Gram Schmidt Process

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1 Purpose

The purpose of the Gram Schmidt algorithm is to derive an orthonormal vector set from a list of linearly independent vectors. Given an input set of vectors $\{\vec{v}_1, \vec{v}_2 ... \vec{v}_n\}$, the algorithm will produce a set $\{\hat{u_1}, \hat{u_2} ... \hat{u_n}\}$. Each vector in the output set will be orthogonal to the other members in the set. Finally, both the input and output set will span the same space.

2 Algorithm Steps

Here are the steps, given a set of input vectors $\{\vec{v}_1, \vec{v}_2 ... \vec{v}_n\}$:

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= v_2 - \text{proj}_{u_1}(v_2) \\ u_3 &= v_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3) \end{aligned}$$

We can generalize u_n:

$$u_n = v_k - \sum_{j=1}^{k-1} \text{proj}_{u_j}(v_k)$$

We conclude with normalizing u_1, u_2, ... u_n to obtain the orthonormal set:

$$\{\frac{u_1}{||\vec{u}_1||}, \frac{u_2}{||\vec{u}_2||}, \frac{u_3}{||\vec{u}_3||} ... \frac{u_n}{||\vec{u}_n||}\} = \{\hat{u_1}, \hat{u_2} ... \hat{u_n}\}$$

3 Intuition Using Two Vectors

Figure (1) shows the geometric intuition of two linearly independent vectors in $\mathbb{R}^2$:

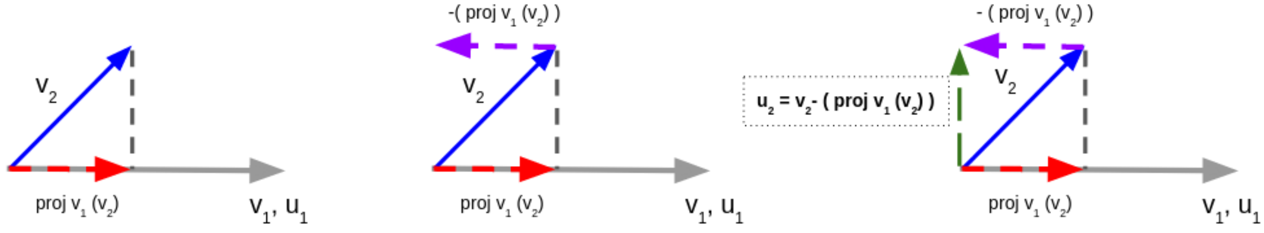

Figure 1

We can also show this algebraically. If $\hat{u_1}$ and $\hat{u_2}$ are orthogonal to each other, then their dot product is zero. Suppose that $\vec{v}_1 =< a, b >$ and $\vec{v}_2 =< c, d >$. The first vector is just $\vec{v}_1$.

We now calculate $\vec{u}_2$:

$$< c, d > -(\frac{ac + bd}{a^2 + b^2} < a, b >)$$

We can now take this result and perform a dot product with $\vec{v}_1$:

$$\begin{aligned}
(< c, d > -(\frac{ac + bd}{a^2 + b^2} < a, b >)) \cdot < a, b > \\
= a(c - \frac{ac + bd}{a^2 + b^2} a) + b(d - \frac{ac + bd}{a^2 + b^2} b) \\
= ac - a^2 \frac{ac + bd}{a^2 + b^2} + bd - b^2 \frac{ac + bd}{a^2 + b^2} \\
= \frac{ac(a^2 + b^2)}{a^2 + b^2} - \frac{a^2(ac + bd)}{a^2 + b^2} + \frac{bd(a^2 + b^2)}{a^2 + b^2} - \frac{b^2(ac + bd)}{a^2 + b^2} \\
= \frac{a^3c + ab^2c - a^3c - a^2bd + a^2bd + b^3d - ab^2c - b^3d}{a^2 + b^2} \\
= 0
\end{aligned}$$

Ex. 1 Given $\vec{v}_1 =< 3, 1 >$ and $\vec{v}_2 =< 2, 2 >$ find an orthonormal set using the Gram Schmidt process.

We have $\vec{u}_1 = \vec{v}_1$, so $\vec{u}_1 =< 3, 1 >$.

We proceed to derive $\vec{u}_2$:

$$\begin{aligned}
\vec{u}_2 = \vec{v}_2 - \text{proj}_{u_1}(v_2) \\
=< 2, 2 > -(\frac{(3)(2) + (1)(2)}{(3)(3) + (1)(1)} < 3, 1 >) \\
=< 2, 2 > - \frac{4}{5} < 3, 1 > \\
=< \frac{-2}{5}, \frac{6}{5} >
\end{aligned}$$

To conclude, normalize $\vec{u}_1$ and $\vec{u}_2$. We find that $||\vec{u}_1|| = \sqrt{10}$ and $||\vec{u}_2|| = \sqrt{\frac{8}{5}}$, so the orthonornmal set is:

$$\begin{aligned}
\hat{u_1} = \frac{1}{\sqrt{10}} < 3, 1 > \\
\hat{u_2} = \frac{\sqrt{5}}{\sqrt{8}} < \frac{-2}{5}, \frac{6}{5} >
\end{aligned}$$

This result can be verified by showing that $\hat{u_1} \cdot \hat{u_2}$ is zero:

$$(\frac{3}{\sqrt{10}})(\frac{\sqrt{5}}{\sqrt{8}})(\frac{-2}{5}) + (\frac{1}{\sqrt{10}})(\frac{\sqrt{5}}{\sqrt{8}})(\frac{6}{5}) = 0$$

Ex. 2 Given $\vec{v}_1 =< 1, 2, 0 >$ and $\vec{v}_2 =< 8, 1, -6 >$ and $\vec{v}_3 =< 0, 0, 1 >$ find an orthonormal set using the Gram Schmidt process:

We have $\vec{u}_1 = \vec{v}_1$, so $\vec{u}_1 =< 1, 2, 0 >$.

Let's calculate $\vec{u}_2$:

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{u_1}(v_2)$$

$$\begin{aligned}\vec{u}_2 =< 8, 1, -6 > - \frac{(1)(8) + (2(1) + (0)(-6)}{(1)(1) + (2)(2) + (0)(0)} < 1, 2, 0 > \\ =< 6, -3, -6 >\end{aligned}$$

Let's calculate $\vec{u}_3$:

$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(\vec{v}_3)$$

$$\begin{aligned}\vec{u}_3 =< 0, 1, 1 > - \frac{(1)(0) + (2)(0) + (0)(1)}{(0)(0) + (0)(0) + (1)(1)} < 1, 2, 0 > - \frac{(1)(8) + (2)(1) + (0)(0)}{(1)(1) + (2)(2) + (0)(0)} < 6, -3, -6 > \\ =< \frac{4}{9}, -\frac{2}{9}, \frac{5}{9} >\end{aligned}$$

We conclude with normalizing our results given that $||\vec{u}_1|| = \sqrt{5}$, $||\vec{u}_2|| = 9$, and $||\vec{u}_3|| = \frac{\sqrt{45}}{9}$:

$$\begin{aligned}\hat{u_1} &= \frac{1}{\sqrt{5}} =< 1, 2, 0 > \\ \hat{u_2} &= \frac{1}{3} < 6, -3, -6 > \\ \hat{u_3} &= \frac{9}{\sqrt{45}} < \frac{4}{9}, -\frac{2}{9}, \frac{5}{9} >\end{aligned}$$