Relating CT and DT frequency for sampled signals

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This is just a cleaner version of the last page of the handwritten notes in `lec4_samplingBoardNotes.pdf`.

Let's call the continuous frequency F (units of Hz, or 1/sec) and the discrete-time frequency ω (units of radians/sample). Picking up from the handwritten notes on page 3, we have that in the frequency domain,

$$\begin{aligned} X_s(F) &= \int_{-\infty}^{\infty} x_s(t) e^{-j2\pi Ft} dt \qquad (0.1) \\ &= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x_a(t)\delta(t - nT) \right] e^{-j2\pi Ft} dt \end{aligned}$$

where the first line is just the CTFT definition and the second line is plugging in for the original signal, multiplied by a pulse train in time (which is a simple math model for what sampling does.)

Then, we can just pull the sum out front:

$$X_s(F) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x_a(t)\delta(t - nT) e^{-j2\pi Ft} dt \qquad (0.2)$$

then use the delta function, which pulls out only times equal to nT from the integral:

$$X_s(F) = \sum_{n=-\infty}^{\infty} x_a(nT) e^{-j2\pi FnT} \qquad (0.3)$$

Now we're just going to play games with notation. For short-hand, let's define $x(n) = x_a(nT)$ as the n^{th} sampled point. Then

$$X_s(F) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi FnT} \qquad (0.4)$$

Next, remember that the sampling period $T = 1/F_s$, where F_s is the sampling frequency. Then,

$$X_s(F) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi(F/F_s)n} \tag{0.5}$$

Now, as a short-hand, let's define a normalized frequency $f = F/F_s$, so we have

$$X_s(F) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn}. \tag{0.6}$$

Ok, so far we have just being playing with rewriting the CTFT. Now, look at the DTFT definition:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}. \tag{0.7}$$

Looking at these, it's pretty clear that they are equal if we set

$$\begin{aligned} \omega &= 2\pi f \\ &= 2\pi(F/F_s) \end{aligned} \tag{0.8}$$

A few interesting frequencies: we know we avoid aliasing for $-F_s/2 < F < F_s/2$. Plugging in, we can see that $-F_s/2$ maps to $\omega = -\pi$, and $F_s/2$ maps to $\omega = \pi$. So, the unaliased region maps to the full unit circle (and any higher frequencies correspond to wrapping around the unit circle in an ambiguous way).