Where the DFT formulas come from

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This is just a cleaner version of the last page of the handwritten board notes.

Let's call the continuous frequency F (units of Hz, or 1/sec) and the discrete-time frequency ω (units of radians/sample).

Finding X(k)

Let's not try a 'derivation' here - the link to the DTFT is pretty close.

The DTFT for $X(\omega)$ is defined as

$$X(\omega) = \sum_{n=0}^{N-1} x(n)exp(-j\omega n)$$

where we can write $\omega = 2\pi F \ F_s$, where F and F_s are in Hz. Note that you'll often see a normalized frequency defined (we did this way back when talking about sampling) as $f = F/F_s$.

The sampled spectrum is periodic every 2π in terms of ω, every F_s in in terms of frequency F in Hz, or has a period of 1 in terms of f.

Let's say we sample the spectrum at N evenly spaced points in frequency. We call each frequency sample a 'bin', and label it with index k.

In terms of ω, these N samples will be spaced at $2\pi/N$ apart. Thus, the k-th bin is at

$$\omega_k = k(2\pi/N)$$

Plugging this particular ω into the DTFT formula we get

$$X(\omega_k) = \sum_{n=0}^{N-1} x(n) exp(-j2\pi kn/N)$$

or for shorthand,

$$X(k) = \sum_{n=0}^{N-1} x(n) exp(-j2\pi kn/N)$$

1 DFT derivation: finding $\mathbf{x}_p(n)$

From the CTFT, we have

$$\begin{aligned}
x_p(n) &= \int_0^1 \left[\frac{1}{N} \sum_{k=0}^{N-1} X(f) \delta(f - \frac{k}{N}) \right] e^{+j2\pi fn} df && (1.1)\\
&= \frac{1}{N} \sum_{k=0}^{N-1} \int_0^1 X(f) e^{+j2\pi fn} \; \delta(f - \frac{k}{N}) df && (1.2)\\
&= \frac{1}{N} \sum_{k=0}^{N-1} X(\frac{k}{N}) e^{+j2\pi \frac{k}{N} n} && (1.3)\\
&= \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{+j2\pi kn/N} && (1.4)
\end{aligned}$$