Gram Schmidt Process

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1   Purpose

The purpose of the Gram Schmidt algorithm is to derive an orthonormal vector set from a list of linearly independent vectors. Given an input set of vectors $\{\vec{v}_1, \vec{v}_2...\vec{v}_n\}$, the algorithm will produce a set $\{\hat{u}_1, \hat{u}_2...\hat{u}_n\}$. Each vector in the output set will be orthogonal to the other members in the set. Finally, both the input and output set will span the same space.

2   Algorithm Steps

Here are the steps, given a set of input vectors $\{\vec{v}_1, \vec{v}_2...\vec{v}_n\}$:

$$\vec{u}_1 = \vec{v}_1$$
$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{u_1}(v_2)$$
$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(v_3)$$

We can generalize $\vec{u}_n$:

$$\vec{u}_n = \vec{v}_k - \sum_{j=1}^{k-1} \text{proj}_{u_j}(v_k)$$

We conclude with normalizing $\vec{u}_1$, $\vec{u}_2$, ... $\vec{u}_n$ to obtain the orthonormal set:

$$\{\frac{\vec{u}_1}{||\vec{u}_1||}, \frac{\vec{u}_2}{||\vec{u}_2||}, \frac{\vec{u}_3}{||\vec{u}_3||}...\frac{\vec{u}_n}{||\vec{u}_n||}\} = \{\hat{u}_1, \hat{u}_2...\hat{u}_n\}$$

3   Intuition Using Two Vectors

Figure (1) shows the geometric intuition using two vectors in $\mathbb{R}^2$:
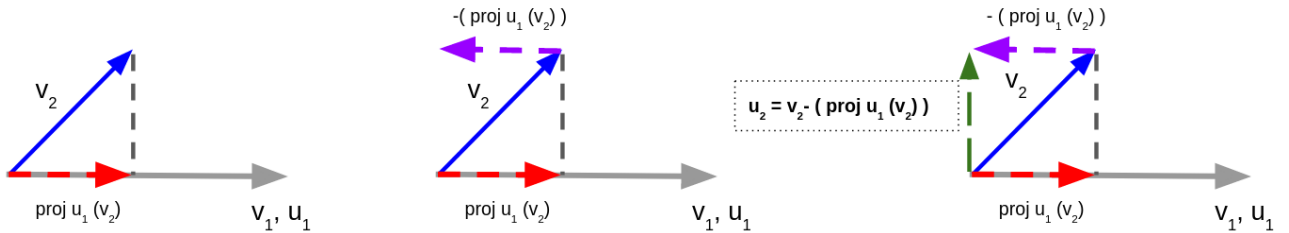

Figure 1

If $\hat{u}_1$ and $\hat{u}_2$ are orthogonal to each other, then their dot product is zero. Suppose that $\vec{v}_1 = <a, b>$ and $\vec{v}_2 = <c, d>$. The first vector is just $\vec{v}_1$.

We now calculate $\vec{u}_2$:

$$<c, d> - (\frac{ac + bd}{a^2 + b^2} <a, b>)$$

We can now take this result and perform a dot product with $\vec{v}_1$:

$$(<c, d> - (\frac{ac + bd}{a^2 + b^2} <a, b>)) \cdot <a, b>$$
$$= a(c - \frac{ac + bd}{a^2 + b^2} a) + b(d - \frac{ac + bd}{a^2 + b^2} b)$$
$$= ac - a^2 \frac{ac + bd}{a^2 + b^2} + bd - b^2 \frac{ac + bd}{a^2 + b^2}$$
$$= \frac{ac(a^2 + b^2)}{a^2 + b^2} - \frac{a^2(ac + bd)}{a^2 + b^2} + \frac{bd(a^2 + b^2)}{a^2 + b^2} - \frac{b^2(ac + bd)}{a^2 + b^2}$$
$$= \frac{a^3 c + ab^2 c - a^3 c - a^2 bd + a^2 bd + b^3 d - ab^2 c - b^3 d}{a^2 + b^2}$$
$$= 0$$

Ex. 1 Given $\vec{v}_1 = <3, 1>$ and $\vec{v}_2 = <2, 2>$ find an orthonormal set using the Gram Schmidt process.

We have $\vec{u}_1 = \vec{v}_1$, so $\vec{u}_1 = <3, 1>$.

We proceed to derive $\vec{u}_2$:

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{u_1}(v_2)$$
$$= <2, 2> - (\frac{(3)(2) + (1)(2)}{(3)(3) + (1)(1)} <3, 1>)$$
$$= <2, 2> - \frac{4}{5} <3, 1>$$
$$= <\frac{-2}{5}, \frac{6}{5}>$$

To conclude, normalize $\vec{u}_1$ and $\vec{u}_2$. We find that $||\vec{u}_1|| = \sqrt{10}$ and $||\vec{u}_2|| = \sqrt{\frac{8}{5}}$, so the orthonornmal set is:

$$\hat{u}_1 = \frac{1}{\sqrt{10}} <3, 1>$$
$$\hat{u}_2 = \frac{\sqrt{5}}{\sqrt{8}} <\frac{-2}{5}, \frac{6}{5}>$$

This result can be verified by showing that $\hat{u}_1 \cdot \hat{u}_2$ is zero:

$$(\frac{3}{\sqrt{10}})(\frac{\sqrt{5}}{\sqrt{8}})(\frac{-2}{5}) + (\frac{1}{\sqrt{10}})(\frac{\sqrt{5}}{\sqrt{8}})(\frac{6}{5}) = 0$$

Ex. 2 Given $\vec{v}_1 = \ <1,2,0> $ and $\vec{v}_2 = \ <8,1,-6>$ and $\vec{v}_3 = \ <0,0,1>$ find an orthonormal set using the Gram Schmidt process:

We have $\vec{u}_1 = \vec{v}_1$, so $\vec{u}_1 = \ <1,2,0>$.

Let's calculate $\vec{u}_2$:

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{u_1}(v_2)$$

$$\vec{u}_2 = \ <8,1,-6> - \frac{(1)(8) + (2)(1) + (0)(-6)}{(1)(1) + (2)(2) + (0)(0)} <1,2,0>$$
$$= \ <6,-3,-6>$$

Let's calculate $\vec{u}_3$:

$$\vec{u}_3 = \vec{v}_3 - \text{proj}_{u_1}(v_3) - \text{proj}_{u_2}(\vec{v}_3)$$

$$\vec{u}_3 = \ <0,1,1> - \frac{(1)(0) + (2)(0) + (0)(1)}{(1)(1) + (2)(2) + (0)(0)} <1,2,0> - \frac{(1)(8) + (2)(1) + (0)(0)}{(6)(6) + (-3)(-3) + (-6)(-6)} <6,-3,-6>$$
$$= \ <\frac{4}{9}, -\frac{2}{9}, \frac{5}{9}>$$

We conclude with normalizing our results given that $||\vec{u}_1|| = \sqrt{5}$, $||\vec{u}_2|| = 9$, and $||\vec{u}_3|| = \frac{\sqrt{45}}{9}$:

$$\hat{u}_1 = \frac{1}{\sqrt{5}} = \ <1,2,0>$$
$$\hat{u}_2 = \frac{1}{3} <6,-3,-6>$$
$$\hat{u}_3 = \frac{1}{\sqrt{45}} <4,-2,5>$$