Curl

Andy Chong Sam

1 Overview

The curl operator describes the rotation tendency of a vector field. Both its input and output are vectors. In the cases when we consider only two dimensions, the result of curl is a scalar.

If a vector field is defined by $F =< P, Q >$ then its two dimdensional curl is defined as:

$$\nabla \times F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \tag{1}$$

If $F =< P, Q, R >$ we can calculate a three dimensional curl with:

$$\nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} \ + \ (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} \ + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k} \tag{2}$$

From Expressions (1) and (2) we again note that the latter produces a vector, and that its third component is just the two dimensional curl.

2 Geometric Intuition

To start we will note that the orthodox direction of rotation is counter clockwise. Fields that induce a counter clockwise rotation will have a positive curl. We can start out by imagining a particle on the xy plane.

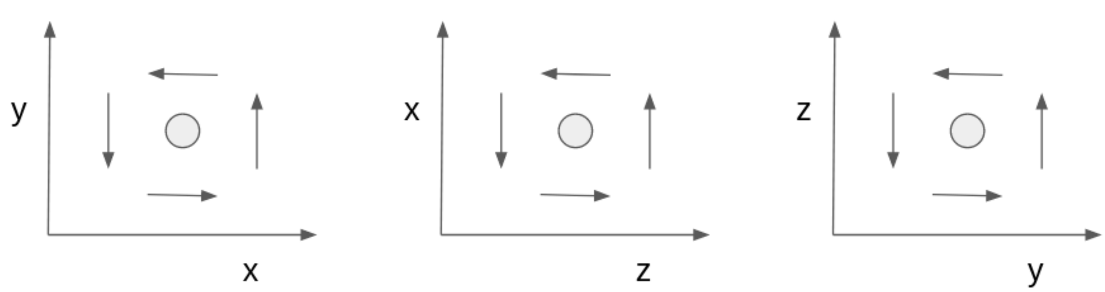

Figure 1

In Figure 1's left graph, we've generalized what happens to a particle when a field induces positive curl. In these diagrams the arrows represent the field. We note that as x increases, the Q component of the field has a tendency to shift from negative to positive. Likewise as as y increases, the P component becomes more negative. In order to guarantee a positive value if the field is inducing a counter clockwise rotation we can take the change of R with respect to y (which will be positive) and subtract from it the change of P with respect to y which will be negativ. From this intuition, we get $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ as the two dimensional curl.

In our analysis so far we've imagined that the field can only induce rotation on the xy plane. Because z is fixed in this scenario, the two dimensional curl is the $\vec{k}$ component of the three dimensional curl. We can think of this as the observed rotation viewed from z's perspective.

Looking again at Figure 1, the central diagram describes a hypothetical counter clockwise rotation from the perspective of y. As z increases, the P component of the field becomes more positive and as x increases, the R compoment becomes more negative. So to guarantee a positive value we can write $\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}$. The same analysis can be performed with Figure 1's right graph.

3 Examples

Ex. 1 Calculate $\nabla \times F$ given $F(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}(x\,\vec{i} + y\,\vec{j} + z\,\vec{k})$

Source:

$$\frac{\partial R}{\partial y} = \frac{zy}{2\sqrt{x^2+y^2+z^2}} \quad \frac{\partial Q}{\partial z} = \frac{zy}{2\sqrt{x^2+y^2+z^2}} \quad \frac{\partial P}{\partial z} = \frac{xz}{2\sqrt{x^2+y^2+z^2}}$$
$$\frac{\partial R}{\partial x} = \frac{xz}{2\sqrt{x^2+y^2+z^2}} \quad \frac{\partial Q}{\partial x} = \frac{xy}{2\sqrt{x^2+y^2+z^2}} \quad \frac{\partial P}{\partial y} = \frac{xy}{2\sqrt{x^2+y^2+z^2}}$$

$$\nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} \ + \ (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} \ + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k}$$

$$\nabla \times F = 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k}$$

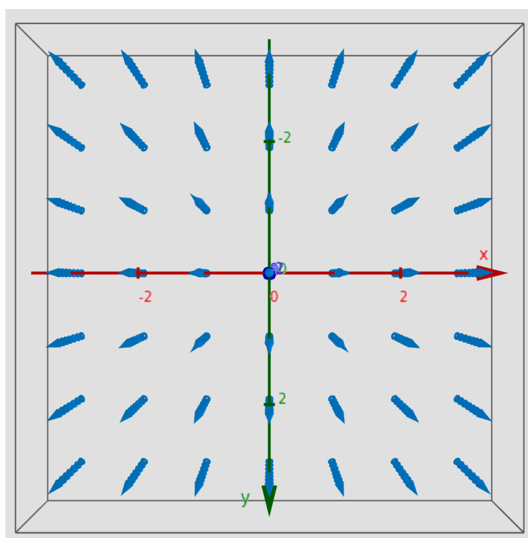

Figure 2

No curl is present. A graph of the field (top down) is shown in Figure 2. We can intuitively see that this field is incapable of inducing rotation.

Ex. 2 Calculate $\nabla \times F$ where $F = xye^z\ \vec{i} + yze^x\ \vec{k}$

Source:

$$\frac{\partial R}{\partial y} = ze^x \quad \frac{\partial Q}{\partial z} = 0 \quad \frac{\partial P}{\partial z} = xye^z$$
$$\frac{\partial R}{\partial x} = yze^x \quad \frac{\partial Q}{\partial x} = 0 \quad \frac{\partial P}{\partial y} = xe^z$$

$$\nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} \ \ + \ \ (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} \ \ + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k}$$

$$\nabla \times F = ze^x\ \vec{i} + xye^z - yze^x\ \vec{j} - xe^z\ \vec{k}$$