Linear Combination

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1 Linear Combinations Visual Intuition

We first start with the idea of a generic vector which can be scaled by a positive or negative k. As seen on Figure 1, scaling the vector positively or negatively through k will result in a line.

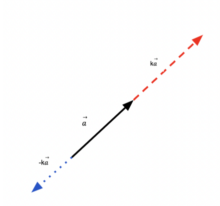

Figure 1

Now suppose we have two vectors $\vec{a}$ and $\vec{b}$. They can each be scaled independently. Both vectors can also be added together. With some visual imagination we can see that by varying the scaling factor of each vector we can reach every point on the two dimensional plane.

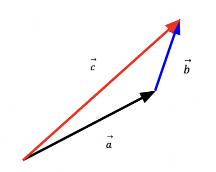

Figure 2

We can of course add multiple vectors together, each with the ability to scale independently. A hypothetical situation with three vectors is shown on Figure 3.

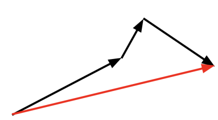

Figure 3

The points that can be reached by adding and scaling a set of vectors is the **span** of the vector set. The set of vectors themselves (along with any scaling and vector addition) is referred to as a **vector space**.

We can describe the summation and scaling of n vectors $< x_n, y_n >$ together like so:

$$k_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + k_3 \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} + ...k_n \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} k_1x_1 + k_2x_2 + k_3x_3... + k_nx_n \\ k_1y_1 + k_2y_2 + k_3y_3... + k_ny_n \end{pmatrix}$$

The expression above can be rewritten in the form of a matrix multiplication:

$$\begin{pmatrix} x_1 & x_2 & x_3 & ... & x_n \\ y_1 & y_2 & y_3 & ... & y_n \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ ... \\ k_n \end{pmatrix}$$

It is for this reason that when vectors are written into a matrix they are often done so vertically down a column.

With this in mind, we can document two important vector spaces, $\mathbb{R}^2$ and $\mathbb{R}^3$.Consider the following linear combination:

$$k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We can visualize how any point on the two dimensional coordinate plane can be reached by simply adjusting the values for k_1 and k_2. This is the vector space known as $\mathbb{R}^2$. Now consider the following combination:

$$k_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

By adjusting the values for k_1, k_2 and k_3 we can reach every point in the three dimensional coordinate system. This vector space is known as $\mathbb{R}^3$.

Problems related to linear combinations involve verifying if a vector is a combination of other vectors. A few examples are provided below.

Write $< 10, 12 >$ as a linear combination of $< 2, 2 >$ and $< 3, 1 >$.

We want to solve for the matrix $< k_1, k_2 >$:

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \end{pmatrix}$$

Which can be solved with the augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 3 & 10 \\ 2 & 1 & 12 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 2 & 3 & 10 \\ 2 & 1 & 12 \end{array}\right) \xrightarrow{R_2 - R_1 = R_2} \left(\begin{array}{cc|c} 2 & 3 & 10 \\ 0 & -2 & 2 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2 = R_2} \left(\begin{array}{cc|c} 2 & 3 & 10 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{-\frac{1}{2}R_1 = R_1}$$

$$\left(\begin{array}{cc|c} 1 & \frac{3}{2} & 5 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{R_1 - \frac{3}{2}R_2 = R_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{13}{2} \\ 0 & 1 & -1 \end{array}\right)$$

Solution:

$$\frac{13}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \end{pmatrix}$$

Write $< -5, 3, 16 >$ as a linear combination of $< 1, -1, 4 >$ and $< -3, 2, 6 >$.

We proceed to setup the augmented matrix and reduce:

$$\left(\begin{array}{cc|c} 1 & -3 & -5 \\ -1 & 2 & 3 \\ 4 & 6 & 16 \end{array}\right) \xrightarrow{R_2 <--> R_3} \left(\begin{array}{cc|c} 1 & -3 & -5 \\ 4 & 6 & 16 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 = R_2} \left(\begin{array}{cc|c} 1 & -3 & -5 \\ 2 & 3 & 8 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{R_1 + R_2 = R_1}$$

$$\left(\begin{array}{cc|c} 3 & 0 & 3 \\ 2 & 3 & 8 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 = R_1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 3 & 8 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{R_2 - 2R_1 = R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 3 & 6 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{\frac{1}{3}R_2 = R_2}$$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{R_3 + R_1 = R_3} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right) \xrightarrow{R_3 - 2R_2 = R_3} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

Solution:

$$\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + 2\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 16 \end{pmatrix}$$

2 Linear Independence

If a vector can be obtained by scaling another vector, then the two vectors are dependent. In a different case, suppose we have a set of three vectors. If one of the vectors can be derived by a combination of scaling and adding the other two vectors, then the set is also dependent.

A set of vectors is dependent if there exists $k_1 ... k_n$ such that $k_1\vec{v}_1 + ...k_n\vec{v}_n = 0$. If the only solution is the trivial solution $k_1 = ...k_n = 0$, then the set is independent.

Suppose we have $\vec{a} =< 1, 1 >$ and $\vec{b} =< 2, 2 >$ as shown on Figure 4. Since $2\vec{a} = \vec{b}$, the set $\{\vec{a}, \vec{b}\}$ is dependent.

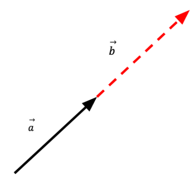

Figure 4

One possible intution is as follows. Suppose that there are two vectors v_1 and v_2. Based on our earlier definition, we know that if either vector can be scaled to obtain the other, then the set is not independent. If the set is dependent, then there is a constant k by which $kv_2 = v_1$ or $kv_1 = v_2$. In the latter case, we end up with $kv_1 - v_2 = 0$. In this very specific example, the coefficient for v_2 turned out to be negative, but we can generalize this whole expression to a general coefficient k_2 which can be either positive or negative: $k_1v_1 + k_2V_2 = 0$.

Here are some example problems:

Determine if the set of vectors $< 2, 2 >$ and $< -1, 5 >$ are independent or dependent.

We are solving for the matrix $< k_1, k_2 >$:

$$\begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & 5 & 0 \end{array}\right) \xrightarrow{R_1 - \frac{1}{2}R_2 = R_1} \left(\begin{array}{cc|c} 1 & -\frac{7}{2} & 0 \\ 2 & 5 & 0 \end{array}\right) \xrightarrow{R_2 - 2R_1 = R_2} \left(\begin{array}{cc|c} 2 & -\frac{7}{2} & 0 \\ 0 & 12 & 0 \end{array}\right) \xrightarrow{\frac{1}{12}R_2 = R_2}$$

$$\left(\begin{array}{cc|c} 2 & -\frac{7}{2} & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 + \frac{7}{2}R_2 = R_1} \left(\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 = R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The set is independent.

Determine if the set of vectors $< -1, 4 >$ and $< 2, -8 >$ are independent or dependent.

We are solving for the matrix $< k_1, k_2 >$:

$$\begin{pmatrix} -1 & 2 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ 4 & -8 & 0 \end{array}\right) \xrightarrow{R_1 <--> R_2} \left(\begin{array}{cc|c} 4 & -8 & 0 \\ -1 & 2 & 0 \end{array}\right) \xrightarrow{R_1 + 3R_2 = R_1} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right) \xrightarrow{R_2 + R_1 = R_2} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

We are left with $k_1 - 2k_2 = 0$, or $k_1 = 2k_2$. Any combination of k_1 and k_2 that meet the established relationship will work, hence there are solutions other than the trivial solution.

The set is dependent.

Here are some examples using vectors with three components:

Determine if the set of vectors $< 1, 2, 3 >$, $< -2, 1, 0 >$, and $< 1, 0, 1 >$ are independent or dependent.

We are solving for the matrix $< k_1, k_2, k_3 >$:

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2-2R_1=R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\frac{1}{5}R_2=R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 3 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_3-3R_1=R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 6 & -2 & 0 \end{array}\right) \xrightarrow{R_3-6R_2=R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & \frac{2}{5} & 0 \end{array}\right) \xrightarrow{R_2+R_3=R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 \end{array}\right) \xrightarrow{R_1+2R_2=R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 \end{array}\right) \xrightarrow{\frac{5}{2}R_3=R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1-R_3=R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The set is independent.

Determine if the set of vectors $< 1, 0, 1 >$, $< 1, 1, 0 >$, and $< 0, 1, -1 >$ are independent or dependent.

We are solving for the matrix $< k_1, k_2, k_3 >$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{R_1-2R_2=R_1} \left(\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{R_1+R_2=R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{R_3-R_1=R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We are left with $k_1 - k_3 = 0$ and $k_2 + k_3 = 0$, or $k_1 = k_3$ and $k_2 = -k_3$. Any combination of k_1, k_2, k_3 that meets the criteria is a possible solution.

The set is dependent.

3 Basis

A set of vectors comprise a **basis** if they are linearly independent and span the vector space being analyzed. The number of vectors in the basis is called the **dimension** of the vector. Here are some sample basis problems:

Is the set $\{ <3,-2>, <4,5> \}$ a basis for $\mathbb{R}^2$?

We first check for independence by using row reduction, we are solving for $< k_1, k_2 >$ in:

$$\begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can arrive at I_2 by applying the following operations: $R_1 + R_2 = R_1$, $R_2 + 2R_1 = R_2$, $\frac{1}{23}R_2 = R_32$, $R_1 - 9R_2 = R_1$

Since the solution is $< k_1, k_2 > = < 0, 0 >$, the system is independent. **The set is a basis for $\mathbb{R}^2$.**

Is the set $\{ <4,2,5>, <3,-1,-2>, <6,2,0> \}$ a basis for $\mathbb{R}^3$?

We check for independence by using row reduction, we are solving for $< k_1, k_2, k_3 >$ in:

$$\begin{pmatrix} 4 & 3 & 6 \\ 2 & -1 & 2 \\ 5 & -2 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We arrive at I_3 by applying the following operations: $\frac{1}{4}R_1 = R_1$, $R_2 - 2R_1 = R_2$, $-\frac{2}{5}R_2 = R_2$, $R1 - \frac{3}{4}R_2 = R_1$, $R_3 - 5R_1 = R_3$, $R_3 + 2R_2 = R_3$, $-\frac{5}{26}R_3 = R_3$, $R_2 - \frac{2}{5}R_3 = R_2$, $R_1 - \frac{6}{5}R_3 = R_1$

Since $< k_1, k_2, k_3 > = < 0, 0, 0 >$, **the set is a basis for $\mathbb{R}^2$.**

Is the set $\{ <4,-3>, <12,-9> \}$ a basis for $\mathbb{R}^2$?

We check for independence by using row reduction, we are solving for $< k_1, k_2 >$ in:

$$\begin{pmatrix} 4 & 12 \\ -3 & -9 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 12 \\ -3 & -9 \end{pmatrix} \xrightarrow{R_1+R_2=R_1} \begin{pmatrix} 1 & 3 \\ -3 & -9 \end{pmatrix} \xrightarrow{R_2+3R_1=R_2} \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$$

There are non-trivial solutions present, as we end up with $k_1 + 3k_2 = 0$. Since the set is dependent, **they do not form a basis for $\mathbb{R}^2$.**

4 Subspace

A **subspace** is a vector space that is a subset of a larger vector space. A subspace V of $\mathbb{R}^n$ must meet the following criterion:

1. It must contain the zero vector.
2. V exhibits closure under addition, if $\vec{a}$ and $\vec{b}$ are part of V, then $\vec{a} + \vec{b}$ is also part of V.
3. V exhibits closure under scalar multiplication, if $\vec{a}$ is in V and k is in R, then $k\vec{a}$ is in V.

Problems on this subject will typically involve proving if a space (defined with variables) is a subspace of $\mathbb{R}^n$:

Is the space defined by y = 2x a subspace of $\mathbb{R}^2$?

To test this, let $a, b, k \in \mathbb{R}$.

(1) The zero vector is part of the space, if $x = 0$ then $y = 0$.

(2) We want to show that this space is closed under addition. We can do so with the following strategy: We make two generic vectors using a and b by plugging them into the definition of the space $y = 2x$. We add the results together.

$$\begin{pmatrix} a \\ 2a \end{pmatrix} + \begin{pmatrix} b \\ 2b \end{pmatrix} = \begin{pmatrix} a+b \\ 2a+2b \end{pmatrix}$$

We now plug in $a + b$ into the space directly:

$$\begin{pmatrix} (a+b) \\ 2(a+b) \end{pmatrix} = \begin{pmatrix} a+b \\ 2a+2b \end{pmatrix}$$

We get the same result with either approach so the second condition is met.

(3) First, we apply ka to the space:

$$\begin{pmatrix} ka \\ 2(ka) \end{pmatrix}$$

Next, we apply a to the space, then multiply the resulting vector by k:

$$k \begin{pmatrix} a \\ 2a \end{pmatrix} = \begin{pmatrix} ka \\ 2(ka) \end{pmatrix}$$

Since we get the same result with either approach, the space is closed under scalar multiplication.

The space is a subspace of $\mathbb{R}^2$.

Is the space defined by $y = 2x + 5$ a subspace of $\mathbb{R}^2$

Since the zero vector is not part of the space, **it is not a subspace of** $\mathbb{R}^2$. When $x = 0$, $y = 5$, and when $y = 0, x = -\frac{2}{5}$.

Is the space defined by $z = xy$ a subspace of $\mathbb{R}^3$?

(1) The zero vector is part of the space, if $x = 0$ and $y = 0$, then $z = 0$.

(2) We'll test closure under addition. Let $a, b, c, d \in \mathbb{R}$. We'll construct two generic vectors by plugging a, b, c, d directly into the space:

$$\begin{pmatrix} a \\ b \\ ab \end{pmatrix} + \begin{pmatrix} c \\ d \\ cd \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \\ ab+cd \end{pmatrix}$$

These results are compared against plugging in $a + c$ and $b + d$ directly into the space:

$$\begin{pmatrix} a+c \\ b+d \\ (a+c)(b+d) \end{pmatrix}$$

Since $ab+cd \neq (a+c)(b+d)$, the space is not closed under addition. **The space is not a subspace of $\mathbb{R}^2$.**

Is the space defined by $z = 3y - 5x$ a subspace of $\mathbb{R}^3$?

(1) The zero vector is part of the space. If $x = 0$, $y = 0$, then $z = 0$.

(2) We'll verify closure under addition. Let $a, b, c, d \in \mathbb{R}$. First, we'll plug in a and b directly into the space and add the two vectors:

$$\begin{pmatrix} a \\ b \\ 3b-5a \end{pmatrix} + \begin{pmatrix} c \\ d \\ 3d-5c \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \\ (3b-5a)+(3d-5c) \end{pmatrix}$$

We will now plug in $a + c$ and $b + d$ directly into the space:

$$\begin{pmatrix} a+c \\ b+d \\ 3(b+d)-5(a+c) \end{pmatrix}$$

Since $3(b + d) - 5(a + c) = (3b - 5a) + (3d - 5c)$, the space is closed under addition.

(3) To test for closure under scalar multiplication we first apply $k < a, b, 3b - 5a >$ to the space.

$$\begin{pmatrix} ka \\ kb \\ 3bk-5ak \end{pmatrix}$$

If we apply $< a, b, 3b - 5a >$ to the space first, then multiply the resulting vector by k the same outcome is observed:

$$k\begin{pmatrix} a \\ b \\ 3b-5a \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ 3bk-5ak \end{pmatrix}$$

The space is a subspace of $\mathbb{R}^3$.

From these examples we observed that a line in $\mathbb{R}^2$ passing through the origin is a subspace. In $\mathbb{R}^3$, a plane passing through the origin is a subspace.