Lecture # outline: Frequency –domain analysis of LTI systems P&M 5.1-5.2

This lecture assumes we have already:

- Covered pole/zero plots
- Studied rational functions in Z transform
- Covered relationship between Z transform, DTFT

After learning this material, students should be able to:

- Calculate magnitude and phase response of H(w)
- Find frequency response of a rational function and have intuition from graphical arguments
- Understand how many systems can lead to same |H(w)| and how this relates to system identification

Skills that will be used in later work include:

- Mag and phase – many times
- Understanding of ambiguity in |H| - min phase discussions

Outline

0) Recap H(z) from last time
    a. Note benefit of pulling out the positive powers: H=1/(1+.5 z^-1)
1) H(w) basics (start 5.1)
    a. Definition
    b. Mag-squared response
    c. Example: multipath interference, direct plus surface bounce
    d.
2) H(w) for rational systems
    a. Basics
    b. Geometric interpretation
        i. 1 real pole, 1 real zero
        ii. Double poles/zeros
        iii. Zeros on unit circle, poles at origin
        iv. Conjugate symmetric poles
3) Write |H(w)|^2 in terms of z, so we can see where the poles and zeros go.
    a. Can we recover H(w) given H(w)^2
4) Link between autocorrelation, magnitude

~~HW from book: 5.4 (at least some), 5.12a-c, 5.25 (graphical interpretation of p/z diag)~~

H(w) basics (5.1)

ppt

Given an impulse response $h(n)$, let's take it's F.T.

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k}$$

→ If $h(k)$ is real

$$= \sum_{k} h(k) \cos \omega k - j \sum h(k) \sin \omega k$$

$$= H_R(\omega) + j H_I(\omega)$$

$$= |H(\omega)| e^{j\theta}$$

board

$$|H(\omega)| = \sqrt{H_R^2 + H_I^2} \quad \text{magnitude response}$$

$$\Theta(\omega) = \operatorname{atan} \frac{H_I(\omega)}{H_R(\omega)} \quad \text{phase response}$$

~~Proakis & Manolakis~~

We can see $H_R(\omega) = H_R(-\omega)$: even

$H_I(\omega) = -H_I(-\omega)$: odd

this means $|H(\omega)|$ is even

$\Theta(\omega)$ is odd

$$H(\omega) = H^*(-\omega) \quad \text{, or} \quad H(-\omega) = H^*(\omega) \quad \text{(same)}$$

On board

if at ω, $H(\omega) = |H(\omega)| e^{j\theta}$

at $-\omega$ $H(-\omega) = \underset{\uparrow\,\text{even}}{|H(\omega)|}\, \underset{\uparrow\,\text{odd}}{e^{-j\theta}} = H(\omega)^*$

$H(\omega)$ – response to inputs

if $y(n) = h * x$,

$$Y(\omega) = H(\omega)\, X(\omega)$$

$$|Y(\omega)| = |H(\omega)|\, |X(\omega)|$$

$$\measuredangle Y(\omega) = \measuredangle H(\omega) + \measuredangle X(\omega)$$

<u>observe</u>:

1) $|H(\omega)|$ boosts or attenuates magnitude at each frequency

2) $\measuredangle H(\omega)$ introduces a freq-dep phase shift

3) H can't create new frequencies in output:
   if $|X(\omega)| = 0$, $|Y(\omega)| = 0$

   only nonlinear or non-TI systems can change frequency

~~notation~~ let's think about the magnitude-squared:

$$|Y(\omega)|^2 = |H(\omega)|^2 \, |X(\omega)|^2$$

notation define $S_{yy}(\omega)\ |Y(\omega)|^2$ = energy spectral density of y

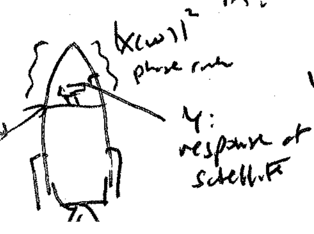
$S_{xx}(\omega) = |X(\omega)|^2$ = " " of x

then

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

Q: why would we just care about $|\ |^2$?

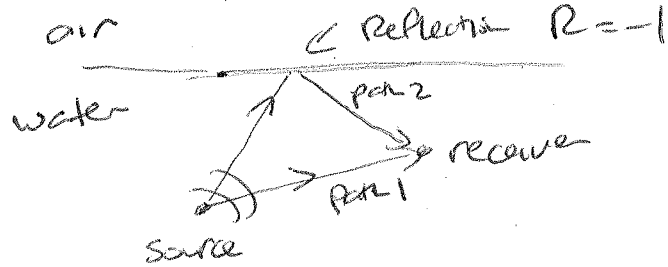
A. Sometimes the input signals are noise-like, w/ basically random phase, then, it's the magnitude$|^2$ that is useful



turbulence → $\boxed{\text{car window}}$ → noise inside car

③ $H(\omega)$ example

ex) multipath fading (like parts of homework problem 5.4)



$$y(n) = x(n) - x(n-M)$$

↑ path 1

↑ path 2 – delay M samples, reflection coef −1

$$Y(\omega) = X(\omega) - e^{-j\omega M} X(\omega)$$
$$= X(\omega)\left(1 - e^{-j\omega M}\right)$$

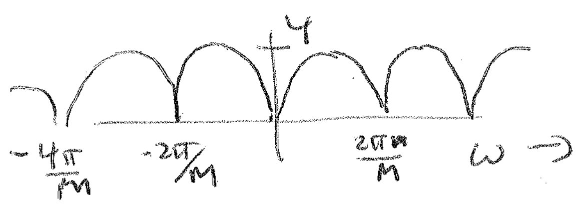
then,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 1 - e^{-j\omega M}$$
$$= e^{-j\omega M/2}\left(e^{+j\omega M/2} - e^{-j\omega M/2}\right)\frac{2j}{2j}$$
$$= 2j\, e^{-j\omega M/2} \sin \omega M/2$$

now look at $|H(\omega)|^2 = |H(\omega) H^*(\omega)| = 4 \sin^2\left(\frac{\omega M}{2}\right)$

← sin = 0 at 0, $\frac{\omega M}{2} = n\pi$

or $\omega = 2n\pi/M$



interpret:

① "best" case → constructive interference: the two paths add to give amplitude 2 or $|H|^2 = 4$

② "worst" case → destructive cancellation $|H|^2 = 0$

③ note we get a sign reversal each time the sine goes through zero – affects phase but not $|\ |^2$ (see PPT)

3.5

multi path example ~~cont~~ continued

Q) what if reflector was $R = +1$?

A) $y(n) = x(n) + x(n-M)$

$\rightarrow$ cosine, not sine

Consider case: input has broad frequency $|X(\omega)|^2 = k$

$$|Y(\omega)|^2 = |H(\omega)|^2 \, |X(\omega)|^2$$
$$= k \, |H(\omega)|^2 \sim k \sin^2\left(\frac{\omega M}{2}\right)$$

see picture in PPT

Writing $H(\omega)$ as a rational function — P&M 5.2

~~Remember~~

see PPT for equations — similar to $H(z)$

remember p_k, z_k can be complex.

But, if $x(n)$ is real, we must have conjugate symmetry

$$H^*(\omega) = H(-\omega)$$

Geometric
single pole

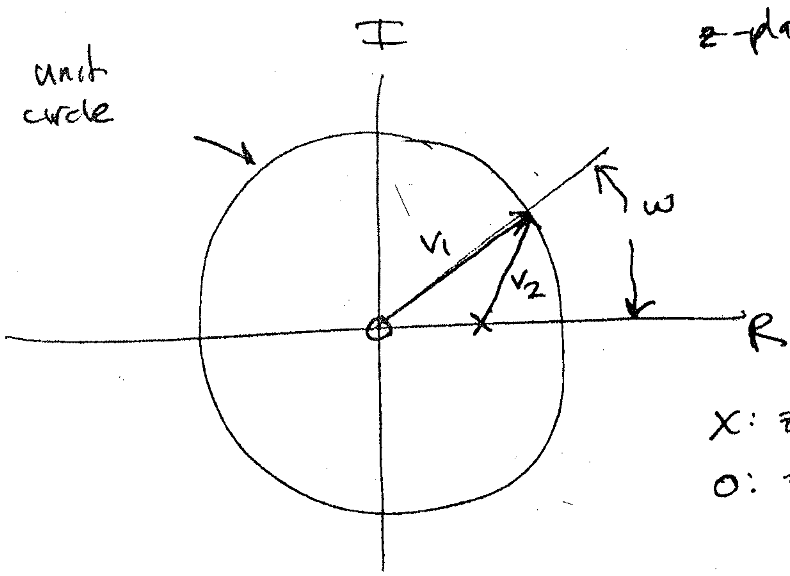
order Systems

$$h[n] = a^n \cdot u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

if $|a| < 1$ ROC includes unit circle

$\Rightarrow$ FT of $h[n]$ converges to $H(z)$ $\quad z = e^{j\omega}$ so $z^{-1} = e^{-j\omega}$

$$\Rightarrow H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$



X: $z = a$ pole
O: $z = 0$ zero

$$|\bar{v}_1| = 1$$

$$\left| H(z)\Big|_{z=e^{j\omega}} \right| = \frac{|\bar{v}_1|}{|v_2|} = \frac{1}{|v_2|}$$

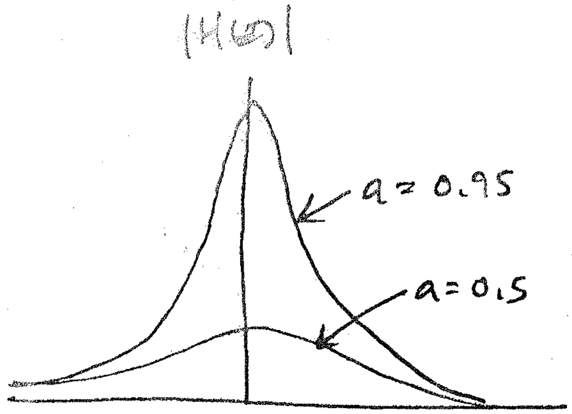
$$\angle H(z)\Big|_{z=e^{j\omega}} = \angle\bar{v}_1 - \angle\bar{v}_2 = \omega - \angle\bar{v}_2$$

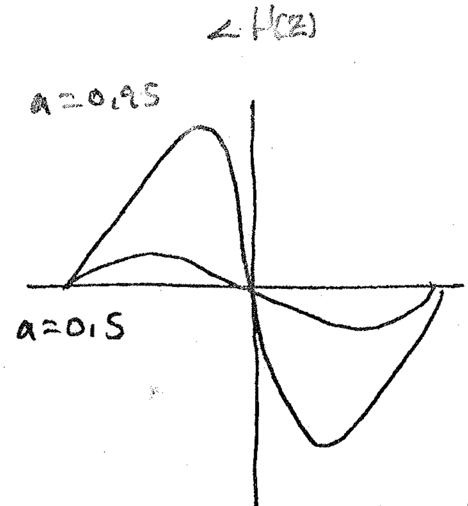
if $0 < a < 1 \Rightarrow$ pole vector minimum length @ $\omega = 0$ & monotonically increases to maximum @ π

geometric - single pole

$\therefore$ max frequency occurs @ π

min frequency occurs @ 0.





a ≡ plays role in DT system of τ (time constant) for CT systems (1st order)

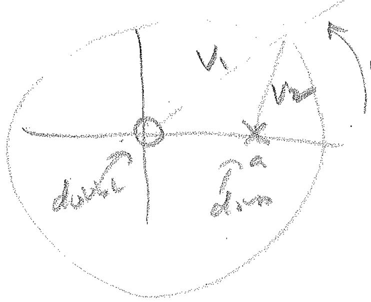
as $|a|$ decreases, the impulse response decreases more sharply

when multiple poles:

speed of response associated with each pole is related to its distance from the origin

closest ⇒ most rapid decaying terms in the impulse response

Geometric - double pole

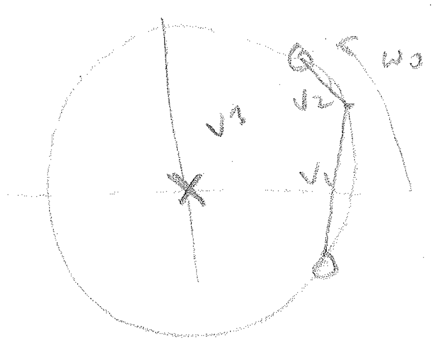


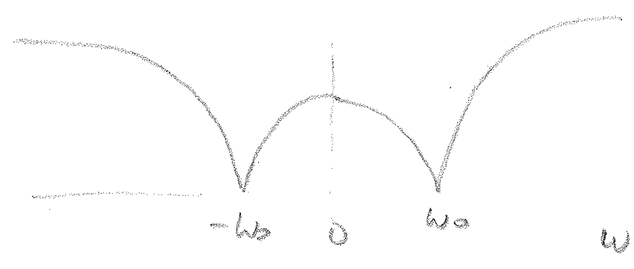
$$|H(z)|\Big|_{e^{j\omega}} = \frac{|V_1|^2}{|V_2|^2} = \frac{1}{|V_2|^2}$$

$$\angle H(z)\Big|_{e^{j\omega}} = 2\angle V_1 - 2\angle V_2$$
$$= 2\omega - 2\angle V_2$$

So changes will occur more rapidly

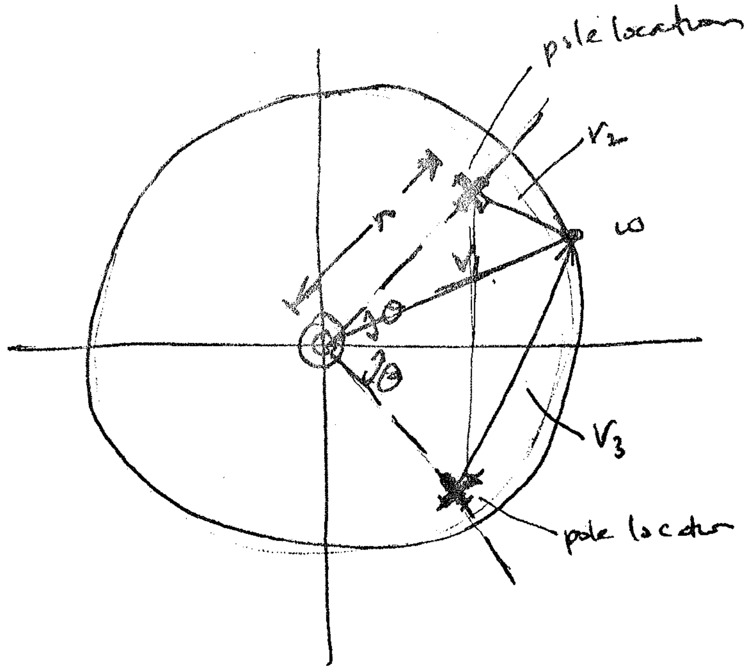
2 zeros on unit circle 2 poles at $z=0$





$$|H| = \frac{|V_1|\,|V_2|}{|V_3|^2} = |V_1|\,|V_2|$$

$$\angle H = \angle V_1 + \angle V_2 - 2\omega$$



$$|H(e^{j\omega})| = \frac{|\bar{V}_1|^2}{|\bar{V}_2| \cdot |\bar{V}_3|}$$

due to double zero @ 0

due to product of magnitudes

since $|\bar{V}_1| = 1$ for all ω

$$|H(e^{j\omega})| = \frac{1}{|\bar{V}_2| \cdot |\bar{V}_3|}$$

$$\angle H(e^{j\omega}) = 2 \cdot \angle \bar{V}_1 - (\angle \bar{V}_2 + \angle \bar{V}_3)$$

think about $|H(\omega)|^2$

$$|H(\omega)|^2 = H(\omega)\, H^*(\omega)$$

but if h is real, ~~$H^*(\omega) = H(-\omega)$~~

$$= H(\omega)\, H(-\omega) \quad \leftarrow \text{conj symmetry}$$

$$= H(z)\, H(z^{-1})\Big|_{z=e^{j\omega}}$$

how to see last point?

$$H(z) = \sum_n h(n)\, z^{-n} \qquad (\text{def } z \text{ transform})$$

so $$H(z^{-1}) = \sum_n h(n)\, z^n$$

$$H(z^{-1})\Big|_{z=e^{j\omega}} = \sum_n h(n)\, e^{j\omega n}$$

$$= \sum_n h(n)\, e^{-(-\omega)n}$$

$$= H(-\omega) \quad \text{from DTFT}$$

If $H(z)$ has ~~poles~~ zeros z_k and poles p_k

$H(z^{-1})$ has zeros $1/z_k$ and poles $1/p_k$

pole - zero confusion

Q) Given $|H(\omega)|^2$, can we determine $H(\omega)$?

say $H(z)$ has zeros $[z_1, z_2]$

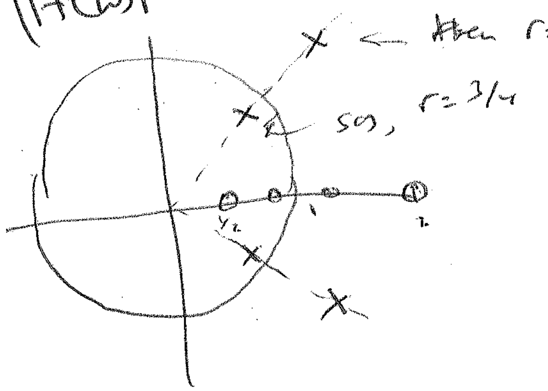
poles $[p_1, p_2]$ ← complex conjugates

$|H(\omega)|^2 = H(z) H(z^{-1})$

$H(z^{-1})$ has zeros $[\frac{1}{z_1}, \frac{1}{z_2}]$

poles $[1/p_1, 1/p_2]$

$|H(\omega)|^2$



given poles / zeros of $|H(\omega)|^2$

$[z_1, 1/z_1, z_2, 1/z_2]$

$[p_1, p_2, 1/p_1, 1/p_2]$

many possible combinations!

even if we pick poles inside unit circle (H stable) still 4 combinations possible

$(z_1, 1/z_1), (z_1, z_2), (1/z_1, z_2) (1/z_1, 1/z_2)$

thus in general, we <u>cannot</u> distinguish $H(\omega)$

there are special conditions where we can - <u>minimum phase</u> systems