Linear Combination

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1 Linear Combinations Visual Intution

We first start with the idea of a generic vector which can be scaled by a positive or negative k. As seen on Figure 1, given positive enough or negative values of k, the vector can occupy a line of space.

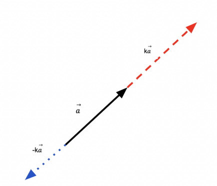

Figure 1

Now suppose we have two vectors $\vec{a}$ and $\vec{b}$. They can each be scaled independently. Both vectors can also be added together. With some visual imagination we how by varying the scaling factor of each vector we can reach every point on the two dimensional plane.

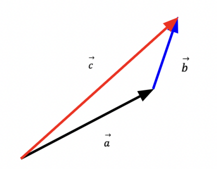

Figure 2

We can of course add multiple vectors together, each with the ability to scale independently. A hypothetical situation with three vectors is shown on Figure 3.

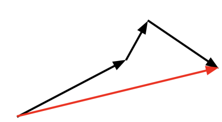

Figure 3

We can describe describe the summation of n vectors $< x_n, y_n >$ together like so:

$$k_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + k_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + k_3 \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} + ...k_n \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} k_1x_1 + k_2x_2 + ... + k_nx_n \\ k_1y_1 + k_2y_2 + ... + k_ny_n \end{pmatrix}$$

The expression above can be rewritten in the form of a matrix multiplication:

$$\begin{pmatrix} x_1 & x_2 & ... & x_n \\ y_1 & y_2 & ... & y_n \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ ... \\ k_n \end{pmatrix}$$

It is for this reason that when vectors are written into a matrix they are often done so vertically down a column.

Problems related to linear combinations involve verifying if a vector is a combination of other vectors. A few examples are provided below.

Write $< 12, 10 >$ as a linear combination of $< 2, 2 >$ and $< 1, 3 >$.

We want to solve for the matrix $< k_1, k_2 >$:

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \end{pmatrix}$$

Which can be solved with the augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 3 & 10 \\ 2 & 1 & 12 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 2 & 3 & 10 \\ 2 & 1 & 12 \end{array}\right) \xrightarrow{R_2 - R_1 = R_2} \left(\begin{array}{cc|c} 2 & 3 & 10 \\ 0 & -2 & 2 \end{array}\right) \xrightarrow{-\frac{1}{2}R_2 = R_2} \left(\begin{array}{cc|c} 2 & 3 & 10 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{-\frac{1}{2}R_1 = R_1}$$

$$\left(\begin{array}{cc|c} 1 & \frac{3}{2} & 5 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{R_1 - \frac{3}{2}R_2 = R_1} \left(\begin{array}{cc|c} 1 & 0 & \frac{13}{2} \\ 0 & 1 & -1 \end{array}\right)$$

Solution:

$$\frac{13}{2}\begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 12 \end{pmatrix}$$

Write $< -5, 3, 16 >$ as a linear combination of $< 1, -1, 4 >$ and $< -3, 2, 6 >$.

We proceed to setup the augmented matrix and reduce:

$$\left(\begin{array}{cc|c} 1 & -3 & -5 \\ -1 & 2 & 3 \\ 4 & 6 & 16 \end{array}\right) \xrightarrow{R_2 <--> R_3} \left(\begin{array}{cc|c} 1 & -3 & -5 \\ 4 & 6 & 16 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 = R_2} \left(\begin{array}{cc|c} 1 & -3 & -5 \\ 2 & 3 & 8 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{R_1 + R_2 = R_1}$$

$$\left(\begin{array}{cc|c} 3 & 0 & 3 \\ 2 & 3 & 8 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{\frac{1}{3}R_1 = R_1} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 3 & 8 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{R_2 - 2R_1 = R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 3 & 6 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{\frac{1}{3}R_2 = R_2}$$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow{R_3 + R_1 = R_3} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array}\right) \xrightarrow{R_3 - 2R_2 = R_3} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

Solution:

$$\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + 2\begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 16 \end{pmatrix}$$

2 Linear Independence

If a vector can be obtained by scaling another vector, then the two vectors are dependent. A set of vectors is dependent if there exists $c_1 ... c_n$ such that $c_1 v_1 + ... c_n v_n = 0$. If the only solution is the trivial solution $c_1 = ... c_n = 0$, then the set is independent.

Suppose we have $\vec{a} =< 1, 1 >$ and $\vec{b} =< 2, 2 >$ as shown on Figure 4. Since $2\vec{a} = \vec{b}$, we know that set $\{\vec{a}, \vec{b}\}$ is dependent.

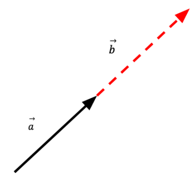

Figure 4

One possible intution is as follows. Suppose that there are two vectors v_1 and v_2. Based on our earlier definition, we know that if either vector can be scaled to obtain the other, then the set is not independent. If the set is dependent, then there is a constant k by which $kv_2 = v_1$ or $kv_1 = v_2$. In the latter case, we end up with $kv_1 - v_2 = 0$. In this very specific example, the coefficient for v_2 turned out to be negative, but we can generalize this whole expression to a general coefficient k_2 which can be either positive or negative: $k_1 v_1 + k_2 V_2 = 0$.

Here are some example problems:

Determine if the set of vectors $< 2, 2 >$ and $< -1, 5 >$ are independent or dependent.

We are solving for the matrix $< k_1, k_2 >$:

$$\begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & 5 & 0 \end{array}\right) \xrightarrow{R_1 - \frac{1}{2}R_2 = R_2} \left(\begin{array}{cc|c} 1 & -\frac{7}{2} & 0 \\ 2 & 5 & 0 \end{array}\right) \xrightarrow{R_2 - 2R_1 = R_2} \left(\begin{array}{cc|c} 2 & -\frac{7}{2} & 0 \\ 0 & 12 & 0 \end{array}\right) \xrightarrow{\frac{1}{12}R_2 = R_2}$$

$$\left(\begin{array}{cc|c} 2 & -\frac{7}{2} & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 + \frac{7}{2}R_2 = R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The set is independent.

Determine if the set of vectors $< -1, 4 >$ and $< 2, -8 >$ are independent or dependent.

We are solving for the matrix $< k_1, k_2 >$:

$$\begin{pmatrix} -1 & 2 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ 4 & -8 & 0 \end{array}\right) \xrightarrow{R_1 <-- > R_2} \left(\begin{array}{cc|c} 4 & -8 & 0 \\ -1 & 6 & 0 \end{array}\right) \xrightarrow{R_1 + 3R_2 = R_1} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right) \xrightarrow{R_2 + R_1 = R_2} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

We are left with $k_1 - 2k_2 = 0$, or $k_1 = 2k_2$. Any combination of k_1 and k_2 that meet the established relationship will work, hence there are solutions other than the trivial solution.

The set is dependent.

Here are some examples using vectors with three components:

Determine if the set of vectors $< 1, 2, 3 >$, $< -2, 1, 0 >$, and $< 1, 0, 1 >$ are independent or dependent.

We are solving for the matrix $< k_1, k_2, k_3 >$:

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2-2R_1=R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\frac{1}{5}R_2=R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 3 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_3-3R_1=R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 6 & -2 & 0 \end{array}\right) \xrightarrow{R_3-6R_2=R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & 0 \\ 0 & 0 & \frac{2}{5} & 0 \end{array}\right) \xrightarrow{R_2+R_3=R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 \end{array}\right) \xrightarrow{R_1+2R_2=R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{5} & 0 \end{array}\right) \xrightarrow{\frac{5}{2}R_3=R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The set is independent.

Determine if the set of vectors $< 1, 0, 1 >$, $< 1, 1, 0 >$, and $< 0, 1, -1 >$ are independent or dependent.

We are solving for the matrix $< k_1, k_2, k_3 >$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{R_1-2R_2=R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow{R_3-R_1=R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We are left with $k_1 - k_3 = 0$ and $k_2 + k_3 = 0$, or $k_1 = k_3$ and $k_2 = -k_3$. Any combination of k_1, k_2, k_3 that meets the criteria is a possible solution.

The set is dependent.