Curl

Andy Chong Sam

1 Overview

The curl operator describes the rotation tendency of a vector field. Both its input and output are vectors. In the cases when we consider only two dimensions, the result of curl is a scalar.

If a vector field is defined by $F =< P, Q >$ then its two dimdensional curl is defined as:

$$\nabla \times F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \tag{1}$$

If $F =< P, Q, R >$ we can calculate a three dimensional curl with:

$$\nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} \ + \ (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} \ + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k} \tag{2}$$

From Expressions (1) and (2) we again note that the latter produces a vector, and that its third component is just the two dimensional curl.

2 Geometric Intuition

To start we will note that the orthodox direction of rotation is counter clockwise. Fields that induce a counter clockwise rotation will have a positive curl. We can start out by imagining a particle on the xy plane.

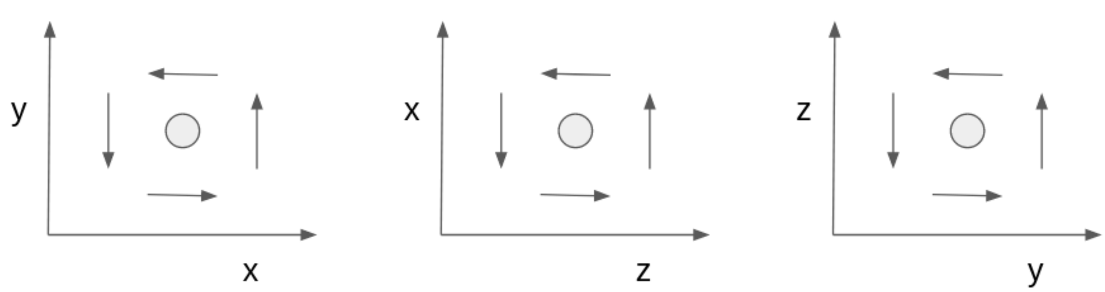

Figure 1

In Figure 1's left graph, we've generalized what happens to a particle when a field induces positive curl. In these diagrams the arrows represent the field. We note that as x increases, the Q component of the field has a tendency to shift from negative to positive. Likewise as as y increases, the P component becomes more negative. In order to guarantee a positive value if the field is inducing a counter clockwise rotation we can take the change of R with respect to y (which will be positive) and subtract from it the change of P with respect to y which will be negative. From this intuition, we get $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ as the two dimensional curl.

In our analysis so far we've imagined that the field can only induce rotation on the xy plane. Because z is fixed in this scenario, the two dimensional curl is the $\vec{k}$ component of the three dimensional curl. We can think of this as the observed rotation viewed from z's perspective.

Looking again at Figure 1, the central diagram describes a hypothetical counter clockwise rotation from the perspective of y. As z increases, the P component of the field becomes more positive and as x increases, the R compoment becomes more negative. So to guarantee a positive value we can write $\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}$. The same analysis can be performed with Figure 1's right graph.

3 Derivation of Two Dimension Curl

Suppose we have a patch of xy like the one described in Figure 2. We have here a simple connected region where the path C can be divided into $C1 + C2 + C3 + C4$. We have also drawn here some vectors that approximate what the field $F =< P, Q, R >$ is doing near these points along the four sub paths.

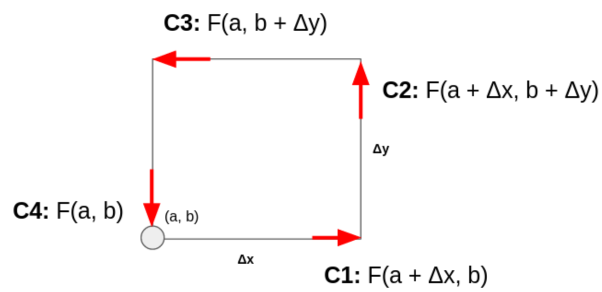

Figure 2

So the line integral for the complete path is:

$$\oint_C F\,dr = \sum_{i=1}^{4} \int_{Ci} F\,dr$$

For very small values of Δx and Δy we can state the following:

$$\int_{C1} F\,dr \approx P(a + \Delta x, b)\Delta x$$
$$\int_{C2} F\,dr \approx Q(a + \Delta x, b + \Delta y)\Delta y$$
$$\int_{C3} F\,dr \approx P(a, b + \Delta y)\Delta x$$
$$\int_{C4} F\,dr \approx Q(a, b)\Delta y$$

Our goal is ultimately to describe the rotational tendency of a field per area of $\Delta x \Delta y$, albeit an infinitesimally small area. Our strategy here is to look at the horizontal (C1, C3) and vertical paths (C2, C4). Let's start with the horizontal:

$$\lim_{(x,y)\to(0,0)} \frac{P(a + \Delta x, b)\Delta x - P(a, b + \Delta x)\Delta y}{\Delta x \Delta y}$$

Note how we treat C3 as a negative because it goes in the opposite direction of C1. Upon canceling out the Δx and evaluating the limit as x approaches zero we are left with:

$$\lim_{y\to 0} \frac{P(a, b) - P(a, b + \Delta y)}{\Delta y}$$

If we modify this result by making it negative, we arrive at one of the two components of the two dimension curve:

$$-\lim_{y\to 0} \frac{P(a, b + \Delta y) - P(a, b)}{\Delta y} = -\frac{\partial P}{\partial y}$$

We can repeat this analysis for the vertical components:

$$\lim_{(x,y)\to(0,0)} \frac{Q(a+\Delta x, b+\Delta y)\Delta x - Q(a,b)\Delta y}{\Delta x \Delta y}$$

We take the limit as y approaches zero first and are left with the definition of a partial derivative with respect to x, the remaining component of two dimension curl:

$$\lim_{x\to 0} \frac{Q(a+\Delta x, b) - Q(a,b)}{\Delta x} = \frac{\partial Q}{\partial x}$$

4 Examples

Ex. 1 Calculate $\nabla \times F$ given $F(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}(x\,\vec{i} + y\,\vec{j} + z\,\vec{k})$

Source:

$$\frac{\partial R}{\partial y} = \frac{zy}{2\sqrt{x^2+y^2+z^2}} \quad \frac{\partial Q}{\partial z} = \frac{zy}{2\sqrt{x^2+y^2+z^2}} \quad \frac{\partial P}{\partial z} = \frac{xz}{2\sqrt{x^2+y^2+z^2}}$$
$$\frac{\partial R}{\partial x} = \frac{xz}{2\sqrt{x^2+y^2+z^2}} \quad \frac{\partial Q}{\partial x} = \frac{xy}{2\sqrt{x^2+y^2+z^2}} \quad \frac{\partial P}{\partial y} = \frac{xy}{2\sqrt{x^2+y^2+z^2}}$$

$$\nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} \;+\; (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} \;+ (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k}$$

$$\nabla \times F = 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k}$$

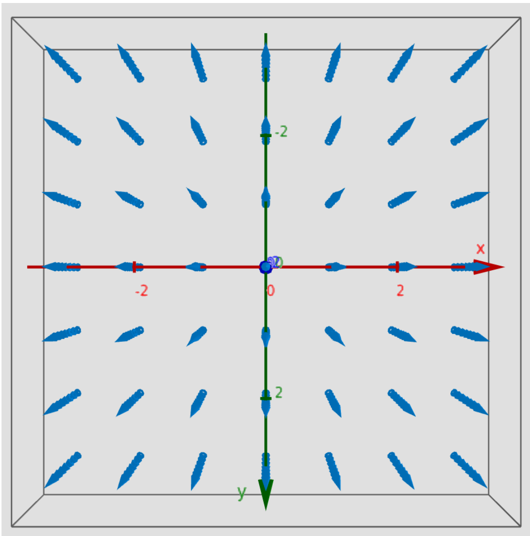

Figure 2

No curl is present. A graph of the field (top down) is shown in Figure 2. We can intuitively see that this field is incapable of inducing rotation.

Ex. 2 Calculate $\nabla \times F$ where $F = xye^z \ \vec{i} + yze^x \ \vec{k}$

Source:

$$\frac{\partial R}{\partial y} = ze^x \quad \frac{\partial Q}{\partial z} = 0 \quad \frac{\partial P}{\partial z} = xye^z$$
$$\frac{\partial R}{\partial x} = yze^x \quad \frac{\partial Q}{\partial x} = 0 \quad \frac{\partial P}{\partial y} = xe^z$$

$$\nabla \times F = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\vec{i} \ + \ (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\vec{j} \ + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\vec{k}$$

$$\nabla \times F = ze^x \ \vec{i} + xye^z - yze^x \ \vec{j} - xe^z \ \vec{k}$$