Administrative

- Matlab 2 – due next Monday! Coding isn't too hard, but interpretations can be tricky. Any questions?
  - Talk about part 1 - aliasing
  - Talk about part 2 with sinc function
- Matlab 1 markup started on Turnitin
  - Grades not released, but you can see feedback / markup – grading in progresss
- Quiz 1 is a week from today, covering up through today's HW plus Matlabs 1& 2
  - There will be a little HW from today's lecture, but it will be light
  - HW solutions for first 2 weeks posted; will post remaining

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Feedback markup examples in Turnitin

Blue text = regular markup

Solid blue boxes are pre-defined comments

- These often (but not always) have extra info that you will see if you **mouse over them**.

- The white thought bubble (top blue box) means the grader added a special comment for your case

EE-125:
Digital Signal Processing

Lecture 7, LTI systems:
DTFT of LTI systems

Professor Tracey

Outline

- Review discussion of $H(z)$ from last lecture
  - and more on stability
- $H(\omega)$ basics (P&M 5.1)
- $H(\omega)$ for rational systems (P&M 5.2)
  - Calculating $|H(\omega)|$^2

Recap - rational functions -1

- Start with a difference equation (often, set a0=1)

$$a_0 y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

- Take the Z-transform, and find H as a *ratio* (rational function):

$$H(z) \;=\; \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Pull out the highest power, and divide through by b0 and a0:

$$H(z) \;=\; \frac{b_0 z^{-M}}{a_0 z^{-N}} \frac{\sum_{k=0}^{M} b_k/b_0 z^{M-k}}{\sum_{k=0}^{N} a_k/a_0 z^{N-k}}$$

Recap - rational functions - 2

- Factor the numerator and denominator into poles, zeros

$$H(z) = \frac{b_0}{a_0} z^{-M+N} \frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)}$$

- Finally, rewrite, giving:

M zeros

$$H(z) = G z^{(-M+N)} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

N poles

Gain G

If we specify poles/zeros, we know H to within a constant. We can find G if given a constraint; for example, H(z=1) = 1

|N-M|-th order zero at z=0, if N>M

|N-M|-th order pole at z=0, if M>N

Counting these, total # poles always = total # zeros

Stability revisited

- The exact statement of stability is: stability is guaranteed if the unit circle is included in the ROC of $H(z)$
  - Causal: poles are all inside u.c., ROC is $|z|$ outside the biggest pole
    - Why the biggest?
  - Anticausal: poles are all outside u.c. ROC is $|z|$ inside the smallest pole
    - Why the smallest?

Stability revisited – poles on the unit circle

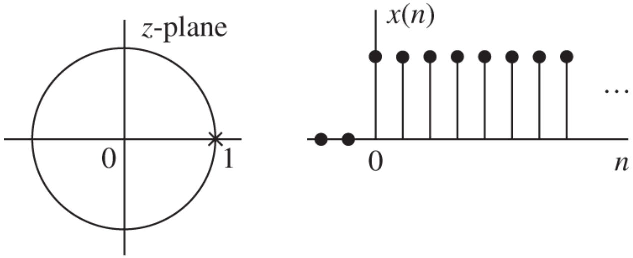
- The exact statement of stability is: stability is guaranteed if the unit circle is included in the ROC of H(z)

- If the pole is on the unit circle, the unit circle can't be in the ROC - so stability is not *guaranteed*. Some systems may be stable while others aren't
  - Single poles on unit circle are conditionally stable (don't grow or decay)
  - Double poles on unit circle are unstable

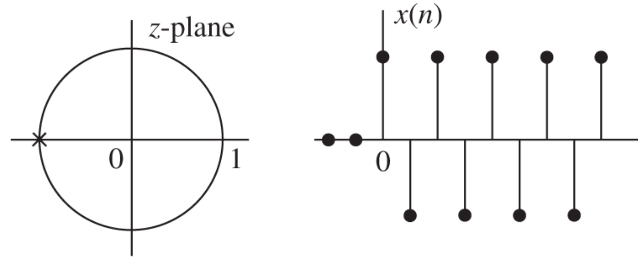
- This is a rather detailed point…

Poles on unit circle: Fig 3.3.5, 3.3.6

System A

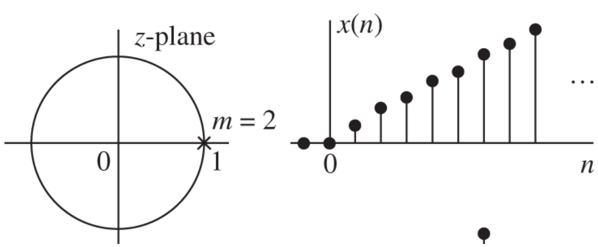


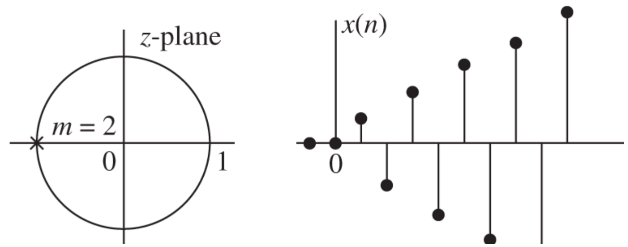
System B



Single pole-
Conditionally
stable

System C





Double pole-
Unstable

Unrelated question:
if I have system A, how can I “make” a System C?

Stability

- We went through proof: "An LTI system is BIBO stable if and only if the ROC for H(z) contains the unit circle".
- Since the unit circle is where frequency response is evaluated, this means "$H(\omega)$ exists if the ROC contains the unit circle".
- When is this true? Fill out table:

	FIR	**IIR**
Causal h(n)	1.	4.
Two-sided h(n) (non-causal)	2.	5.
Anti-causal h(n) (non-causal)	3.	6.

Stability

- We went through proof: "An LTI system is BIBO stable if and only if the ROC for $H(z)$ contains the unit circle".
- Since the unit circle is where frequency response is evaluated, this means "$H(\omega)$ exists if the ROC contains the unit circle".
- When is this true? Fill out table:

	FIR	IIR
Causal h(n)	1. Always true	4. True if poles inside unit circle
Two-sided h(n) (non-causal)	2. Always true	5. Need to check (ROC is annulus)
Anti-causal h(n) (non-causal)	3. Always true	6. True if poles outside unit circle

Outline

- Finish discussion of $H(z)$ from last lecture
- $H(\omega)$ basics (P&M 5.1)
- $H(\omega)$ for rational systems (P&M 5.2)
  - Calculating $|H(\omega)|^2$
  - Geometric interpretation – poles and zeros

Link between last lecture, today

Link between them

$$H(\omega) = H(z)\,\big|_{z=e^{j\omega}}$$

Frequency response of the system, from DTFT
P&M 5.1, 5.2 (today)

Good: Just need to sample frequency axis

Z-transform of the system
P&M 3.3 (last lecture)

Good: Z-transform pole/zero representation is very compact

Bad: don't want to evaluate everywhere in z plane

It's convenient to think about systems using poles / zeros (from Z) but evaluate the frequency response (DTFT)

Geometric interpretation – |H(z)| looks like this on z plane …

- We can think of evaluating X(z) or H(z) in the complex plane.

  To calculate figures like this one, could write code like:

```
Npts=100;
Zreal=linspace(-2,2,Npts);
Zimag=linspace(-2,2,Npts);

for izR=1:Npts
  for izI = 1:Npts
      z =Zreal(izR)+j*Zimag(izI);
      H = (system function using z)
      Hmag(izR,izI) = abs(H);
  end
end
```

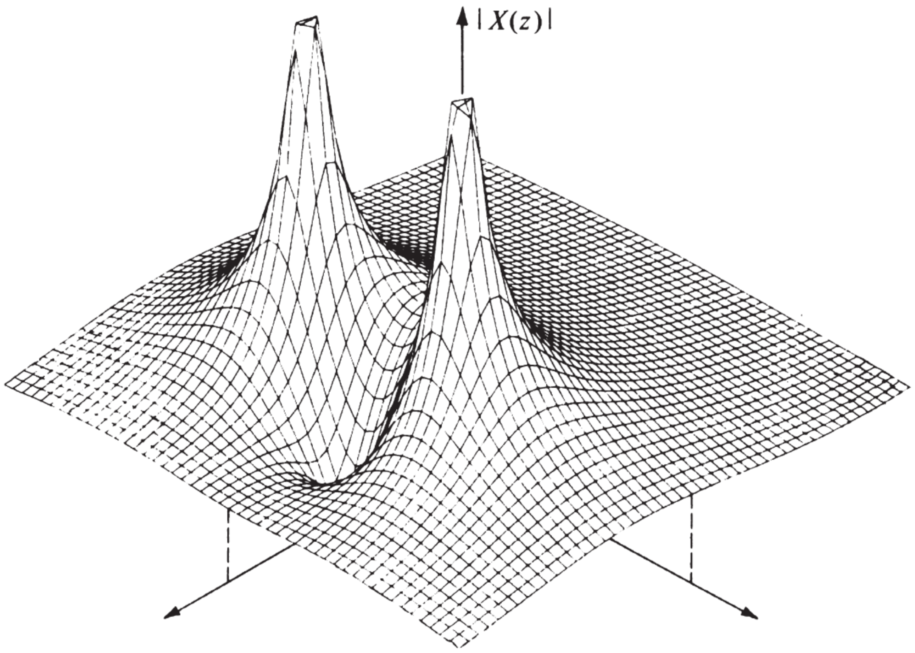


Figure 3.3.4 Graph of $|X(z)|$ for the z-transform in (3.3.3).

...and DTFT just evaluates it along the unit circle

- We can think of evaluating X(z) or H(z) in the complex plane. Then, frequency response is found by tracing out the unit circle

  To calculate figures like this one, could write code like:

```
Npts=100;
Zreal=linspace(-2,2,Npts);
Zimag=linspace(-2,2,Npts);

for izR=1:Npts
  for izI = 1:Npts
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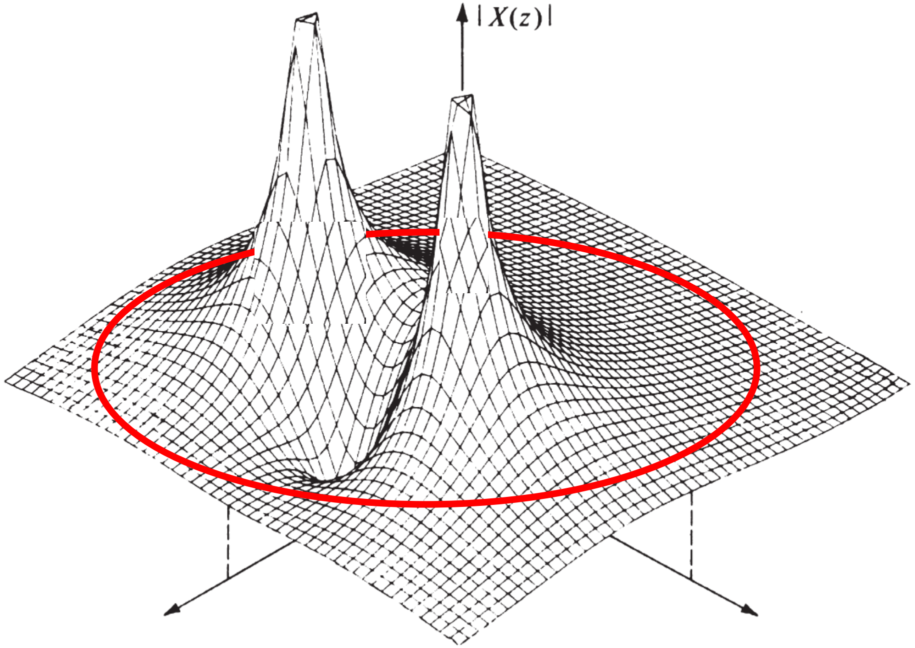

Figure 3.3.4 Graph of $|X(z)|$ for the z-transform in (3.3.3).

Definition of H(ω)

- From the definition of the DTFT,

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

- If h(n) is real, then

$$\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{\infty} h(n)\cos(\omega n) - j\sum_{n=-\infty}^{\infty} h(n)\sin(\omega n) \\ &= H_R(\omega) + jH_I(\omega) \\ &= |H(\omega)|e^{j\theta(\omega)} \end{aligned}$$

where H_R is even (cosine), H_I is odd (sine). This means that H(ω) is conjugate symmetric for real signals

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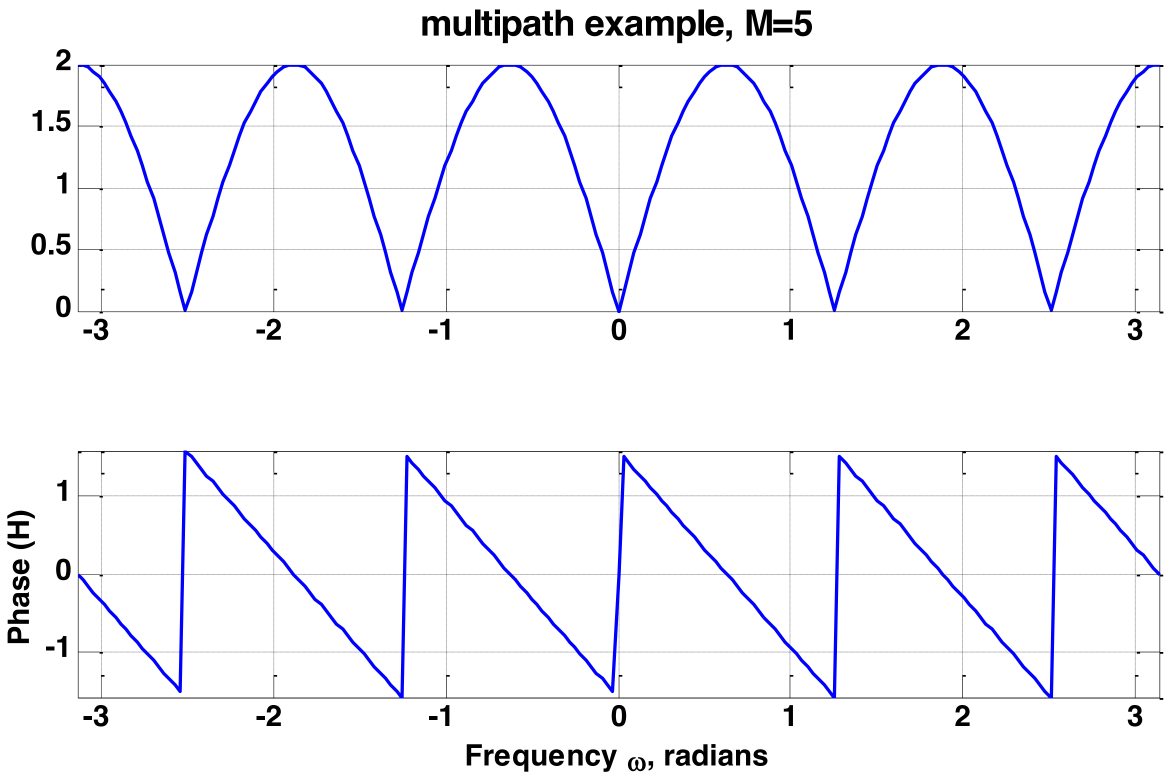
Multipath interference example

- A good model of sound arriving directly from a source + sound bouncing off ocean surface is:

  $y(n)=x(n)-x(n-M)$

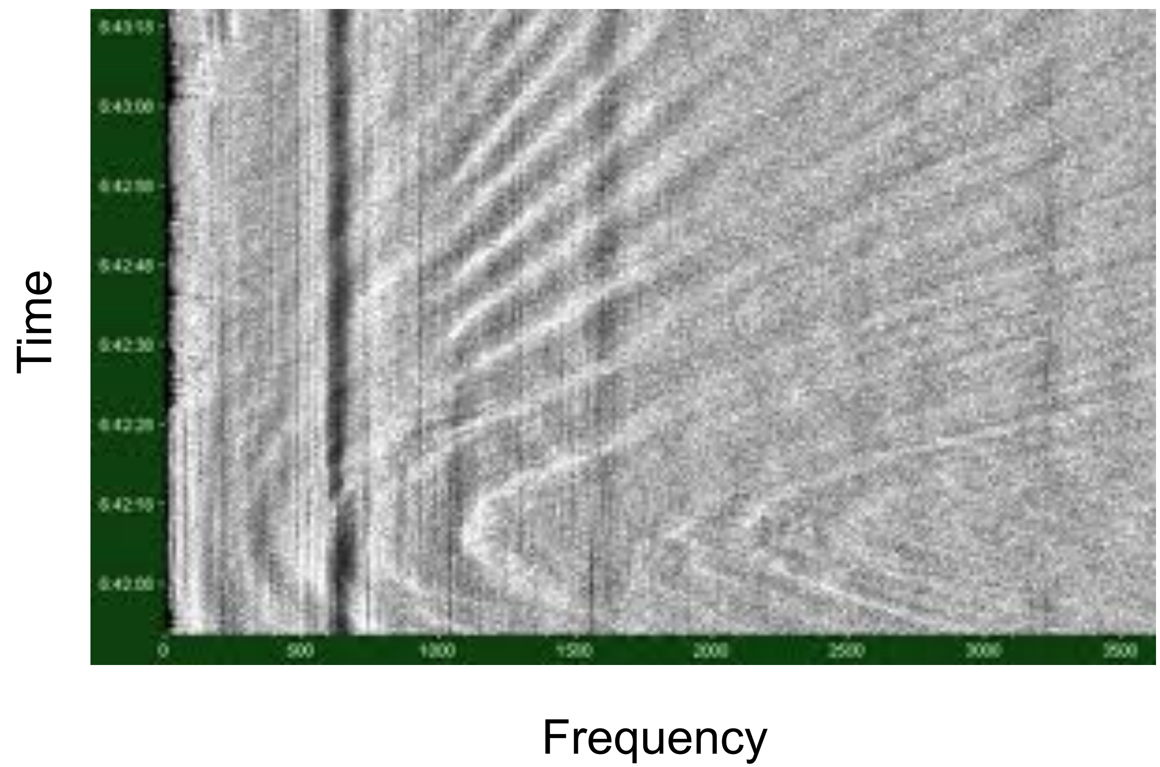
- Calculated H is shown here
  - Periodic "fades" in frequency
  - Note phase jump of pi every time H goes through zero - due to sign change



Real data: ocean multipath interference example

- This plots shows frequency vs. time – basically |Y(ω)|^2 vs time
- Source is moving, so M changes over time

- Source has non-zero X(ω) over many frequencies - fairly constant at higher frequencies
- Thus Y=H X shows multipath fades in frequency, with pattern that changes over time



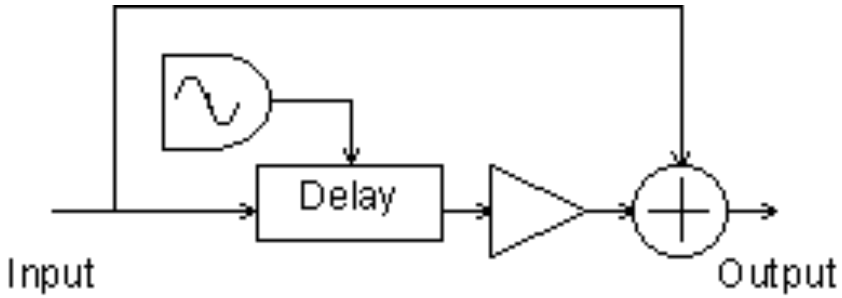
Exact same idea: flanging effect

- See https://en.wikipedia.org/wiki/Flanging

- Insert a time-varying delay into one branch of the signal:

$$y(n) = x(n) + x(n-k(n))$$

Where k(n) is a delay that changes slowly in time – thus boosting or cutting different frequencies



- 1960's implementation: play back a taped recording, with the engineer's finger on the tape to slow it down tiny bit

- 1980's: popular digital effect for lots of bands

H(ω) for rational systems 5.2

- From last lecture, we had H(z) for rational systems; we can just evaluate that result at z = exp(j ω) to give:

$$H(\omega) = Ge^{-j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

- In section 5.2 the book gives the formula:

$$H(\omega) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - p_k e^{j\omega})}$$

This is the same as above, if 1) a0=1 and 2) we don't pull out the positive powers of z

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Geometric interpretation of H(ω)

- The magnitude of H is the *product* of the individual magnitudes:

$$|\ H(\omega)\ | \ = \ |G|\frac{\prod_{k=1}^{M}|e^{j\omega}-z_k|}{\prod_{k=1}^{N}|e^{j\omega}-p_k|}$$

The individual magnitudes are the distances between the pole or zero and points on the unit circle.

- The phase of H is the *sum* of the individual phases:

$$\angle H(\omega) \ = \ \angle G - \omega(N-M) + \sum_{k=1}^{M}\angle\left(e^{j\omega}-z_k\right) - \sum_{k=1}^{N}\angle\left(e^{j\omega}-p_k\right)$$

The individual phases are the angles between each pole or zero and points on the unit circle

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