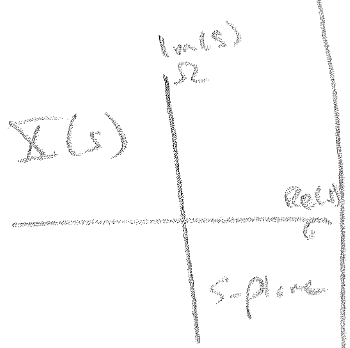
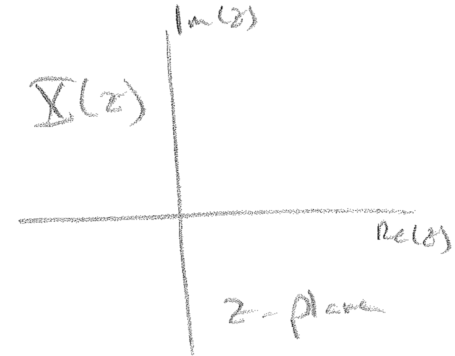
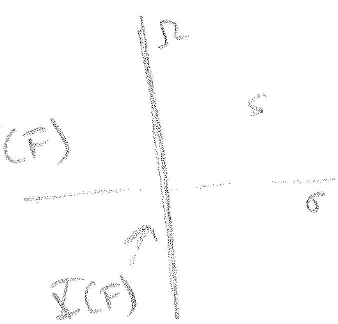
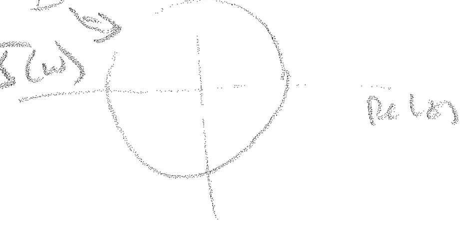
z transform

High-level

The z-transform is like the Laplace transform, but for discrete-time signals $x(n)$ instead of CT signals $x(t)$

See PPT slides – used to solve difference equations

Laplace	z-transform
$x(t) \underset{\text{back}}{\overset{\text{Fwd}}{\rightleftarrows}} X(s)$ ($\mathcal{L}$)	$x(n) \overset{\text{z-xform}}{\rightleftarrows} X(z)$
	
$s = \sigma + j\Omega$	
if we just care about frequency response look on y axis	If we just care about frequency response, look on unit circle (we'll see why soon)
$x(t) \rightleftarrows X(F)$	$x(n) \rightleftarrows X(\omega)$
CT Fourier transform	DTFT
CTFT	$X(\omega)$ defined on axis
	
$X(F)$ or $X(\Omega)$ here ($\Omega = 2\pi F$)	

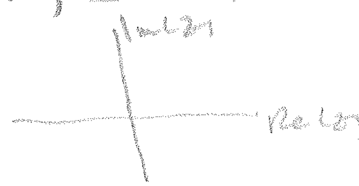
z-transform definition & Region of convergence (ROC)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \cdots$$
$$+ x(-1) z + x(-2) z^{2} + \cdots$$

a power series in z^{-1} with $x(n)$ as the coefficients //

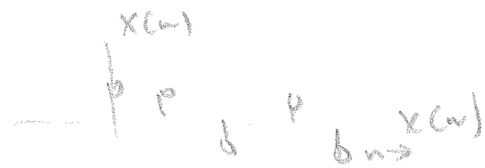
⇒ z is a complex number $x = Re(z) + j\, Im(z)$



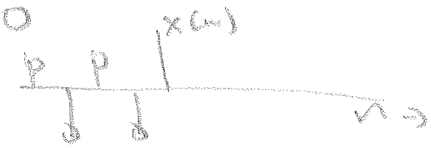
⇒ ROC are the collection of points in the z-plane where $|X(z)| < \infty$

Quick review: types of signals / system responses

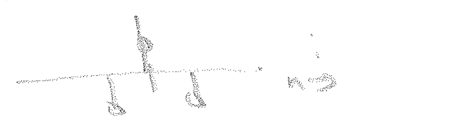
1) Causal signals: $x(n) = 0$ for $n < 0$



2) anti-causal signals: $x(n) = 0$ for $n \geq 0$



3) non-causal or two-sided: non zero on both sides



example z-transform: $\boxed{\text{finite-length sequence}}$

$$x_1(n) = [\underset{\uparrow}{3}, 42, 1, -17, 0.1]$$

$$X_1(z) = 3 + 42 z^{-1} + z^{-2} - 17 z^{-3} + 0.1 z^{-4}$$

ROC: $|X_1(z)|$ is finite everywhere except $z = 0$

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Same signal, shifted in time:

$$x_2(n) = [3, 42, \underset{\downarrow}{1}, -17, 0.1]$$

$$X_2(z) = 3z^2 + 42z + 1 - 17z^{-1} + 0.1z^{-2}$$

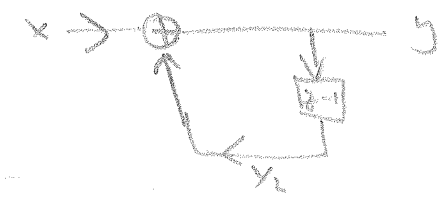
ROC: finite everywhere except $z=0$, $z=\infty$

$\boxed{\text{Result}}$ For finite-length sequence ROC is the whole z-plane except <u>possibly</u> $z=0$ and $z=\infty$

$\boxed{\text{Infinite-length sequences}}$

$$y(n) = (\tfrac{1}{2})^n u(n)$$

← we saw this in CCDE example: <u>impulse</u> response of



$$Y(z) = 1 + \tfrac{1}{2}z^{-1} + \tfrac{1}{4}z^{-2} + \tfrac{1}{8}z^{-3} + \ldots$$

$$= 1 + A + A^2 + A^3 + \ldots$$

if we define $A = \tfrac{1}{2}z^{-1}$

<u>Math identity</u>: $1 + A + A^2 + \ldots = \dfrac{1}{1-A}$ if $|A| < 1$

(what if $A \geq 1$?)

So, $Y(z) = \dfrac{1}{1 - \tfrac{1}{2}z^{-1}}$, ROC is $|z| > \tfrac{1}{2}$

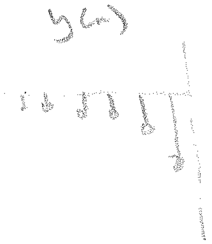
Why is this the ROC? need $|A| < 1$

$$|\tfrac{1}{2}z^{-1}| < 1$$

$$\tfrac{1}{2}\frac{1}{|z|} < 1$$

$$|z| > \tfrac{1}{2}$$

<u>anti-causal systems</u> : follow similar approach. See example 3.1.4 in book, find



$$y(n) = -\left(\tfrac{1}{2}\right)^n u(-n-1) \Longleftrightarrow Y(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

ROC $|z| < 1/2$

same $Y(z)$ (!) but different ROC

<u>takeaways</u>

→ always need to specify ROC as well as $Y(z)$

→ Distinct patterns to ROC

1) finite length: ROC is all z except possibly $z=0$ & $|z|=\infty$

2) infinite, causal: ROC is $|z| > r_1$ r_1 positive (outside of a circle).

3) infinite, anticausal, ROC is $|z| < r_2$ r_2 positive (inside of circle)

4) non-causal: ROC is annulus $r_1 < |z| < r_2$

why is 4) true?

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

need both parts finite! $r_2 < |z| < r_1$ ⇐

so $$|X(z)| \le \left|\sum_{-\infty}^{-1} x(n) z^{-n}\right| + \left|\sum_{n=0}^{\infty} x(n) z^{-n}\right|$$

anti causal ROC $|z| < r_1$

causal ROC $|z| > r_2$

Q) : what is a finite length signal whose ROC is entire z-plane

z-transform properties

1) linearity

$$Z\{a x_1(n) + b x_2(n)\} \longrightarrow a Z(x_1(n)) + b Z(x_2(n))$$

$$= a X_1(z) + b X_2(z)$$

2) time shift $\quad Z(x(n-k)) = z^{-k} X(z)$

ex2) $y(n) = (1/2)^n u(n) \iff \dfrac{1}{1-\frac{1}{2}z^{-1}}$ ROC $|z| > 1/2$

so, $y(n-2) \iff z^{-2} \dfrac{1}{1-\frac{1}{2}z^{-1}}$, ROC $|z| > 1/2$ and $z \neq 0$

Proof: $X(z) = \sum_n x(n) z^{-n}$ (definition)

let $y(n) = x(n-k)$

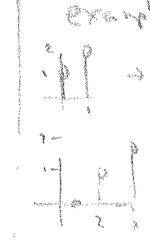
so, $Y(z) = \sum_{n=-\infty}^{\infty} x(n-k) z^{-n}$

let $m = n-k \implies n = m+k$

then, $Y(z) = \sum_{m=-\infty}^{\infty} x(m) z^{-(m+k)}$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x(m) z^{-m} = z^{-k} X(z) \;//$$

ex1) example



$\rightarrow 1 + 2z^{-1} = X(z)$

$\rightarrow z^{-2} + 2z^{-3} = z^{-2} X(z)$

3) time reversal $\quad Z(x(-k)) = X(z^{-1})$

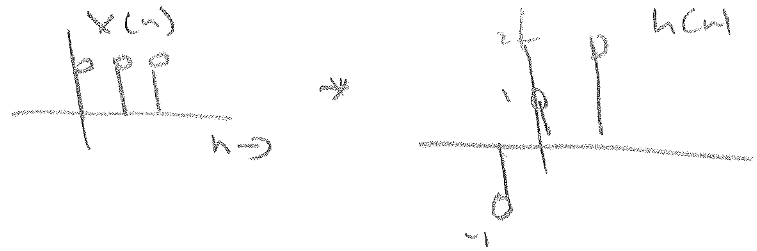
intuition: think about flipping a finite-length sequence. coefficients of $z \rightarrow$ coeff of z^{-1}, etc

4) convolution (important)

$$Z\{x_1(n) * x_2(n)\} = X_1(z)\, X_2(z)$$

→ proof in book

example:



$$X(z) = 1 + z^{-1} + z^{-2}$$

$$H(z) = -z + 1 + 2z^{-1}$$

$$\begin{array}{rrrrrrr}
X(z)\,H(z) = & -z & +1 & +2z^{-1} & & \\
 & & -1 & z^{-1} & +2z^{-2} & \\
 & & & -z^{-1} & +z^{-2} & +2z^{-3} \\
\hline
Y(z) & -z & +0 & +2z^{-1} & +3z^{-2} & +2z^{-3}
\end{array}$$

by inspection

$$y(n) = \{-1, \underset{\uparrow}{0}, 2, 3, 2\}$$

which matches regular convolution result

A few questions

① what is $X(z)$ if $x(n) = 2\delta(n-4)$?

2 ways: properties or direct computation

② what is $X(z)$ if $x(n) = u(n)$?

$a^n u(n) \Longleftrightarrow \frac{1}{1-az^{-1}} \quad |z|>a$

set $a=1$

$x(n) = u(n) \Longleftrightarrow X(z) \; \frac{1}{1-z^{-1}} \quad |z|>1$