(F1)

Lecture 3: Fourier transform

$$x(t) \longrightarrow \boxed{A/D} \longrightarrow x[n]$$

continuous time (CT) — next lecture! — discrete-time (DT)

stability: use Laplace

stability: use z

frequency analysis (CT)

most general, CTFT

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

note $1/2\pi$ conventions differ

if periodic, can use Fourier series (FS)

frequency analysis (DT)

most general: DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

if periodic, discrete FS.

Three big questions

1) how to relate CT f, t to DT ω, n?
   → need to understand sampling theory: next lecture

2) why so many exponentials?
   → constant-coefficient ODE's (wave eqn, vibration, engineering system) solved by $e^{j\omega t}$
   → many physical systems (which create data we measure) are described by this type of eqn

3) how to keep all the transforms straight?

(PPT)

relationship of z to DTFT

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

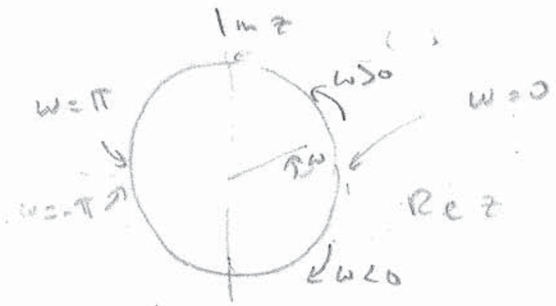
but $z = re^{j\omega}$

$$= \sum x(n) r^{-n} e^{-j\omega n}$$

if $r = 1$, z transform = DTFT!

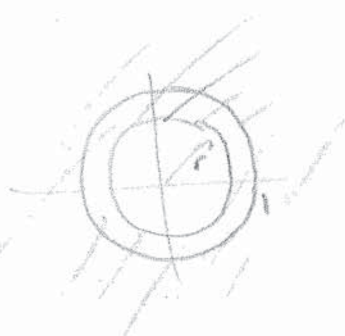
or, $$X(z)\Big|_{z=e^{j\omega}} = X(\omega)$$

show Fig 4.2.9 $\in$ PPT



ROC: remember for z transform, for causal signal ROC should be $|z| > r$

If Fourier transform exists (stable frequency response) unit circle must be in ROC



note z is more compact:
we may have some

F2

Fourier transform taxonomy

	periodic	aperiodic
CT	CT Fourier Series $x(t)$, $c_k \Leftarrow$ F.S. coef.	CTFT $x(t)$, $X(f)$
DT	DT Fourier Series $x[n]$, c_k	DTFT $x[n]$, $X(\omega)$ ✓

note for a computer, <u>discretized</u> quantities are natural.

DTFT: time is discrete, frequency ω is continuous

Good for analysis

Later in class (Chap 7-8) we'll discretize <u>frequency</u>

also $\Rightarrow$ Discrete Fourier Transform (DFT)

$x[n] \Leftrightarrow X[k]$

FFT is fast algorithm for doing the DFT

(PPT)

(FS)

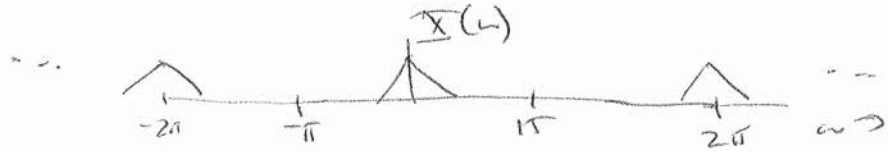
DTFT $\quad X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \equiv \mathcal{F}\{x(n)\}$ analysis

$$x(n) = \int_{-\pi}^{\pi} X(\omega)\, e^{+j\omega n}\, d\omega \equiv \mathcal{F}^{-1}\{X(\omega)\} \quad \text{synthesis}$$

notes:

(1) $X(\omega)$ is 2π periodic

$X(\omega) = X(\omega + 2\pi k)$ (plug in to see)



(2) $X(\omega)$ is generally complex

$$X(\omega) = X_R(\omega) + j\, X_I(\omega)$$

$$= |X(\omega)|\, e^{j[\measuredangle X(\omega)]}$$

phase spectrum

magnitude spectrum

$$X(\omega) = \sum_n x(n)\, e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

FS

example DTFT pairs

impulses

1) time impulse $x(n) = \delta(n)$

$X(\omega) = e^0 = 1$

reconstructing a perfectly sharp transition requires all frequencies

2) $X(\omega) = \delta(\omega)$; only DC

$x(n) = \frac{1}{2\pi} e^0 = 1/2\pi$

sines / cosines

First, consider $X(\omega) = 2\pi\,\delta(\omega - \omega_0)$

$$x(n) = \frac{2\pi}{2\pi} \int \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega$$

$$= \exp(j\omega_0 n)$$

then, by basic math

$$\cos \omega_0 n = \tfrac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)$$

$$\sin \omega_0 n = \frac{1}{2j}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)$$

by inspection,

$$\cos \omega_0 n \Leftrightarrow \frac{2\pi}{2}\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)$$

$$\sin \omega_0 n \Leftrightarrow \frac{2\pi}{2j}\left(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right)$$

cos

sin

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(3) 'rect' box car $\Longleftrightarrow$ sinc

$$X(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0 & \text{else} \end{cases}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$n \neq 0$: $$x[n] = \frac{1}{2\pi}\frac{1}{jn} e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = \frac{1}{\pi n}\frac{1}{2j}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right)$$

$$= \frac{\sin(\omega_c n)}{n\pi} \qquad n \neq 0$$

$n = 0$: $$x[0] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

$$x[n] = \begin{cases} \frac{\sin \omega_c n}{n\pi} & \text{else} \\ \omega_c/\pi & n = 0 \end{cases}$$

sinc

(1) first zero: $\sin \omega_c n = \sin \pi$

$n = \pi/\omega_c$

$X(\omega)$

(2) magnitude $\sim 1/n$

☆ so as $\omega_c \uparrow$ (bigger in ω) sinc width $\downarrow$
true in general

Converse:

$$x(n) = \begin{cases} A, & |n| \le L \\ 0 & \text{else} \end{cases}$$

ex. 4.4.2 in book

$$X(\omega) = A\frac{\sin(L + 1/2)\omega}{\sin \omega/2}$$

(4) Pulse trains

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nM) \Longleftrightarrow X(\omega) = \frac{2\pi}{M}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0)$$

$$\omega_0 = 2\pi/M$$

(check)

Matlab examples of above

→ ~~vary~~ pulse train, vary timing
box car <=> sinc, vary width

Gibb's phenomenon

instead of $X(\omega) = \sum_{n=-\infty}^{\infty} \frac{\sin \omega_c n}{\pi n}$

consider $X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin \omega_c n}{\pi n}$

Symmetry

Here, limit discussion to Real signals

$X(\omega) = \sum_{n=-\infty}^{\infty} x[n](\cos \omega n - j \sin \omega n)$ as $e^{-j\omega} = \cos\omega - j\sin\omega$

$= \sum_n x[n] \cos \omega n - j \sum x(n) \sin \omega n$

$= \underbrace{\;}_{X_R(\omega)} + \underbrace{\;}_{X_I(\omega)}$

X_R is even in ω (cosine)
X_I is odd in ω (sine)

$|X(\omega)| = \sqrt{X_R^2 + X_I^2}$ = even ($|X(\omega)| = |X(-\omega)|$)

so we just need to plot pos. freq. 1-sided

$\angle X(\omega) = \tan^{-1}\left(\frac{X_I}{X_R}\right)$ is odd

F.T. is complex conjugate (Hermitian conjugate)

(picture) $\boxed{H(\omega)^* = H(-\omega)}$ $|H(\omega)|e^{j\theta}$ pos ω $|H(\omega)|e^{-j\theta}$

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Further, if $x(n)$ is even, $(x(-n) = x(n))$

then $X(\omega)$ is real

(because $X_I(\omega) = -\sum_{n=-\infty}^{\infty} x(n) \sin \omega n = \sum (\text{even})(\text{odd}) = 0$

if $x(n)$ is odd, $X(\omega)$ imaginary

can list more properties, but these are the most frequently used (see Table 4.4 - Fig 4.4.2)

Properties : show Table 4.5 electronically

(1) Linearity: very important mathematically.
for problem solving, split a problem into several we can solve

ex) $x(n) = a^{|n|}, a > 1$

$= a^n u(n) + a^{-n} u(-n-1)$

(3) time reversal: similar to z

(2) time shifting $\rightarrow x(n-k) \Longleftrightarrow e^{-j\omega k} X(\omega)$

freq shifting

very important in filter design
linear phase shift $\Rightarrow$ time delay; no distortion
desirable in a filter

(4) Parseval's theorem: energy is preserved

most commonly used as: $\sum_{n=-\infty}^{\infty} x(n)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$

($x_1 = x_2$ in Table 4.5)

PPT + board

F9

5. Correlation & Wiener-Khintchine — ignore for now (we'll talk about correlation next week)

6. convolution / multiplication: multiplication/convolution summary:

$$x_1 * x_2 \Longleftrightarrow X_1(\omega)\,X_2(\omega)$$

$$x_1(n)\,x_2(n) \Longleftrightarrow X_1(\omega) * X_2(\omega)$$

very, very important!

<u>convolution</u>: preassessment example

<u>note</u>: $X_1(\omega)X_2(\omega)$ means convolution / LTI systems output doesn't create any new frequencies

<u>multiplication</u>: special case is modulation/mixing by a carrier:

$y(n) = x(n)\cos\omega_0 n \Longrightarrow$

$$Y(\omega) = X(\omega-\omega_0) + X(\omega+\omega_0)$$

=

(F10)

Energy vs. power spectral densities - Notation

pm 2.1.2 energy signals vs. power signals

signal <u>energy</u> is $E \equiv \sum_{n=-\infty}^{\infty} |x(n)|^2$

signal <u>power</u> is $P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

a signal w/ finite energy is an <u>energy signal</u>

" " " power is a power signal

<u>examples</u>: a finite-length sequence has finite energy
energy signal

a periodic signal has ∞ ~~...~~ energy
but finite power - power signal

section 4.2.5 defines the
energy density spectrum of nonperiodic x(n) as:
$$S_{xx}(\omega) = |X(\omega)|^2 \quad \leftarrow \text{magnitude info}$$

section 4.2.2 defined a
<u>power density</u> <u>spectrum</u> for periodic signals.

More on this in November...