

Linear Transformations

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1 Introduction

To introduce this topic we start with the idea of a function. The generic function $f(x) = y$ contains an independent variable x and a dependent variable y . Assigning to x some number and using it in the function's calculation gives us a unique value y .

This concept extends to the realm of vectors. Instead of treating an individual number as an independent variable, we use an input vector. In this document we'll call this \vec{v} . A matrix will encode the calculation being performed. This is traditionally called matrix A . The result of this operation is matrix B . This is written in the form of a matrix multiplication:

$$A\vec{v} = B$$

We can also describe the environment in which this operation takes place. This is often written in the following form:

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

The letter T denotes that this is describing a transformation. The input vector is from the vector space \mathbb{R}^n . This space is known as the **domain**. If we have, for instance, \mathbb{R}^2 we are saying that this transformation accepts a two dimensional vector with a real x and y component. Next we define what the transformation itself does.

Here is an example, where $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$$

In this example, feeding the transformation a two dimension vector, say $\langle 1, 3 \rangle$ results in another two dimension vector $\langle 2, 9 \rangle$.

It's also possible to rewrite the transformation in its standard form $A\vec{v} = B$. To do this we will use the standard coordinate vectors:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \dots \\ 0 \end{pmatrix}$$

The number of vectors needed is the same as the dimensions of the input vector. The matrix A , which encapsulates the transformation is composed of the columns that result from $T(e_n)$.

We can now rewrite the following transformation:

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$$

$$T(e_1) = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, T(e_2) = T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

So we get:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

We can now write the transformation in its standard form:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = B$$

2 Image and Kernel

The image is the vector space that the transformation will end up occupying. If the definition of the transformation is given, this is a trivial process. In the example we've been using:

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$$

We can describe the **image** as: $x = 2s$ and $y = 2t$, where $s, t \in \mathbb{R}$. Now suppose you were not given the transformation definition but only matrix A. We can arrive at the transformation by simply carrying out the matrix multiplication of the standard form:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$$

We can describe the **kernel** as the solution to $A\vec{v} = \vec{0}$.

For the transformation we've been looking at, the only way to obtain the zero vector is if $x = 0$ and $y = 0$. We can therefore say:

$$Ker(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For more complex transformations, applying row elimination can be used to formulate the kernel. Consider the following transformation:

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - z \\ x + 2y - 4z \end{pmatrix}$$

To determine the kernel we solve for the system of equations comprised of $x - z = 0$ and $x + 2y - 4z = 0$. Row reduction can be used here:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & -4 \end{pmatrix} \xrightarrow{R_2 - R_1 = R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -3 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2 = R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{3}{2} \end{pmatrix}$$

$$Ker(T) = \left\{ \begin{array}{l} x = s \\ y = \frac{3}{2}s \\ z = s \end{array} \right\}, s \in \mathbb{R}$$

The key point is that for any value of s the x, y, z values produced, when plugged into the transformation will always result in the zero vector.

3 Set Visualization

We can now draw a diagram that brings together all these terms:

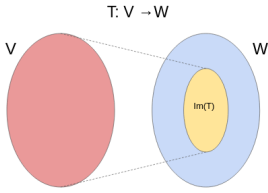


Figure 1

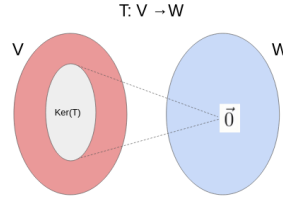


Figure 2

In figure 1, V represents the domain and W the codomain. The possible set of vectors produced by the transformation are represented by the yellow subset within W . This subset is the image of T . In figure 2, there is a subset within the domain that will always be transformed into the zero vector, thus the gray subset within V is the kernel of T .

4 Linear Transformation

Although we can use this terminology to describe all transformations, not all transformations are linear. A **linear transformation** must meet certain conditions.

$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if the following conditions hold:

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$, where $\vec{u}, \vec{v} \in \mathbb{R}^m$
- $T(k\vec{v}) = kT(\vec{v})$ for a scalar k .

Let's demonstrate that the two transformations we've examined are indeed linear transformations. Let $\vec{u} = \langle a, b \rangle$ and $\vec{v} = \langle c, d \rangle$ and k a scalar:

$$\begin{aligned} T\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2x \\ 3y \end{pmatrix} \\ T(\vec{u}) + T(\vec{v}) &= \begin{pmatrix} 2a \\ 3b \end{pmatrix} + \begin{pmatrix} 2c \\ 3d \end{pmatrix} = \begin{pmatrix} 2(a+c) \\ 3(b+d) \end{pmatrix} \\ T(\vec{u} + \vec{v}) &= \begin{pmatrix} 2(a+c) \\ 3(b+d) \end{pmatrix} \end{aligned}$$

First condition met

$$\begin{aligned} T(k\vec{u}) &= k\begin{pmatrix} 2a \\ 3b \end{pmatrix} = T(k\vec{u}) = \begin{pmatrix} 2ak \\ 3bk \end{pmatrix} \\ kT(\vec{u}) &= k\begin{pmatrix} 2a \\ 3b \end{pmatrix} = \begin{pmatrix} 2ak \\ 3bk \end{pmatrix} \end{aligned}$$

Second condition met

$$\begin{aligned} T\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x - z \\ x + 2y - 4z \end{pmatrix} \\ T(\vec{u}) + T(\vec{v}) &= \begin{pmatrix} a - c \\ a + 2b - 4c \end{pmatrix} + \begin{pmatrix} d - f \\ d + 2e - 4f \end{pmatrix} \\ &= \begin{pmatrix} a - c + d - f \\ a + 2b - 4c + d + 2d - 4f \end{pmatrix} \\ T(\vec{u} + \vec{v}) &= \begin{pmatrix} (a + d) - (c + f) \\ (a + d) + 2(b + e) - 4(c + f) \end{pmatrix} \\ &= \begin{pmatrix} a - c + d - f \\ a + 2b - 4c + d + 2d - 4f \end{pmatrix} \end{aligned}$$

First condition met

$$T(k\vec{u}) = \begin{pmatrix} ak - ck \\ ak + 2yk - 4zk \end{pmatrix} = k\begin{pmatrix} a - c \\ a + 2y - 4z \end{pmatrix}$$

$$kT(\vec{u}) = k\begin{pmatrix} a - c \\ a + 2y - 4z \end{pmatrix}$$

Second condition met

5 Additional Problems

Ex. 1 Is the following a linear transformation?

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$$

Let $\vec{u} = \langle a, b \rangle$ and $\vec{v} = \langle c, d \rangle$.

We test the first condition:

$$T(\vec{u}) + T(\vec{v}) = \begin{pmatrix} a^2 + c^2 \\ b^2 + d^2 \end{pmatrix}$$

$$T(\vec{u} + \vec{v}) = \begin{pmatrix} (a+c)^2 \\ (b+d)^2 \end{pmatrix} = \begin{pmatrix} a^2 + 2ac + c^2 \\ b^2 + 2bd + d^2 \end{pmatrix}$$

Since $T(\vec{u}) + T(\vec{v}) \neq T(\vec{u} + \vec{v})$, **this is not a linear transformation.**

Ex. 2 Is the following a linear transformation?

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^4$$

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ y + z \\ x - z \\ z - y \end{pmatrix}$$

Let $\vec{u} = \langle a, b \rangle$ and $\vec{v} = \langle c, d \rangle$.

We test the first condition:

$$T(\vec{u}) + T(\vec{v}) = \begin{pmatrix} a + b + d + e \\ b + c + e + f \\ a - c + d - f \\ c - b + f - e \end{pmatrix}$$

$$T(\vec{u} + \vec{v}) = \begin{pmatrix} a + d + b + e \\ b + e + c + f \\ (a + d) - (c + f) \\ (c + f) - (b + e) \end{pmatrix} = \begin{pmatrix} a + b + d + e \\ b + c + e + f \\ a - c + d - f \\ c - b + f - e \end{pmatrix}$$

Condition Met

We test the second condition:

$$T(k\vec{u}) = \begin{pmatrix} ak + bk \\ bk + ck \\ ak - ck \\ ck - bk \end{pmatrix} = k \begin{pmatrix} a + b \\ b + c \\ a - c \\ c - b \end{pmatrix}$$

$$kT(\vec{u}) = k \begin{pmatrix} a + b \\ b + c \\ a - c \\ c - b \end{pmatrix}$$

Condition Met

T is a linear transformation.

Ex. 3 For T in example 2, determine the kernel.

We start by reducing the transformation using a matrix:

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ x-z \\ z-y \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2 = R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1 = R_3} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_2 + R_4 = R_2}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{\frac{1}{2} R_2 = R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_1 + R_2 = R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_4 - R_2 = R_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

So

$$Ker(T) = \left\{ \begin{array}{l} x = 0 \\ y = 0 \\ z = 0 \end{array} \right\}$$

Ex. 4 Given the following matrix A , find the kernel:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{pmatrix}$$

The kernel is the solution to the following system:

$$x + 2y + 3z = 0$$

$$4x + 5y + 6z = 0$$

$$7x + 8y + 9z = 0$$

... which can be achieved through row reduction:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 - 4R_1 = R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{-\frac{1}{3} R_2 = R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_3 - 7R_1 = R_3}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{pmatrix} \xrightarrow{R_3 + 6R_2 = R_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 - 2R_2 = R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

We are left with a system of equations that contains $x - z = 0$ and $y + 2z = 0$, so the solution is:

$$Ker(T) = \left\{ \begin{array}{l} x = s \\ y = -2s \end{array} \right\}, s \in \mathbb{R}$$