Reconstruction of sampled signals

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Let's say we have a signal that has a bandwidth of at most $2B$ Hz (i.e., its energy is between -B and B Hz). To recover this, we'll do two steps:

- Apply a perfect (hint: non-real-world) filter with response H(F), that passes all signals between -Fs/2 and Fs/2, with gain. (Q: Why Fs/2? A: these are the unaliased components)
- Apply the inverse continuous-time Fourier transform

These two steps can be written as (multiplying by H(F), taking inverse xform):

$$x_a(t) = \int_{-\infty}^{\infty} X_s(F) H(F) e^{j2\pi Ft} dF \tag{0.1}$$

But, $H(F) = T$ in the range $(-F_s/2, F_s/2)$, and 0 otherwise, so plugging in:

$$x_a(t) = T \int_{-F_s/2}^{F_s/2} X_s(F) e^{j2\pi Ft} dF \tag{0.2}$$

Now, what is $X_s(F)$? It's just the (forward) CTFT of the sampled data, $x(n) = x_a(nT)$. The result for that was found in the last lecture. Plugging in, and reorganizing (2nd line):

$$\begin{aligned} x_a(t) &= T \int_{-F_s/2}^{F_s/2} \left[\sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi F/F_s n} \right] e^{j2\pi Ft} dF \qquad (0.3) \\ x_a(t) &= \sum_{n=-\infty}^{\infty} x(n) T \int_{-F_s/2}^{F_s/2} e^{j2\pi F(t-n/F_s)} dF \end{aligned}$$

We can (hopefully) recognize that the integral above as leading to a sinc. In fact, remembering that $1/F_s = T$, we get

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left(\pi(t-nT)/T\right)}{\pi(t-nT)/T} \tag{0.4}$$

which has a couple of nice features:

- at the times $(t - nT)$, the sinc cancels perfectly so $x_a(t) = x(n)$
- we get a smooth intepolation for times in between

Which is really nice! The only problem is that we have to sum over n for an infinite number of values, so it will take us an infinite amount of time to reconstruct the signal. However, this result gives insight into an answer that might be more realistic (for example, what if we summed over some large but not infinite range?).

A more generic way of writing this is

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) g(t - nT) \tag{0.5}$$