Essence of Single Variable Calculus

Andy Chong Sam

1 Derivatives

We will first summarize derivatives. Given a function $f(x)$, its derivative $f'(x)$, describes the instantaneous rate of change at x. The formal definition of the derivative is as follows:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

From the expression above we can derive all the derivatives found on a textbook table.

2 Integrals

A definite integral across a continuous interval $[a, b]$ is defined as:

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(t)\Delta x$$

t is a point within an interval of width Δx

From a practical standpoint, the definite integral tells us the area under the curve from point a to b.

In Figure 1, we try and approximate the area under x^2 for the interval $[0, 1]$ by creating rectangles with heights of $f(t)$ and width of Δx. Using more rectangles, with smaller values of Δx gets us an increasingly accurate measure of the area.

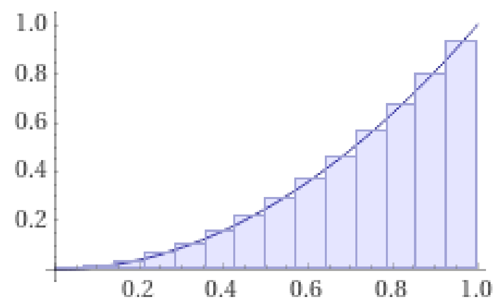

Figure 1

Given a function $f(x)$ and $F(x)$, if $F'(x) = f(x)$ then we say that $F(x)$ is the anti-derivative of $f(x)$. This is one key connection between derivatives and integrals. Now we can define an **indefinite integral** as:

$$\int f(x)dx = F(x) + C$$

In this expression C is an arbitrary constant. This is needed for the following situation: Suppose we have $x^2 + 3$ or $x^2 + 5$, taking the derivative of either expression will result in $2x$.

Using the Second Fundamental Theorem of Calculus, which we will discuss later, the **definite integral** can be defined as:

$$\int_a^b f(x)dx = F(b) - F(a)$$

3 Mean Value Theorem

Given a continuous function f on a closed interval $[a, b]$, then there is a point c, such that $f'(c)$ is equivalent to the slope of the secant line that runs from points a and b.

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

In Figure 2 the line that runs through points a and b has the same slope as the tangent line that runs through point c

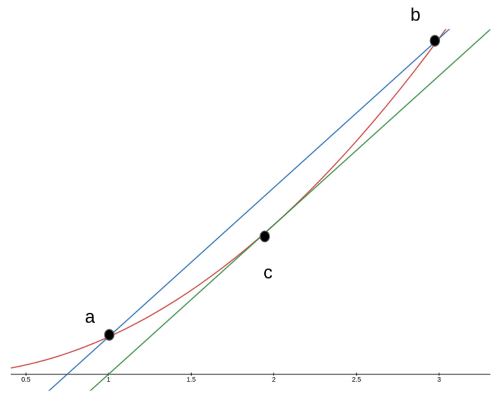

Figure 2

4 Mean Value Theorem of Integrals

Given a continuous function f there exists a value c such that

$$\int_a^b f(x)dx = f(c)(b - a)$$

In Figure 3 the mean value theorem of integrals predicts that the area under the curve from points a to b is equivalent to the purple edged rectangle with a height of $f(c)$ and length $b - a$.

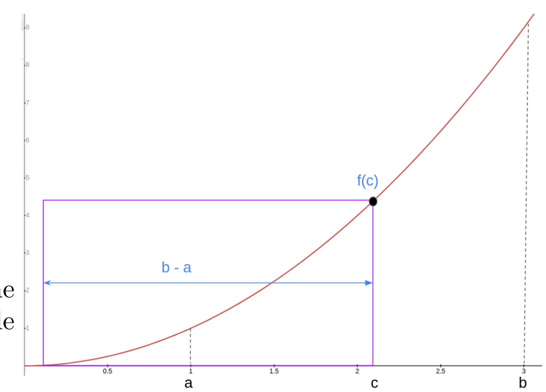

Figure 3

5 First Fundamental Theorem of Calculus

If $f(x)$ is continuous over $[a, b]$, and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$ over $[a, b]$. We say that F is an antiderivative of f.

Derivation:

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x}$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x}$$

$$F'(x) = \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x}$$

By the **Mean Value Theorem of Integrals** we know that there is a value c such that:

$$f(c)\Delta x = \int_{x}^{x+\Delta x} f(t)dt$$

We are left with evaluating:

$$F'(x) = \lim_{\Delta x \to 0} f(c)\Delta x$$

By the **Squeeze Theorem** we know that as Δx approaches zero, c approaches x, so we are left with:

$$F'(x) = \lim_{\Delta x \to 0} f(c)\Delta x = f(x)$$

6 Second Fundamental Theorem of Calculus

If $f(x)$ is continuous over $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Derivation:

There are infinite real numbers between a and b, so we can say that:

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$$

We can replace a and b with x_0 and x_n respectively:

$$F(b) - F(a) = F(x_n) - F(x_0)$$

Next, we insert a series of self-cancelling expressions:

$$\begin{aligned} F(b) - F(a) &= F(x_n) + (-F(x_{n-1}) + F(x_{n-1})) + (-F(x_{n-2}) + F(x_{n-2})) + ... + (-F(x_1) + F(x_1)) - F(x_0) \\ &= (F(x_n) - F(x_{n-1})) + (F(x_{n-1}) - F(x_{n-2})) + ... + (F(x_2) - F(x_1)) + (F(x_1) - F(x_0)) \end{aligned}$$

This result can be summarized with the summation:

$$F(b) - F(a) = \sum_{i=1}^{n} (\ F(x_i) - F(x_{i-1})\ )$$

By the Mean Value Theorem of Integrals we know that there is a value c such that $F(x_i) - F(x_{i-1}) = F'(c)(x_i - x_{i-1})$. We can now rewrite $F(b) - F(a)$ as:

$$F(b) - F(a) = \sum_{i=1}^{n} (\ F'(c)(x_i - x_{i-1})\ )$$

By the First Fundamental Theorem of Calculus we know that $F'(c) = f(c)$ so we now have:

$$F(b) - F(a) = \sum_{i=1}^{n} (\ f(c)(x_i - x_{i-1})\ )$$

If we take the limit for both sides we will be left with definition of the definite integral on the right hand side. This is the same definition we saw in section 2:

$$\begin{aligned} \lim_{n\to\infty} (\ F(b) - F(a)\ ) &= \lim_{n\to\infty} \sum_{i=1}^{n} (\ f(c)(x_i - x_{i-1})\ ) \\ F(b) - F(a) &= \lim_{n\to\infty} \sum_{i=1}^{n} (\ f(c)(x_i - x_{i-1})\ ) \end{aligned}$$

We are left with:

$$\int_a^b f(x)dx = F(b) - F(a)$$

7 Product Rule

Given $h(x) = f(x)g(x)$, $h'(x) = f'(x)g(x) + f(x)g'(x)$

$$h'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

We'll add the following two terms which cancel each other out, $-f(x+h)g(x) + f(x+h)g(x)$:

$$\begin{aligned}
h'(x) = \lim_{h \to 0} &\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)(g(x+h) - g(x))}{h} + \lim_{h \to 0} \frac{g(x)(f(x+h)) - f(x)}{h}
\end{aligned}$$

The limit $\lim_{h \to 0} f(x+h)$ is just $f(x)$. We can also see that $\lim_{h \to 0} \frac{g(x+h)-g(x)}{h}$ is $g'(x)$.

We also observe that $\lim_{h \to 0} g(x) = g(x)$ and $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ is the definition of $f'(x)$.

8 Quotient Rule

If $h(x) = \frac{f(x)}{g(x)}$ then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

We will apply the cross multiplication property. Given $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$:

$$\begin{aligned}
h'(x) &= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \frac{1}{h}
\end{aligned}$$

We'll add to the numerator two terms that cancel each other out $-f(x)g(x) + f(x)g(x)$:

$$\begin{aligned}
h'(x) &= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \frac{1}{h} \\
&= \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \\
&= \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \left(\frac{f(x+h)g(x) - f(x)g(x)}{h} + \frac{f(x)g(x) - f(x)g(x+h)}{h}\right) \\
&= \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \left(g(x)\frac{f(x+h) - f(x)}{h} + f(x)\frac{g(x) - g(x+h)}{h}\right)
\end{aligned}$$

From these results we can see that:

$\lim_{h \to 0} \frac{1}{g(x+h)g(x)}$ is $\frac{1}{g(x)^2}$

$\lim_{h \to 0} g(x)\frac{f(x+h)-f(x)}{h}$ is $g(x)f'(x)$.

$\lim_{h \to 0} f(x)\frac{g(x+h)-g(x)}{h}$ is $f(x)g'(x)$.

9 Chain Rule

Given a composition relationship $h(x) = f(g(x))$ then $h'(x) = f'(g(x))g'(x)$.

Let's suppose that we're interested in the derivative at a specific point a, so we can say:

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

If we multiply the entire expression by $\frac{g(x)-g(a)}{g(x)-g(a)}$:

$$\begin{aligned} &\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \frac{g(x) - g(a)}{g(x) - g(a)} \\ &= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \frac{g(x) - g(a)}{x - a} \end{aligned}$$

The left fraction corresponds to the rate of change $f'(g(a))$, while the right fraction corresponds to the rate of change $g'(a)$

10 Power Rule

Given $f(x) = x^n$, its derivative is $f'(x) = nx^{n-1}$

Derivation:

$$f'(x) = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}$$

By the Binomial Theorem, we know that:

$$(a + b)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^{n-k} b^k$$

Expanding the numerator we have:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(x^n + nx^{n-1}h + \frac{n(n-1)}{2!} x^{n-2}h^2 + ... + nxh^{n-1} + h^n) - x^n}{h} \\ &= \lim_{h \to 0} (nx^{n-1} + \frac{(n)(n-1)}{2!} x^{n-2}h + ... + nxh^{n-2} + h^{n-1}) \end{aligned}$$

With the exception of $n(x)^{n-1}$, the limit as h approaches zero of all other components is zero, for instance:

$$\lim_{h \to 0} (\frac{(n)(n-1)}{2!} x^{n-2}h) = 0 \qquad \lim_{h \to 0} (nxh^{n-2}h) = 0 \qquad \lim_{h \to 0} (h^{n-1}) = 0$$

Since $n(x)^{n-1}$ does not depend on h, we have:

$$\lim_{h \to 0} (n(x)^{n-1}) = n(x)^{n-1}$$

So we are left with:

$$f'(x) = n(x)^{n-1}$$

The First Fundamental Theorem of calculus predicts the existence of a function $F(x)$ such that $F'(x) = f(x)$.

$$\frac{d}{dx}[\frac{x^{n+1}}{n+1}] = \frac{n+1}{n+1}x^{n+1-1} = x^n$$

So we have:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

11 Sine

If $f(x) = \sin x$, then $f'(x) = \cos x$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

Using the trig identity $\sin(x+h) = \sin x \cos h + \cos x \sin h$ we can rewrite the numerator:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h} \\
&= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h} \\
&= -\sin x \lim_{h \to 0} \frac{1 - \cos h}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}
\end{aligned}$$

We will apply two known special limits here, namely that $\lim_{h \to 0} \frac{1-\cos h}{h} = 0$ and $\lim_{h \to 0} \frac{\sin h}{h} = 1$. So we are left with:

$$f'(x) = \cos x$$

The First Fundamental Theorem of calculus predicts the existence of a function $F(x)$ such that $F'(x) = f(x)$.

$$\frac{d}{dx}\left[-\cos x\right] = -(-\sin x) = \sin x$$

Note: derivative of cosine is discussed in the next section.

So therefore:

$$\int \sin x dx = -\cos x + C$$

12 Cosine

If $f(x) = \cos x$ then $f'(x) = -\sin x$

$$f'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

We can use the following trig identity, $\cos(x+h) = \cos x \cos h - \sin x \sin h$, to rewrite the numerator:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(\cos x\cos h - \sin x\sin h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x\sin h}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h\to 0} \frac{\sin x\sin h}{h} \\
&= \cos x \lim_{h\to 0} \frac{(\cos h - 1)}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h} \\
&= -\cos x \lim_{h\to 0} \frac{1-\cos h}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}
\end{aligned}
$$

We will apply two known special limits here, namely that $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$. So we are left with:

$$f'(x) = -\sin x$$

13 Tangent

If $f(x) = \tan x$, then $f'(x) = \sec^2 x$

$$f'(x) = \lim_{h\to 0} \frac{\tan(x+h) - \tan x)}{h}$$

If we apply the trig identity, $\tan(x+h) = \frac{\tan x + \tan h}{1-\tan x\tan h}$, we get:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\frac{\tan x + \tan h - \tan x(1-\tan x\tan h)}{1-\tan x\tan h}}{h} \\
&= \lim_{h\to 0} \frac{\tan h + \tan^2 x\tan h}{h(1-\tan x\tan h)} \\
&= \lim_{h\to 0} \frac{\tan h(1+\tan^2 x)}{h(1-\tan x\tan h)} \\
&= \lim_{h\to 0}(\frac{\tan h}{h}) \lim_{h\to 0}(\frac{1+\tan^2 x}{1-\tan x\tan h}) \\
&= \lim_{h\to 0}(\frac{\sin h}{h\cos h}) \lim_{h\to 0}(\frac{1+\tan^2 x}{1-\tan x\tan h}) \\
&= \lim_{h\to 0} \frac{\sin h}{h} \lim_{h\to 0} \frac{1}{\cos h} \lim_{h\to 0}(\frac{1+\tan^2 x}{1-\tan x\tan h}) \\
&= 1 + \tan^2 x
\end{aligned}
$$

We can simplify the result further:

$$f'(x) = 1 + \tan^2 x = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{\cos^2 x + sin^2 x}{cos^2 x}$$

Since $\cos^2 x + \sin^2 x = 1$, we are left with $\frac{1}{cos^2 x}$ or $sec^2 x$.

$$f'(x) = \sec^2 x$$

There is a function $F(x)$ such that $F'(x) = f(x)$. Let's differentiate $\ln(\sec x)$:

$$\begin{aligned}\frac{d}{dx}[\ln(\sec x)] &= (\frac{1}{\frac{1}{\cos x}})(-\frac{1}{\cos^2 x})(-\sin x) \\ &= \frac{(\cos x)(-\sin x)}{-\cos^2 x} = \frac{\sin x}{\cos x} \\ &= \tan x\end{aligned}$$

We can therefore say that:

$$\int \tan x dx = \ln(\sec x) + C$$

14 Natural Log

If $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$

We can apply the following rule of natural logs: $\ln(x+h) - \ln x = \ln(\frac{x+h}{x})$:

$$f'(x) = \lim_{h\to 0} \frac{1}{h} \ln(\frac{x+h}{x})$$

Applying the power rule of natural logs we get:

$$f'(x) = \lim_{h\to 0} \ln(\ \ (\frac{x+h}{x})^{\frac{1}{h}}\ \ )$$

Let $u = \frac{h}{x}$, so $h = ux$:

$$\begin{aligned}f'(x) &= \lim_{u\to 0} \ln(\ \ (1+u)^{\frac{1}{ux}}\ \ ) \\ &= \lim_{u\to 0} \ln(\ \ (1+u)^{\frac{1}{u}\frac{1}{x}}\ \ )\end{aligned}$$

We apply the power rule one more time:

$$\begin{aligned}f'(x) &= \lim_{u\to 0} \frac{1}{x} \ln(\ \ (1+u)^{\frac{1}{u}}\ \ ) \\ &= \frac{1}{x} \lim_{u\to 0} \ln(1+u)^{\frac{1}{u}}\end{aligned}$$

The expression $\lim_{u\to 0}(1+u)^{\frac{1}{u}}$ is the definition of Euler's number, e, so we are left with:

$$f'(x) = \frac{1}{x}\ln e$$
$$f'(x) = \frac{1}{x}$$

Let's take the derivative of $\ln x$. We know that $y = \ln x$ is just stating that $e^y = x$, to which we'll apply implicit differentiation:

$$
\begin{aligned}
\frac{d}{dx}[e^y] &= \frac{d}{dx}[x] \\
e^y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{e^y} \\
\text{(If we plug in } y = \ln x\text{)} \quad \frac{dy}{dx} &= \frac{1}{e^{\ln x}} = \frac{1}{x}
\end{aligned}
$$

So we have:

$$\int \ln x dx = \frac{1}{x} + C$$

15 e

If $f(x) = e^x$, then $f'(x) = e^x$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x e^h - e^x}{h}
\end{aligned}
$$

Since e^x does not depend on h we can take it out of the limit evaluation:

$$f'(x) = e^x \lim_{h\to 0} \frac{e^h - 1}{h}$$

Let $y = e^h - 1$, or $e^h = y + 1$, we observe that $\lim_{h\to 0} y = 0$. Taking the natural log of boths sides:

$$
\begin{aligned}
\ln(e^h) &= \ln(y+1) \\
h &= \ln(y+1)
\end{aligned}
$$

We can substitue the above results into h and $e^h - 1$:

$$
\begin{aligned}
f'(x) = e^x \lim_{h\to 0} \frac{e^h - 1}{h} &= e^x \lim_{y\to 0} \frac{y}{\ln(y+1)} \\
&= e^x \lim_{y\to 0} \frac{1}{\frac{1}{y}\ln(y+1)} \\
&= e^x \lim_{y\to 0} \frac{1}{\ln(\ (y+1)^{\frac{1}{y}}\ )}
\end{aligned}
$$

The expression $\lim_{y\to 0}(y+1)^{\frac{1}{y}}$ is the definition of Euler's number, so we are left with:

$$
\begin{aligned}
f'(x) &= e^x \frac{1}{\ln e} \\
f'(x) &= e^x
\end{aligned}
$$

It can also be said that:

$$\int e^x dx = e^x + C$$