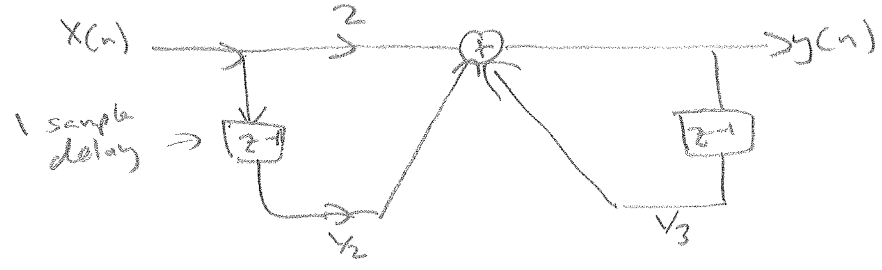
Lecture: LTI system analysis using z-transforms

Example: consider system $\boxed{y(n) = \frac{1}{3} y(n-1) + 2x(n) + \frac{1}{2} x(n-1)}$

$\boxed{y(n) = 0, \; n < 0}$ starts at rest

picture:



this is an LTI system : $x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$, so $y(n) = h * x$

Question what is $H(z)$? what are its characteristics?

option 1: we could find $h(n)$ by plug-and-chug

$y(0) = 2$, $y(1) = \frac{1}{3}(2) + 0 + \frac{1}{2}$, etc.

then take z-transform.

option 2: use z-transform properties

$$y(n) = x(n) * h(n) \overset{z}{\Longleftrightarrow} Y(z) = H(z) X(z)$$

transform the equation using time-shift property $x(n-k) = z^{-k} X(z)$

$$y(n) = \frac{1}{3} y(n-1) + 2x(n) + \frac{1}{2} x(n-1)$$

$$\Downarrow$$

$$Y(z) = \frac{1}{3} z^{-1} Y(z) + 2X(z) + \frac{1}{2} z^{-1} X(z)$$

$$Y(z)\left(1 - \frac{1}{3} z^{-1}\right) = X(z)\left(2 + \frac{1}{2} z^{-1}\right)$$

but, if $Y(z) = H(z) X(z) \rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\text{output}}{\text{input}}$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + \frac{1}{2} z^{-1}}{1 - \frac{1}{3} z^{-1}}$$

yay!

(page for extra work)

motivating example cont'

multiply $H(z)$ by $\frac{z}{z}$

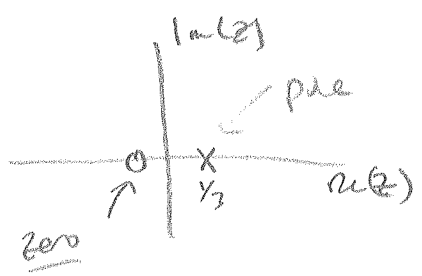
$$H(z) = \frac{2z + 1/2}{z - 1/3}$$

notice : at $z = 1/3$, $|H(z)| \to \infty$; response blows up.
this is an undamped resonance (physically) or a pole

notice : at $z = -1/4$, top $\to 0$;
$|H(z)| \to 0$ zero of function

Here, $H(z)$ is a rational function (ratio of 2 functions) with (here) 1 pole + 1 zero

we can draw a pole-zero diagram



↑ note, same pole/zero plot for

$$H(z) = 1000\left(\frac{2z + 1/2}{z - 1/2}\right)$$

so, from pole-zero we can see behaviour up to a constant

$$H(z) = G\,\frac{(z - z_1)}{(z - p_1)}$$

Continuous Random Variables

A random variable is **continuous** if it can take any value in an interval.

3

last example was for a system, but same math holds true for signals

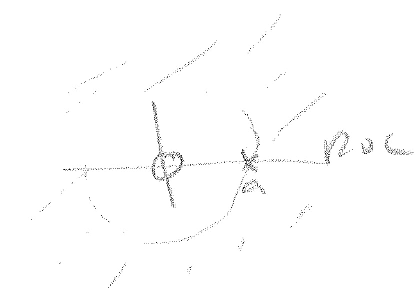
example from before: $x(n) = a^n u(n)$, $a > 0$

we saw before

$$X(z) = \frac{1}{1-az^{-1}} \quad \text{ROC } |z|>a$$

multiply by z/z → positive powers of z

$$X(z) = \frac{z}{z-a} \quad \begin{array}{l} \leftarrow \text{zero at } z=0 \\ \leftarrow \text{pole at } z=a \end{array}$$



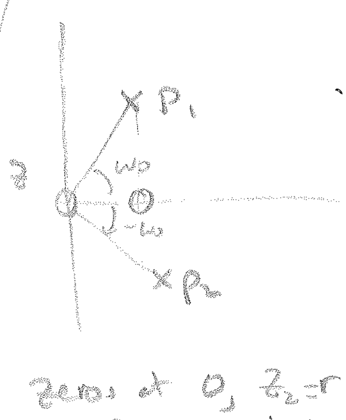
See PPT: time-domain response linked to pole location

complex-valued poles → describe sinusoids.

z-transform properties: if $x(n)$ is real, then $X(z)$ must be either real-valued (last case) or conjugate-symmetric

↳ so if pole p_1 is at $re^{j\omega_0}$
p_2 must be at $re^{-j\omega_0}$

consider this diagram:



Ex. 3.3.3 in book

$$X(z) = \underset{\substack{\uparrow \\ \text{unknown}}}{G} \frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = G \frac{z(z - r\cos\omega_0)}{(z-re^{j\omega_0})(z-re^{-j\omega_0})}$$

$$= G \frac{z(z-r\cos\omega_0)}{z^2 - rz(e^{j\omega_0}+e^{-j\omega_0}) + r^2}$$

zeros at 0, $z_2 = r\cos\omega_0$
$p_1 = re^{j\omega_0}$, $p_2 = p_1^*$

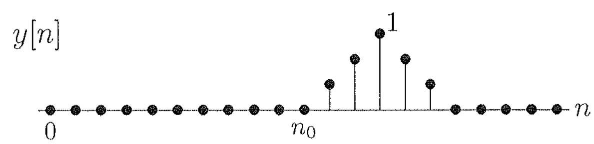
↳ algebra

$$x(n) = G(r^n \cos\omega_0 n)\, u(n)$$

see PPT

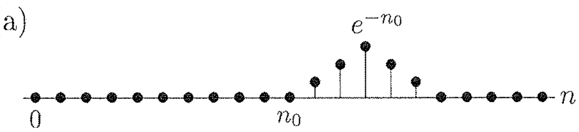
Question 22

The plot below shows the output $y[n]$ of an LTI system that is known to have the frequency response $H(e^{j\omega}) = e^{-j\omega n_0}$ for $-\pi < \omega \leq \pi$.

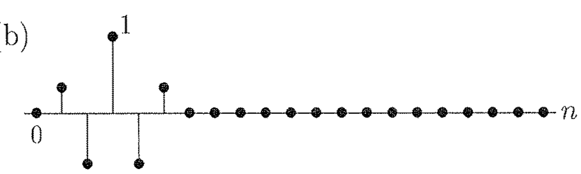


Which of the plots below is the input $x[n]$?

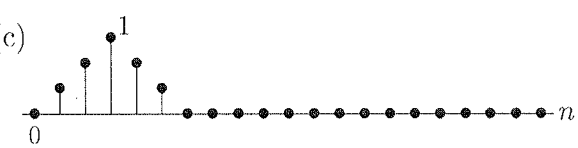
(a)



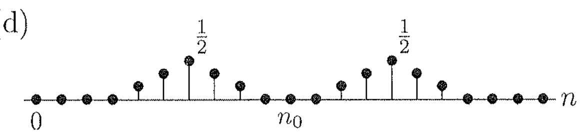
(b)



(c)



(d)



see PPT slides on "General case for rational functions"

pull out highest power to make things positive.

an example:

$$\underline{X}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 + a_1 z^{-1}}$$

or $H(z)$
if a signal

$$= \frac{b_0 z^{-2}}{a_0 z^{-1}} \left(\frac{z^2 + b_1/b_0 z + b_2/b_0}{z + a_1/a_0} \right)$$

$$= G\, z^{(1-2)} \left(\frac{z^2 + b_1' z + b_2'}{z + a_1'} \right)$$

$b_1' = b_1/b_0$
etc.

$$= G\, z^{-1} \left(\frac{z^2 + b_1' z + b_2'}{z + a_1'} \right)$$

2 zeros:
solve quadratic
$z^2 + b_1' z + b_2' = 0$

poles
$z = 0,\ z = -a_1'$

FINALS: May 5-12

KEY

~~Lecture: LTI analysis using z-transform~~

~~P+M 3.3, also 3.5.3 + 3.5.4~~

NOW - why do we care about rational functions??

3) LTI system functions + rational function (3.3.3)

assume our system is at rest (relaxed). Then

$$y(n) = h(n) * x(n)$$

$$\Updownarrow$$

$$Y(z) = H(z)X(z)$$

$h(n) =$ unit sample response

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \text{ is } \underline{\text{system function}}$$

a) consider a $\underline{\text{constant-coefficient difference eqn}}$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

take the z-xform:

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$Y(z)\left(1 + \sum a_k z^{-k}\right) = X(z)\left(\sum_k b_k z^{-k}\right)$$

any CCDE gives this

so $$H(z) = \frac{Y}{X} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{\sum b_k}{\sum_{k=0}^{N} a_k z^{-k}} \text{ if } a_0 \equiv 1$$

$\boxed{\text{Ratio or Rational function}}$

b) ~~all pole~~ Case 1: all $a_k = 0$

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

M zeros, and M^{th} order pole at $z = 0$

~~(why? example: all pole?)~~ ~~if $b_0 = 0$, $H(z) = b_1 z^{-1}$~~ $= \frac{b_1}{z}$

example ~~consider~~ $H(z) = b_0 + b_1 z^{-1}$

$$= \frac{b_0 z + b_1}{z}$$

← zero at $z = -b_1/b_0$

← pole at $z = 0$

<u>case 1 con't</u>

This is <u>all</u> - <u>zero</u> system

finite impulse response (FIR)

also called moving average

c) all-pole : case 2

all $b_k = 0$ (except b_0)

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=0}^{N} a_k z^{N-k}} \qquad a_0 \equiv 1$$

N poles determined by a_k

N^{th} order ~~pole~~ zero at 0 $\qquad \Leftarrow \quad x(n) = a^n u(n)$

example $\qquad H_z = \dfrac{b_0}{1 + a_1 z^{-1}} = \dfrac{b_0 z}{z + a_1}$

~~2) factoring & ROC of rational z-transforms~~ ~~(3.3.1)~~

5) Causality & stability (3.5.3)

a) <u>Causal</u> means: $h(n) = 0, \quad n<0$

the ROC of z-xform of a causal <u>signal</u> is exterior of a circle. example: causal FIR, causal IIR

⇒ Thus, LTI system is causal iff ROC of $H(z)$ is <u>exterior</u> of a circle, radius $r < \infty$

b) <u>stability</u> means $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$ ← from Chapter 2 2.3.6 *

we can z-transform above:

$$H(z) = \sum h(n) z^{-n}$$

$$|H(z)| \leq \sum |h(n) z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)| \, |z^{-n}|$$

= if all add in phase

if we consider unit circle, $|z^{-n}| \to 1$

$$|H(z)|\Big|_{|z|=1} \leq \sum |h(n)| < \infty$$

from *

⇒ thus if system is BIBO stable, ROC includes unit circle!

b.c. * say $\sum |h(n)| < \infty$; thus H can't blow up on unit circle.

⇒ causality doesn't imply stability, & vice versa

⇒ but, a causal LTI system is stable if & only if all poles of $H(z)$ are inside unit circle

(as we saw before)

6) Pole-zero cancellation

when analyzing a system, there may be a pole + zero at same point, that will cancel each other out:

$$H(z) = G \frac{(z-a)\,z}{(z-a)(z-b)}$$

or, ~~that~~ poles + zeros in input + system can cancel:

$$Y(z) = [H(z)]\,X(z)$$
$$= \left[\frac{z-a}{z-b}\right]\left[\frac{z-b}{z-c}\right]$$

but trying to use this can be tricky... finite numerical precision means we might get

$\frac{z-b+\delta_1}{z-b+\delta_2}$ that don't quite cancel

7) what are we skipping in chapter 3?

a) ~~[illegible]~~ in this class we are mostly focusing on case where system
   1) starts from rest (relaxed)
   2) has been running for a long time - steady state

b) however ~~that~~ system may not start from rest.
   - parts of 3.5A, all of 3.6 deal w/ this

c) we're also skipping details of 2nd order system stability. 2nd-order systems are interesting as they are building blocks for filter implementation