Essence of Single Variable Calculus

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1 Derivatives

We will first summarize derivatives. Given a function $f(x)$, its derivative $f'(x)$, describes the instantaneous rate of change at x. The formal definition of the derivative is as follows:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1}$$

From Expression (1) we can derive all the derivatives found on a textbook table:

2 First Fundamental Theorem of Calculus

If $f(x)$ is continuous over $[a, b]$, and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$ over $[a, b]$. We say that F is an antiderivative of f.

Derivation:

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x}$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x}$$

By the **Mean Value Theorem** we know that there is a value c such that:

$$f(c)\Delta x = \int_x^{x+\Delta x} f(t)dt$$

We are left with evaluating:

$$F'(x) = \lim_{\Delta x \to 0} f(c)\Delta x$$

By the **Squeeze Theorem** we know that as Δx approaches zero, c approaches x, so we are left with:

$$F'(x) = \lim_{\Delta x \to 0} f(c)\Delta x = f(x)$$

3 Second Fundamental Theorem of Calculus

If $f(x)$ is continuous over $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Derivation:

Let $g(x) = \int_a^x f(t)dt$. Applying the First Fundamental Theorem we know that $g'(x) = f(x)$. If $F(x)$ is any antiderivative of $f(x)$ then $F(x) = g(x) + c$. Let's evaluate $F(b) - F(a)$:

$$\begin{aligned}
&F(b) - F(a) \\
&= (g(b) + c) - (g(a) + c) \\
&= g(b) - g(a) \\
&= \int_a^b f(t)dt - \int_a^a f(t)dt \\
&= \int_a^b f(t)dt - 0 \\
&= \int_a^b f(t)dt
\end{aligned}$$

4 Product Rule

Given $h(x) = f(x)g(x)$, $h'(x) = f'(x)g(x) + f(x)g'(x)$

$$h'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

We'll add two terms that cancel each other out (but will help us derive the rule): $-f(x+h)g(x)+f(x+h)g(x)$:

$$\begin{aligned} h'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ = \lim_{h \to 0} \frac{f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))}{h} \\ = \lim_{h \to 0} \frac{f(x+h)(g(x+h) - g(x))}{h} + \lim_{h \to 0} \frac{g(x)(f(x+h)) - f(x)}{h} \end{aligned}$$

From this result, we see that $\lim_{h \to 0} f(x+h)$ is just $f(x)$. We can also see that $\lim_{h \to 0} \frac{g(x+h)-g(x)}{h}$ is $g'(x)$.

We also observe that $\lim_{h \to 0} g(x) = g(x)$, and $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ is the definition of $f'(x)$.

5 Quotient Rule

If $h(x) = \frac{f(x)}{g(x)}$ then $h'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$

We will apply the cross multiplication property. Given $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$:

$$\begin{aligned} h'(x) = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\ = \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \frac{1}{h} \end{aligned}$$

We'll add two terms that cancel each other out (but will help us derive the rule): $-f(x)g(x) + f(x)g(x)$:

$$\begin{aligned} h'(x) = \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \frac{1}{h} \\ = \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \\ = \lim_{h \to 0} \frac{1}{g(x+h)g(x)} (\frac{f(x+h)g(x) - f(x)g(x)}{h} + \frac{f(x)g(x) - f(x)g(x+h)}{h}) \\ = \lim_{h \to 0} \frac{1}{g(x+h)g(x)} (g(x)\frac{f(x+h) - f(x)}{h} + f(x)\frac{g(x) - g(x+h)}{h}) \end{aligned}$$

From these results we can see that:

$\lim_{h \to 0} \frac{1}{g(x+h)g(x)}$ is $g'(x)^2$

$\lim_{h \to 0}(g(x)\frac{f(x+h)-f(x)}{h}$ is $g(x)f'(x)$.

$\lim_{h \to 0}(f(x)\frac{g(x+h)-g(x)}{h}$ is $f(x)g'(x)$.

6 Power Rule

Given $f(x) = x^n$, its derivative is $f'(x) = nx^{n-1}$

Derivation:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

By the Binomial Theorem, we know that:

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k$$

Expanding the numerator we have:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + ... + nxh^{n-1} + h^n) - x^n}{h} \\ &= \lim_{h \to 0} (nx^{n-1} + \frac{(n)(n-1)}{2!}x^{n-2}h + ... + nxh^{n-2} + h^{n-1}) \end{aligned}$$

With the exception of $n(x)^{n-1}$, the limit as h approaches zero of all other components is zero, for instance:

$$\lim_{h \to 0} (\frac{(n)(n-1)}{2!}x^{n-2}h) = 0 \qquad \lim_{h \to 0} (nxh^{n-2}h) = 0 \qquad \lim_{h \to 0} (h^{n-1}) = 0$$

Since $n(x)^{n-1}$ does not depend on h, we have:

$$\lim_{h \to 0} (n(x)^{n-1}) = n(x)^{n-1}$$

So we are left with:

$$f'(x) = n(x)^{n-1}$$

The First Fundamental Theorem of calculus predicts the existence of a function $F(x)$ such that $F'(x) = f(x)$.

$$\frac{d}{dx}[\frac{x^{n+1}}{n+1}] = \frac{n+1}{n+1}x^{n+1-1} = x^n$$

So we have:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

7 Sine

If $f(x) = \sin x$, then $f'(x) = \cos x$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

Using the trig identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$, we can rewrite the numerator:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h} \\
&= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} \\
&= -\sin x \lim_{h \to 0} \frac{1 - \cos h}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}
\end{aligned}$$

We will apply two known special limits here, namely that $\lim_{h \to 0} \frac{1-\cos h}{h} = 0$ and $\lim_{h \to 0} \frac{\sin h}{h} = 1$. So we are left with:

$$f'(x) = \cos x$$

The First Fundamental Theorem of calculus predicts the existence of a function $F(x)$ such that $F'(x) = f(x)$.

$$\frac{d}{dx}\left[-\cos x\right] = -(-\sin x) = \sin x$$

Note: derivative of cosine is discussed in the next section.

So therefore:

$$\int \sin x dx = -\cos x + C$$

8 Cosine

If $f(x) = \cos x$ then $f'(x) = -\sin x$

$$f'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

We can use the following trig identity, $\cos(x+h) = \cos x \cos h - \sin x \sin h$, to rewrite the numerator:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x \sin h}{h} \\
&= \cos x \lim_{h\to 0} \frac{(\cos h - 1)}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\
&= -\cos x \lim_{h\to 0} \frac{1 - \cos h}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}
\end{aligned}$$

We will apply two known special limits here, namely that $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$. So we are left with:

$$f'(x) = -\sin x$$

9 Tangent

If $f(x) = \tan x$, then $f'(x) = \sec^2 x$

$$f'(x) = \lim_{h\to 0} \frac{\tan(x+h) - \tan x)}{h}$$

If we apply the trig identity, $\tan(x+h) = \frac{\tan x + \tan h}{1 - \tan x \tan h}$, we get:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{\tan x + \tan h - \tan x(1 - \tan x \tan h)}{1 - \tan x \tan h}}{h} \\
&= \lim_{h\to 0} \frac{\tan h + \tan^2 x \tan h}{h(1 - \tan x \tan h)} \\
&= \lim_{h\to 0} \frac{\tan h(1 + \tan^2 x)}{h(1 - \tan x \tan h)} \\
&= \lim_{h\to 0} (\frac{\tan h}{h}) \lim_{h\to 0} (\frac{1 + \tan^2 x}{1 - \tan x \tan h}) \\
&= \lim_{h\to 0} (\frac{\sin h}{h \cos h}) \lim_{h\to 0} (\frac{1 + \tan^2 x}{1 - \tan x \tan h}) \\
&= \lim_{h\to 0} \frac{\sin h}{h} \lim_{h\to 0} \frac{1}{\cos h} \lim_{h\to 0} (\frac{1 + \tan^2 x}{1 - \tan x \tan h}) \\
&= 1 + \tan^2 x
\end{aligned}$$

We can simplify the result further:

$$f'(x) = 1 + \tan^2 x = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + sin^2 x}{cos^2 x}$$

Since $\cos^2 x + \sin^2 x = 1$, we are left with $\frac{1}{cos^2 x}$ or $sec^2 x$.

$$f'(x) = \sec^2 x$$

There is a function $F(x)$ such that $F'(x) = f(x)$. Let's differentiate $\ln(\sec x)$:

$$\begin{aligned}\frac{d}{dx}[\ln(\sec x)] &= (\frac{1}{\frac{1}{\cos x}})(-\frac{1}{\cos^2 x})(-\sin x) \\ &= \frac{(\cos x)(-\sin x)}{-\cos^2 x} = \frac{\sin x}{\cos x} \\ &= \tan x\end{aligned}$$

We can therefore say that:

$$\int \tan x dx = \ln(\sec x) + C$$

10 Natural Log

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$

We can apply the following rule of natural logs: $\ln(x+h) - \ln x = \ln(\frac{x+h}{x})$:

$$f'(x) = \lim_{h\to 0} \frac{1}{h} \ln(\frac{x+h}{h})$$

Applying the power rule of natural logs we get:

$$f'(x) = \lim_{h\to 0} \ln(\ (\frac{x+h}{x})^{\frac{1}{h}} \ )$$

Let $u = \frac{h}{x}$, so $h = ux$:

$$\begin{aligned}f'(x) &= \lim_{u\to 0} \ln(\ (1+u)^{\frac{1}{ux}} \ ) \\ &= \lim_{u\to 0} \ln(\ (1+u)^{\frac{1}{u}\frac{1}{x}} \ )\end{aligned}$$

We apply the power rule one more time:

$$\begin{aligned}f'(x) &= \lim_{u\to 0} \frac{1}{x} \ln(\ (1+u)^{\frac{1}{u}} \ ) \\ &= \frac{1}{x} \lim_{u\to 0} \ln(1+u)^{\frac{1}{u}}\end{aligned}$$

The expression $\lim_{u\to 0}(1+u)^{\frac{1}{u}}$ is the definition of Euler's number, e, so we are left with:

$$\begin{aligned}f'(x) &= \frac{1}{x} \ln e \\ f'(x) &= \frac{1}{x}\end{aligned}$$

Let's take the derivative of $\ln x$. We know that $y = \ln x$ is basically stating that $e^y = x$, to which we'll apply implicit differentiation:

$$
\begin{aligned}
\frac{d}{dx}[e^y] &= \frac{d}{dx}[x] \\
e^y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{e^y} \\
\text{(If we plug in } y = \ln x\text{)} \quad \frac{dy}{dx} &= \frac{1}{e^{\ln x}} = \frac{1}{x}
\end{aligned}
$$

So we have:

$$
\int \ln x dx = \frac{1}{x} + C
$$

11 e

If $f(x) = e^x$, then $f'(x) = e^x$.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h \to 0} \frac{e^x e^h - e^x}{h}
\end{aligned}
$$

Since e^x does not depend on h we can take it out of the limit evaluation:

$$
f'(x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h}
$$

Let $y = e^h - 1$, or $e^h = y + 1$ we observe that $\lim_{h \to 0} y = 0$. Taking the natural log of boths sides:

$$
\begin{aligned}
\ln(e^h) &= \ln(y+1) \\
h &= \ln(y+1)
\end{aligned}
$$

We can substitue the above results into h and $e^h - 1$:

$$
\begin{aligned}
f'(x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h} &= e^x \lim_{y \to 0} \frac{y}{\ln(y+1)} \\
&= e^x \lim_{y \to 0} \frac{1}{\frac{1}{y} \ln(y+1)} \\
&= e^x \lim_{y \to 0} \frac{1}{\ln(\ (y+1)^{\frac{1}{y}} \ )}
\end{aligned}
$$

The expression $\lim_{y \to 0} (y+1)^{\frac{1}{y}}$ is the definition of Euler's number, so we are left with:

$$
\begin{aligned}
f'(x) &= e^x \frac{1}{\ln e} \\
f'(x) &= e^x
\end{aligned}
$$

It can also be said that:

$$
\int e^x dx = e^x + C
$$