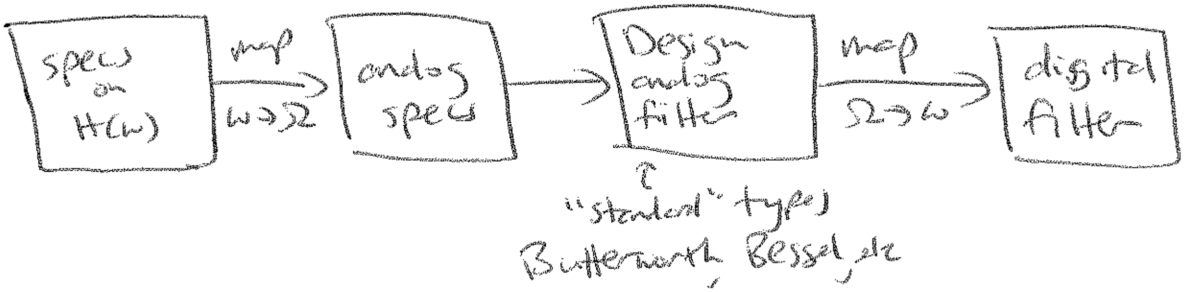
# IIR Filter Design (PoM 10.3.3)

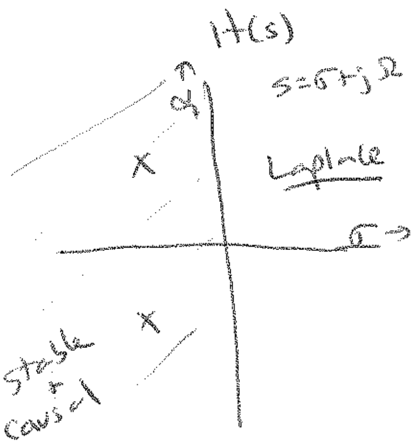
## Basic idea

1) state specifications for $H(\omega)$
2) map specs to continuous frequency ($\omega \to \Omega$), $\Omega = 2\pi f$
3) design a CT (analog filter) that meets specs
4) transform back to DT filter ($\Omega \to \omega$)

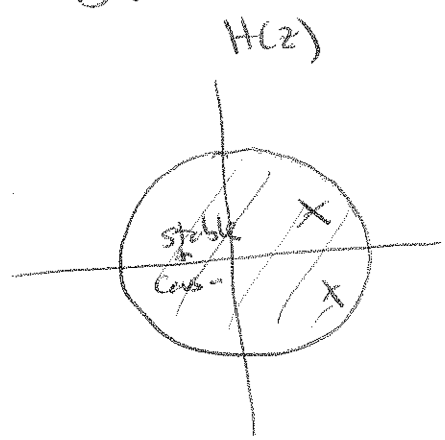



This seems strange. Why do it?
→ there are many really good well-understood analog filters
→ because filters are well understood, no need for iterative design

**Problem**: how to map CT freq response $H(s)$ (Laplace) to DT " " $H(z)$ ?




y-axis is frequency → $H(\Omega)$




unit circle is frequency → $H(\omega)$

Three main mappings

1) approx by derivatives
2) impulse invariance
3) bilinear transform (best)

note: $\Omega$ is $\infty$ long
$\omega$ is $2\pi$ long

# Approximation by derivative (1)

Idea: 1) represent a derivative in continuous time vs discrete time

2) transform each into freq domain

3) get a mapping

$$\frac{dy}{dt} \approx \frac{y(nT) - y(nT - T)}{T} = \frac{\Delta y}{\Delta t}$$

$T$ = sampling interval

$$\frac{dy}{dt} \approx \frac{y(n) - y(n-1)}{T}$$

Laplace transform ↙ ↓ z transform

$$sY(s) \approx \frac{1}{T}\left(Y(z) - z^{-1}Y(z)\right) = \frac{1 - z^{-1}}{T}\,Y(z)$$

or, map

$$\boxed{s = \frac{1 - z^{-1}}{T}}$$

maps $s$ to $z$

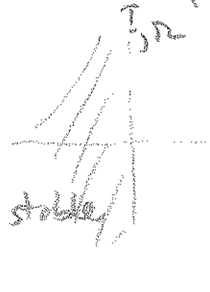
So approximate: $H(z) = H(s)\Big|_{s = \frac{1 - z^{-1}}{T}}$ plug in for $s$

where do CT freq map to?

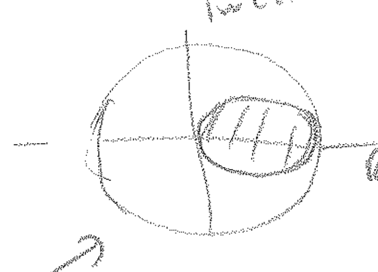
$$s = \frac{1 - z^{-1}}{T} \Longrightarrow z = \frac{1}{1 - Ts} \rightarrow$$

so on $j\Omega$ axis,

$$z = \frac{1}{1 - Tj\Omega} = \frac{1}{1 + \Omega^2 T^2} + j\,\frac{\Omega T}{1 + \Omega^2 T^2}$$




stable




stable but limited

at $\Omega = 0$, $z = \frac{1}{1}$

as $\Omega \to \pm\infty$, $z \to \frac{1}{\pm\infty} = 0$

in between, maps on circle

# Approx by derivative - 2

ex) Say we have analog filter

$$H_a(s) = \frac{1}{(s+0.1)^2 + 9}$$

this mapping gives

$$H(z) = \frac{1}{\left(\frac{1-z^{-1}}{T} - 0.1\right)^2 + 9}$$

We start with a system (in Laplace-land)

$$H(s) = \frac{b}{s+a}$$

but $H(s) = Y(s)/X(s)$, so

$$Y(s)(s+a) = bX(s)$$

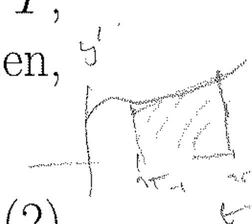
As you hopefully remember from a linear systems class, 's' in the Laplace transform corresponds to a time derivative, so in the time domain we get

$$y'(t) + ay(t) = bx(t) \qquad (1)$$

In general, it's true from calculus that

$$y(t) = y(t_0) + \int_{t_0}^{t} y'(\tau)d\tau$$

We approximate the integral by the trapezoidal rule, and evaluate at the points $t = nT$ and and $t_0 = (n-1)T$ (notice that by moving to time sampled at intervals of $T$, we are moving from continuous to sampled time). Then, the above becomes



$$y(nT) \approx y(nT - T) + T\left(\frac{y'(nT) + y'(nT - T)}{2}\right) \qquad (2)$$

However, Eq. 1 gives a result for the derivative. We can plug this into Eq 2, and do a bunch of algebra. If we define $y(n) = y(nT)$ for simplicity, we have

i.e. $y'(t) = bx(t) - ay(t)$

$$(1 + aT/2)y(n) - (1 - aT/2)y(n-1) = bT/2(x(n) + x(n-1))$$

If we take the z-transform (as we are now in discrete time), we get

$$(1 + aT/2)Y(z) - (1 - aT/2)z^{-1}Y(z) = bT/2(1 + z^{-1})X(z)$$

Collecting $Y(z)$ terms and rearranging, we can find

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + a}$$

but

$$H(z) \approx H(s) = \frac{b}{s + a}$$

(the reason it's approximate is that the trapezoidal rule only approximated the integral). Thus

$$s \approx \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

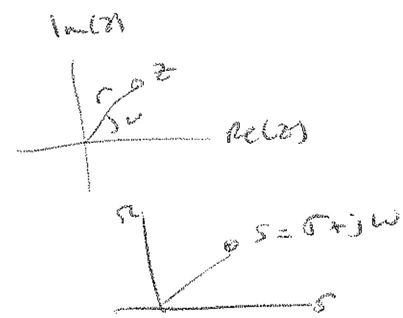
which is the bilinear transform.

(2)

Bilinear transform → see pdf for derivation

get $\boxed{s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}}$

where $T = 1/F_s$




Characteristics

First, we write $z = r e^{j\omega}$

$s = \sigma + j\Omega$

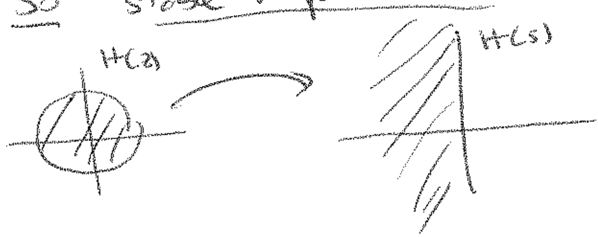
after algebra, (see book) get

a) $\sigma = \frac{2}{T} \frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}$

b) $\Omega = \frac{2}{T} \frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}$

from a) If $r < 1$, then $\sigma < 0$ ; if $r > 1$, $\sigma > 0$

So stable maps to stable

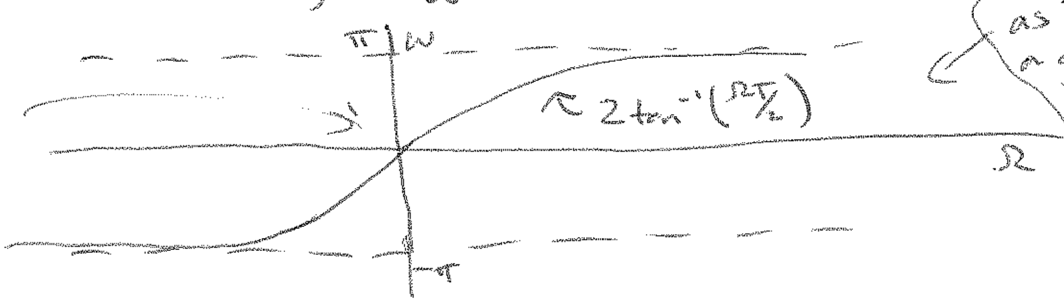



from b), look at freq response. on unit circle, $r = 1$

$\Omega = \frac{2}{T} \frac{2\sin\omega}{2 + 2\cos\omega} = \frac{2}{T} \frac{\sin\omega}{1 + \cos\omega} = \boxed{\frac{2}{T} \tan \omega/2}$

or, $\omega = 2\tan^{-1}(\Omega T/2)$

nearly linear near $\Omega T = 0$




as $T\downarrow$ ($F_s\uparrow$) a given $\Omega$ maps closer to linear region

## Some examples

**Case 1** : specify $\begin{cases} \text{general form for } H(s) \\ \text{specifications in } \omega \end{cases}$

$T$ will divide out

ex) $\to$ $H(s) = \dfrac{\Omega_c}{s+\Omega_c}$ $\quad$ $\Omega_c$ = −3dB point
form $\quad$ lowpass

problem statement

$\to$ Say we want digital cutoff (−3 dB point)
at $0.2\pi$; $\quad \omega_c = 0.2\pi$

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step 1) warp $\omega_c$ to $\Omega_c$

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = \frac{2}{T}\tan\frac{0.2\pi}{2} = \frac{0.65}{T}$$

so $H(s) = \dfrac{\Omega_c}{s+\Omega_c} = \dfrac{0.65/T}{s + 0.65/T}$

bilinear map: $\quad s = \dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}$

$$H(z) = \frac{0.65/T}{2/T\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.65/T} = \frac{0.65}{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.65}$$

$T$ divides out

now we could find poles/zeros, etc.

Case 2 sometimes $H(s)$ is specified — not just in general terms ($\Omega_c$, etc) but in more detail (say, CT ~~as~~ frequencies are fixed). ← trying to translate a specific filter, not a type of filter

Then, we need to pick $T$ to match our digital spec.

ex) problem statement: given $H(s) = \dfrac{s+0.1}{(s+0.1)^2+16}$ and want resonant freq at $\omega = \pi/2$

this analog filter is a resonator w/ resonance at
$$\Omega = 4, \quad \text{or} \quad 2\pi F = 4, \quad F = \frac{4}{2\pi} = \frac{2}{\pi}\ \text{Hz}$$

the mapping is still
$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

but now $\Omega$ & $\omega$ are set; find $T$

$$4 = \frac{2}{T}\tan\left(\frac{\pi/2}{2}\right) \Rightarrow T = \frac{2}{4}\tan\left(\frac{\pi}{4}\right)$$
$$T = 1/2 \qquad F_s = 2\ \text{Hz}$$

then,
$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

plug in:
$$H(z) = \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1}{\left(4\frac{(1-z^{-1})}{(1+z^{-1})} + 0.1\right)^2 + 16}$$

Simplify, find poles / zeros

inverse z transform to get $h(n)$

In matlab (or simulink), this is easy:

[b, a] = butter(n, Wn)     for lowpass

n = filter order

Wn = normalized freq: $W_n = f_c/(f_s/2)$

cutoff

Q) how to pick n?

A) use 'buttord' command

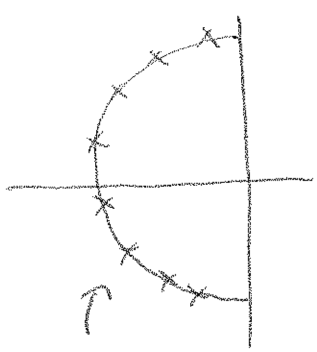
Q) what does that do?

A) look at Butterworth response:

$$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2(\Omega/\Omega_p)^{2N}}$$

specific forms of Nth order Butterworth

H(s)



Butterworth poles are on unit circle in s

$\Omega_c$ = −3 dB cutoff frequency

$\Omega_p$ = passband frequency

alternate spec: $\frac{1}{1+\varepsilon^2}$ is magnitude when $\Omega = \Omega_p$

to meet an attenuation $\delta_2^2$ at some frequency $\Omega_s$ (stop freq)

$$\frac{1}{1+\varepsilon^2\left(\Omega_s/\Omega_p\right)^{2N}} = \delta_2^2$$

if we specify $\delta_2$, $\varepsilon$, $\Omega_s$, $\Omega_p$, we can solve for N that is big enough to meet specs.

→ Similar formulas for other filters

4) You are trying to build a digital filter to mimic a simple RC lowpass filter. This RC filter has the Laplace transform response:

H(s) = (1/RC) / (s+ (1/RC))

i.e., it is lowpass with cutoff frequency $\Omega_c$ = 1/RC. The filter you are trying to match has a 1000 ohm resistor and a 1e-6 Farad capacitor.

a) What is the cutoff frequency, in Hz, of the RC filter? If your digital system has a sampling frequency of 3000 Hz, what radian frequency $\omega_c$ does this correspond to?

$F_c = \frac{1}{2\pi RC} = \frac{1000}{2\pi}$ Hz $\qquad \omega_c = 2\pi \frac{F_c}{F_s} = 2\pi \frac{1000}{2\pi 3000} = \frac{2\pi}{3}\frac{1}{2\pi} = 1/3$

c) Assuming you are using the bilinear transform to do the mapping, find the $\Omega_c$ corresponding to $\omega_c$.

$$\Omega_c = \frac{2}{T} \tan \frac{\omega_c}{2} = 1009$$

$T = 1/F_s$

d) take the first step in finding H(z) – you do not need to do any algebraic simplifications.

plug in bilinear transform

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{\Omega_c}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + \Omega_c}$$