①

Lecture 4 - sampling theory

→ to implement DSP algorithms on a computer, we can only work w/ a finite # of samples.

→ how to do this? a) sample the CT signal

$x(t) \Rightarrow x(n) = x(\underset{\substack{\uparrow \\ \text{sampling period } T}}{nT})$

ex if $T = 0.1$ sec,

~~$x(0).\ x(3)$~~ $x(0), x(.1), x(.2), \ldots$

b) consider a finite # of samples

$x(n) \quad n = 0, 1, 3 \ldots N-1$

CT Fourier review. - building blocks needed

Recall in CT: ① $x_1(t) * x_2(t) \Longleftrightarrow X_1(\Omega) \cdot X_2(\Omega)$

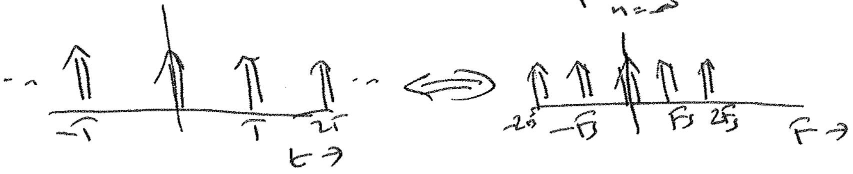
convolution ⟺ multiplication

② $x_1(t)\, x_2(t) \Longleftrightarrow X_1(\Omega) * X_2(\Omega)$

$\Omega = 2\pi F$

also, state without proof:

③ $D(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \Leftrightarrow D(F) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(F - nF_s)$



these are CT version of the DT properties we just discussed

DT property:

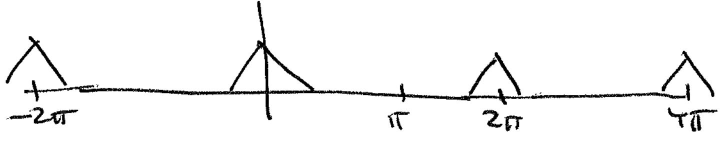
$X(\omega)$ is 2π periodic

$X(\omega) = X(\omega + 2\pi k)$

def:

$X(\omega) = \sum_n x(n) e^{-j\omega n}$

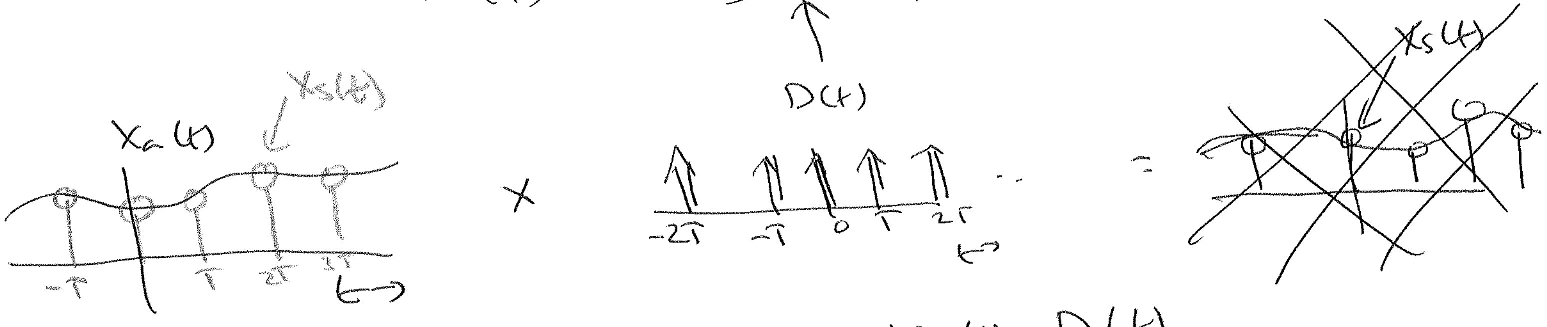
$X(\omega + 2\pi) = \sum x(n) e^{-j\omega n} \cancel{e^{-j2\pi n}}$



now, we can start sampling

$$x_a(t) \longrightarrow \boxed{\text{sampler}} \longrightarrow x(n) = x_a(nT)$$

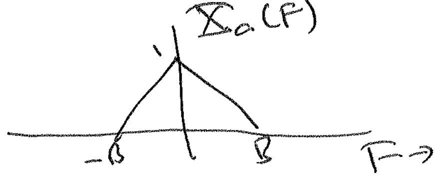
$$\begin{tikzcd} x_a(t) \arrow[r] & \otimes \arrow[r] & x_s(t) \\ & D(t) \arrow[u] & \end{tikzcd}$$



$$x_s(t) = x_a(t)\cdot D(t)$$
$$= \sum_{n=-\infty}^{\infty} x_a(t)\,\delta(t-nT)$$
$$= \begin{cases} x_a(nT), & t = nT \\ 0 & \text{else} \end{cases}$$

Q1) Ok, what happens in frequency domain?

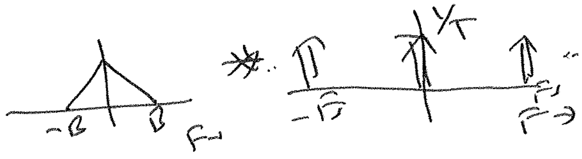
let's say our signal has non-zero energy over some frequency region.



By conv ⟺ mult duality

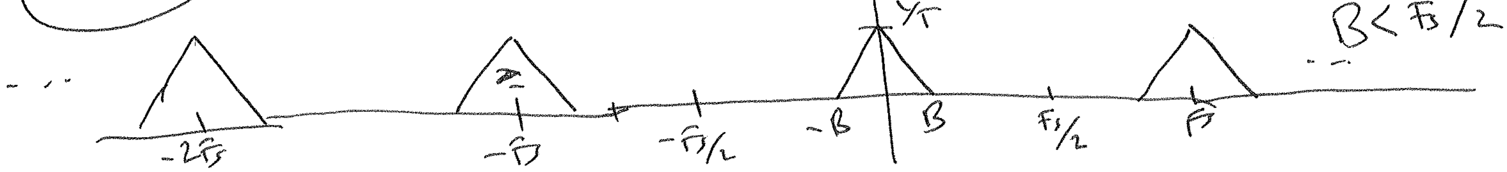
$$X_s(F) = X_a(F) * D(F)$$

=

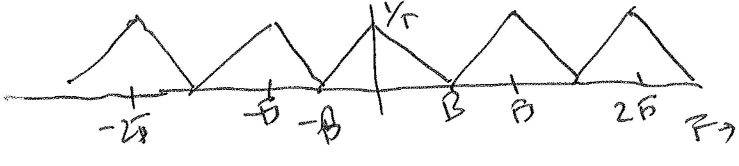


Case 1

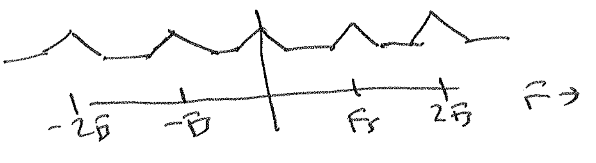
$B < F_s/2$



Case 2 $B = F_s/2$



Case 3 $B > F_s/2$



So, no overlap if $F_s > 2B$

this is the Nyquist criterion; we will see why it's important when we do reconstruction

Question 2) we use the CTFT to represent CT signals
DTFT to " DT signals

How is F (CT frequency, Hz) related to ω (DT freq, radians/sec?)

need to do math...

CTFT: $$X_s(F) = \int_{-\infty}^{\infty} x_s(t) e^{-j2\pi F t} dt$$

$$= \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} x_a(t) \delta(t-nT) \right] e^{-j2\pi F t} dt$$

$$X_s(F) = \sum_{n=-\infty}^{\infty} \int x_a(t)\, e^{-j2\pi F t}\, \delta(t-nT)\, dt \quad \text{(rearrange)}$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT)\, e^{-j2\pi F T n} \quad \text{(use } \delta \text{ function)}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi (F/F_s) n} \quad (\star) \quad \left(\begin{array}{l}\text{use:} \\ x(n) = x_a(nT) \\ T = 1/F_s\end{array}\right)$$

↑ used on p. 3

$$= \sum_{n=-\infty}^{\infty} X(n)\, e^{-j2\pi f n}$$

In the last line, we have defined a normalized frequency $f = F/F_s$.

So far, this is all CTFT. Compare to the DTFT:

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

equating them, we get
$$\omega = 2\pi f$$
$$= 2\pi\, F/F_s$$

Some interesting points: Nyquist frequency is $F = F_s/2$ (this is the highest frequency w/o aliasing)

From the formula, $F = F_s/2$ gives:

$$\omega = 2\pi \frac{F_s/2}{F_s} = \pi \;//$$

similarly, $F = -F_s/2 \Rightarrow \omega = -\pi$