④ Filter design using pole / zero placement

→ ~~Based on what we've done, we can~~
we can start designing filters!

→ methods in 5.4 are a bit crude but are used in practice

BASIC concept

1) put pole near ~~things~~ ω's we'd like to enhance, zeros near ω's we'd like to suppress.

2) stable : all poles should be inside $|z|=1$
zeros can be anywhere

3) real : poles and zeros should occur in conjugate-symmetric pairs or on the real axis.

0) decide if want FIR (no feedback) or IIR (feedback)

Examples

single pole system: $H_1(z) = \dfrac{1-a}{1-az^{-1}}$

numerator $1-a$ picked so at $\omega=0$, $z=1$
$H_1(1) = 1$

see Fig 5.4.3 for $a=0.9$

single-pole, single zero: $H_2(z) = \dfrac{1-a}{2}\,\dfrac{1+z^{-1}}{1-az^{-1}}$

see Fig 5.4.3 again

again, at $\omega=0$ $z=1$
$H_2(1) \to \dfrac{(1-a)(2)}{2(1-a)} = 1$

General formula:

→ decide on form of function (# poles, # zeros) and write $H(z)$ in terms of parameters

→ impose a constraint on response for each free variable

→ solve

Examples: 5.4.1, 5.4.2 — review from book (scan in pages)

5 Practical example: notch filter

→ system w/ very deep null at "problem frequency" i.e. 60 Hz, 50 Hz

→ simplest: ~~[illegible]~~ (FIR): null out frequency ω_0 by placing conjugate symmetric poles at $\pm\omega_0$

→ problem: too wide. Solution: add a pole

show eqns on p. 339 (scanned)

→ MATLAB examples

→ show $\measuredangle H$, $|H|$, grdelay

show time to settle as notch deepens

show distortion on nearby signal

Comb filters

from PPT $\quad y(n) = \frac{1}{M+1} \sum_{k=0}^{M} x(n-k)$

board

$$H(z) = \frac{1}{M+1} \sum_{k=0}^{M} z^{-k}$$

use geometric series identity:

$$\sum_{i=0}^{n} x^i = \frac{1-x^{(n+1)}}{1-x}$$

~~plus~~

z^{-1} looks like 'x' above

then get PPT result: $\quad H(z) = \frac{1}{M+1} \frac{\left[1 - z^{-(M+1)}\right]}{1-z^{-1}}$

$$H(\omega) = H(z)\Big|_{e^{j\omega}}$$

$$= \frac{1}{M+1} \frac{\left[1 - e^{-j\omega(M+1)}\right]}{1 - e^{-j\omega}} = \frac{1}{M+1} \frac{e^{-j\omega(M+1)/2}\left[e^{+j\omega(M+1)/2} - e^{-j\omega\frac{M+1}{2}}\right]}{e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]}$$

~~put out $e^{-j\omega(M+1)/2}$~~

$$= \frac{1}{M+1} e^{-j\omega M/2} \frac{\sin \omega (M+1)/2}{\sin \omega/2}$$

Lecture: All-pass and minimum-phase systems

reading: P&M 5.4.6 (AP) and 5.5

Warmup problem

we have a system $H(z) = \frac{B(z)}{A(z)}$. We'd like to find the inverse system $H^{-1}(z) = \frac{A(z)}{B(z)}$.

→ what is required for both H and H^{-1} to be stable & causal?

→ why might we care about H^{-1}?

P&M 5.4.6

1) All pass systems

a) definition: $|H_{AP}(\omega)| = 1$ (phase will shift)

b) simplest example: time delay

$$H(z) = z^{-k}$$

$$|H(\omega)| = 1, \quad \measuredangle H(\omega) = \omega k$$

even tho simple, this is probably the most widely used AP... compensate for time delay of a linear phase filter

ppt slide for speed?

c) more interesting example: 1 zero, 1 pole

$$H(z) = \frac{z^{-1} - a}{1 - az^{-1}}$$

← zero at $z^{-1} = a$, or $z = 1/a$

← pole at $z^{-1} = 1/a$, $z = a$

look at the magnitude:

$$|H(e^{j\omega})|^2 = |H(e^{j\omega}) H^*(e^{j\omega})| = |H(e^{j\omega}) H(e^{-j\omega})|$$

$$= H(z) H(z^{-1}) \Big|_{z = e^{j\omega}}$$

here, $$|H(\omega)|^2 = \left(\frac{z}{z}\right) \frac{z^{-1} - a}{1 - az^{-1}} \cdot \frac{z - a}{1 - az} = \left(\frac{1 - az}{z - a}\right)\left(\frac{z - a}{1 - az}\right) = 1 \Big|_{z = e^{j\omega}}$$

multiply by

This works b/c we have a reciprocal real pole + zero
could also have reciprocal complex-conjugate P + Z

show figures 5.4.16, 5.4.17

d) general form - Eq 5.4.45

$$H_{ap}(z) = \prod_{k=1}^{N_R} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}} \prod_{k=1}^{N_C} \frac{(z^{-1} - \beta_k)(z^{-1} - \beta_k^*)}{(1 - \beta_k z^{-1})(1 - \beta_k^* z^{-1})}$$

e) why do we care?
→ gives a way to compensate for undesired phase w/o changing magnitude → "phase equalizer"
→ more uses ~~below~~ coming in lecture...

---

2) Minimum phase systems

a) Motivation - deconvolution
→ often, we have a measurement y, $y = h * x$
we want to recover $x(n)$ or $X(\omega)$
→ x is a comms signal, h is receiver
→ x is an image, h is optical system

→ thanks to F.T., this can be easy

$$Y(\omega) = H(\omega) X(\omega)$$

$$X(\omega) = \frac{Y(\omega)}{H(\omega)} \quad \text{or} \quad X(z) = \frac{Y(z)}{H(z)}$$

→ but for this to work, H^{-1} must be stable + causal

(note: could do example 5.5.2 here; but what's wrong w/ $|z| > 1/2$ answer?)

think about $|H(\omega)|^2$

$$|H(\omega)|^2 = H(\omega)\, H^*(\omega)$$

but if h is real, $\cancel{H^*(\omega) = H(-\omega)}$

$$= H(\omega)\, H(-\omega) \quad \leftarrow \text{conj symmetry}$$

$$= H(z)\, H(z^{-1})\Big|_{z=e^{j\omega}}$$

how to see last point?

$$H(z) = \sum_n h(n)\, z^{-n} \qquad \text{(def z transform)}$$

so

$$H(z^{-1}) = \sum_n h(n)\, z^{n}$$

$$H(z^{-1})\Big|_{z=e^{j\omega}} = \sum_n h(n)\, e^{j\omega n}$$

$$= \sum_n h(n)\, e^{-(-\omega)n}$$

$$= H(-\omega) \qquad \text{from DTFT}$$

If $H(z)$ has ~~pol~~ zeros z_k and poles p_k

$H(z^{-1})$ has zeros $1/z_k$ and poles $1/p_k$

pole - zero confusion

Q) Given $|H(\omega)|^2$, can we determine $H(\omega)$?

Say $H(z)$ has zeros $[z_1, z_2]$

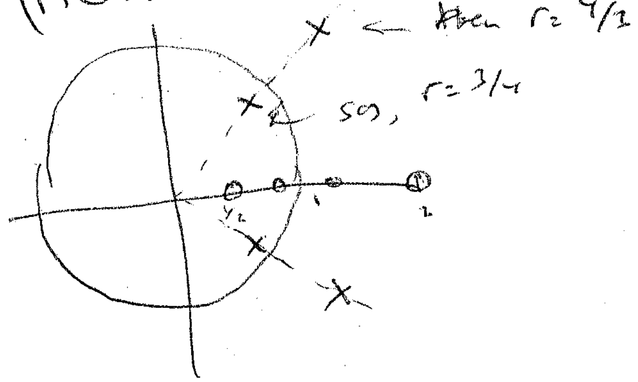
poles $[p_1, p_2]$ ← complex conjugates

$|H(\omega)|^2 = H(z)H(z^{-1})$

$H(z^{-1})$ has zeros $[\frac{1}{z_1}, \frac{1}{z_2}]$

poles $[1/p_1, 1/p_2]$

$|H(\omega)|^2$



given poles / zeros of $|H(\omega)|^2$

$[z_1, 1/z_1, z_2, 1/z_2]$

$[p_1, p_2, 1/p_1, 1/p_2]$

16 possible combinations!

even if we pick poles inside unit circle (H stable) still 4 combinations possible

$(z_1, 1/z_1), (z_1, z_2), (1/z_1, z_2) (1/z_1, 1/z_2)$

thus in general, we cannot distinguish $H(\omega)$

there are special conditions where we can - minimum phase systems

next lecture