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TRANSACTIONS COSTS AND THE THEORY OF PORTFOLIO SELECTION

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INTRODUCTION

IN MOST WORK IN PORTFOLIO THEORY, transactions costs, if not neglected, are only an afterthought, and are treated in a qualitative or mechanical manner.[1] Here we shall examine, and generally try to quantify, the ways in which transactions costs make the level of an investor's wealth affect the selection of his portfolio. In Section I we analyze a model in which the opportunity set consists only of risky securities. The main result is that the optimal number of securities to hold in a portfolio increases as the square root of an investor's initial wealth. In Section II a riskless security is added to the opportunity set, and it is shown that the presence of transactions costs make the fraction of a portfolio invested in risky securities rise with wealth. The optimal diversification question is also considered and related to the discussion of Section I. The extension of Section III to portfolios consisting of many classes of securities is especially important for Section IV, where we use all three variants of the model to interpret certain aspects of the portfolio behavior of households, commerical banks, and common stock mutual funds.

I.

We consider a one-period world in which investors possess initial cash endowments and securities are issued by exogenous agencies. Investors purchase securities at the beginning of the period and redeem them at the end of the period with no intermediate receipts. The return from a security or portfolio is defined to be the change in price divided by the purchase price and is often referred to as the "rate of return." The securities are assumed to be perfectly divisible as they may be purchased and redeemed in any dollar amount. We come next to our strategic assumption. The expected return and variance in return of each security, and the correlation between the returns of any two securities, are assumed to be equal. We may designate the expected return of any security by E, the variance in return of any security by σ^2, and the correlation between the returns of any two securities by ρ.

Investors choose portfolios on the basis of their expected return, E_p, and variance in return, V_p. Each investor maximizes an objective function U of the form

$$U = E_p - ZV_p, \tag{1.1}$$

* An earlier version of this paper was prepared while the author was an undergraduate at Princeton University. The author is now a graduate student in Economics at Harvard University and under the supervision of Professor John Lintner.

1. See for example Markowitz [6], Tobin [13] and Mossin [9]. Zabel [14] and Leland [4] devise models showing that transactions costs tend to inhibit portfolio adjustment. Goldsmith [3] covers roughly the same ground as we do here and Goldsmith [2] corresponds to Section I of the present paper.

where Z is a positive constant and a measure of his aversion to risk. By defining the objective function over "rate of return" we are assuming that in the absence of transactions costs the composition of an investor's portfolio chosen from an arbitrary opportunity set is independent of his initial wealth.[2]

Investors may incur a variety of transactions costs. They include costs of gathering and reviewing information about security prospects, and the cost of administering security transactions and ownership, as well as direct costs such as brokerage commissions.

It may be supposed that the activities giving rise to these costs occur from just before the opening of the "purchase market" to the time the "redemption market" closes. Though security transactions take place both at the beginning and end of the period, we assume for algebraic simplicity that actual payments of transactions costs occur only at the end of the period.

Rather than work with the above descriptive classification, we shall categorize transactions costs on the basis of the way in which they are calculated. In general, transactions costs would depend on the number of portfolio shifts; the number of different securities bought, sold, or held; and the dollar values of the securities bought sold, or held.[3] To keep our problem manageable we assume each of these relationships to be one of direct proportionality.[4]

It should be apparent that only one of the three categories of transactions costs has any importance in our model. Each investor makes two portfolio shifts,[5] one at the beginning and one at the end of the period; the first type of transactions cost thus represents a fixed cost and may be dropped from the analysis because of the form of the investors' objective functions. Nor is it necessary to deal explicitly with the third type of transactions cost: we shall assume the variables describing each security's return characteristics to be defined net of this cost. This is possible because its effect on the return characteristics of each security is independent of the dollar amount invested, and therefore it is the same for every investor. Freedom to drop these types of transactions cost would be lost if the investors' initial wealth was not in the form of cash, if there was a choice of holding period, or if investors could issue securities. We shall use the term transactions cost *only* to refer to those costs which increase with the number of securities bought, sold, or held; in our model they are of course equal. Investors are assumed to pay at the end of the period a composite transactions fee of a dollars for each distinct security included in their portfolios.

2. Assuming the portfolio returns to be normally distributed, the objective function in the text results if an investor maximizes $E[u(R)]$, where "E" denotes expectation, $u(R) = -e^{-2ZR}$, and "R" is the portfolio "rate of return". An investor employing this utility function obviously has a preference ordering over terminal wealth which depends upon initial wealth (see Mossin [8]), but his Pratt [10]–Arrow [1] proportional risk aversion at any particular level of *initial* wealth is constant and equal to $2Z$.

3. See Tobin [13]. Tobin does not explicitly consider holding costs and information costs.

4. Information costs are a problem. We would prefer not to have investors' expenditures on information gathering and processing straitjacketed by a set formula; but providing investors with a degree of freedom in this area would not only be difficult to do in a formal model, it would also be incompatible with the concept of homogeneous expectations.

5. Remember that we are only discussing investors, those individuals who find it worthwhile to invest.

In computing the expressions for the expected return and variance in return of a portfolio, let T be an investor's initial wealth or the amount invested in the portfolio at the beginning of the period, n the number of different securities purchased, and X_i the fraction of the portfolio invested in the ith security. We then have

$$E_p = E - \frac{an}{T} \tag{1.2}$$

$$V_p = \sum_{i=1}^{n} X_i^2\sigma^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} X_i X_j \sigma^2 \rho^2. \tag{1.3}$$

Note that the X_i do not appear in (1.2) since each security has the same expected return. To maximize U with respect to each X_i we need only minimize V_p, subject to the constraint $\sum_{i=1}^{n} X_i = 1$. Forming the Lagrangian

$$V_\lambda = \sum_{i=1}^{n} X_i^2\sigma^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} X_i X_j \sigma^2 \rho + \lambda\left(1 - \sum_{i=1}^{n} X_i\right), \tag{1.4}$$

the first order necessary conditions[6] with respect to any X_i, say X_a or X_b are

$$\frac{\partial V_\lambda}{\partial X_a} = 2X_a\sigma^2 + 2\sum_{j=1, j \neq a}^{n} X_j \sigma^2 \rho - \lambda = 0 \tag{1.5}$$

$$\frac{\partial V_\lambda}{\partial X_b} = 2X_b\sigma^2 + 2\sum_{j=1, j \neq b}^{n} X_j \sigma^2 \rho - \lambda = 0. \tag{1.6}$$

Simultaneous solution of (1.5) and (1.6) yields $X_a = X_b$. As X_a and X_b represent any pair of X_i, all the X_i are equal to each other. Optimally, the same fraction of initial wealth is invested in each security. Each X_i is then equal to $1/n$, giving us

$$V_p = \frac{\sigma^2(1-\rho)}{n} + \sigma^2\rho.^7 \tag{1.7}$$

Turning to the basic optimization, an investor now has one decision variable, n, the number of securities to hold in his portfolio. Clearly n must be an integer. We ignore this constraint as an integer adjacent to any non-integer solution will provide the correct solution. The number of securities held must be greater than or equal to one and less than or equal to the number of securities issued. This gives us the constraints $n \geqslant 1$, $n \leqslant n_I$, where n_I is the number of securities issued, but for now we shall assume that the values of the parameters are such that the constraints are

6. If $\rho = 1$, $V_p = \sigma^2$ no matter what values are assigned to the X_i. The necessity of the first order conditions follows because we are implicitly assuming $\rho < 1$. We know that we indeed have a minimum because portfolio variance in return is bounded from below by zero.

7. Markowitz [6] sets $X_i = 1/n$ and gives this formula.

not binding: $1 < n^* < n_I$. (A star attached to a decision variable denotes its optimal value.) Substituting from (1.2) and (1.7) into (1.1), the objective function is

$$U = E - \frac{an}{T} - Z\left[\frac{\sigma^2(1-\rho)}{n} + \sigma^2\rho\right]. \tag{1.8}$$

A necessary condition for an interior maximum is

$$\frac{dU}{dn} = -\frac{a}{T} + \frac{Z\sigma^2(1-\rho)}{n^2} = 0, \tag{1.9}$$

which is easily solved[8] for

$$n^* = \sqrt{\frac{ZT\sigma^2(1-\rho)}{a}}\ . \tag{1.10}$$

The wealth elasticity of diversification is one-half: the optimal number of securities to include in a portfolio increases as the square root of initial wealth.

Other comparative statics properties of the model are obvious, but must be interpreted with some care. A larger a, a greater transactions fee, reduces the number of securities held because it makes diversification more costly. This transactions fee, though, does not have an easily recognizable real world analogue since it is only designed to represent that part of transactions cost (information, administrative, and brokerage costs) which increases with the number of securities held. It may also be seen that n^* increases with $\sigma^2(1-\rho)$ because the marginal reduction in risk through diversification increases with the magnitude of this term, which we may call the "diversifiable risk" of each security to distinguish it from the "systematic risk" of the portfolio, $\sigma^2\rho$. This nomenclature seems appropriate because the "diversifiable risk" may be expressed as $\sigma^2 - \sigma^2\rho$ and is then easily seen to represent the difference between the variance in return of a portfolio holding only one security and that of an "infinitely diversified" portfolio.[9] In the real world, all securities don't have the same return characteristics so that increases in $\sigma^2(1-\rho)$ can only be given the interpretation of a generalized rise in the diversifiable risk of security Einvestment as perceived by the investors. Finally, we consider a change in taste, say a rise in Z, an investor's aversion to risk. The model is not well suited to handle this sort of experiment in comparative statics, nor is it intended to. It only predicts a rise in the number of securities held, while our intuition suggests that an investor would also want to substitute less risky securities

8. This assumes that $\rho < 1$. If $\rho = 1$ the fact that (1.9) has no solution means that an interior maximum does not exist.

9. This suggests that the analysis could also be carried out in terms of the "index model" developed by Sharpe [11]. It would be assumed that every security had the same unsystematic variance and the same beta coefficient. The unsystematic variance of each security would replace $\sigma^2(1-\rho)$ in our equations, and the product of beta squared and the variance in the index would replace $\sigma^2\rho$.

into his portfolio, reducing and perhaps reversing the necessity of greater diversification. This difficulty serves to stress the fact that we have developed a pure optimal diversification model, for there is no room for adjusting the return characteristics of the securities included in the portfolio.[10]

An alternative diagramatic presentation should further clarify the process of portfolio selection.

Equation (1.9) may be written

$$\frac{dU}{dn}=\frac{dE_p}{dn}-\frac{ZdV_p}{dn}=0 \qquad \text{with} \tag{1.11}$$

$$\frac{dE_p}{dn}=\frac{-a}{T} \qquad \text{and} \qquad \frac{-ZdV_p}{dn}=\frac{Z\sigma^2(1-\rho)}{n^2}.$$

Call $-dE_p/dn$ the marginal cost of diversification and $-ZdV_p/dn$ the marginal benefit from diversification. An investor equates marginal cost, MC, to marginal benefit, MB, to determine the optimal number of securities to hold in his portfolio. This solution is shown in Figure 1.

An increase in wealth or decrease in transactions costs increases the optimal number of securities to include in a portfolio by lowering the marginal cost of diversification schedule, while an increase in risk aversion or "diversifiable risk" produces the same result by raising the marginal benefit of diversification schedule.

The parameters may be such that a boundary solution results. If $(dU/dn)\geqslant 0$ when evaluated at $n=n_I$, an investor will hold every security (See (1.9)). In fact, if there are no transactions costs ($a=0$), dU/dn is always positive implying that every investor regardless of his wealth holds every (risky) security just as in the Sharpe-Lintner-Mossin Capital Asset Pricing Model.[11]

Similarly, if $(dU/dn)\leqslant 0$ when evaluated at $n=1$, an investor will hold only a single security. In a world of transactions costs ($a>0$) in which all security returns are perfectly correlated ($\rho=1$), dU/dn is always negative. Diversification reduces expected return without reducing the variance in return, so every investor holds only one security.

Returning to consideration of the interior solution, let S be the size of each security holding. Since $S=T/n$, the optimal size of each security holding is given by

$$S^*=\sqrt{\frac{Ta}{Z\sigma^2(1-\rho)}}\ . \tag{1.12}$$

10. We could interpret $\sigma^2(1-\rho)$ in (1.10) as an endogenously determined average diversifiable risk dependent upon an investor's aversion to risk; as well as his initial wealth, the risk-return characteristics of the securities in the opportunity set and the magnitude of the relavant transactions costs. This approach could be made more rigorous, but since it would cease to yield a closed-form expression as in (1.10), we shall not pursue it any further here. It might prove useful, however, for some empirical tests of the model.

11. See Sharpe [12], Lintner [5], and Mossin [7].

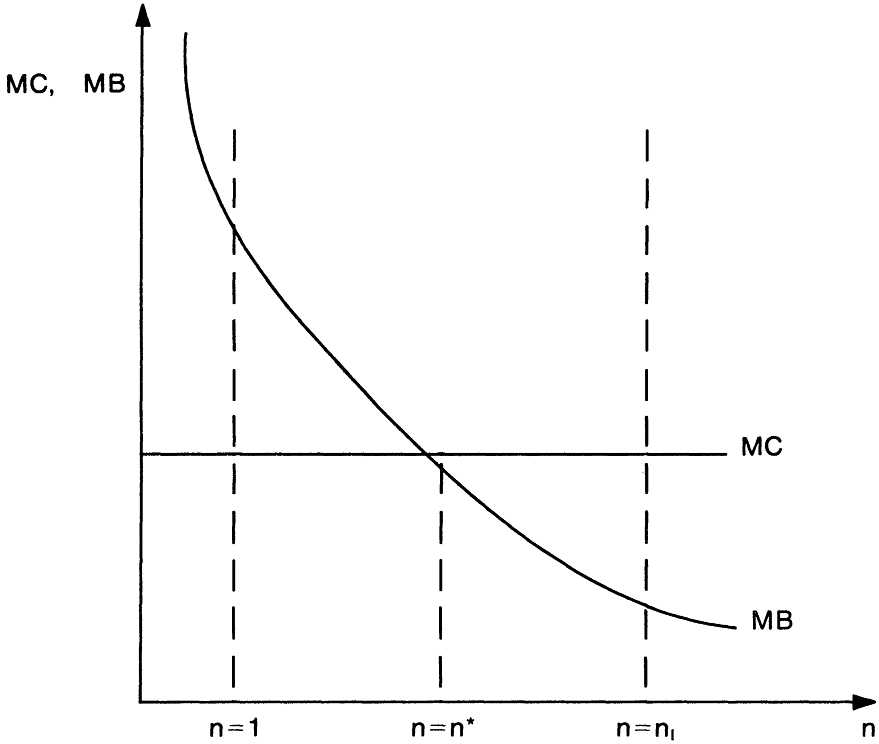

FIGURE 1

Thus, the size of each security holding as well as the number of securities held increase as the square root of initial wealth. An investor finding his wealth doubled will not continue to hold the same number of securities merely doubling the amount invested in each. Nor will he double the number of securities held keeping the dollar size of each security holding constant. Instead, he will take a more plausible course of action, one lying between these extreme possibilities.[12]

12. It's interesting to see how this appealing result is modified by changing the objective function. In particular suppose it takes the form: $\hat{U} = \hat{E}_p - \hat{Z}\hat{V}_p$, where $\hat{E}_p$ and $\hat{V}_p$ are the mean and the variance of terminal wealth and $\hat{Z}$ is a new measure of risk aversion. It is then easy to show that the optimal number of securities to include in a portfolio is given by:

$$n^* = T\sqrt{\frac{\hat{Z}\sigma^2(1-\rho)}{a}}\ .$$

The new objective function implies that the number of securities included in a portfolio increases in direct proportion to initial wealth, the size of each security holding remaining constant. But such an implausible result should not be surprising: assuming portfolio returns to be normally distributed, the new objective function results if an investor maximizes expected utility and has constant absolute risk aversion equal to $2\hat{Z}$.

We can make the following generalization: constant absolute risk aversion corresponds to a wealth elasticity of diversification of one; constant proportional risk aversion corresponds to a wealth elasticity of diversification of one-half (see footnote 2); risk neutrality corresponds to a wealth elasticity of diversification of zero.

II.

We allow for the existence of a riskless security which investors may purchase or issue. This security carries a certain return of P, and taking a position in this security does not cause investors to incur any additional transactions costs. The assumption of no transactions costs associated with a position in the riskless security is quite harmless in the context of what we're trying to accomplish. Its effect is to remove any bias against borrowing or lending by making the effective borrowing and lending rates equal and independent of the amount borrowed or lent.

An investor must decide upon the allocation of his investment funds between the riskless security and the risky portion of his portfolio. Let the fraction of a portfolio invested in the risky securities described in Section I be equal to Q and the fraction invested in the riskless security be $1-Q$. We then have

$$E_p = QE + (1-Q)P - \frac{an}{T} \tag{2.1}$$

$$V_p = Q^2\left[\sum_{i=1}^{n} X_i^2\sigma^2 + \sum_{i=1}^{n} \sum_{\substack{j=1, \\ i\neq j}}^{n} X_i X_j \sigma^2\rho\right], \tag{2.2}$$

where n is now the number of *risky* securities held, and X_i is the fraction of the *risky part* of the portfolio invested in the ith risky security.

As in Section I, the expression for the expected return of the portfolio does not include the X_i. To maximize U with respect to each X_i, we need only minimize V_p. The right hand side of (2.2) is equal to that of (1.3) multiplied by Q^2. It immediately follows that portfolio variance in return is minimized with respect to each X_i when all the X_i are equal to each other. Any $X_i = 1/n$ as in Section I. We then have

$$V_p = Q^2\left[\frac{\sigma^2(1-\rho)}{n} + \sigma^2\rho\right]. \tag{2.3}$$

Investors maximize U with respect to n and Q. We shall assume that investors take a nonzero position in the riskless security, and hold positive fractions of their portfolios in risky securities ($Q \neq 1, Q > 0$). The constraints $1 \leqslant n \leqslant n_I$ will be considered later. Substituting from (2.1) and (2.3) into (1.1) the objective function is

$$U = QE + (1-Q)P - \frac{an}{T} - ZQ^2\left[\frac{\sigma^2(1-\rho)}{n} + \sigma^2\rho\right]. \tag{2.4}$$

Necessary conditions for an interior maximum are

$$\frac{\partial U}{\partial Q}=E-P-2ZQ\left[\frac{\sigma^2(1-\rho)}{n}+\sigma^2\rho\right]=0 \tag{2.5}$$

$$\frac{\partial U}{\partial n}=\frac{-a}{T}+\frac{ZQ^2\sigma^2(1-\rho)}{n^2}=0. \tag{2.6}$$

Simultaneous solution of (2.5) and (2.6) gives us

$$Q^*=\frac{E-P-2\sqrt{\dfrac{Z\sigma^2(1-\rho)a}{T}}}{2Z\sigma^2\rho}. \tag{2.7}$$

We must have $0<\rho<1$ for a solution of the form of (2.7) to result.[13] We can then see that the elasticity of Q^* with respect to Z is negative: the fraction of a portfolio invested in risky securities declines as risk aversion rises. More interesting is the fact that the elasticity of Q^* with respect to T is positive and decreasing: the fraction of a portfolio invested in risky securities rises, though at a declining rate, with increases in wealth. This occurs because the necessary diversification of risky security holdings is relatively less costly to more wealthy investors, while there is no diversification with respect to the riskless security.

As in Section I an investor will hold every risky security if $(\partial U/\partial n)\geqslant 0$ when evaluated at $n=n_I$ and Q^* (See (2.6)). If there are no transacted costs ($a=0$), $\partial U/\partial n$ is always positive implying that every investor regardless of his wealth holds every risky security. Substituting $n^*=n_I$ into (2.5), we get

$$Q^*=\frac{E-P}{2Z\left[\dfrac{\sigma^2(1-\rho)}{n_I}+\sigma^2\rho\right]}. \tag{2.8}$$

The fraction of a portfolio invested in risky securities is independent of wealth when there are no transactions costs.

Similarly, if $(\partial U/\partial n)\leqslant 0$ when evaluated at $n=1$ and Q^*, an investor will hold only one risky security. In a world of transactions costs ($a>0$) in which all security

13. The parameter ρ must be non-negative because a second necessary condition for an interior maximum is that the Hessian matrix associated with the objective function $U(n,Q)$ be negative semi-definite, a requirement which fails if $\rho<0$. If $\rho=0$, (2.5) and (2.6) are either inconsistent–implying that an interior maximum does not exist, or yield only a solution in terms of n/Q. In neither case do we get a solution of the form of (2.7). We really shouldn't worry too much about a nonpositive ρ because the "possibility" becomes exceedingly unrealistic as the number of securities issued grows. Formally, the positive semi-definiteness of the covariance matrix of the returns of all securities issued implies that the lower bound on ρ rises to zero as n_I becomes very large. If $\rho=1$, (2.6) has no solution which means that an interior maximum does not exist.

returns are perfectly correlated ($\rho = 1$), $\partial U / \partial n$ is always negative, implying that every investor holds a single security. Substituting $n^* = 1$ into (2.5) we get

$$Q^* = \frac{E - P}{2Z\sigma^2}, \tag{2.9}$$

so that the fraction of a portfolio invested in risky securities is independent of initial wealth. An identical result would also be obtained if only one risky security was issued, for then there would be no choice about n^*.

For the fraction of a portfolio invested in risky securities to increase with wealth there must exist the opportunity for diversification, diversification must produce benefits and diversification must produce costs. This result is summarized by (2.7), (2.8) and (2.9) and follows from defining investors' objective functions over "rate of return."

Returning to our consideration of the interior solution, note that it is possible to solve for n^* in terms of the parameters by substituting the expression for Q^* in (2.7) into (2.5) or (2.6). The resulting equation is complicated and we will not bother to write it down. It will suffice to write (2.6) as

$$n^* = Q^* \sqrt{\frac{ZT\sigma^2(1 - \rho)}{a}} \ . \tag{2.10}$$

The elasticity of n^* with respect to T exceeds one-half by the elasticity of Q^* with respect to T and therefore declines toward one-half as wealth increases. The elasticity of n^* with respect to Z is equal to one-half plus the elasticity of Q^* with respect to Z. Since this latter elasticity will always be algebraically smaller than negative one, the number of risky securities held declines with risk aversion, a more intuitive conclusion than the one reached in Section I.

The differences between these results and those of Section I are a consequence of the marginal benefit of diversification being equal to $(ZQ^2\sigma^2(1-\rho)/n^2)$ instead of $(Z\sigma^2(1-\rho)/n^2)$, cf. (2.6) and (1.11), and can best be understood by reference to Figure 1. An increase in T not only increases n^* by lowering the MC schedule, but also by raising the MB schedule through increasing Q^*. The direct effect on the marginal benefit of diversification of an increase in Z is more than offset by the fact that it also decreases Q^* so that the MB schedule falls instead of rises, making n^* decrease instead of increase.

It is also true that n^*/Q^* in (2.10) bears the same relationship to the parameters as n^* in (1.10). The "reason" is that n/Q is the same natural measure of diversification as n was in Section I. They are both equal to the ratio of total wealth to the dollar amount invested in any single risky security: T/S. When there is investment in a riskless security, $S = QT/n$ and $T/S = T/QT/n = n/Q$. In a model without a riskless security, Q is identical to one. In both cases, designating $D = T/S$ as the measure of diversification, optimal diversification is given by

$$D^* = \sqrt{\frac{ZT\sigma^2(1 - \rho)}{a}} \ . \tag{2.11}$$

Equation (2.10) can also be solved for the optimal size of each security holding:

$$S^* = \sqrt{\frac{Ta}{Z\sigma^2(1-\rho)}}\;, \tag{2.12}$$

which is identical to the result given in (1.12) for portfolios composed only of risky securities.

III.

We have considered the notion of optimal diversification with and without the opportunity for investment in a riskless security, but in both cases optimal diversification was considered with respect to all risky securities. In many portfolio problems, however, it is more desireable or data is only available to analyze the diversification of a particular class of risky securities, that is a subset of an investor's risky security holdings.

Thus we extend the model to allow for various classes of securities. Each of the postulated m classes is distinguished by its own risk-return-transactions fee characteristics. Consider the Kth class. Each security in this class has an expected return E_K, variance in return σ_K^2, and transactions fee a_K. The correlation between the returns of any two securities in this class is given by ρ_K. The covariance between the returns of any securityin the Kth class and any security in the Lth class is given by Cov_{KL}. Let n_K be the total number of securities in the Kth class held in the portfolio and Q_K be the fraction of an investor's portfolio invested in the Kth class of securities. We may then write

$$E_p = \sum_{K=1}^{m} Q_K E_K - \sum_{K=1}^{m} \frac{a_K n_K}{T}. \tag{3.1}$$

Optimally, the fraction of the portfolio invested in any pair of securities within a given class is equal, so

$$V_p = \sum_{K=1}^{m} Q_K^2\left(\frac{\sigma_K^2(1-\rho_K)}{n_K} + \sigma_K^2\rho_K\right) + \sum_{\substack{K=1\\K\neq L}}^{m} \sum_{L=1,}^{m} Q_K Q_L \text{Cov}_{KL}. \tag{3.2}$$

Investors maximize U with respect to each n_k and Q_K subject to the constraints $1 \leqslant n_K \leqslant n_{KI}$, $Q_K \geqslant 0$ and $\sum_{K=1}^{m} Q_K = 1$, where n_{KI} is the number of securities issued in the Kth class and the first constraint is inoperative if $Q_K = 0$. A simple Lagrangian is

$$U_\lambda = \sum_{K=1}^{m} Q_K E_K - \sum_{K=1}^{m} \frac{a_K n_K}{T} - Z\left[\sum_{K=1}^{m} Q_K^2\left(\frac{\sigma_K^2(1-\rho_K)}{n_K} + \sigma_K^2\rho_K\right)\right.$$

$$\left. + \sum_{\substack{K=1\\K\neq L}}^{m} \sum_{L=1,}^{m} Q_K Q_L \text{Cov}_{KL}\right] + \lambda\left(1 - \sum_{K=1}^{m} Q_K\right), \tag{3.3}$$

and some necessary conditions for a maximum are

$$\frac{\partial U_\lambda}{\partial n_K} = \frac{-a_K}{T} + \frac{ZQ_K^2\sigma_K^2(1-\rho_K)}{n_K^2} = 0 \qquad \{K: Q_K > 0, 1 < n_K < n_{KI}\} \tag{3.4}$$

$$\frac{\partial U_\lambda}{\partial Q_K} = E_K - 2Z\left[Q_K\left(\frac{\sigma_K^2(1-\rho_K)}{n_K} + \sigma_K^2\rho_K\right) + \sum_{L=1, K \neq L}^{m} Q_L \operatorname{Cov}_{KL}\right] - \lambda = 0 \quad \{K: Q_K > 0\} \tag{3.5}$$

$$\frac{\partial U_\lambda}{\partial \lambda} = 1 - \sum_{K=1}^{m} Q_K = 0 \tag{3.6}$$

We have not bothered to designate a riskless security, for without putting additional restrictions on the return parameters, we couldn't use it to derive precise results like those summarized by (2.7); we shall work only with the first order conditions represented by (3.4), which may be written

$$n_{K*} = Q_{K*}\sqrt{\frac{ZT\sigma_K^2(1-\rho_K)}{a_K}} \; . \tag{3.7}$$

The elasticity of n_K^* with respect to T is equal to one-half plus the elasticity of Q_K^*, with respect to T, a result analagous to the one summarized by (2.10).

Let the size of each security holding in the Kth class be $S_K = Q_K T / n_K$ and the diversification measure be $D_K = T / S_K$, the ratio of total wealth to the size of each security holding in the class. The optimal size of each security holding in the Kth class is then

$$S_K^* = \sqrt{\frac{a_K T}{Z\sigma_K^2(1-\rho_K)}} \; , \tag{3.8}$$

and optimal diversification is given by

$$D_K^* = \sqrt{\frac{ZT\sigma_K^2(1-\rho_K)}{a_K}} \; . \tag{3.9}$$

For any class, the size of each security holding and our diversification measure both increase as the square root of initial wealth, results analagous to those summarized by (2.12) and (2.11), respectively. Two points deserve emphasis. While the fraction of the portfolio invested in any class depends upon the risk-return-transactions costs characteristics of all the classes of securities, the optimal size of each security holding and diversification measure for a particular class depend only upon the risk-return-transactions costs characteristics of that class. More important, our analysis of individual *parts* of a portfolio is based upon the selection of an optimal *total* portfolio. In particular, the relevant relationship is between the size of each security holding in a class and *total wealth*, not just wealth invested in that class of securities.

Of course in some applications one may not know the size of the total portfolio and the best one can do is analyze a specific part of a portfolio as if it were a total portfolio, using the simple model developed in Section I. Letting $T_K = Q_K T$ be the wealth invested in the Kth class of securities, (3.4) can also be written

$$n_K^* = \sqrt{\frac{Z T_K^* Q_K^* \sigma_K^2 (1-\rho_K)}{a_K}} \quad . \tag{3.10}$$

By comparing (3.10) with (1.10) we see that any comparative statics predictions would only be approximations, since Q_K^* will in general vary with the unknown total wealth. Moreover, if a relevant riskless security exists, Model II suggests that the approximation may be biased: the fraction of the unknown total wealth invested in all risky securities should increase with total wealth implying that Q_K^* will tend to increase with T_K^*.

IV.

In Sections I–III we developed several models "explaining" some effects of transactions costs on portfolio selection: here, we will point out some implications of these models for the behavior of real world investors, though we must make some modifications to deal with the abstraction introduced to produce these models. Instead of referring to the size of each security holding, it's necessary to use the average size of the security holdings. We also take wealth, specifically, investment in "riskless securities," to be measured exclusive of cash: noninterest bearing assets held for the purpose of meeting liquidity and transactions requirements.[14] Most importantly, the implications drawn below are based upon the assumption that there is not a significant systematic relationship between wealth and risk aversion in cross-sections of investors. This is, of course, a somewhat different assumption than we have made in the above comparative statics analysis.

Consider portfolios of households. The "riskless security" can be taken to be a composite of savings accounts, government securities of appropriate maturity, etc.[15] Risky securities include common stocks, real estate investments, oil wells, etc.

14. The separation of the decision regarding the optimal cash holding and the composition of the rest of the portfolio is a useful and often used simplification. Rigorous justification of this approach would appear to require two assumptions: All noncash assets are perfectly illiquid. The cash holding in contrast to the rest of the portfolio is not intended for future consumption.

15. The conceptualization of the "riskless security" used here and in the following paragraph will of course be flawed to the extent to which investors consider the uncertainty as to the rate of inflation when choosing their portfolios.

"Model II" predicts that the fraction of a portfolio invested in risky securities will rise with wealth, at a declining rate. "Model III" tells us that the size of the average common stock holding, or the size of the average security holdings in any other class, will increase approximately as the square root of total wealth. An individual's investment in a single real estate project will generally be higher relative to total wealth than an investment in a single common stock because of the higher transactions costs. An investment in any one oil well is usually small relative to a person's wealth because of the high risks and great possibility for risk reduction by investing in many different wells.

Looking at commerical bank portfolios, the "riskless security" is a composite of government securities. Risky securities are mainly various kinds of loans: commercial, real estate, consumer, as well as any other type of relatively risky security in which banks invest. "Model II" predicts that larger banks will invest a greater percentage of their assets in risky securities than smaller banks. According to "Model III" the size of the average commercial loan, for instance, will tend to increase as the square root of a bank's assets.

"Model I" tells us that the number of securities held by a common stock mutual fund should increase roughly as the square root of a fund's assets.

These are just a few of the many possible applications of our approach. Hopefully, others will contribute further.

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