Mean-Variance, Single Period Trading, and Multi-Period Trading via Convex Optimization

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***Abstract*—The abstract goes here.**

I. Background

In 1952 Harry Markowitz published his seminal paper Portfolio selection and gave birth to the field of portfolio theory. In the preceding years many developments have been built on his work, yielding increasing levels of insight into financial market functionality and financial decision making. The fundamental question answered by his paper is: How should an investor allocate funds among a possible set of asset choices?

To address this question Markowitz suggested two novel approaches:

1) A method to quantify the risk and return of an asset using statistics (variance, expected value)
2) Investors should consider risk and return together and determine the allocation of funds based off of a risk-return profile of their portfolio.

Before the introduction of his theory, investors used risk and return in a disjointed fashion, rather than treating them as a connected trade-off between one measurement and the other. This approach has provided great value to investors. By quantifying and treating risk and return as a single metric, investors could then evaluate a portfolios return based off of both the individual potential returns of an asset and the relationship of assets returns to each other, rather than viewing the returns of a portfolio as the weighted average of the individual returns.

A simple example of the pre and post portfolio theory approach:

> Take a portfolio composed of oil, transportation, and industrial manufacturing securities. The pre portfolio theory approach would be analyzing the individual returns of the assets. Our qualitative analysis of the companies indicates that all three of the securities will increase in value, and when we quantify and project the financial growth, we find the companies are projected to grow by the same amount. In this case we would want to purchase only one of the securities to minimize the cost of purchasing the security (a broker is required to link a buyer and seller in the market known as market makers, and they charge a fee per order for their service).
>
> Using the methods of portfolio theory an investor would likely find a connection between the three companies. In this case, if the price of oil increases, the cost of transportation will increase and decrease the value or return of our transportation security. The rise of oil price will likely increase the value of our oil drilling component manufacturer and the share price will rise together. Our analysis of the individual securities implied that all three will grow, but the connection between the industries implies that their growths are linked to each other and our financial analysis is likely risky for each of them. In order to reduce the risk of the portfolio we would want to distribute roughly half our funds to the transportation company and half between the oil and oil component manufacturer. In this case we hedge our projections against each other and reduce the risk of an overall loss, while maximizing the return.

What portfolio theory allows us to do is extrapolate this approach to a wide variety of assets in a highly complicated and inter-connected system. This is especially useful where the connections between assets may not be as readily apparent, e.g. adding corn to your portfolio may reduce the risk associated with investing in an aerospace security because there may be an illogical correlation between the two. Applying this method across a portfolio allows investors to generate higher returns at a lower risk.

Markowitz formulated portfolio theory as an optimization problem as mean variance optimization (MVO). The MVO has an infinite number of potential portfolios to select from to generate a desired return, and then from there the risk is minimized. The formulation of this problem is in the appendix. To apply portfolio theory in practice investors will utilize quantitative trading algorithms. These algorithms have become increasingly popular over the past decade as the cost of computation has decreased. One of the problems to develop from portfolio MVO is that the model does not include the cost of transaction for each of the stocks.

To address this a series of new trading algorithms have been developed. Most of the work done in this field has been on approaching the problem from a single period point of view,

This tutorial paper was written as part of EE109: Convex Optimization, taught by Professor Mai Vu, Spring 2019

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referred to as Single Period Optimization (SPO). These SPO techniques give investors guidance as to which and how much of the funds should be distributed to a selection of assets, effectively performing the same analysis as MVO but adding additional information on the transaction and hold costs of assets. A limitation in the technique is it does not take into account the future returns of the portfolio. To address this a new field of Multi-Period Optimization (MPO) has been created to address this. It answers the question for investors of when each of the assets should be purchased and has become more viable as computation has become less costly.

II. Mean-Variance Optimization

A. Assumptions

Markowitzs theory of portfolio selection is based on several assumptions:

1) Risk of a portfolio is based on the variability of returns from the said portfolio
2) An investor is risk averse
3) An investor is rational
4) An investor prefers to increase consumption
5) An investors utility function is concave and increasing, due to his risk aversion and consumption preference
6) A portfolio is selected for a single period
7) An investor either minimizes risk for a specified return or maximizes returns for a specified risk

B. Problem Formulation

We consider a portfolio of n assets $S_1, S_2, ..., S_n$, with returns $r_1, r_2, ..., r_n$. We represent the returns as $r = [r_1, r_2, ..., r_n]^T$. We have a total amount of funds to invest, F, and distribute it across the portfolio of our securities. We use $h = [h_1, h_2, ..., h_n]^T$ to denote the vector of our holdings, where h_i represents the dollar amount of each holding. We want to view these holdings proportional to our total portfolio, so we define a weight vector $w = [w_1, w_2, ..., w_n]^T$, where $w_i = \frac{h_i}{1^T h}$. We can find the returns of the entire portfolio as:

$$r_p(w) = w_1 r_1 + \cdots + w_n r_n = w^T r \tag{1}$$

The goal of this problem is based on the fact that the risk of the investment is related to the variance of the entire portfolio. To build on this we take the σ_i as the standard deviation of r_i, and ρ_{ij} as the correlation coefficient between assets S_i and S_j. We can create an $n \times n$ co-variance matrix, Σ defined as:

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$$

Where $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$. We can compute the variance of the portfolio as:

$$V(w) = w^T \Sigma w \tag{2}$$

We can represent the expected return of each of the assets in the portfolio as: $\mu = [\mu_1, \mu_2, ...\mu_n]^T$ where μ_i is the return of asset i.

There are now three ways to formulate this problem. First, if an investor has a minimum return required and want to minimize risk the problem becomes:

$$\begin{array}{ll} \underset{w}{\text{minimize}} & w^T \Sigma w \\ \text{subject to} & \mu^T w \geq R_{min}. \\ \text{or} & \\ \underset{w}{\text{maximize}} & \mu^T w \\ \text{subject to} & w^T \Sigma w \leq \sigma_{max}^2. \end{array}$$

Where R_{min} represents the minimum return required by an investor and σ_{max}^2 represents the maximum risk for the portfolio. A portfolio manager can combine these two metrics by incorporating a risk aversion parameter γ, and the problem becomes:

$$\underset{w}{\text{maximize}} \quad \mu^T w - \gamma w^T \Sigma w$$

To form an unconstrained maximization problem. All of these problems can be solved by casting them as a quadratic program.

C. Implementation and Experimental Results

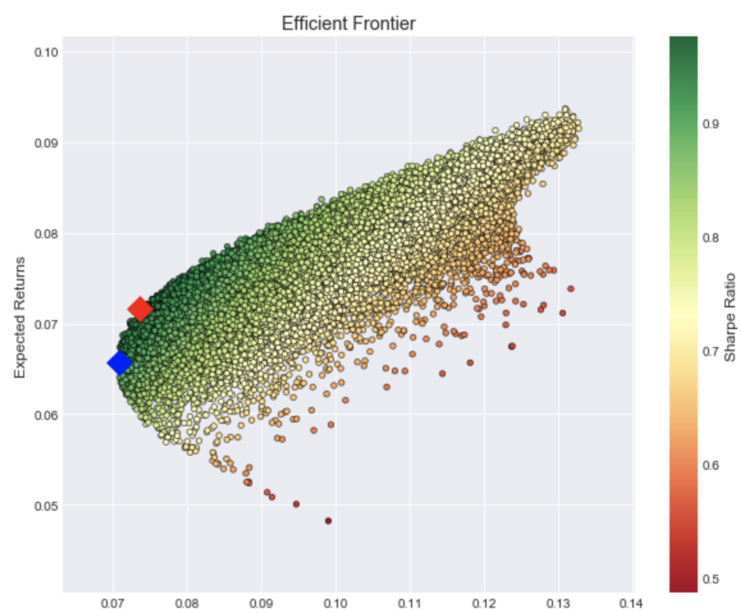

Fig. 1. Efficient Frontier Graph, Red diamond indicates maximum sharpe ratio, blue point indicated the minimized risk.

I implemented the mean-variance optimization technique in two ways for this project. I used a data set collected from yahoo finance of the returns of several major index funds over the past several years. I computed the variance and returns based off of historical results. Additionally I computed the portfolio returns and the portfolio variance for 10,000 simulated portfolios of random weights for each of the assets. I then minimized the variance and maximized the risk adjusted return by maximizing the sharpe ratio for the portfolio (returns/risk). In the case of the sharpe ratio, a concave function is maximized and in the case of the risk minimization, the risk function is maximized. This is much easier to understand when visualized via the efficient frontier graph.

III. Single Period Optimization

A. Assumptions

For the single period trading algorithms, all of the same assumptions are made as for mean variance optimization, however, the investor preference to increased consumption is removed. This is done to allow investors to analyze trading and holding costs for assets, making long-short mixed portfolios an option (because MVO often leads to portfolios where large shorts are taken to increase long positions). In the multi-period framework, the same assumptions as the single period hold true less the single period constraint.

B. Problem Formulation

For this problem we want to take the MVO problem and build on it. We denote the trade of asset i in a portfolio, in dollar value as x_i, with $x = [x_1, x_2, ..., x_n]^T$. We normalize our trade vector $z = [z_1, z_2, ..., z_n]^T$ by taking $z = x/F$. We denote the post trade portfolio as $h^+ = h + x$. This translates to updating the weights as $w^+ = w + z$. We can denote the transaction costs as $\phi^{trade}(x)$. The transaction costs are separable, meaning that the transaction cost of the portfolio is simply the sum of the transaction costs for each of the assets. The transaction costs are here to represent the cost of placing a trade with a broker. A reasonable general model for the scalar function would look something like:

$$x \mapsto a|x| + b\sigma \frac{|x|^{3/2}}{V^{1/2}} + cx \tag{3}$$

In this case a is one half the bid-ask spread for the asset. This represents the difference the buyer and seller are willing to pay from the midpoint between offers. b is a positive constant chosen by fitting the estimated transaction costs for historical data, V is the market volume of the asset being traded, and σ is the volatility of the asset. c is used to create asymmetry in the model, in this case that means the cost difference when using the broker to buy and sell the assets. In the case that buying and selling are the same price, $c = 0$. These cost functions are dependent on which asset you are calculating the transaction cost for. The holding costs of the assets are defined as $\phi^{hold}(h^+)_- = s^T(h^+)$. These holding costs represent the price that is paid to short an asset. When a buyer thinks the price of an asset will decrease they can borrow the asset and sell it immediately, and pay back the person they borrowed the asset from back later. Until they pay back the borrower, they pay a fee proportional to the amount borrowed. In this case, s_i represents the cost to hold asset i, and the function $(z)_- = \max\{-z, 0\}$. We also impose a self-financing constraint on the portfolio. This simply means we are not going to be adding cash into or out of the portfolio in the optimization. This makes the equation:

$$1^T x + \phi^{trade}(x) + \phi^{hold}(h^+) = 0 \tag{4}$$

We can also express this self financing condition in terms of the normalized vectors as:

$$1^T z + \phi^{trade}(z) + \phi^{hold}(w + z) = 0 \tag{5}$$

We can also model the portfolio returns, R^p as:

$$R^p = r^T w + r^T z - \phi^{trade}(z) - \phi^{hold}(w + z) \tag{6}$$

In this equation the first term represents the returns of the portfolio, the second term represents the returns on the trades, and the cost functions represent their respective costs on returns of the portfolio.

In this model we also generalize the risk function from the Mean-Variance Optimization as $\psi(w + z)$. We can translate these updated functions into the old optimization problem. We will also update the terms to represent the estimates by adding the ˆ over our variables. In many of the cases where this is relevant, the portfolio is large enough that the transaction and holding costs are irrelevant in the self-financing condition. Additionally in implementation the error of the estimates for the cost functions are much greater than theor impact on the trade volume. Therefore we can drop the costs in the condition and the problem becomes:

$$\begin{array}{ll} \underset{w,z}{\text{maximize}} & \hat{r}^T z - \hat{\phi}^{trade}(z) - \hat{\phi}^{hold}(w + z) - \gamma\psi(w + z) \\ \text{subject to} & 1^T z = 0 \end{array}$$

Dropping the cost functions from the equality in the constraints allows for the equality to be a convex condition. Additionally, the first term of the objective function is concave, and the other terms are all negative times convex function, therefore the objective is concave and the entire problem is a concave maximization problem.

If you were to want to keep the cost functions in the self-financing constraint the problem would no longer be convex. In order to preserve the convexity of the problem you would turn the function into:

$$1^T z + \phi^{trade}(z) + \phi^{hold}(w + z) \leq 0 \tag{7}$$

In this case when the problem is optimized, the inequality becomes tight. This makes sense intuitively as the optimal solution for any efficient portfolio you will be using all of your cash, otherwise the portfolio is inefficient.

IV. Multi-Period Optimization

A. Assumptions

For the single period case, there is an assumption that the returns from period to period are independent and drawn from the same distribution, and that the historical data from period to period can give insight into the distribution. These assumptions are then lifted in the multi-period model.

B. Problem Formulation

In this case we want to view the returns of the portfolio over multiple periods. We denote all of the functions and vectors we had previously with an indicator for the period within the planning horizon, H, of periods $t, t+1, ..., t+H-1$. For example, we now have a bunch of trades planned out over the planning horizon as $z_t, z_{t+1}, ..., z_{t+H-1}$. We will also denote our returns of estimate made at time t for period τ as $\hat{r}_{\tau|t}$. Taking these changes into account transforms our optimization problem to:

$$\begin{array}{ll} \underset{w_\tau, z_\tau}{\text{maximize}} & \displaystyle\sum_{\tau=t}^{t+H-1} \hat{r}_{\tau|t}^T (w_\tau + z_\tau) - \hat{\phi}_\tau^{trade}(z_\tau) \\ & -\hat{\phi}_\tau^{hold}(w_\tau + z_\tau) - \gamma_\tau \psi_\tau (w_\tau + z_\tau), \\ \text{subject to} & 1^T z_\tau = 0, \\ & w_{\tau+1} = w_\tau + z_\tau, \\ & \tau = t, t+1, ..., t+H-1. \end{array}$$

In actuality the weights are updated according to a dynamics equation

$$w_{t+1} = \frac{1}{1 + R_t^P}(1 + r_t) \circ (w_t + z_t) \tag{8}$$

However, in order to make this problem simpler and easily solvable by numerical methods, the returns are assumed to be small relative to one, so they can be set to zero in the dynamics equation and the resulting equation can be used:

$$w_{t+1} = w_t + z_t \tag{9}$$

The cost functions and holding functions are held to the same conditions as before, being convex functions, and with the constraints still convex, and the multi-period optimization problem becomes the positively weighted sum of concave functions and is therefore a concave optimization problem.

V. Problems

In the implementation of the SPO and MPO algorithms I ran into a number of issues, all of which were with the data and the trading functions. I couldn't find any suitable trading functions and data sets with comprehensive information on the bid-ask spread for the assets. Additionally, I couldn't find any good methods for projecting the trading and holding cost functions into the future periods. For these reasons I could not collect any meaningful results for the SPO and MPO trading algorithm simulations.

Acknowledgment

Thank you to Professor Vu and Alireza for making this class and therefore this paper possible.

Additionally, the tutorial paper by Boyd, et al. was extremely helpful in the development of this paper. Many of the equations and parts of the problem formulation are drawn from his work.

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