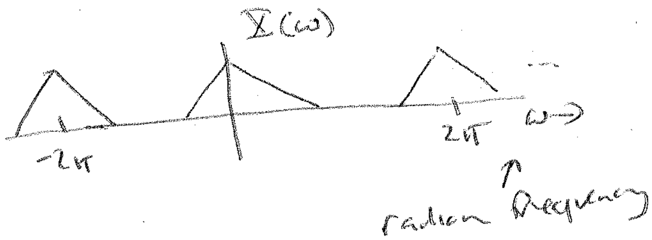
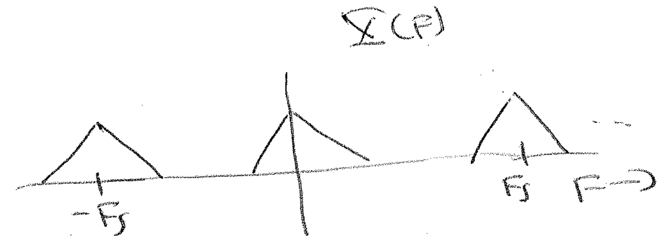
Lecture notes (1)

DFT and sampling in frequency (P+M 7.1)

Let's start with the spectrum of a sampled sequence $x(n)$. Because we've already sampled in time, the spectrum is periodic in frequency. We can draw it as:



or



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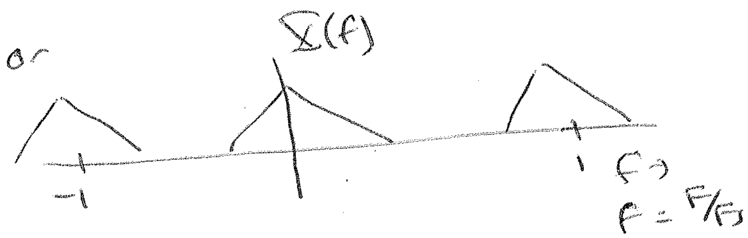


where $\omega = 2\pi (F/F_s)$
$= 2\pi f$

$\Rightarrow$ For understanding sampling in frequency, working in f will be convenient. (Note that the approach in these notes is different from P+M)

Now, let's say we will choose <u>N evenly-spaced points</u> in f. Because $X(f)$ is periodic, over a single period

Sampling a single period is enough – we could pick $[0, 1]$ or $[-1/2, 1/2]$. These samples will be spaced $\frac{1}{N}$ apart.

$\rightarrow$ Sampling these points in frequency is like multiplying $X(f)$ by a delta function to select out these frequencies.

notation shortcut: k^{th} sample of X

thus,
$$X(k) \equiv X(f)\Big|_{(\frac{1}{N})k} = X(f)\cdot \overbrace{\frac{1}{N}\sum_k \delta\left(f-\frac{1}{N}k\right)}^{\text{delta function train}}$$

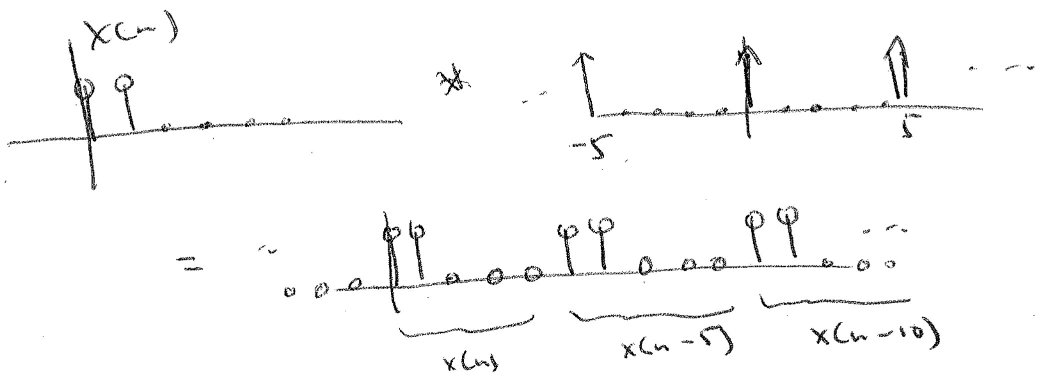
applying the inverse fourier transform, we get
$$x_p(n) = x(n) * \sum_k \delta(n-kN) = \sum_{k=-\infty}^{\infty} x(n-kN)$$

☆ [So sampling evenly in frequency corresponds to making the original signal periodic.

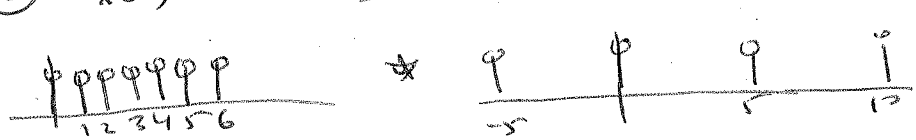
There's a strong link to the Fourier series - if we had a truly periodic signal, it's transform is given by a few coefficients. Here, we <u>make</u> the signal periodic.

<u>Some examples</u>

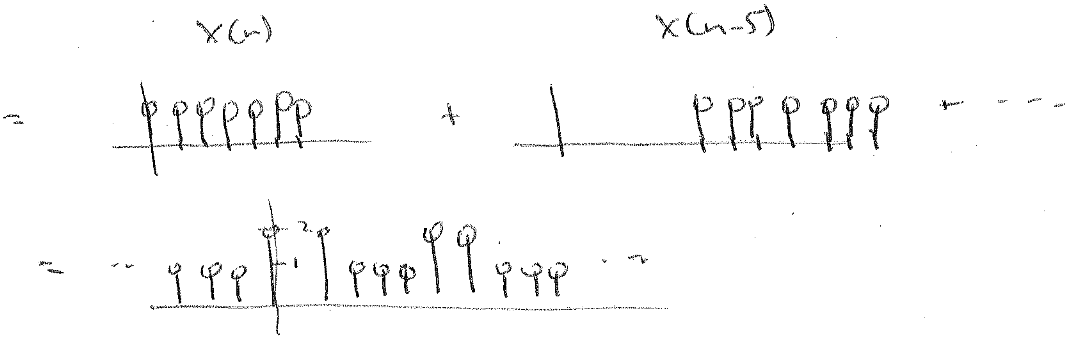
(1) start with a signal of length $L=2$. Sample with $N=5$ points in frequency.



(2) $x(n)$ has length $L=7$. Keep $N=5$ for sampling



= (OVER)



In this case ($N < L$) we get time-domain aliasing — the repeated copies lie on top of each other.

Requirement: To avoid time-domain aliasing, we must sample densely enough in frequency, so (# frequency samples N) $\geq$ (# time samples L)

→ This is similar to sampling in time; avoid aliasing by closer-spaced samples.

→ We'll see later how to handle very long signals — basically, by breaking them into blocks.

Math for finding $x_p(n)$

CTFT has $e^{j2\pi ft}$ but here $t = nT = n/f_s$

From the Fourier transform,

$$x_p(n) = \int_0^1 \underbrace{\left[\frac{1}{N}\sum_{k=-\infty}^{\infty} X(f)\,\delta(f-k/N)\right]}_{\text{samples of } X(f)} e^{+j2\pi fn}\,df$$

since periodic over [0,1]

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty}\int_0^1 X(f)\,e^{j2\pi fn}\,\delta(f-k/N)\,df$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k/N)\,e^{j2\pi \frac{k}{N} n}$$ ← since only N δ's over range $f = [0,1)$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,e^{j2\pi kn/N}$$ ← relabel $X(k)$ as k^{th} sample.

repeating:

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}$$

similarly,

$$X(k) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}$$

DFT pairs page 456 in book

Observation

1) Both $x_p(n)$ and $X(k)$ are described by N numbers (over 1 period)
2) Both are periodic
3) Recovery of ~~x_p~~ $x(n)$ from $x_p(n)$ is easy:
If no aliasing, just take the first L samples!

$$x(n) = \begin{cases} x_p(n), & n = 0, 1, \ldots, L-1 \\ 0 & \text{else} \end{cases}$$

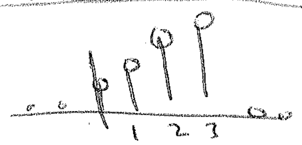
Notation for the rest of the course

1) Consider finite length signals $x(n)$, length L
2) unless explicitly noted, we'll discuss $x(n)$, not $x_p(n)$, with the assumption that we've sampled enough to avoid aliasing,

→ Thus, the DFT pair on p. 456 is in terms of $x(n)$

Relationship between $x(n)$ and $x_p(n)$

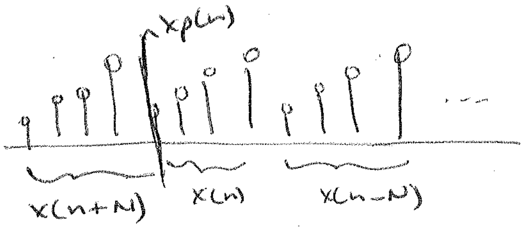
$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$ =



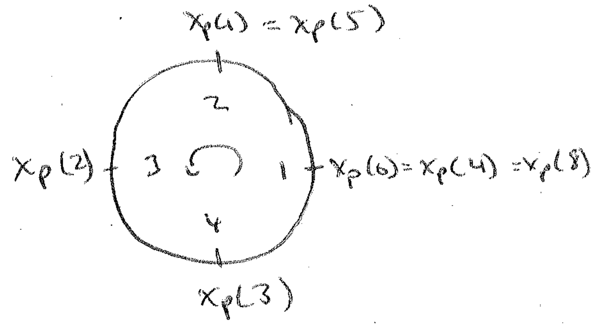
$L=4$

now, say $x(n)$ is one period of a signal $x_p(n)$ that has period $N \geq L$. How do we picture this?

Case 1: $N = L$



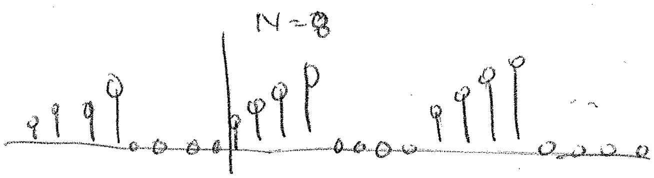
equivalent picture

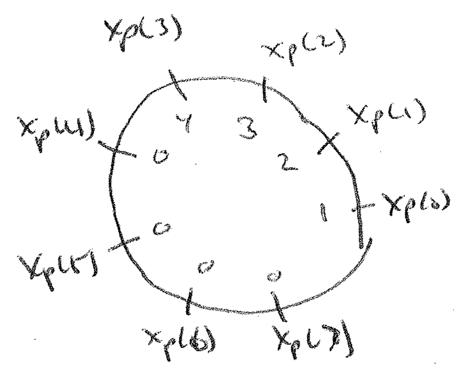


This is a handy way to think of the periodic signal. Time goes counter-clockwise.

Case 2: $N > L$

here, we "pad" $x(n)$ with zeros





why not always $N = L$?

→ sometime have 2 signals of different lengths

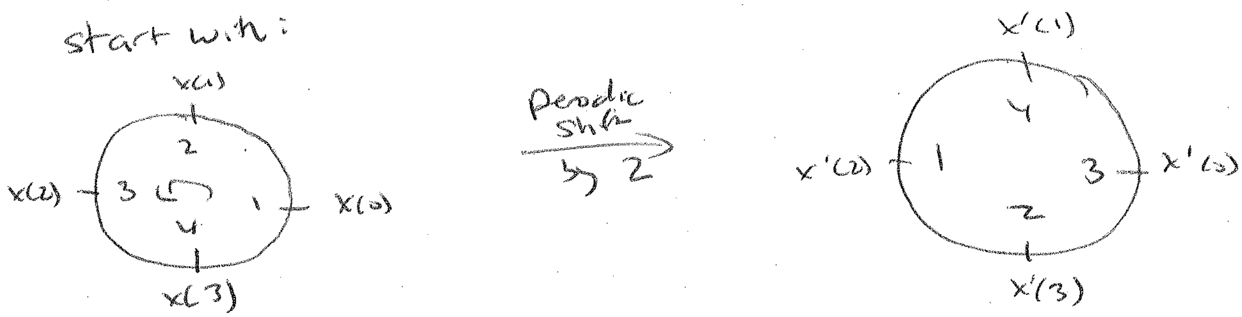
→ see example 7.1.2 in book / class slides

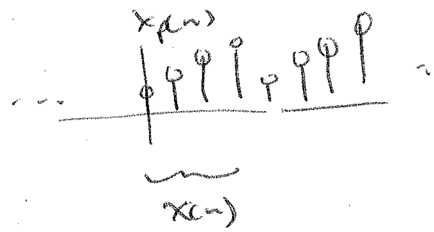
Shifting periodic sequences

→ For linear time shift (used before) samples just slide over.

→ for a periodic time shift, now signals rotate in:

start with:





$x'(n) = \{3, 4, 1, 2\}$

Time reversal – see class ppt