FIR filter design

I) What structures give us linear phase filters?

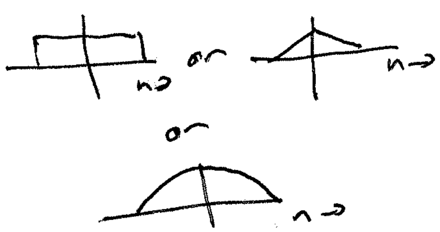
Recall Fourier transform properties:

1) real + even $h(n)$ $\Longleftrightarrow$ real $H(\omega) = H_R(\omega)$

2) real + odd $h(n)$ $\Longleftrightarrow$ imaginary $H(\omega) = H_I(\omega)$

3) n-sample delay $\Longleftrightarrow$ $e^{-j\omega n}$ phase shift

Symmetric filters



$\Longleftrightarrow$ real $H(\omega)$

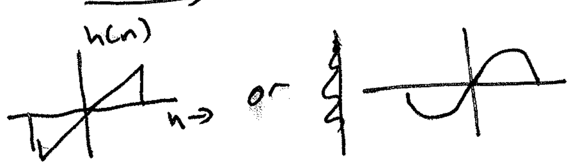
If we shift these filters to be causal (start at $n=0$) by half-width, time shift gives us linear phase

if M samples, M even, shift by $M/2$

$$H(\omega) = e^{-j\omega M/2} H_R(\omega)$$

if M odd, shift by $(M-1)/2$ – similar

Antisymmetric filters



$\Longleftrightarrow$ purely imaginary $H(\omega)$

if shift by $M/2$ to make causal (M even case)

$$H(\omega) = e^{-j\omega M/2} H_I(\omega)$$

but, if purely imaginary, we can ~~put out~~ write

$$H_I(\omega) = j\, H_R(\omega) = e^{j\pi/2} H_R(\omega)$$

then,
$$H(\omega) = e^{j(\pi/2 - (M - \frac{1}{2})\omega)} H_R(\omega)$$

this is <u>generalized</u> linear phase

Still gives constant group delay.

thus in general, $H(\omega) = H_R(\omega)\, e^{j(\beta\pi/2 - \omega M/2)}$ M even

$= H_R(\omega)\, e^{j(\beta\pi/2 - \omega\frac{(M-1)}{2})}$ M odd

for even symmetry, $\beta = 0$; for odd, $\beta = 1$

even symmetry is most common.

Roots (zeros) of $H(z)$

→ because $h(n)$ is real, zeros are conjugate symmetric; so on unit circle, we get z_0 and z_0^*

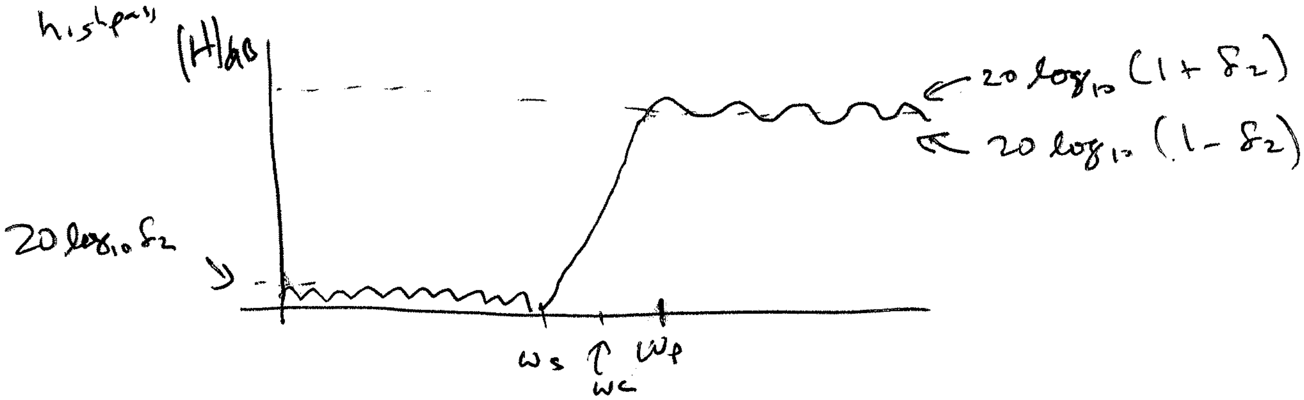
→ because of symmetry, it turns out that if z_0 is a root, $1/z_0$, z_0^*, $1/z_0^*$ are also roots.

↑ by conj.

→ Math: book, 10.2.1

→ intuition: if symmetric, we should be able to time-reverse w/o changing answer; time reversal turns a z_0 into $1/z_0$

filter specifications:



in dB: $|H|_{dB} = 20\log_{10}(|H(\omega)|)$ (freqz plots this)

power

Some filters are more naturally specified in terms of a cutoff frequency ω_c; this is the effective edge of the filter. We could say, for example,

$\omega_c = \frac{\omega_p + \omega_s}{2}$; do mainly by trial & error.

Get different behavior for even & odd →

for odd-valued n, $H(\pi) = 0$

for antisymmetric, $H(0) = 0$

Now that we have structures in place, we can go on to <u>filter design</u>

<u>Theme</u>:
1. we have some desired frequency response $H_d(\omega)$
2. we choose a structure for the filter that will give us a FIR filter w/ constant group delay
3. we pick coefficients (weights) to approximate $H_d(\omega)$

Three main methods
- → windowing
- → frequency sampling
- → optimizing a cost function

<u>Windowing</u>

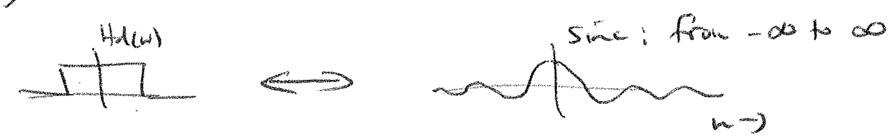
we specify some desired response $H_d(\omega)$; then we can find filter from the inverse transform

If $$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n)\, e^{-j\omega n}$$ ← $H_d(\omega)$ is F.T. of some causal impulse response

↑ causal!

then
$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

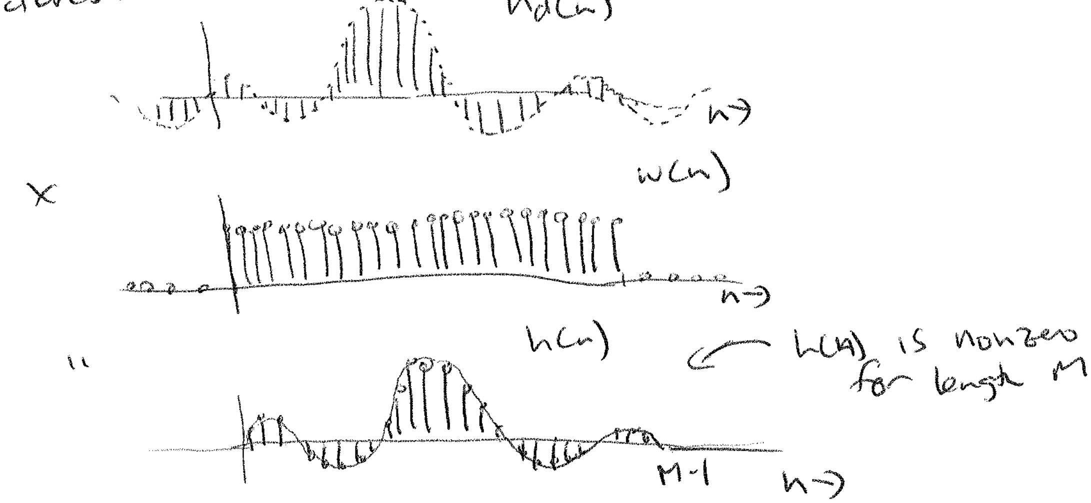
<u>Problem</u>: in general, finding $h_d(n)$ from above will give us an <u>infinitely long</u> $h_d(n)$

example: 

Solution: We truncate $h_d(n)$ by multiplying it by a window function $w(n)$ of finite length M.

then $h(n) = h_d(n)\, w(n) \quad , \; 0 \le n \le M-1$

in pictures:



What does this do in frequency domain?

We can transform the window:

$$W(\omega) = \sum_{n=0}^{M-1} w(n)\, e^{-j\omega n}$$

(above is a causal window: we could also transform a noncausal window, then shift it)

now,

$$h_d(n)\, w(n) \Longleftrightarrow H_d(\omega) * W(\omega)$$

so $$H(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\lambda)\, W(\omega-\lambda)\, d\lambda$$

linear phase is preserved, if $W(\omega)$ is linear phase

is a smoothed-out version of $H_d(\omega)$

If $H_d(\omega)$ & $W(\omega)$ are both linear phase $\left(H_d = H_r(\omega)\, e^{-j\omega M/2}, \; W = W_r e^{-j\omega M/2}\right)$

then we can plug in, find

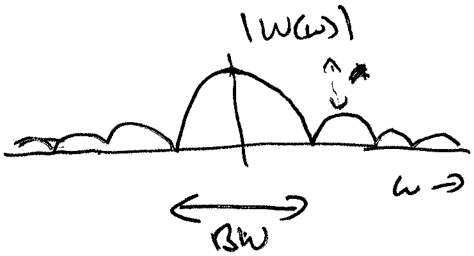
$$H(\omega) = e^{-j\omega M/2} \frac{1}{2\pi} \int H_r(\omega)\, W_r(\omega-\lambda)\, d\lambda$$

Ex): rectangular window:

$$W(\omega) = e^{-j\omega(M-1)/2}\,\frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

$$|W(\omega)| = \left|\frac{\sin \omega M/2}{\sin \omega/2}\right|$$

main lobe width:



BW ~ 2× distance to first zero

1st zero: when $\omega M/2 = \pi$

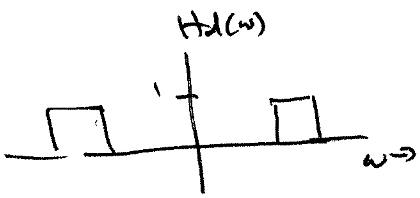
$\omega = 2\pi/M$

so $BW \sim 4\pi/M$

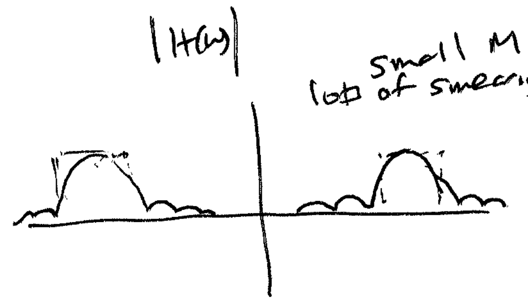
sidelobe height: independent of M, ratio of main lobe to sidelobe ~ −13 dB.

Consider bandpass filter:

$H(\omega) = H_d(\omega) * \text{sinc}$



$*$ sinc =



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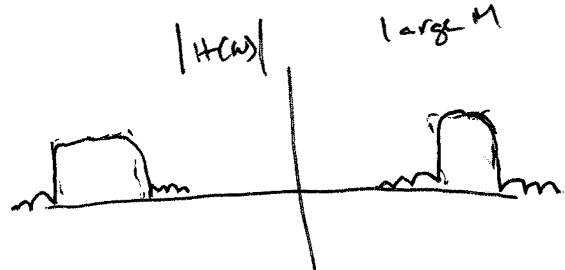


edges better preserved
less smearing,
still high
sidelobes.

see PPT for discussion of window tradeoffs:
tradeoff narrow mainlobe vs. lower sidelobes.

One step beyond the rectangular window:

Bartlett or triangular window

note a triangle is the convolution of two boxcars, with half the length:

In the frequency domain, its the $|\text{sinc}|^2$

$$|W_{tri,M}(\omega)| = \frac{|\sin \omega M_1/2|^2}{|\sin \omega/2|^2}$$

here, $M = 2M_1 - 1$

Review

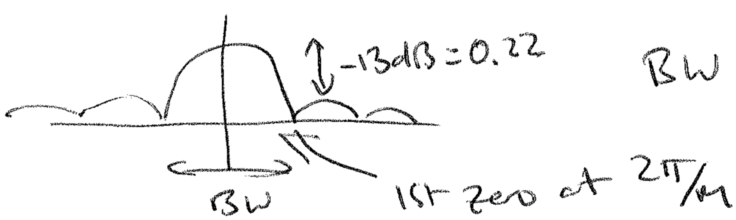
→ slides: review filter spes

→ slides: review basic idea of windowing

most basic window: rectangular

$$W(\omega) = e^{-j\omega(M-1)/2} \frac{\sin \omega M/2}{\sin \omega/2}$$

$$|W(\omega)| = \frac{|\sin \omega M/2|}{|\sin \omega/2|}$$

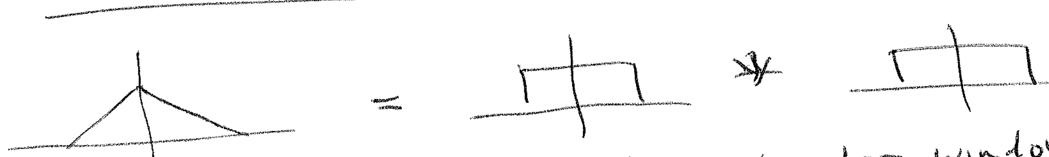


$BW \approx 4\pi/M$

bigger M → smaller mainlobe

but no change in sidelobe ratio

another window: Bartlett / rectangular



So a length M (odd) rectangular window comes from convolving two $\left(\frac{M+1}{2}\right)$ rectangular windows

odd: two $\frac{M+1}{2}$ rectangular windows give M-length Bartlett odd

frequency response is $$\frac{|\sin c\, \omega \left(\frac{M+1}{2}\right)/2|^2}{|\sin \omega/2|^2}$$

sidelobe at -26 dB, mainlobe twice as wide

Filter design examples

windowDesign.m

rectangular: note that window sidelobes at -13 dB, but filter sidelobes may be different - depends on convolution

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Kaiser window

$$w(n) = \begin{cases} I_0\left(\beta\left(1-\left(\frac{n-\alpha}{\alpha}\right)^2\right)^{1/2}\right) / I_0(\beta) & 0 \le n \le M-1 \\ 0 & \text{else} \end{cases}$$

$\alpha = M/2$

I_0 = Bessel function

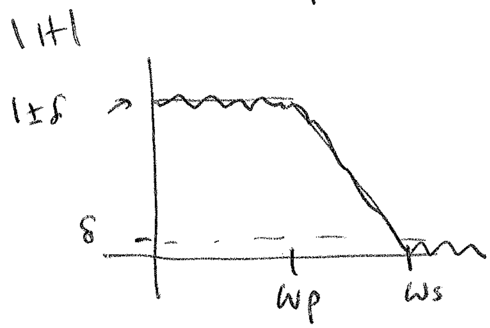
Two parameters:

window length M → gives narrower main lobe as $M \uparrow$

β: higher β widens the main lobe & decreases sidelobes.

This window is nice as the tradeoff is explicit

Also, parameters can be empirically related to low pass design parameters:



ω_p = largest ω s.t. $|H(\omega)| \ge 1-\delta$

set $\Delta\omega$ = transition region = $\omega_s - \omega_p$

then $A = -20\log_{10}\delta$

① $\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.58(A-21)^{0.4} + 0.07(A-21) & 21 < A < 5 \\ 0 & A < 21 \end{cases}$

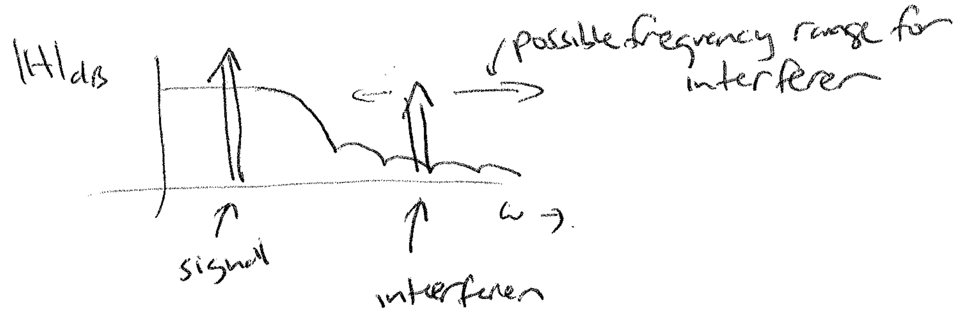
② $M = \frac{A-8}{2.28\,\Delta\omega}$

↑ see matlab help

Filter design scenarios

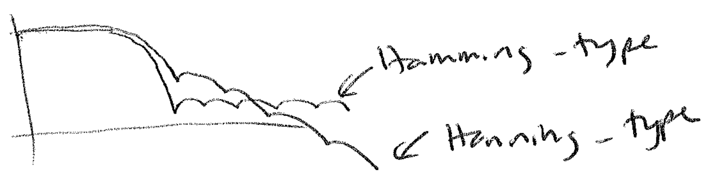
1) • <u>narrowband interferers</u>

We may have the case that our signals are in a known band, and there may be other strong signals fairly nearby



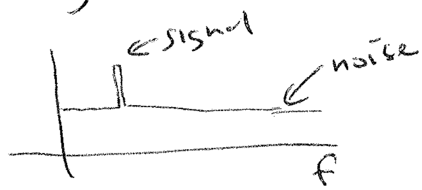
In this case we may want to design for the worst-case; interferer close to the pass band

→ want a very quick dropoff; Hamming or similar rather than Hanning



→ want a fast transition; means large M

2) Signal in broadband noise



Here, minimizing the attenuation across the full stopband may be key; continuous falloff (Hanning-style) is attractive