonal Vectors

The dot product of two orthogonal (perpendicular) vectors is zero. Suppose again that we have two vectors $\vec{a}$ and $\vec{b}$, and they are perpendicular to each other, then the angle θ between them is $\frac{\pi}{2}$. This results in the dot product equaling zero:

$$\vec{a}\cdot\vec{b} = ||\vec{a}||\,||\vec{b}||\cos(\frac{\pi}{2}) = 0$$

The cross product between two vectors results in a third vector that is orthogonal to the two original vectors. Suppose now that $\vec{(a)} =< a_x, a_y, a_z >$ and that $\vec{(b)} =< b_x, b_y, b_z >$. The cross product $\vec{a}\times\vec{b}$ is:

$$\vec{a}\times\vec{b} = det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = (a_yb_z - a_zb_y)\vec{i} - (a_xb_z - a_zb_x)\vec{j} + (a_xb_y - a_yb_x)\vec{k}$$

Restated in vector notation, we have $\vec{a}\times\vec{b} =< a_yb_z - a_zb_y, a_xb_z - a_zb_x, a_xb_y - a_yb_x >$. We can prove that vector $\vec{a}\times\vec{b}$ is orthogonal to both vectors a and b, by showing that $\vec{a}\cdot(\vec{a}\times\vec{b}) = 0$ and $\vec{b}\cdot(\vec{a}\times\vec{b}) = 0$

$$\begin{aligned} &\vec{a}\cdot(\vec{a}\times\vec{b}) \\ =&< a_x, a_y, a_z > \cdot < a_yb_z - a_zb_y, a_xb_z - a_zb_x, a_xb_y - a_yb_x > \\ &= a_x(a_yb_z - a_zb_y) - a_y(a_xb_z - a_zb_x) + a_z(a_xb_y - a_yb_x) \\ &= a_xa_yb_z + a_xa_zb_y - a_ya_xb_z + a_ya_zb_x + a_za_xb_y - a_za_yb_x \\ &= 0 \end{aligned}$$

The same exercise can be done with vector b:

$$\begin{aligned} &\vec{b}\cdot(\vec{a}\times\vec{b}) \\ =&< b_x, b_y, b_z > \cdot < a_yb_z - a_zb_y, a_xb_z - a_zb_x, a_xb_y - a_yb_x > \\ &= b_x(a_yb_z - a_zb_y) - b_y(a_xb_z - b_zb_x) + b_z(a_xb_y - b_yb_x) \\ &= b_xa_yb_z + b_xa_zb_y - b_ya_xb_z + b_ya_zb_x + b_za_xb_y - b_za_yb_x \\ &= 0 \end{aligned}$$