Lecture 8 outline: Linear phase, Group delay, and simple filter design

Reading: P&M section 5.4

This lecture assumes we have already:

- Covered magnitude and phase response H(z) of LTI systems
- Reviewed pole/zero descriptions of H(z)

After learning this material, students should be able to:

- Calculate group delay
- Understand why linear phase systems are preferable
- Use pole/zero placement to design simple filter types

Skills that will be used in later work include:

- Understanding of group delay / linear phase concepts

Warm –up question:
Assume we have a real-valued input signal x[n], and want a real-valued output signal y[n]. What does this imply for pole-zero placement of H(z)?

Outline

1) Brief review of general types of ideal filters and their magnitude response
2) Ideally, filters would also have *linear phase*
    a. Systems with pure delay
    b. Symmetric system responses
    c. Concept of generalized linear phase
3) Group delay: the time delay of different frequencies moving through system
    a. Definition
    b. Some physical insight
4) Filter design using pole/zero placement
    a. Basic idea: put poles near frequencies where large response is desired, zeros where low response is desired
    b. Constraints include: real response, stable response, specified magnitude at certain frequencies
5) Example: notch filter design

Lecture: linear phase, group delay, simple filter design

1) Brief review - ideal filters

For LTI system $y[n] = h[n] * x[n]$

$$Y(\omega) = \underbrace{H(\omega)}\, X(\omega)$$

↑ passes some freq, attenuates others

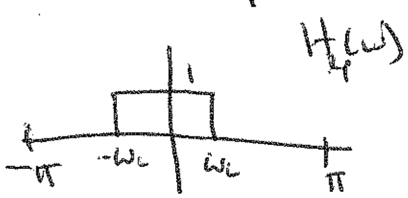
thus

$$|Y(\omega)| = |H(\omega)|\, |X(\omega)|$$

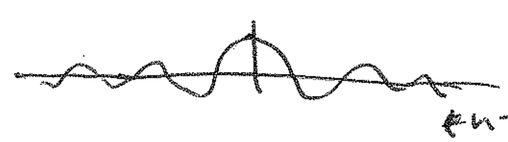
$$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$$

Example ideal filters:

① low pass

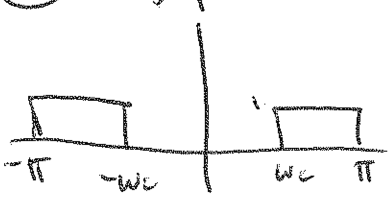


$\rightarrow$ $h_{lp}(n) = \dfrac{\sin \omega_c n}{\pi n}$



* note this is not physically realizable - ∞ long $h[n]$

② high pass



$$|H_{hp}(\omega)| = 1 - |H_{lp}(\omega)|$$

$$\rightarrow h_{hp}[n] = \delta(n) - \frac{\sin \omega_c n}{\pi n}$$

(Q)

~~b~~ Ideally, what would phase be? ~~often~~, zero or easily explained / compensated

a) If goal is frequency suppression

remember F.T. pair: $\delta(n-n_d) \Leftrightarrow e^{-j\omega n_d}$

→ ~~[illegible]~~ <u>Simple example</u>: delay system

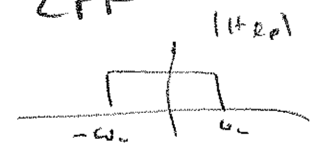
$h = \delta(n-n_d) \rightarrow y(n) = h * x = x(n-n_d)$

so $H(\omega) = e^{-j\omega n_d}$

$|H| = 1$, $\measuredangle H = -\omega n_d \Leftarrow$ <u>Linear phase</u> $(\sim \omega)$

<u>next example</u> : linear phase, ~~[illegible]~~ ideal LPF

$$H_{lp}(\omega) = \begin{cases} e^{-j\omega n_d} & |\omega| < \omega_c \\ 0 & \text{else} \end{cases}$$



$$\measuredangle H_{lp} = \begin{cases} -\omega n_d, & |\omega| < \omega_c \\ 0 & \text{else} \end{cases}$$ again linear

$$h_{lp}(n) = \frac{\sin \omega_c (n-n_d)}{\pi (n-n_d)}$$ = delayed version of first lpf

<u>Idea</u> : linear phase → delay in time.

Otherwise the signal phase is not changed.

Often, we want to just attenuate magnitude in noise regions; in these cases, linear phase/ time delay leaves desired signal untouched

- Examples of distortion later

③ Group delay.

→ ~~Now~~ if we plot $\measuredangle H$ vs. ω, we can see if phase is linear.

→ otherwise, hard to learn much from $\measuredangle H$

→ a more useful way to capture distortion is <u>group delay</u>

$$\tau(\omega) = \text{grd}[H(\omega)] = -\frac{d}{d\omega}\arg[H(\omega)]$$

→ has units of ~~delay~~ time / delay $\left(\frac{1}{1/s} = s\right)$

→ so $1/\tau$ is like velocity (how fast different frequencies move through system)

<u>Some insight (hopefully)</u>

→ For pure delay, we saw $H(\omega) = e^{-j\omega n_d}$

$$\measuredangle H = -\omega n_d$$

$$-\frac{\partial \measuredangle H}{\partial \omega} = n_d \quad \Leftarrow \text{ everything has delay } n_d \ \forall \omega$$

→ group delay / group velocity is a concept from physics – the rate at which an envelope of the signal (information) propagates

→ example of mechanical dispersive system:

→ ~~hit~~ on frozen pond, hit the ice. ~~Faster~~ Higher-freq sounds travel quicker, lower-freq slower, so original signal is smeared out.

more general:

$$\measuredangle H(\omega) = \tan^{-1}\left(\frac{H_I(\omega)}{H_R(\omega)}\right)$$

$$\equiv \tan^{-1}(A(\omega))$$

remember (or look up) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

then use chain rule

$$\tau(\omega) = \frac{1}{1+A^2(\omega)}\frac{dA(\omega)}{d\omega}$$

Other examples of linear phase:

a) pure delay – above

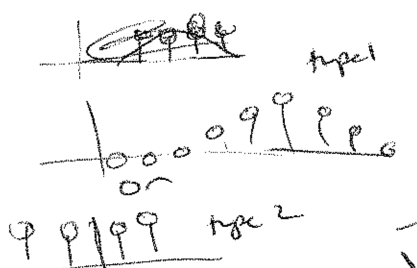
b) symmetric system:

version 1: $h_1[n] = h_1[-n]$

as h_1 is odd, $H_1(\omega)$ is real

$\measuredangle = 0 \rightarrow \tau = 0$

version 2: symmetric, about a shifted point



$h_2[n] = h_1[n - n_0]$

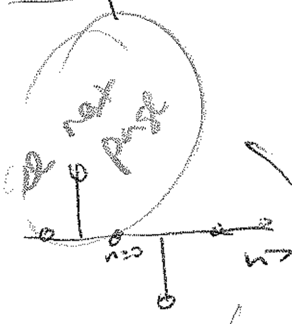
$H_2(\omega) = e^{j\omega n_0} H_1(e^{j\omega})$

$\measuredangle H_2 = \cancel{e}^{j\omega n_0}$

c) Generalized linear phase:

these systems are of form

$$H(\omega) = e^{j\beta} e^{-j\omega\alpha} \underset{\uparrow \text{ real}}{I(\omega)}$$



here,

$$\measuredangle H = \underset{\nwarrow \text{ constant}}{\beta} - \omega\alpha$$

$$-\frac{\partial H}{\partial \omega} = \alpha \quad \leftarrow \text{constant delay}$$

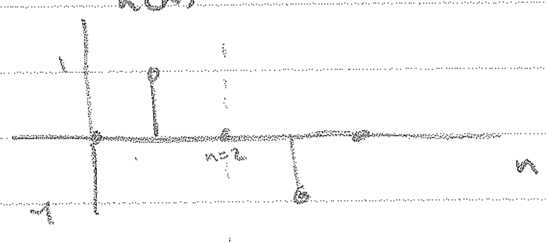
$H(\omega) = j e^{-j\omega} \sin(\omega n)$

$= e^{j\pi/2} e^{-j\omega} \sin(\omega n)$

lecture 7:
multiplying
$H(\omega) = 2j e^{j\omega M/2} \sin \omega M/2$
$= 2 e^{j\pi/2} e^{-j\omega M/2} \sin \omega M/2$

Some ~~[illegible]~~ filters can achieve generalized linear phase ~~[illegible]~~ in their passband – ~~good enough~~

★ Linear phase is easy to do w/ FIR, hard w/ IIR example generalized linear phase system



anti-symmetric about n = 2

$$H(z) = (1)\, z^{-1} - (1)\, z^{-3}$$

$$= z^{-2}\left(z + z^{-1}\right)$$

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = e^{-2j\omega}\left(e^{j\omega} - e^{-j\omega}\right)$$

$$= 2j\, e^{-j2\omega}\left(\frac{e^{j\omega} - e^{-j\omega}}{2j}\right)$$

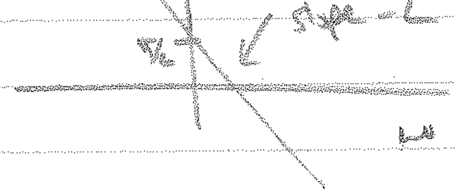
$$= 2j\, e^{-j2\omega} \sin(\omega)$$

but $j = e^{j\pi/2}$, so $H(\omega) = 2\, e^{j(\pi/2 - 2\omega)} \sin(\omega)$

real func

$$\measuredangle H(\omega) = \pi/2 - 2\omega$$

$$\tau(\omega) = -\frac{d}{d\omega} \measuredangle H(\omega) = 2 \; // \quad \text{non-dispersive}$$



3.30: show if $z = re^{j\theta}$ is zero so is $z = \frac{1}{r} e^{j\theta}$

like 3.30

ZERO LOCATIONS OF LINEAR-PHASE FILTERS

Sec 4.4.4 in Mitra

The zeros of the transfer function $H(z)$ of a linear-phase filter lie in specific configurations.

$h(n) = a\delta(n) + b\delta(n-1) + c\delta(n-2)$

ex $H(z) = a z^{0} + b z^{-1} + c z^{-2}$

$H(z^{-1}) = a + b z^{1} + c z^{2}$

$H(z) = z^{-2} H(1/z)$

N=3 points 0, 1, 2; N-1=2

We can write the symmetry condition

$$h(n) = h(N - 1 - n)$$

in the Z domain. Taking the Z-transform of both sides gives

$$H(z) = z^{-(N-1)} H(1/z). \tag{25}$$

Recall that we are assuming that $h(n)$ is real-valued. If z_o is a zero of $H(z)$,

So, if $H(z) \to 0$ at $z = z_0 = re^{j\theta}$

$H(1/z_0) \to 0$

so zero at $z = \frac{1}{z_0} = \frac{1}{re^{j\theta}} = \frac{1}{r} e^{-j\theta}$

$$H(z_o) = 0,$$

then

$$H(z_o^*) = 0.$$

main fact

(Because the roots of a polynomial with real coefficients exist in complex-conjugate pairs.)

Using the symmetry condition (25), it follows that

$$H(z_o) = z_o^{-(N-1)} H(1/z_o) = 0$$

and

$$H(z_o^*) = (z_o^*)^{-(N-1)} H(1/z_o^*) = 0$$

or

$$H(1/z_o) = H(1/z_o^*) = 0.$$

> If z_o is a zero of a (real-valued) linear-phase filter, then so are
> z_o^*, $1/z_o$, $1/z_o^*$.

TYPE I: ODD-LENGTH SYMMETRIC

The frequency response of a length $N = 5$ FIR Type I filter can be written as follows.

$$\begin{aligned}
H^f(\omega) &= h_0 + h_1 e^{-j\omega} + h_2 e^{-2j\omega} + h_1 e^{-3j\omega} + h_0 e^{-4\omega} && (1)\\
&= e^{-2j\omega}\left(h_0 e^{2j\omega} + h_1 e^{j\omega} + h_2 + h_1 e^{-j\omega} + h_0 e^{-2j\omega}\right) && (2)\\
&= e^{-2j\omega}\left(h_0(e^{2j\omega} + e^{-2j\omega}) + h_1(e^{j\omega} + e^{-j\omega}) + h_2\right) && (3)\\
&= e^{-2j\omega}\left(2h_0 \cos(2\omega) + 2h_1 \cos(\omega) + h_2\right) && (4)\\
&= A(\omega) e^{j\theta(\omega)} && (5)
\end{aligned}$$

where

$$\theta(\omega) = -2\omega, \qquad A(\omega) = 2h_0 \cos(2\omega) + 2h_1 \cos(\omega) + h_2.$$

Note that $A(\omega)$ is real-valued and can be both positive and negative. In general, for a Type I FIR filters of length N:

$$H^f(\omega) = A(\omega) e^{j\theta(\omega)}$$

$$\boxed{A(\omega) = h(M) + 2\sum_{n=0}^{M-1} h(n)\cos((M-n)\omega).}$$

$$\boxed{\theta(\omega) = -M\omega}$$

$$\boxed{M = \frac{N-1}{2}.}$$