Line Integrals

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1 Line Integral for Functions

1.1 Overview

For a function, $f(x, y)$ along a path S, the most general form of a line integral is:

$$\lim_{n\to\infty} f(x_i, y_i)\Delta S_i = \int_C f(x, y)\, dS \tag{1}$$

In Figure 1 we have a path S denoted by the red line. The path is divided into fairly large segments of length ΔS, but we can of course take infinitesimally small segments of length dS.

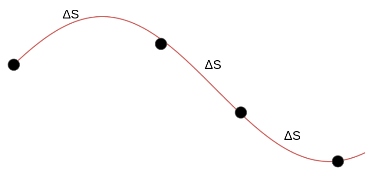

Figure 1

Not shown in Figure 1 is the function f. We can imagine that f blankets the path traversed by S. The line integral tells us the area underneath the path formed by S.

If x, and y can be described parametrically, then the line integral can be defined as:

$$\int_C f(x, y)\, dS = \int_C f(x, y)\sqrt{x'(t)^2 + y'(t)^2}\, dt \tag{2}$$

1.2 Derivation

We will derive Expression (2).

In figure 2, we consider the path again, and we start by approximating the length through secant lines. We know that length of each secant line will be:

$$\overline{P_{i-1}P_i} = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

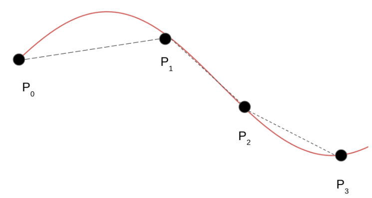

Figure 2

If the curve is defined by a function f, we know that through the mean value theorem there is a value c such that:

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(c)$$
$$f(x_i) - f(x_{i-1}) = f'(c)(x_i - x_{i-1})$$

Let's say that $x_i - x_{i-1} = \Delta x$. If $y = f(x)$, then $y_i - y_{i-1} = f'(c)(x_i - x_{i-1})$. We can now rewrite the secant line:

$$\begin{aligned} \overline{P_{i-1}P_i} &= \sqrt{\Delta x^2 + (f'(c)\Delta x)^2} \\ &= \Delta x\sqrt{1 + f'(c)^2} \end{aligned}$$

The estimated length of the entire curve is therefore:

$$L = \sum_{i=1}^{n} \overline{P_{i-1}P_i} = \sum_{i=1}^{n} \Delta x\sqrt{1 + f'(c)^2}$$

If we draw more secant lines, with smaller lengths for $\overline{P_{i-1}P_i}$ we will improve the length estimate. If we take infinitesimally small segments we get an integral. This expression gives us the length of a curve:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\sqrt{1 + f'(c)^2}$$

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx \tag{3}$$

Now let's suppose that the path is not a function where y depends on x but is instead one in which x and y are represented parametrically, depending on t.

One way to visualize this is to imagine that you are a satellite viewing straight down at a specific location. The land, hills, rivers, valleys, and other features are defined by f, but for whatever reason we are only interested in a specific path S on this terrain. If S crosses a hill at $t = 1$, we expect this point to have its own x and y values. If at $t = 5$ it passes over a plain, we would expect that area to have a lower y value than when it passed the hill. The line integral gives us the area underneath this path.

To complete the integration, we restate Expression (3) in terms of t, so if $f(x, y) = (\ x(t), y(t)\ )$, through implicit differentiation we get:

$$
\begin{aligned}
L = \int_a^b \sqrt{1+f'(x)^2}\,dx &= \int_a^b \sqrt{1+(\frac{dy/dt}{dx/dt})^2}\,\frac{dx}{dt}dx \\
&= \int_a^b \sqrt{(\frac{dx/dt}{dx/dt})^2+(\frac{dy/dt}{dx/dt})^2}\,\frac{dx}{dt}dx \\
&= \int_a^b \frac{1}{dx/dt}\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\frac{dx}{dt}dt \\
&= \int_a^b \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt
\end{aligned}
$$

The last step is to take into account the value of function f (in our analogy, the area underneath a segment of S will be different if we're looking at a hill vs a plain). We are left with expression 2:

$$\int_C f(x,y)\,dS = \int_a^b f(x,y)\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt = \int_C f(x,y)\sqrt{x'(t)^2+y'(t)^2}\,dt$$

1.3 Examples

Ex. 1 Evaluate $\int_C 6x\,dS$ where C is $y=x^2$, where $x(t)=t$ and $y(t)=t^2$ and $-1 \le t \le 2$.

Source: Paul's Online Notes
https://tutorial.math.lamar.edu/Problems/CalcIII/LineIntegralsPtI.aspx

$$x'(t) = 1 \quad y'(t) = 2t$$
$$\text{Because } x = t,\ 6x = 6t$$

So this is the integral we want to evaluate:

$$\int_{-1}^{2} 6t\sqrt{1+(2t)^2}\,dt$$

We'll do a u-substitution with $u = 1+4t^2$:

$$\int_{-1}^{2} 6t\sqrt{1+(2t)^2}\,dt = 6\frac{(1+4t^2)^{\frac{3}{2}}}{12}\Big|_{-1}^{2} = \frac{17^{\frac{3}{2}}-5^{\frac{3}{2}}}{12}$$

Ex. 2 Evaluate $\int_C x^2 + y^2 \, dS$, where C is the line segment that runs from $(0,0)$ to $(1,1)$

Source: Larson, Calculus 6th Edition, pg 1061

We can easily parameterize a line segment with the following formula:

$$x(t) = x_1(1-t) + x_2(t)$$
$$y(t) = y_1(1-t) + y_2(t)$$

This gets us:

$$x(t) = (0)(1-t) + (1)(t) = t$$
$$y(t) = (0)(1-t) + (1)(t) = t$$

We observe that $x'(t) = y'(t) = 1$. Since $x(t) = y(t) = t$, f can be rewritten as $2t^2$. We can now work on the integral:

$$\int_0^1 2t^2\sqrt{2}\, dt = 2\sqrt{2}\, \frac{t^3}{3}\Big|_0^1 = \frac{2}{3}\sqrt{2}$$

Ex. 3 Evaluate $\int_C x^2 + y^2 + z^2 \, dS$, where C is defined by $x(t) = \sin t$,$y(t) = \cos t$, and $z = 2$, $0 \leq t \leq \frac{\pi}{2}$.

Source: Larson, Calculus 6th Edition, pg 1061

$$x'(t) = \cos t \quad y'(t) = -\sin t \quad z'(t) = 0$$

Restating f in terms of t we get:

$$x^2 + y^2 + z^2 = \sin^2 t + cos^2 t + 4 = 5$$

We can now evaluate the integral:

$$\int_0^{\frac{\pi}{2}} 5\sqrt{\cos^2 t + \sin^2 t}\, dt = \int_0^{\frac{\pi}{2}} 5 \, dt = 5t\Big|_0^{\frac{\pi}{2}} = \frac{5\pi}{2}$$

Ex. 4 Evaluate $\int_C xy\ dS$, C is defined by $x(t) = 4t$ and $y(t) = 3t$, $0 \leq t \leq 1$.

Source: Larson, Calculus 6th Edition, pg 1061

$$x'(t) = 4 \quad y'(t) = 3$$

We can restate f in terms of t:

$$xy = (4t)(3t) = 12t^2$$

The integral can now be setup:

$$\int_0^1 12t^2(5)\ dt = 20t^3\Big|_0^1 = 20$$

Ex. 5 Evaluate $\int_C xy^4\ dS$ where C is defined as the right half of the circle defined by $x^2 + y^2 = 16$.

Source: Stewart, Calculus 8th Edition pg 1084

Since we are dealing with the right half of the circle, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

$$x(t) = 4\cos t \quad x'(t) = -4\sin t$$
$$y(t) = 4\sin t \quad y'(t) = 4\cos t$$

Let's redefine f in terms of t:

$$xy^4 = (4\cos t)(4\sin t)^4$$

The full integral to evaluate is now:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\sin t)^4 \sqrt{16\sin^2 t + 16\cos^2 t}\ dt$$

This is a good candidate for integration by parts. We'll pick $u = \sqrt{16sin^2t + 16cos^2t}$ and $v' = (4\cos t)(4\sin t)^4$. Let's figure out u':

$$u = \sqrt{16\sin^2 t + 16\cos^2 t} = \sqrt{16(\sin^2 t + \cos^2 t)} = \sqrt{16} \quad \text{So } u' = 0$$

Let's determine v from v':

$$\int (4\cos t)(4\sin t)^4\ dt = \frac{1024}{5}\sin^5 t + C$$

We can now assemble the final integral. Since u' is 0, we only need to integrate uv:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4\cos t)(4\sin t)^4 \sqrt{16\sin^2 t + 16\cos^2 t}\ dt = \frac{1024\sqrt{16}}{5}\sin^5 t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2048\sqrt{16}}{5}$$

2 Line Integral for Vector Fields

Here are three equivalent definitions of the line integral for vector fields:

$$\int_C \vec{F} \cdot \vec{T}\, dS = \int_C F(\ r(t)\ )\ r'(t)\ dt = \int_C \vec{F}\ dR \tag{4}$$

To assist with seeing the intuition of Expression (4) we start with the left-most definition and Figure 3. The dotted black lines represent a vector field $\vec{F}$. We want to study the effect of $\vec{F}$ on a particle, the black circle, along path S. We will define S with a vector function $r(t)$.

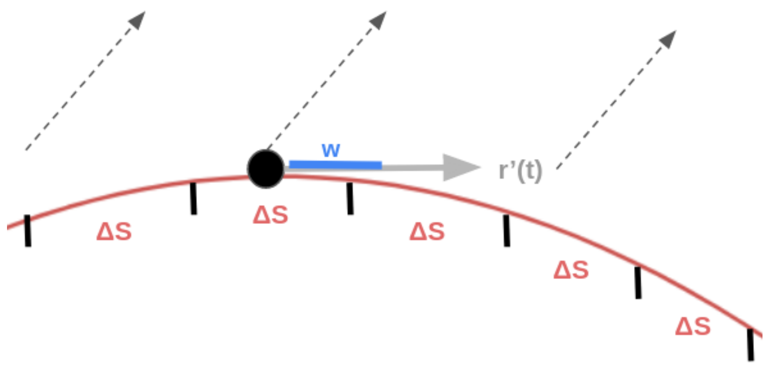

Figure 3

To understand the effect of $\vec{F}$ on the particle, we also need to consider the tangent vector $r'(t)$, and the projection of $\vec{F}$ onto $r'(t)$. In Figure 3, this is denoted by w, and we can calculate it with a dot product and the magnitude of the tangent vector:

$$w = \frac{\vec{F} \cdot r'(t)}{||\ r'(t)\ ||}$$

Symbolically, we refer to the projection as the dot product of $\vec{F}$ and $\vec{T}$, where $\vec{T} = \frac{r'(t)}{||\ r'(t)\ ||}$.

In Figure 3, the curve is arbitrarily subdivided into chunks of ΔS. We can now imagine infinitesimally small segments of dS instead. From Expression (2) we know that $dS = \sqrt{x'(t)^2 + y'(t)^2}\ dt$. If $r(t)$ is a vector function describing S made up of x and y, then dS is nothing more than the magnitude $r'(t)$ along with dt:

$$dS = \sqrt{x'(t)^2 + y'(t)^2}\ dt = ||\ r'(t)\ ||\ dt$$

We've now derived the practical form for the line integral of vector fields:

$$\int_C F \cdot T\ dS = \int_C F(\ r(t)\ )\ r'(t)\ dt$$

The expression $r'(t)dt$ is sometimes stated as dS, which gets us the right most equation of Expression (4):

$$\int_C F(\ r(t)\ )\ r'(t)\ dt = \int_C \vec{F}\ dR$$

Ex. 6 Evaluate $\int_C \vec{F} dR$ where $\vec{F} = xy\,\vec{i} + xz\,\vec{j} + yz\,\vec{k}$ and the path C is defined by $t\,\vec{i} + t^2\,\vec{j} + 2t\,\vec{k}$.

Source: Larson, Calculus 6th Edition, pg 1062

We first define F in terms of t using $r(t)$:

$$< xy, yz, yz > = < (t)(t^2), (t)(2t), (t^2)(2t) > = < t^3, 2t^2, 2t^3 >$$

Next we evaluate $r'(t)$:

$$r'(t) = < 1, 2t, 2 >$$

Assemble and evaluate the integral:

$$\int_0^1 < t^3, 2t^2, 2t^3 > \cdot < 1, 2t, 2 > \ dt$$
$$\int_0^1 9t^3 dt = \frac{9}{4}t^4\Big|_0^1 = \frac{9}{4}$$

Ex. 7 Evaluate $\int_C \vec{F} dR$ where $\vec{F} = xy^2\,\vec{i} - x^2\,\vec{j}$ and C is defined by $t^3\,\vec{i} + t^2\,\vec{j}$, where $0 \leq t \leq 1$.

Source: Stewart, Calculus 8th Edition pg 1085

We first define F in terms of t using $r(t)$:

$$< xy, yz > = < t^7, -t^6 >$$

Evaluate $r'(t)$:

$$r'(t) = < 3t^2, 2t >$$

Evaluate the integral:

$$\int_0^1 < t^7, -t^6 > \cdot < 3t^2, 2t > \ dt$$
$$\int_0^1 3t^9 - 2t^7 \ dt = (\frac{3}{10}t^{10} - \frac{1}{4}t^8)\Big|_0^1 = \frac{1}{20}$$