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SINGLE PERIOD PORTFOLIO OPTIMIZATION WITH FUZZY TRANSACTION COSTS

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Abstract: This paper is concerned with the single period portfolio that consists of holdings in *n* risky assets. The goal is to choose the optimal portfolio to maximize the expected value of the end of period wealth in the presence of transaction costs, while satisfying a set of constraints on the portfolio. The case of a portfolio optimization problem with fuzzy transaction costs is considered. Computational results, which facilitate comparison between the proposed models, are presented.

Keywords: portfolio optimization, transaction costs, fuzzy portfolio selection model.

1. Introduction

In this paper, we consider an investment portfolio that consists of holdings in *n* assets. This portfolio is to be adjusted by performing a number of transactions, after which the portfolio will be held over a fixed time period. The investor's goal is to maximize the expected wealth at the end of period, while taking transaction costs into account and satisfying a set of constraints on the portfolio, which typically include limits on exposure to risk and bounds on the amount held in each asset. The problem is also considered in a fuzzy context.

Recent years have seen a growing interest in portfolio optimization problem. The paper of (Best and Hlouskova, 2003) deals with the portfolio selection problem of risky assets with a diagonal covariance matrix, upper bounds on all assets and transactions costs. (Blog *et al.*, 1983) consider the specific optimal selection problem of small portfolios. (Kellerer *et al.*, 2000) introduce mixed-integer linear programming models dealing with fixed costs and minimum lots and propose heuristic procedures based on the construction and optimal solution of mixed integer subproblems. (Konno and Wijayanayake, 2001) propose a branch and bound algorithm for calculating a globally optimal solution of a portfolio construction/rebalancing problem under concave transaction costs and minimal transaction unit constraints. (Schattman, 2000) develops an iterative heuristic for finding a suboptimal solution for the portfolio problem. If the portfolio optimization problem is nonlinear, the algorithm presented in (Fulga, 2006) that combines penalty concepts and sequential quadratic programming techniques can be used.

In the next section, we present a single-period portfolio selection problem. Transaction cost functions are described, fixed costs are included and it is shown how to obtain a feasible suboptimal portfolio. Section 3 deals with the portfolio optimization problem with fuzzy transaction costs. Computational results are given in Section 4 and conclusions in Section 5.

2. The portfolio selection problem

We are concerned with the single-period portfolio that consists of holdings in *n* risky assets. The portfolio is adjusted at the beginning of the time-period. The goal is to choose the optimal portfolio to maximize the expected value of the end of period wealth in the presence of transaction costs, while satisfying a set of constraints on the portfolio.

2.1. The model

The current holdings in each asset are $w = (w_1,\dots,w_n)^T$. The total current wealth is then $\sum_{i=1}^{n} w_i$. The amount of money transacted in each asset i, $i=\overline{1,n}$, is denoted by x_i, with $x_i > 0$ for buying, $x_i < 0$

for selling and $x=(x_1,...,x_n)^T \in R^n$ is the vector of transactions. After transactions, the adjusted portfolio is $w+x$. Representing the sum of all transaction costs associated with x by $f(x)$, the budget, or self-financing constraint is $\sum_{i=1}^{n} x_i + f(x)=0$. The adjusted portfolio $w+x$ is then held for a fixed period of time. At the end of that period, the return on asset i is the random variable $\tilde{r}_i, i=\overline{1,n}$. All random variables are on a given probability space. We assume knowledge of the first and second moments of the joint distribution of $\tilde{r}=(\tilde{r}_1,...,\tilde{r}_n)^T$, $E(\tilde{r})=r, r=(r_1,...,r_n)^T \in R^n, E\left((\tilde{r}-r)(\tilde{r}-r)^T\right)=C$. A riskless asset can be included, in which case the corresponding r_i is equal to its (certain) return, and the ith row and column of C are zero.

The end of period wealth is a random variable, $\tilde{w}=\tilde{r}^T(w+x)$, with expected value and variance given by $E(\tilde{w})=r^T(w+x), E\left((\tilde{w}-E(\tilde{w}))^2\right)=(w+x)^T C(w+x)$. The budget constraint can also be written as an inequality, $\sum_{i=1}^{n} x_i + f(x) \le 0$. With some obvious assumptions ($f \ge 0, r_i > 0, i=\overline{1,n}$), solving an expected wealth maximization problem with either form of the budget constraint yields the same result. The inequality form is more appropriate for use with numerical optimization methods. For example, if f is convex, the inequality constraint defines a convex set, while the equality constraint does not. We summarize the portfolio selection problem as

$$(PSP)\begin{cases} \max r^T(w+x) \\ s.t. \sum_{i=1}^{n} x_i + f(x) \le 0 \\ w+x \in X, \end{cases}$$

where $r=(r_1,...,r_n)^T \in R^n$ is the vector of expected returns on each asset, $w=(w_1,...,w_n)^T \in R^n$ is the vector of current holdings in each asset, $x=(x_1,...,x_n)^T \in R^n$ is the vector of amounts transacted in each asset, $f: R^n \to R$ is the transaction cost function, $X \subset R^n$ is the set of feasible portfolios.

2.2. Transaction costs

Transaction costs can be used to model a number of costs, such as brokerage fees, bid-ask spreads, taxes, or even fund loads. In this paper, we assume the transaction costs to be separable, i.e., the sum of the transaction costs associated with each trade is $f(x)=\sum_{i=1}^{n} f_i(x_i)$, where f_i is the transaction cost function for asset $i, i=\overline{1,n}$.

The simplest model for transaction costs is that there are none, i.e., $f(x)=0$. In this case the original portfolio is irrelevant, except for its total value. We can make whatever transactions are necessary to arrive at the optimal portfolio.

A better model of realistic transactions costs is a linear one, with the costs for each transaction proportional to the amount traded $f_i(x_i)=\begin{cases} a(x_i)|x_i|, \ x_i \ne 0 \\ 0, \ x_i=0 \end{cases}, i=\overline{1,n}$, where $a(x_i)=\begin{cases} a_i^{buy}, \ x_i>0 \\ -a_i^{sell}, \ x_i<0 \end{cases}$. Here $a_i^{buy}>0$ and $a_i^{sell}>0$ are the cost rates associated with buying and selling asset $i, i=\overline{1,n}$. We will consider a model that includes fixed plus linear costs, but our method is readily extended to handle more complex transaction cost functions. In this case, the transaction cost function is given by

$$f_i(x_i)=\begin{cases} a(x_i)|x_i| + b(x_i), \ x_i \ne 0 \\ 0, \ x_i=0 \end{cases}, \ i=\overline{1,n},$$

where $b(x_i)=\begin{cases} b_i^{buy}, & x_i>0 \\ b_i^{sell}, & x_i<0 \end{cases}$ and $b_i^{buy}>0$ and $b_i^{sell}>0$ are the fixed costs associated with buying and selling asset $i, i=\overline{1,n}.$ Evidently the function f_i is not convex, unless the fixed costs are zero.

We assume from now on equal costs for buying and selling, the extension for nonsymmetric costs being straightforward. The transaction cost function is then

$$f(x)=\sum_{i=1}^{n} f_i(x_i),\, f_i(x_i)=\begin{cases} a_i|x_i|+b_i, & x_i\neq 0 \\ 0, & x_i=0 \end{cases},\ i=\overline{1,n}.$$

In the general case, costs of this form lead to a hard combinatorial problem.

The simplest way to obtain an approximate solution is to ignore the fixed costs, and solve with $f_i(x_i)=a_i|x_i|$. If the b_i are very small, this may lead to an acceptable approximation. In general, however, it will generate inefficient solutions with too many transactions. Note that if this approach is taken and the solution is computed disregarding the fixed costs, some margin must be added to the budget constraint to allow for the payment of the fixed costs.

On the other hand, by considering the fixed costs, we discourage trading small amounts of a large number of assets. Thus, we obtain a sparse vector of trades; i.e., one that has many zero entries. This means most of the trading will be concentrated in a few assets, which is a desirable property.

We assume that lower and upper bounds for x_i are known i.e., there exist m_i^l and m_i^u such that $m_i^l\leq x_i\leq m_i^u$. We denote by f_i^c the convex envelope of f_i, which is the largest convex function which is lower or equal to f_i in the interval $\left\lfloor m_i^l, m_i^u\right\rfloor$. For $m_i^l\neq 0$ and $m_i^u\neq 0$, the function f_i^c is given by

$$f_i^c(x_i)=\begin{cases} \left(\frac{b_i}{m(x_i)}+a_i\right)|x_i|, & x_i\neq 0 \\ 0, & x_i=0 \end{cases},\ i=\overline{1,n},$$

where $m(x_i)=\begin{cases} m_i^u, & x_i>0 \\ m_i^l, & x_i<0 \end{cases}$. Using f_i^c for f_i relaxes the budget constraint, in the sense that it enlarges the search set. Following the approach in (Lobo *et al.*, 2007) we consider the portfolio selection problem *(PSP)* with f_i^c replaced for f_i,

$$\left(PSP^c\right)\begin{cases} \max r^T(w+x) \\ s.t. \sum_{i=1}^{n} x_i+f^c(x)\leq 0 \\ w+x\in X, \end{cases}$$

where $f^c(x)=\sum_{i=1}^{n} f_i^c(x_i)$. This corresponds to optimizing the same objective, the expected end of period wealth, subject to the same portfolio constraints, but with a looser budget constraint. Therefore the optimal value of $\left(PSP^c\right)$ is an upper bound on the optimal value of the unmodified problem (PSP). Since the problem $\left(PSP^c\right)$ is convex, we can compute its optimal solution, and hence the upper bound on the optimal value of the original problem (PSP), very efficiently.

3. The portfolio optimization problem with fuzzy transaction costs

In the classical problems of operations research generally, and in the optimization models in particular, the coefficients of the problems are assumed to be exactly known. However in practice this assumption is seldom satisfied by great majority of real-life problems. The modeling of input data inaccuracy can be made by means of the fuzzy set theory. Generally, two types of problems implying fuzzy uncertainty are studied. Fuzzy approaches to solve deterministic problems could be developed and also fuzzy models, implying fuzzy goals and fuzzy coefficients, could be defined.

In (Vercher *et al.*, 2007) two fuzzy portfolio selection models are presented. Models objective are to minimize the downside risk constrained by a given expected return, the rates of returns on securities are approximated as *LR*-fuzzy numbers of the same shape, and the expected return and risk are evaluated by interval-valued means. The portfolio selection problem is formulated as a linear program when the returns on the assets are of trapezoidal form. In (Inuiguchi, Ramik, 2000) some fuzzy linear programming methods and techniques from a practical point of view are reviewed. Using a numerical example, some models of fuzzy linear programming are described and advantages and disadvantages of fuzzy mathematical programming approaches are exemplified in the setting of an optimal portfolio selection problem. Some newly developed ideas and techniques in fuzzy mathematical programming are also briefly took into consideration. In (Lacagnina, Pecorella, 2006) a multistage stochastic soft constraints fuzzy program in order to capture both uncertainty and imprecision as well as to solve a portfolio management problem is developed.

3.1. Fuzzy model

The model of a portfolio optimization problem with fuzzy transaction costs is formally similar to $\left(PSP^c\right)$ and it is presented below. In the following, triangular fuzzy numbers will be used in order to describe fuzzy quantities.

$$\left(FPSP^c\right)\begin{cases} \max r^T(w+x) \\ s.t. \sum_{i=1}^{n} x_i + \overline{f^c}(x) \le 0 \\ w+x \in X, \end{cases}$$

where function $\overline{f_i^c}$ is defined using fuzzy coefficients $\overline{a_i}, \overline{b_i}, i=\overline{1,n}.$ We rely on the definition of a triangular fuzzy number given in (Dubois and Prade, 1987). Moreover, the definition

$$\overline{f_i^c}(x_i) = \begin{cases} \left(\dfrac{\overline{b_i}}{m(x_i)} + \overline{a_i}\right)|x_i|, \; x_i \ne 0 \\ 0, \; x_i = 0 \end{cases}, \; i=\overline{1,n}$$

which describes fuzzy transaction costs has to be interpreted according to extension's principle for aggregating fuzzy quantities. The extension principle was formulated by Zadeh, see (Zimmermann, 1985), in order to extend the known models implying fuzzy elements to the case of fuzzy entities.

3.2. Solving method

We suppose that transaction costs are separable. It means that the transaction cost function is the sum of the transaction cost functions associated with each trade. Consequently, transaction cost function $\overline{f}(x)$ is computed as $\sum_{i=1}^{n} \overline{f_i^c}(x_i).$ In order to compute $\overline{f_i^c}$ we can use fuzzy transaction costs with $\overline{a_i} = \left(a_i^1, a_i^2, a_i^3\right),\; \overline{b_i} = \left(b_i^1, b_i^2, b_i^3\right)$ defined by real parameters $a_i^1, a_i^2, a_i^3, b_i^1, b_i^2, b_i^3 \in R.$ Considering $\overline{f}(x) = \left(f_1(x), f_2(x), f_3(x)\right)$ we have $f_k(x) = \sum_{i=1}^{n}\left(\dfrac{b_i^k}{m(x_i)} + a_i^k\right)|\, x_i \,|, k=\overline{1,3}.$ First, we apply Kerre's method to transform fuzzy inequalities in disjunctive deterministic constraints. Next, according to the method described in (Patkar and Stancu-Minasian, 1982), we consider the indicator variables $\delta^1, \delta^2, \delta^3$ in order to eliminate the disjunctivity and to obtain the following system of conjunctive constraints *(SCC)*:

$$\sum_{i=1}^{n} x_i + f_3(x) \le \left(1-\delta^1\right)M, f_1(x) + \sum_{i=1}^{n} x_i \le \left(1-\delta^2\right)M, f_2(x) + \sum_{i=1}^{n} x_i \ge \left(1-\delta^2\right)M,$$

$$\left(\sum_{i=1}^{n} x_i + f_2(x)\right)\left(\sum_{i=1}^{n} x_i + f_3(x)\right) - \left(\sum_{i=1}^{n} x_i + f_1(x)\right)\left(3\sum_{i=1}^{n} x_i + f_1(x) + f_2(x) + f_3(x)\right) \le \left(1-\delta^2\right)M$$

$$f_2(x)+\sum_{i=1}^{n} x_i \le \left(1-\delta^3\right)M, f_3(x)+\sum_{i=1}^{n} x_i \ge \left(1-\delta^3\right)M,$$

$$(\sum_{i=1}^{n} x_i + f_3(x))(3\sum_{i=1}^{n} x_i + f_1(x)+f_2(x)+f_3(x)) - (\sum_{i=1}^{n} x_i + f_1(x))(\sum_{i=1}^{n} x_i + f_2(x)) \le \left(1-\delta^3\right)M$$

$$\delta^1+\delta^2+\delta^3 \ge 1, \delta^1,\delta^2,\delta^3 \in \{0,1\},$$

$$w+x \in X.$$

The parameter M represents an upper bounds for all expressions which appear in constraints.

Computing $\max\left(r^T(w+x)\right)$ subject to *(SCC)* will allow us to obtain the solution $x^* = \left(x_1^*,...,x_n^*\right)$, $\delta^1,\delta^2,\delta^3$. Components of x^* represent the solution of Problem $\left(FP^c\right)$.

4. Computational results

We consider the model problem, the crisp (nonfuzzy) form, *(PSP)* with the data: $n=5$, $w=(102,104,106,108,110)$, $a_i=0.04, b_i=4, i=\overline{1,5}$, $r=(0.4,0.6,1.1,1.6,1.8)$. We recall that problem *(PSP^c)* corresponds to optimizing the same objective (the expected end of period wealth), subject to the same portfolio constraints as in *(PSP),* but with a looser budget constraint. Therefore the optimal value of *(PSP^c)* is an upper bound on the optimal of the unmodified problem *(PSP).* This is apparent in Table 1. In the fuzzy models, the parameters a_i and b_i were replaced with symmetric triangular fuzzy numbers, see (Inuiguchi, Ramik, 2000). The center of each of it is a_i, respectively b_i, and the spreads are $s_a = 0.02, s_b = 0.015$. The solving of the problems was performed by using a genetic algorithm implemented in Mathematica package.

Table 1. Experimental results of the optimal value function

	Crisp Form	Fuzzy Form
Convex Problem	890.2	873.0
Initial Problem	861.2	854. 5

Figure 1 shows the computational results for the optimal value of fuzzy *(PSP)* and *(FPSP^c)* for different values of fuzzy numbers spreads. The spread is proportional to α (we selected eleven discrete values for $\alpha \in [0;2]$), α = 0 corresponding to the nonfuzzy case.

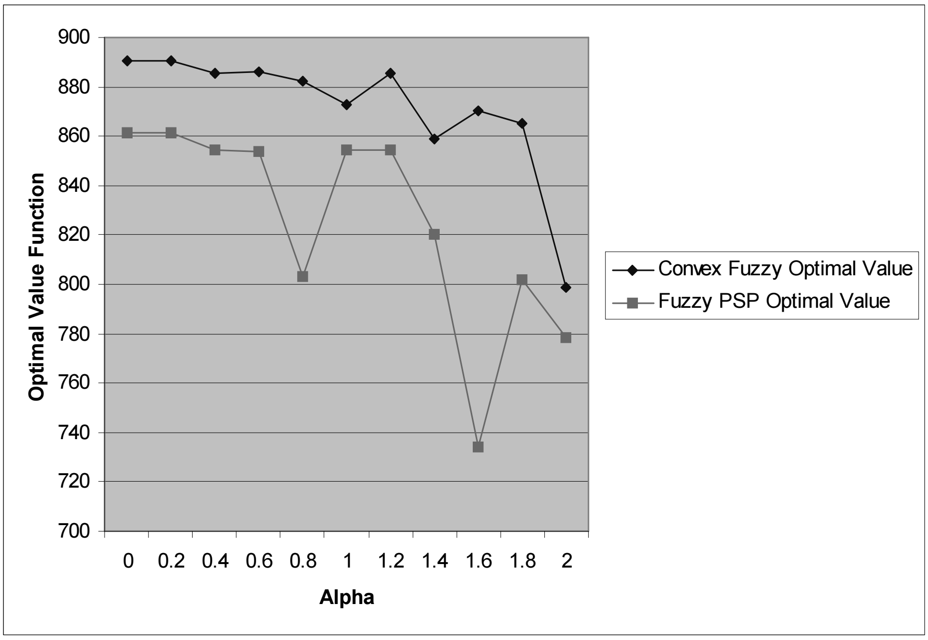

Fig. 1. Optimal value of fuzzy *(PSP)* and *(FPSP^c)*

5. Conclusions

Many mathematical models for the real world use constraints whose coefficients are supposed to be fixed characteristics of modelled reality. Unfortunately, these parameters are often not known exactly because they are variable, unreliable or imprecise in some way. In this paper, this imperfect knowledge is introduced by means of fuzzy transaction costs for the single period portfolio selection problem. The (PSP^c) is a convex optimization problem, which provides a suboptimal solution; the advantage of this model is the use of the powerful algorithms available for the convex optimization problems. The use of fuzzy constraints in the (PSP^c) model gives a more realistic solution for this problem.

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