Deriving convolution: summary of P&M 2.3

EE-125

In the lecture we just went over an example to give intuition into convolution. How can we formally derive it? Here we give a quick synopsis of P&M 2.3. The basic concept is:

- we write the input signals as the sum of a set of elementary signals. Here, we will use delta functions (elsewhere in signal processing, we may use exponentials (Fourier transform), or wavelets, or...)
- calculate the system response to each elementary signal. If our elementary signal is the **impulse**, the response is the **impulse response**, which we usually call h.
- use **linearity** to scale the response to each impulse
- use **time invariance** to shift the response of each impulse in time
- add up all of these scaled, shifted impulse responses to get the answer.

Mathematically, we start by writing the input as a sum of elementary functions x_k:

$$x(n) = \sum_k c_k x_k(n)$$

If we take $x_k(n) = \delta(n-k)$, then the weights c_k will just be the values of the signal at each sample, giving:

$$x(n) = \sum_k x(k)\delta(n-k)$$

To check this is correct, we can see that for $n = k$, the delta function is 1, and otherwise it is zero. Thus the equation gives $x(n) = x(n) * 1 + 0 = x(n)$. To see why this is useful, consider that we have some system to transform x into y, so that

$$y(n) = T[x(n)].$$

Then plugging in and using **linearity** gives

$$\begin{aligned} y(n) &= T[\sum_k x(k)\delta(n-k)] \\ &= \sum_k x(k)T[\delta(n-k)] \\ &= \sum_k x(k)h(n,k) \end{aligned}$$

where we have written the impulse response as $h(n,k) = T[\delta(n-k)]$. So far we haven't made any assumptions about time-invariance, so the impulse response depends on both n and k. We are forced to assume this when the system changes over time, and it makes our life a lot harder.

However, in this class (and a lot of engineering) we can assume this system is **time invariant**. Then if the response to an impulse $\delta[n]$ is defined as $h(n)$, the response to a shifted impulse $\delta(n-k)$ is just $h(n-k$; i.e. it just depends on the *relative* times. With this assumption we get the convolution equation:

$$y(n) = \sum_k x(k)h(n-k).$$