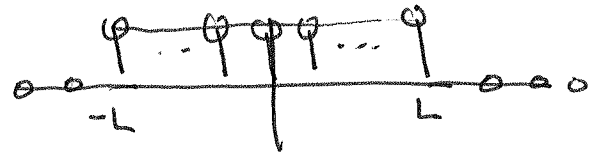
Transform pair: boxcar in time → sinc in ω

Say we have the signal $x(n)$:

$$x(n) = \begin{cases} 1, & |n| \le L \\ 0 & \text{else} \end{cases}$$

or,



"boxcar"

What is $X(\omega)$? From DTFT definition,

$$\underbrace{X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}}_{\text{definition}} = \underbrace{\sum_{n=-L}^{L} (1)\, e^{-j\omega n}}_{\text{this case}} \quad \text{(Eq. 1)}$$

now, we need a geometric series result!
for some #'s a and r, it's a fact that

$$\sum_{k=0}^{N-1} a r^k = a\left(\frac{1-r^N}{1-r}\right)$$

let's set $a=1$, and $r = e^{-j\omega}$. Then we get

$$\sum_{k=0}^{N-1} e^{-j\omega k} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \quad \text{(Eq 2)}$$

this is close to Eq 1, but limits are different (actually this is Ex 4.2.4 in book)

→ we need to shift the limits in Eq 2 to match Eq 1 define a new variable : $\boxed{m = n + L}$

thus, $n = m - L$.

In Eq 1, we summed from $n = -L$ to $n = L$

$n = -L \longrightarrow m = (-L) + L = 0$

$n = L \longrightarrow m = L + L = 2L$

plus in the 3 circled results, and Eq 1 becomes:

$$X(\omega) = \sum_{m=0}^{2L} e^{-j\omega(m-L)} = e^{j\omega L} \sum_{m=0}^{2L} e^{-j\omega m}$$

This is in form of Eq 2! so, we get

$$X(\omega) = e^{j\omega L} \left(\frac{1 - e^{-j\omega(2L+1)}}{1 - e^{-j\omega}} \right) \qquad (\text{Eq } 3)$$

$$\cancel{= e^{j\omega L}\; e^{-j\omega(2L+1)/2}}$$

simplification steps

★ Here is a useful math "trick" for EE-125

$$1 - e^{-j\omega k} = e^{-j\omega k/2} \left(e^{+j\omega k/2} - e^{-j\omega k/2} \right)$$

$$= e^{-j\omega k/2} \, \frac{2j}{2j} \left(e^{j\omega k/2} - e^{-j\omega k/2} \right)$$

$$= 2j \, e^{-j\omega k/2} \sin(\omega k/2)$$

We can use this trick to solve Eq 3, by turning top & bottom into sin functions. It's just a lot of algebra:

$$X(\omega) = e^{j\omega L}\left(\frac{1 - e^{-j\omega(2L+1)}}{1 - e^{-j\omega}}\right) \qquad \text{(repeat Eq 3)}$$

$$= e^{j\omega L}\,\frac{e^{-j\omega(2L+1)/2}\left(e^{j\omega(2L+1)/2} - e^{-j\omega(2L+1)/2}\right)}{e^{-j\omega/2}\left(e^{+j\omega/2} - e^{-j\omega/2}\right)}$$

$$= \left(e^{j\omega L}\, e^{-j\omega(L+1/2)}\, e^{j\omega/2}\right)\frac{e^{j\omega(L+1/2)} - e^{-j\omega(L+1/2)}}{e^{j\omega/2} - e^{-j\omega/2}} \qquad \text{use } \frac{2L+1}{2} = L+1/2$$

$$= \left(e^{j\omega L}\, e^{-j\omega L}\, e^{-j\omega/2}\, e^{+j\omega/2}\right)\frac{2j}{2j}\,\frac{\left(e^{j\omega(L+1/2)} - e^{-j\omega(L+1/2)}\right)}{\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

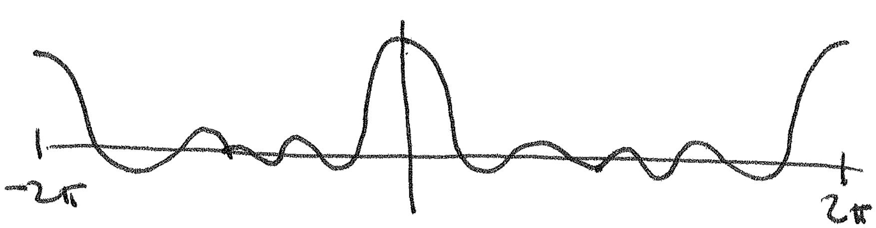
$$= \cancel{e^{0}}^{\,1}\;\frac{\sin \omega(L+1/2)}{\sin \omega/2} \qquad // \qquad \text{phew!}$$

notice: top $\to 0$ when $\omega(L+1/2) = n\pi$

bot $\to 0$ when $\omega/2 = n\pi$

or $\omega = n\,2\pi$

so we get 0/0, or a peak, every 2π



$\Leftarrow X(\omega)$ is 2π periodic, as required