Recitation 2

1D Fourier Transform

Complex Numbers

1. Introduction
2. Periodic Functions
3. Back to Fourier
4. Non-Stationary Signals
5. Sound
6. STFT
7. Spectrograms

Introduction

Periodic Functions
Back to Fourier
Non-Stationary Signals
Sound

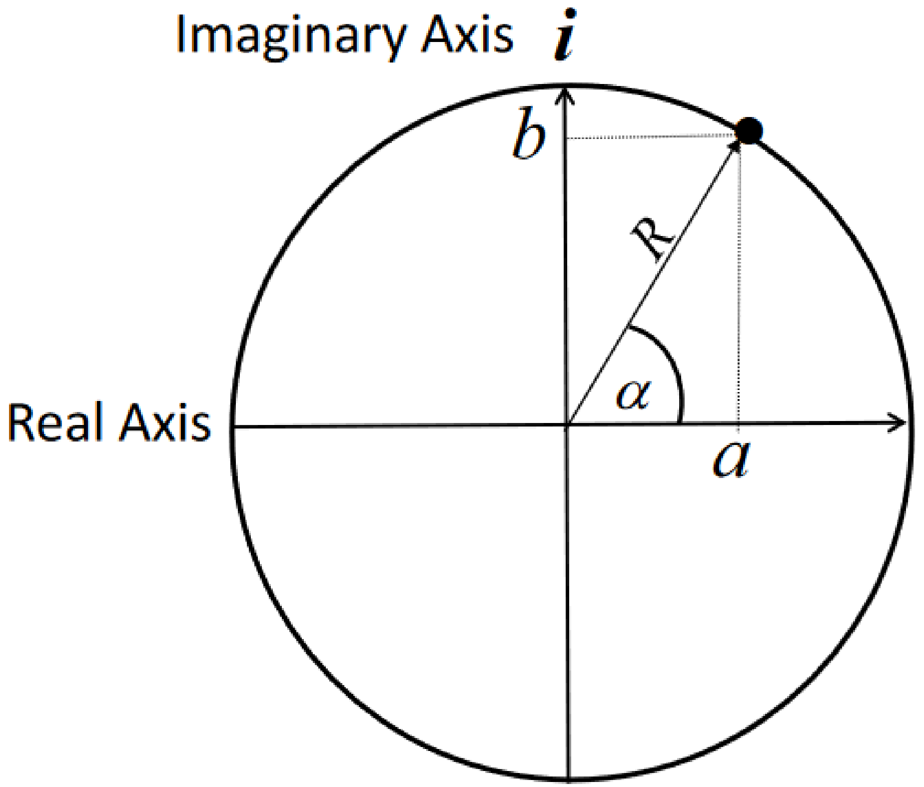
Fourier

- Any periodic function (signal) *f* (*x*) can be decomposed into F - a set of *sin* and *cos* periodic functions of different frequencies. (Fourier series)
- *f* can be reconstructed from *F* without any loss of data!
- Transform Fourier enables us to use the unique properties of decomposition into *sin* and *cos* on **any** function

Jean Baptiste Joseph Fourier
(1768 – 1830)

Complex Numbers

$i^2 = -1$



$$c = a + bi = R \cdot e^{i\alpha}$$

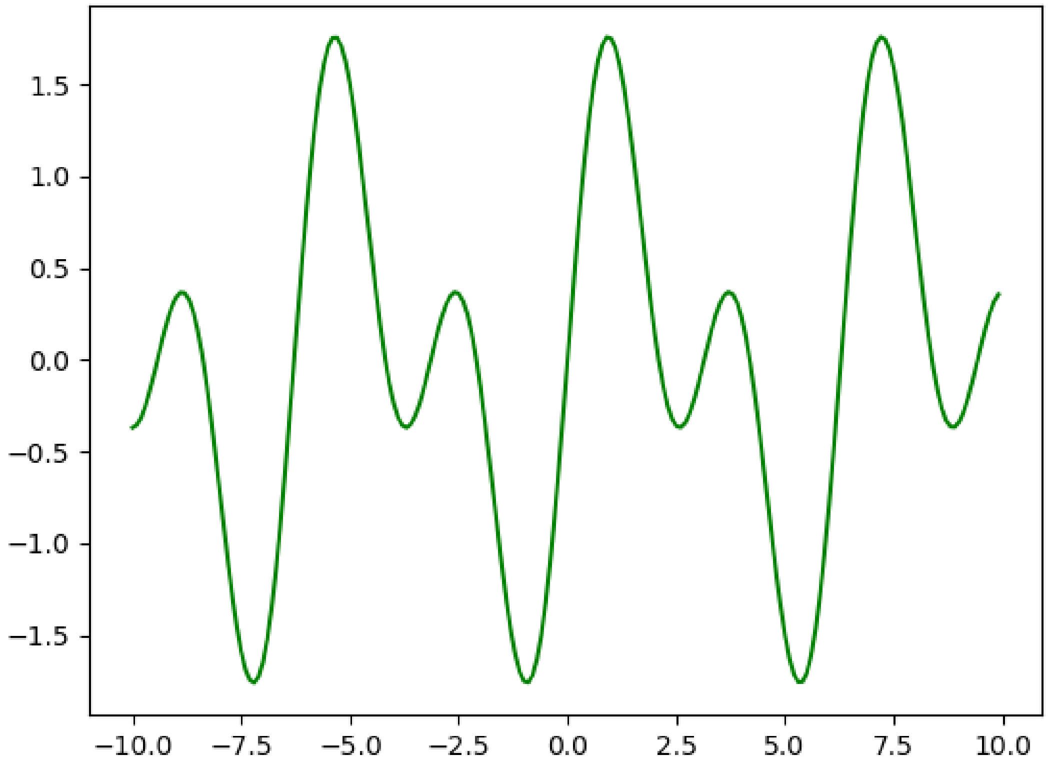
$$e^{i\alpha} = \cos(\alpha) + i \cdot \sin(\alpha)$$

Absolute Value: $|c| = R = \sqrt{a^2 + b^2}$

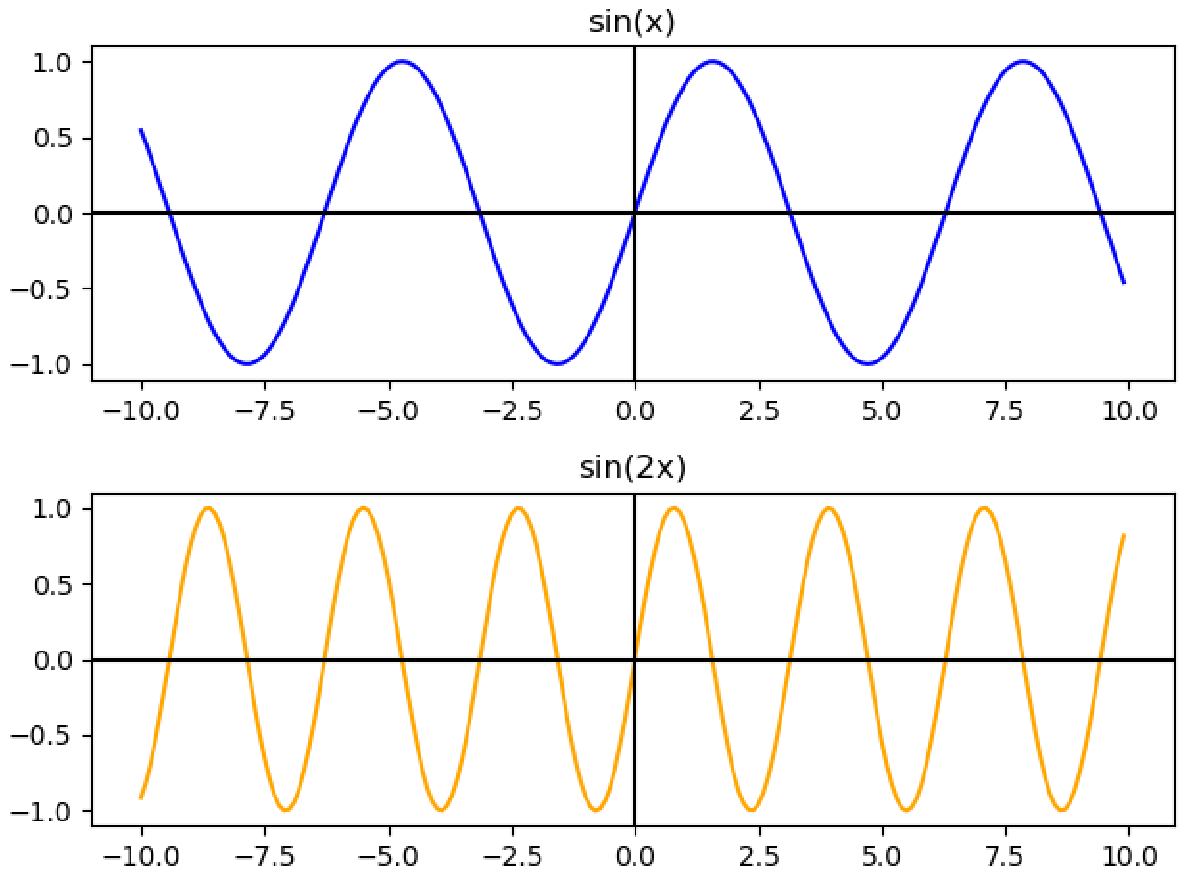
Phase: $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

Conjugate: $\bar{c} = c^* = a - bi$

Example 1

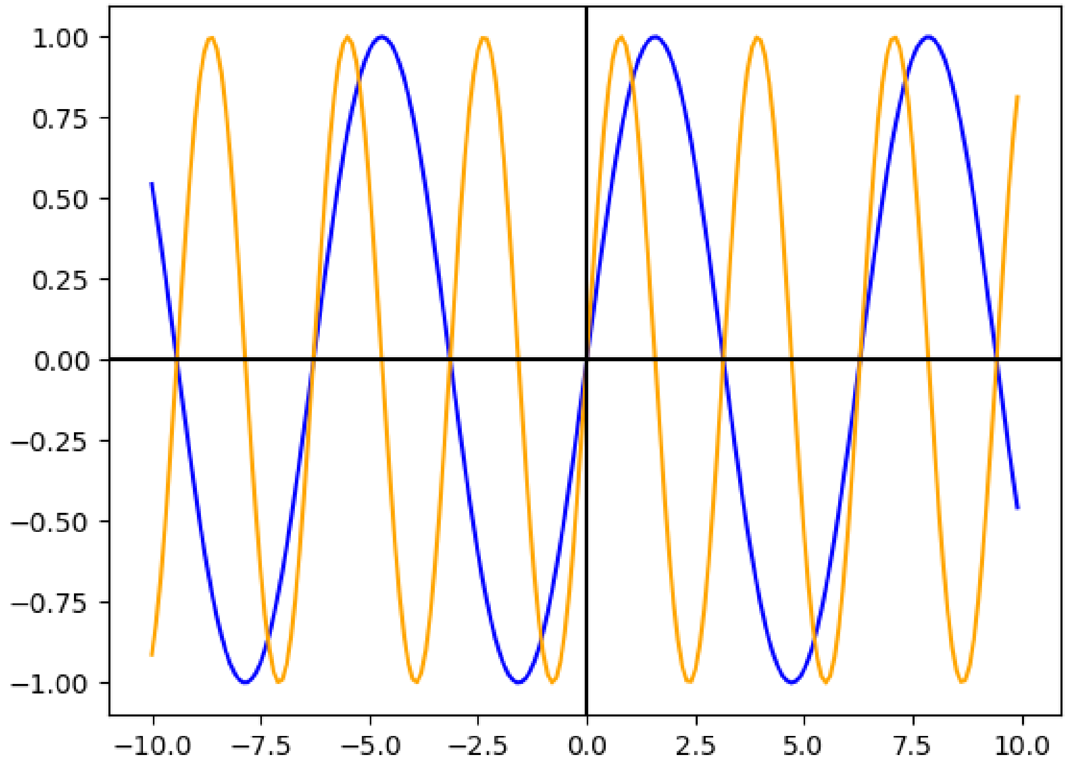


Example 1

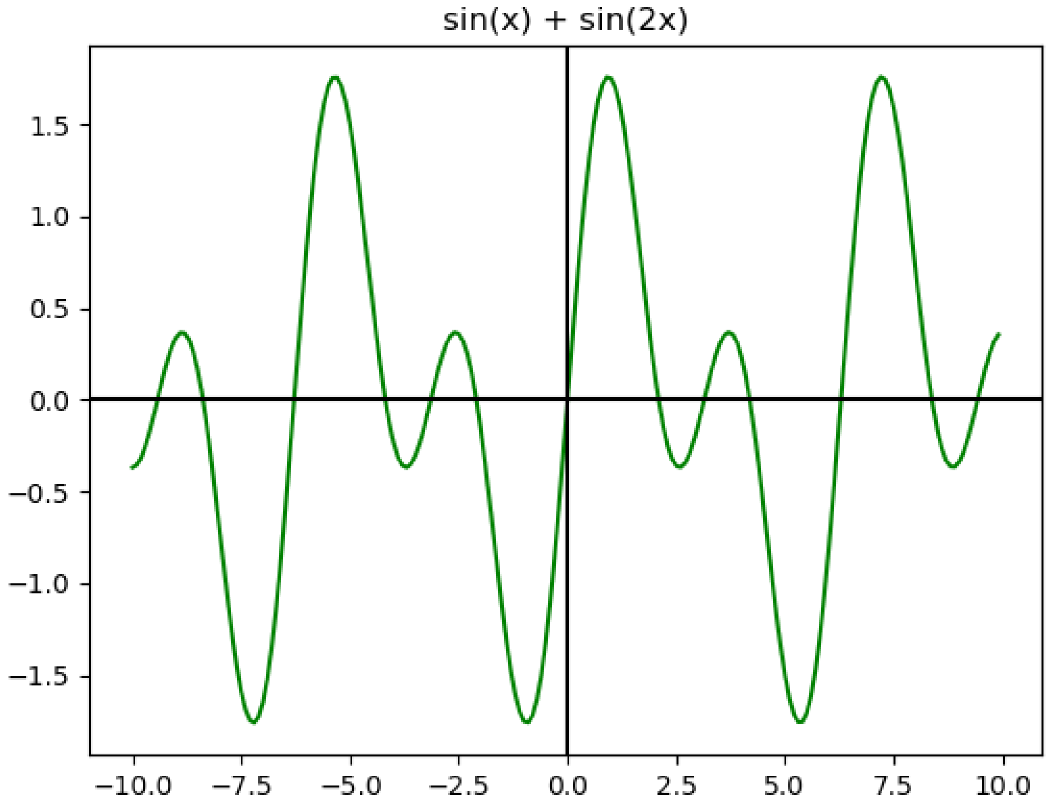


sin(x)
1.0
0.5
0.0
−0.5
−1.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
sin(2x)
1.0
0.5
0.0
−0.5
−1.0
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0

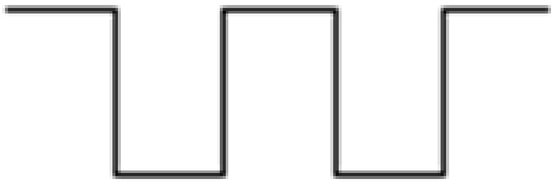
Example 1

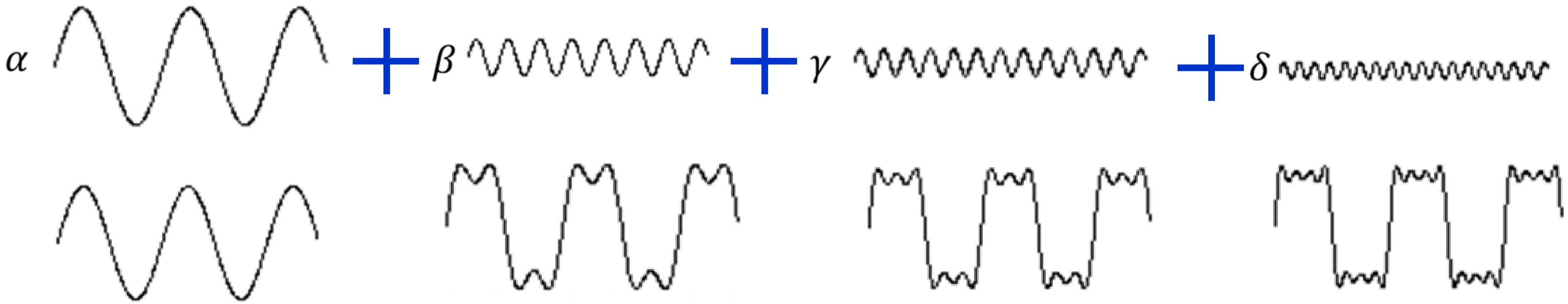


Example 1

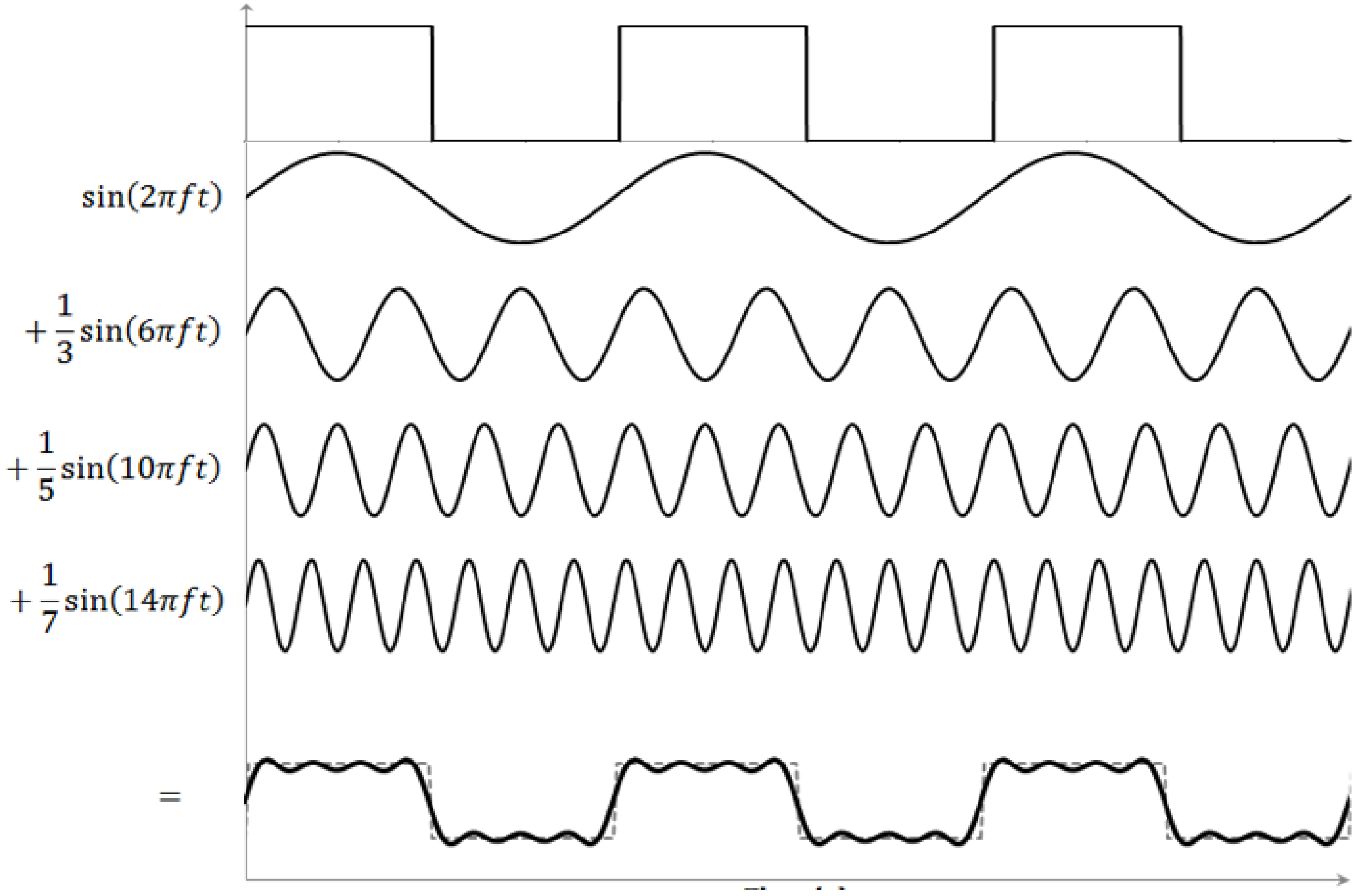


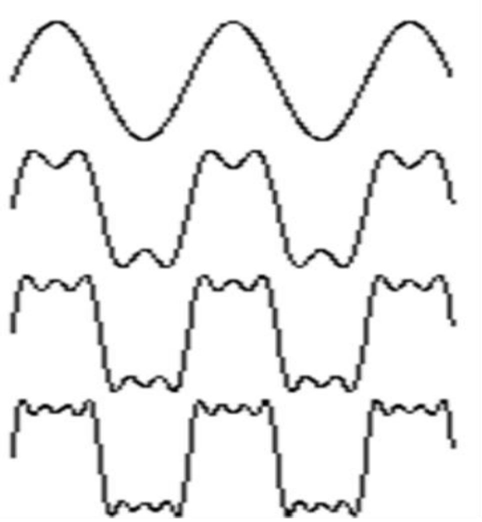
Example 2



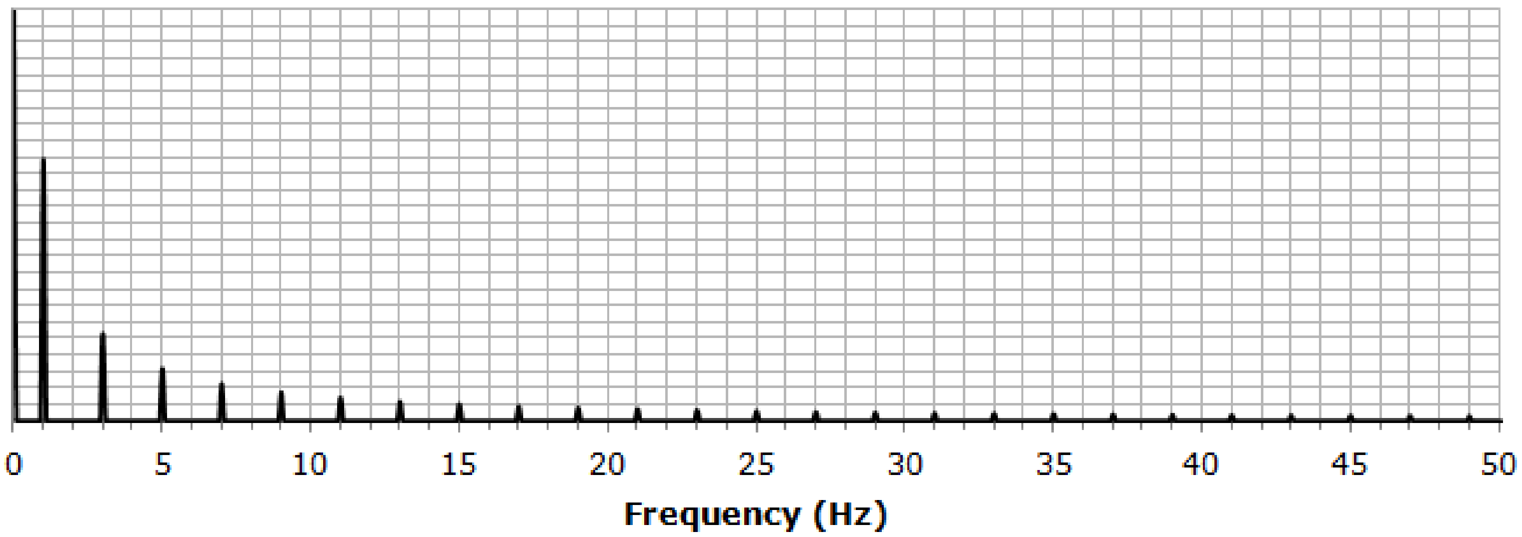


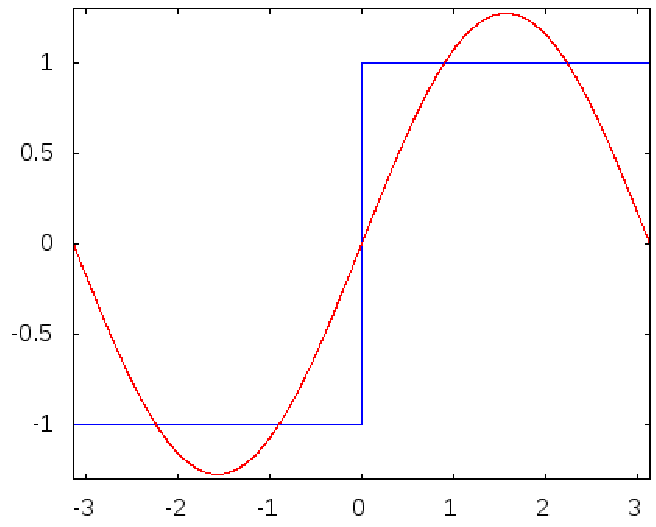
Example 2 – Square Wave





Transform Fourier is (0, 1, 0, 1/3, 0, 1/5, 0, 1/7, ...)





Introduction

Periodic Functions

Back to Fourier

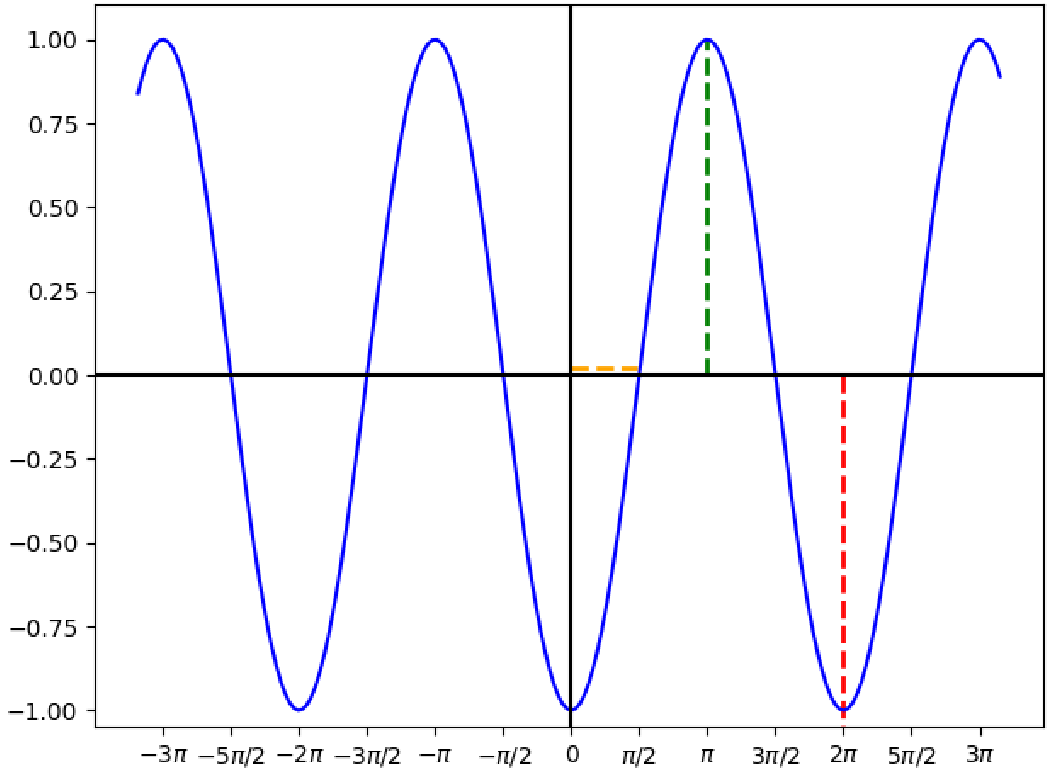
Non-Stationary Signals

Sound

STFT

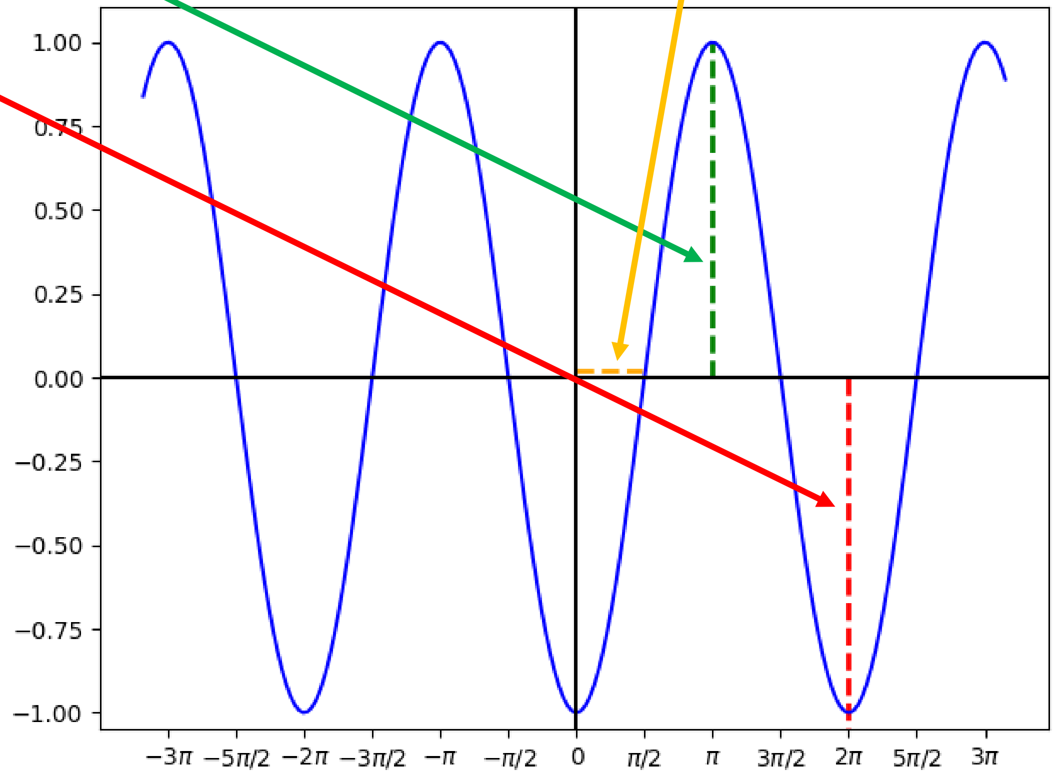
Periodic Functions

A periodic function can be described using its **frequency**, **amplitude** and **phase shift**



Periodic Functions

A periodic function can be described using its **frequency**, **amplitude** and **phase shift**

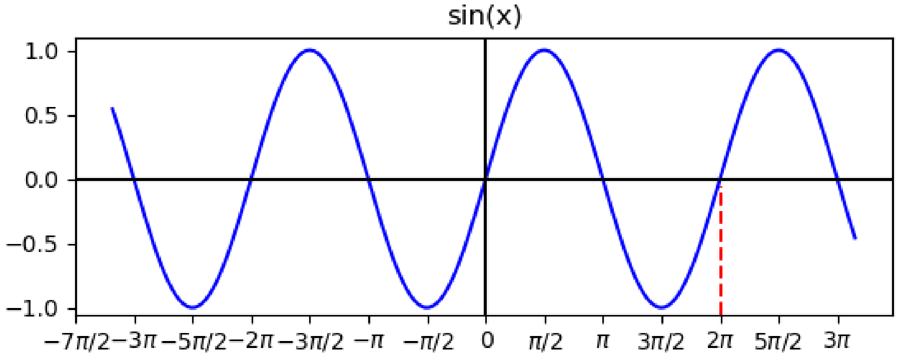


Wavelength and Frequency

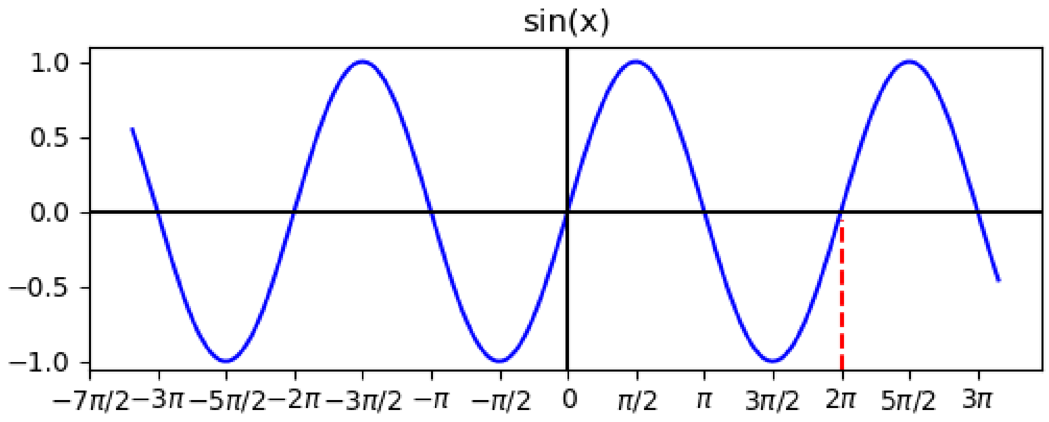
Wavelength - the distance over which a periodic wave's shape repeats

Frequency - the number of waves per unit

$$frequency = \frac{1}{wavelength}$$



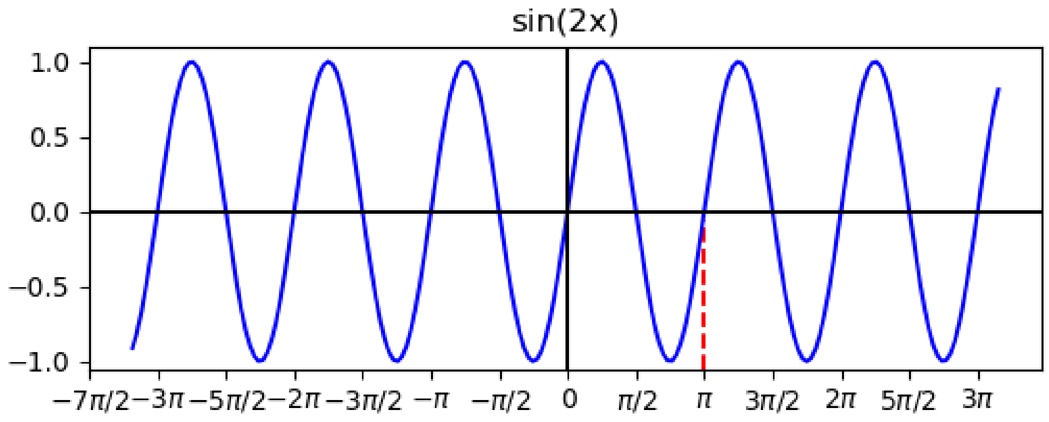
Wavelength and Frequency



The wavelength of $sin(x)$ is 2π

The frequency of $sin(x)$ is $\frac{1}{2\pi}$ Hz

Wavelength and Frequency



The wavelength of *sin(ax)* is $\frac{2\pi}{a}$

The frequency of *sin(ax)* is $\frac{a}{2\pi}$ Hz

Wavelength and Frequency

Another way to write this –

$2\pi\omega x$

- The wavelength of $sin(\cancel{a}x)$ is $\frac{2\pi}{\cancel{a}}$ → $\frac{1}{\omega}$

- The frequency of $sin(a\cancel{x})$ is $\frac{a}{\cancel{2\pi}}$ Hz → $\omega\ Hz$

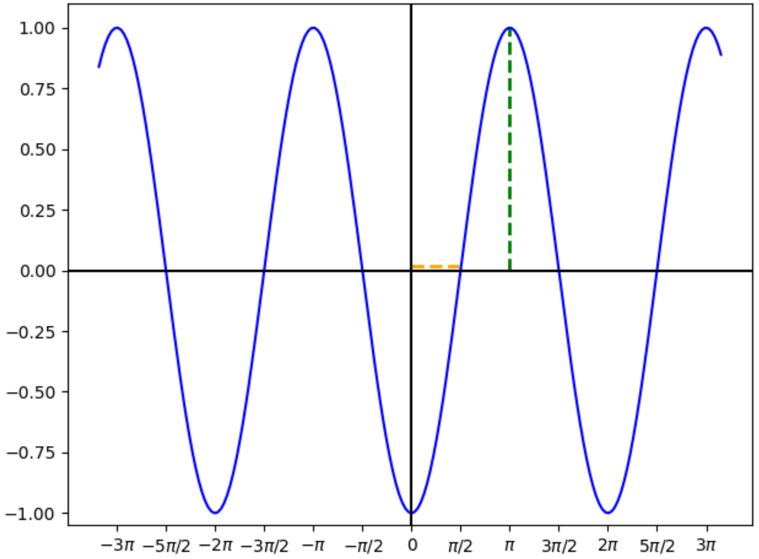
Wavelength and Frequency

Another way to write this -

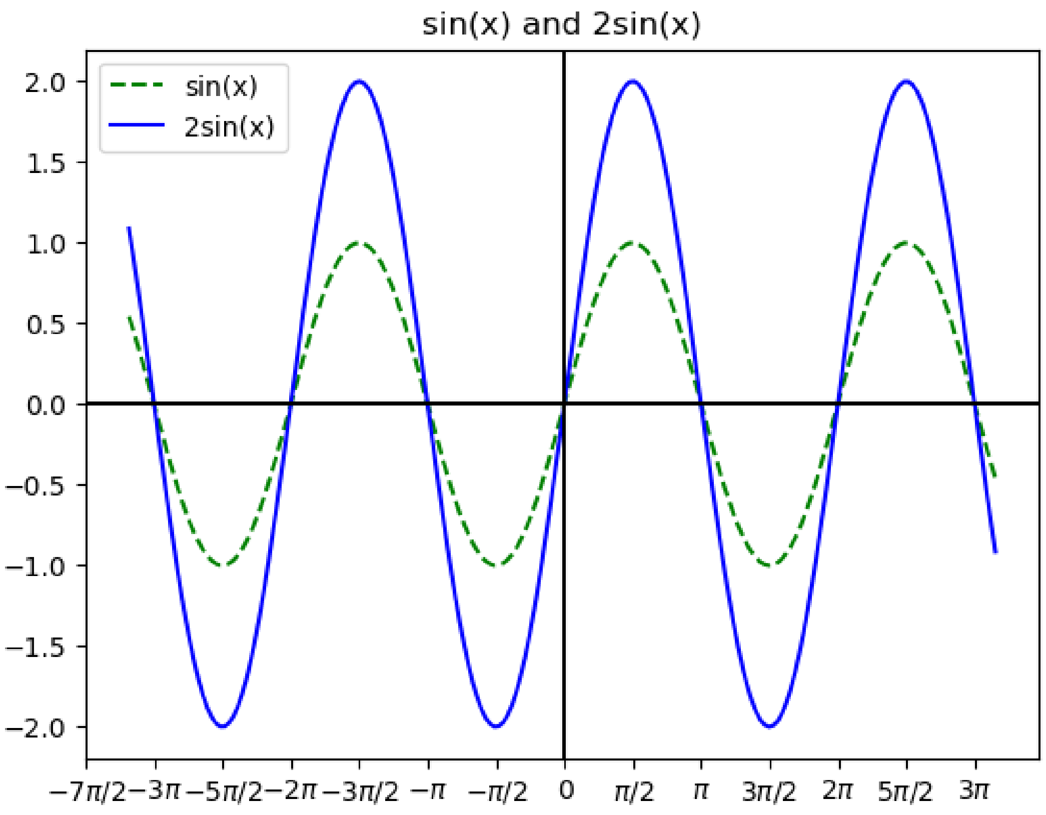
- The wavelength of $sin(2\pi\omega x)$ is $\frac{1}{\omega}$
- The frequency of $sin(2\pi\omega x)$ is ω Hz

Amplitude and Phase

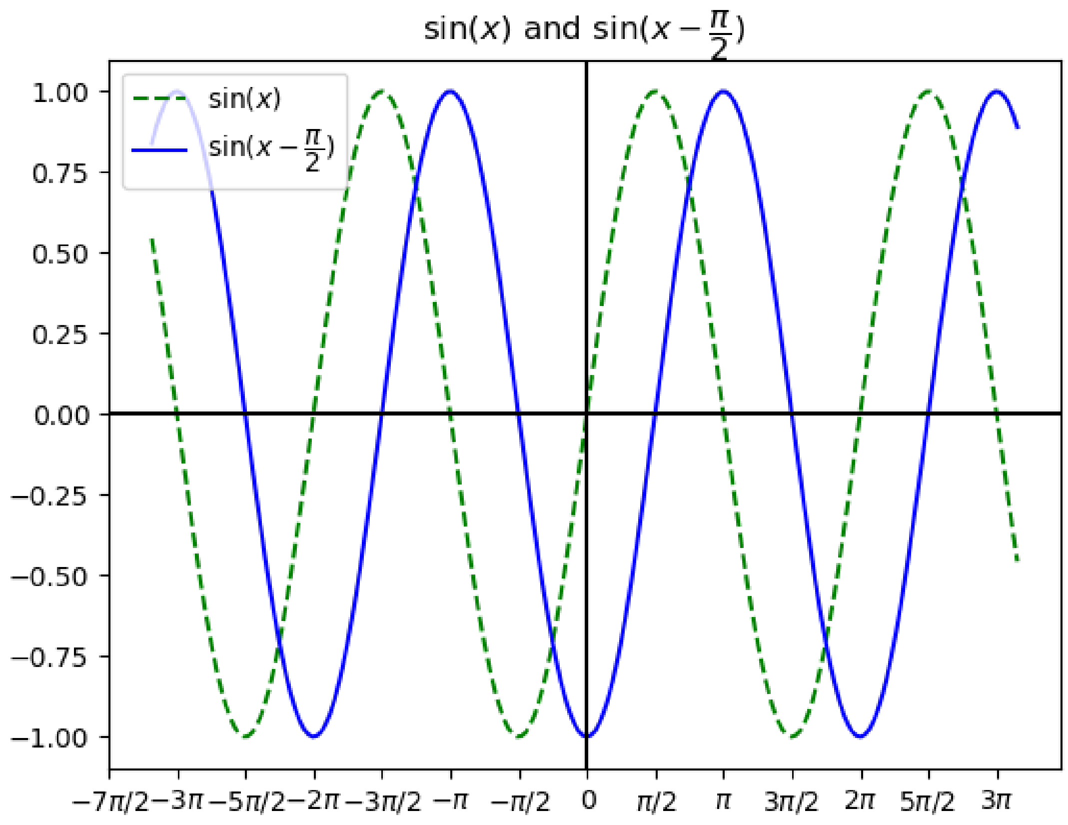
- **<u>Amplitude</u>** – the maximum absolute value of the signal
- **<u>Phase</u>** – how far along the wavelength we are
- **<u>Phase shift</u>** – how far the signal is shifted horizontally



Changing Amplitude



Phase shift


$\sin(x)$ and $\sin(x-\frac{\pi}{2})$
$\sin(x)$
$\sin(x-\frac{\pi}{2})$
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
$-7\pi/2$ -3π $-5\pi/2$ -2π $-3\pi/2$ $-\pi$ $-\pi/2$ 0 $\pi/2$ π $3\pi/2$ 2π $5\pi/2$ 3π

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Why is that important?

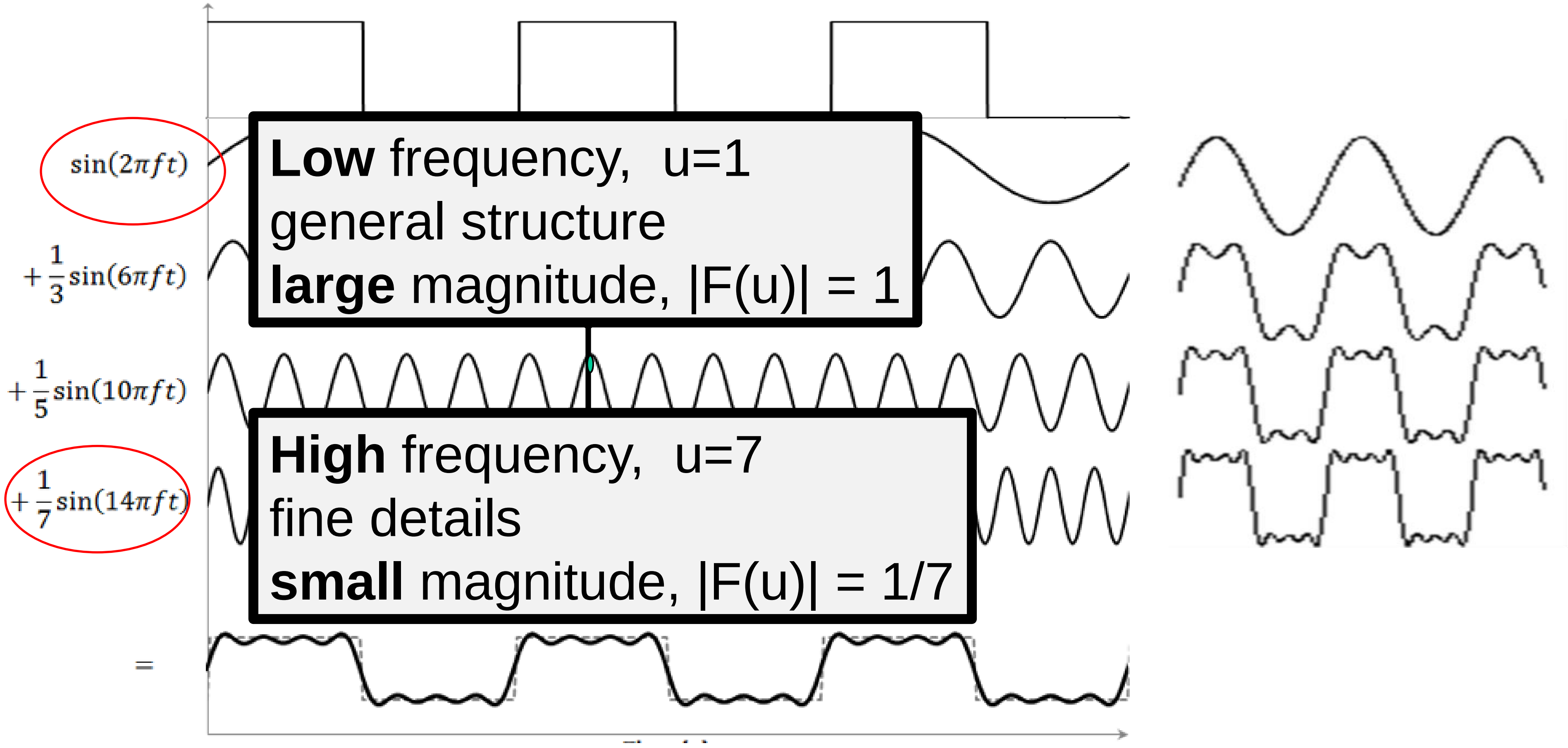
Using the Fourier transform, we are moving from the ***time*** domain to the ***frequency*** domain

Why is that important?

- Decomposition into different resolutions
  - **Low frequencies**: rough general structure
  - **High frequencies**: fine detail

- Very useful for signal understanding and processing
  - Filtering, denoising, compression…

Example 2 – Square Wave



How do we do this?

We want to decompose a function $f(x)$ to a set of sin and cos functions of different frequencies –

$$f(x) = \sum_{\omega} a_{\omega} \cos\left(\frac{2\pi\omega x}{N}\right) + b_{\omega} \sin\left(\frac{2\pi\omega x}{N}\right)$$

We need to find a_{ω} and b_{ω}

$* \;\; N \; data \; points$

We will find them using Fourier transform!

How do we do this?

Discrete Fourier transform (DFT):

$$F(\omega) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x \omega}{N}}$$

Euler's formula:

$$e^{i\theta} = \cos(\theta) + i sin(\theta)$$

How do we do this?

$$F(\omega) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x\omega}{N}}$$

$$= \sum_{x=0}^{N-1} f(x) \left[\cos\left(-\frac{2\pi x\omega}{N}\right) + i \cdot sin(-\frac{2\pi x\omega}{N})\right]$$

$$= \sum_{x=0}^{N-1} f(x) \left[\cos\left(\frac{2\pi x\omega}{N}\right) - i \cdot sin(\frac{2\pi x\omega}{N})\right]$$

a_ω b_ω

$$= \boxed{\sum_{x=0}^{N-1} f(x) \cos\left(\frac{2\pi x\omega}{N}\right)} - i \boxed{\sum_{x=0}^{N-1} f(x) \sin\left(\frac{2\pi x\omega}{N}\right)}$$

Discrete Fourier Transform (DFT)

Discrete Fourier transform (DFT):

$$F(\omega) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x \omega}{N}}$$

Inverse Discrete Fourier transform (IDFT):

$$f(x) = \frac{1}{N} \sum_{\omega=0}^{N-1} F(\omega)\, e^{\frac{2\pi i x \omega}{N}}$$

- In real life we use FFT (Fast Fourier Transform) – takes only $O(NlogN)$ instead of $O(N^2)$

But Why?

$$i^2 = -1$$

$$c = a + bi = R \cdot e^{i\alpha} = R \cdot \cos(\alpha) + i \cdot R \cdot \sin(\alpha)$$

$$f(x) = \frac{1}{N} \sum_{\omega=0}^{N-1} F(\omega)\, e^{\frac{2\pi i x \omega}{N}}$$

$$F(\omega) \in \mathbb{C}, F(\omega) = Re^{i\alpha}$$

$$\downarrow$$

$$F(\omega)e^{\frac{2\pi i x \omega}{N}} = Re^{i\alpha}e^{\frac{2\pi i x \omega}{N}} = Re^{i\left(\frac{2\pi x \omega}{N}+\alpha\right)} =$$

$$R\left(\cos\left(\frac{2\pi x \omega}{N} + \alpha\right) + i \cdot \sin\left(\frac{2\pi x \omega}{N} + \alpha\right)\right)$$

Discrete Fourier Transform (DFT)

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FT Properties

1. Linearity:

$$\Phi\big(f(x)+g(x)\big)=\Phi\big(f(x)\big)+\Phi\big(g(x)\big)$$
$$\Phi\big(a\cdot f(x)\big)=a\cdot\Phi\big(f(x)\big)$$

2. Scaling: if $f(x)\xrightarrow[Fourier]{}F(u)$

then $f(ax)\xrightarrow[Fourier]{}\frac{1}{|a|}\cdot F(\frac{u}{a})$

FT Properties

3. Periodicity:

$$\forall k \in Z \qquad F(u) = F(u + kN)$$

$$\Longrightarrow \quad \begin{aligned} F(u) &= F^*(-u) = F^*(N-u) \\ |F(u)| &= |F(-u)| = |F(N-u)| \end{aligned}$$

4. Symmetry:

$$F(-u) = F^*(u)$$

$$|F(u)| = |F(-u)|$$

How are the Values Real?

Because the signal is real ($\mathbb{R}$)

$$F(u) = F^*(-u) \longrightarrow F(-u) = a - bi = F^*(u) \Rightarrow F(u) = a + bi$$

$$F(u) = Re^{i\alpha}, \qquad F(-u) = Re^{-i\alpha}$$

$$f(x) = \cdots + F(\omega)e^{\frac{2\pi i x\omega}{N}} + F(-\omega)e^{\frac{-2\pi i x\omega}{N}} + \cdots$$

$$F(\omega)e^{\frac{2\pi i x\omega}{N}} + F(-\omega)e^{\frac{-2\pi i x\omega}{N}} = Re^{i\alpha}e^{\frac{2\pi i x\omega}{N}} + Re^{-i\alpha}e^{\frac{-2\pi i x\omega}{N}}$$

How are the Values Real?

$$F(\omega)e^{\frac{2\pi i x\omega}{N}} + F(-\omega)e^{\frac{-2\pi i x\omega}{N}} = Re^{i\alpha}e^{\frac{2\pi i x\omega}{N}} + Re^{-i\alpha}e^{\frac{-2\pi i x\omega}{N}}$$

$$= Re^{\frac{2\pi i x\omega}{N}+i\alpha} + Re^{-\left(\frac{2\pi i x\omega}{N}+i\alpha\right)}$$

$$= R\left(\cos\left(\frac{2\pi x\omega}{N}+\alpha\right) + i\cdot\sin\left(\frac{2\pi x\omega}{N}+\alpha\right) + \cos\left(\frac{-2\pi x\omega}{N}-\alpha\right) + i\cdot\sin\left(\frac{-2\pi x\omega}{N}-\alpha\right)\right)$$

$$= R\left(\cos\left(\frac{2\pi x\omega}{N}+\alpha\right) + i\cdot\sin\left(\frac{2\pi x\omega}{N}+\alpha\right) + \cos\left(\frac{2\pi x\omega}{N}+\alpha\right) - i\cdot\sin\left(\frac{2\pi x\omega}{N}+\alpha\right)\right)$$

$$= 2R\cos\left(\frac{2\pi x\omega}{N}+\alpha\right)$$

F(0)?

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{\frac{-2\pi i u x}{N}}$$

$$F(0) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\, e^{\frac{-2\pi i 0 x}{N}}$$

$$F(0) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)$$ $\approx$ **Signal average**

FT Presentation

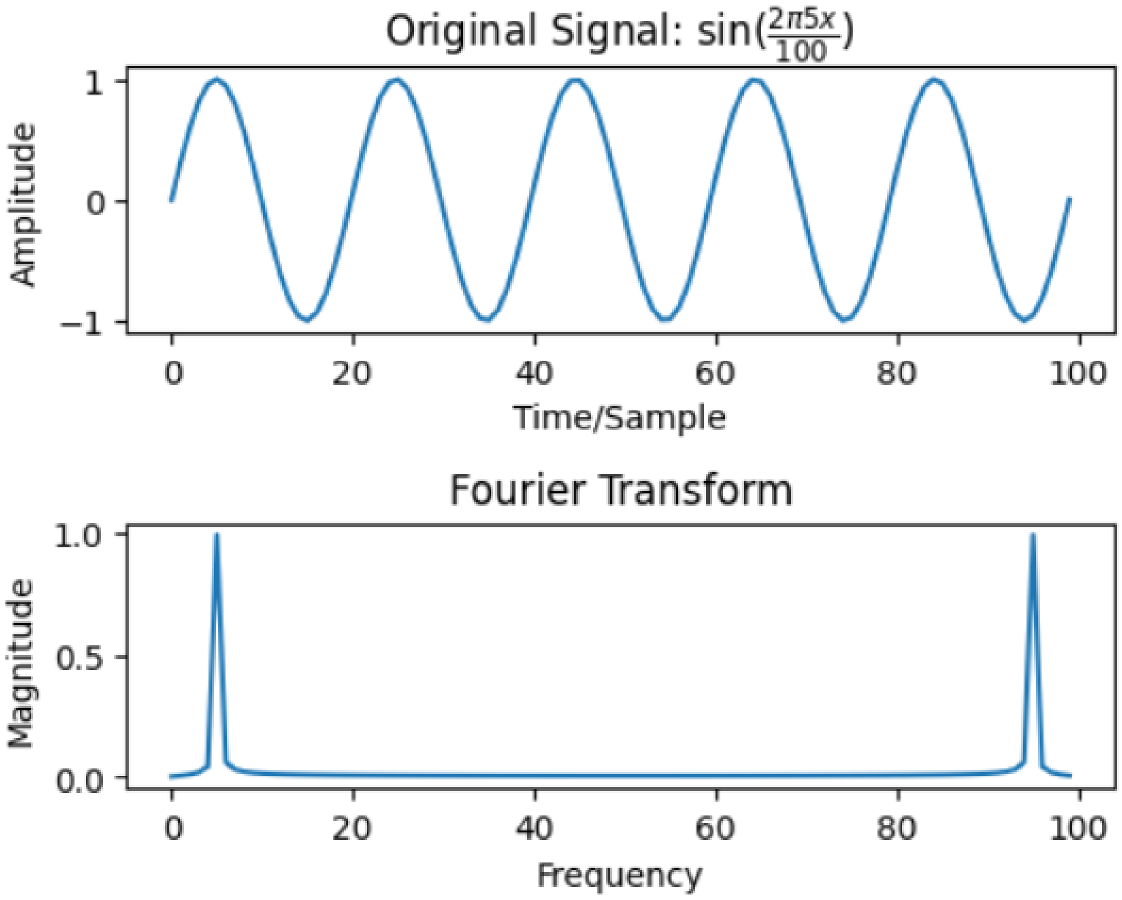
- FT returns complex numbers

$$F(u) = R(u) + i \cdot I(u)$$

- To visualize we use the amplitude

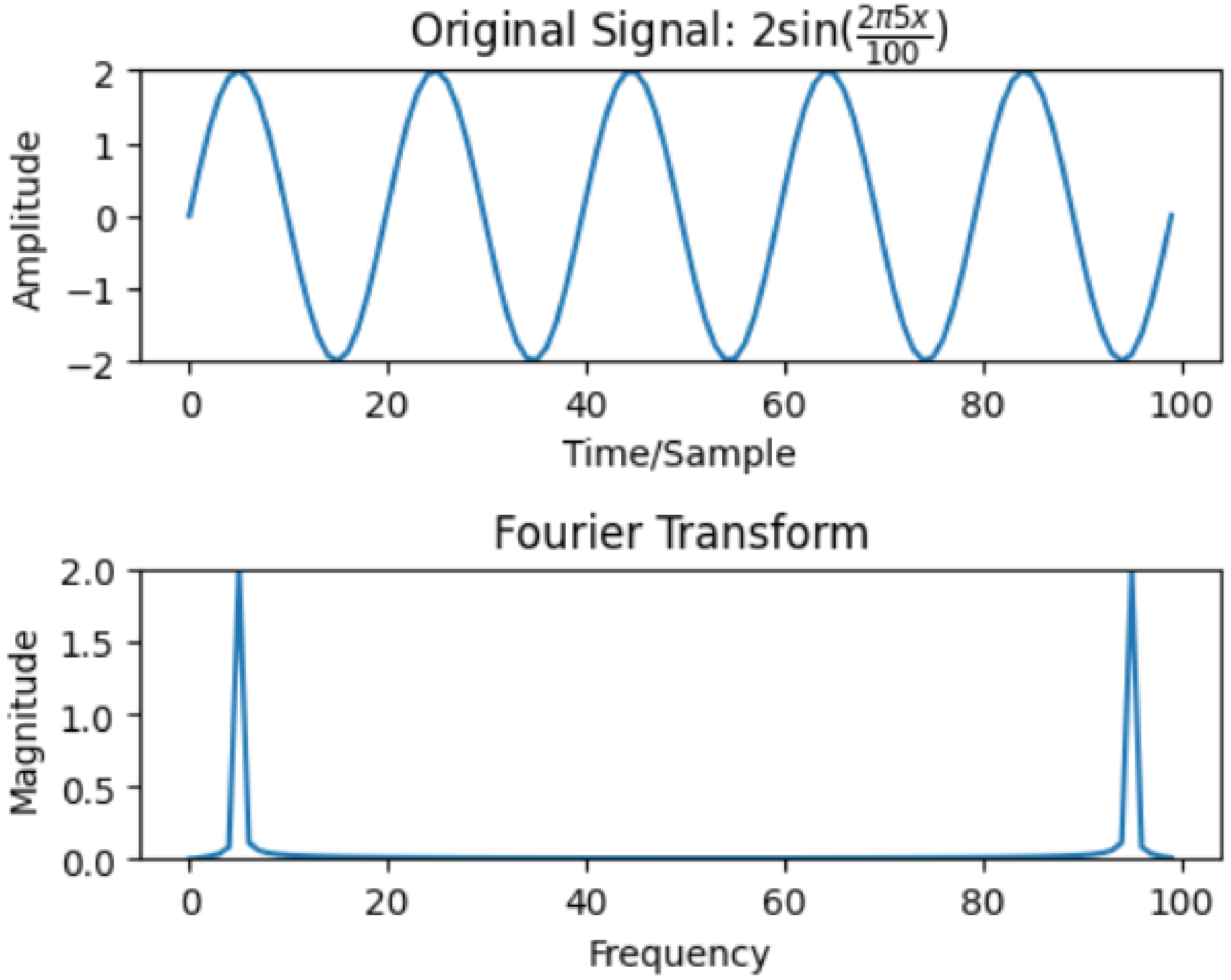
$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

Examples

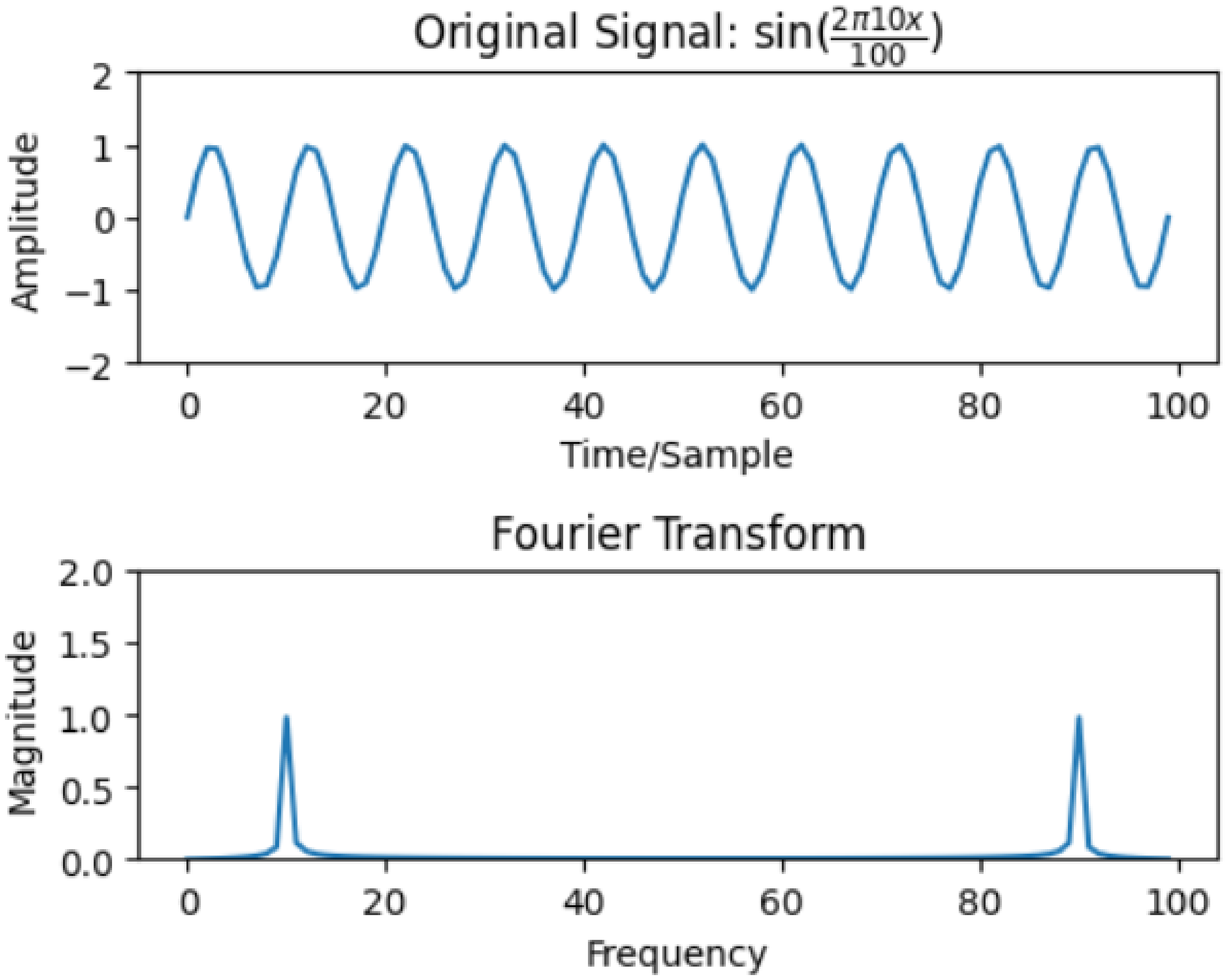


Original Signal: $\sin(\frac{2\pi 5x}{100})$
Amplitude
1
0
−1
0
20
40
60
80
100
Time/Sample
Fourier Transform
Magnitude
1.0
0.5
0.0
0
20
40
60
80
100
Frequency

Examples

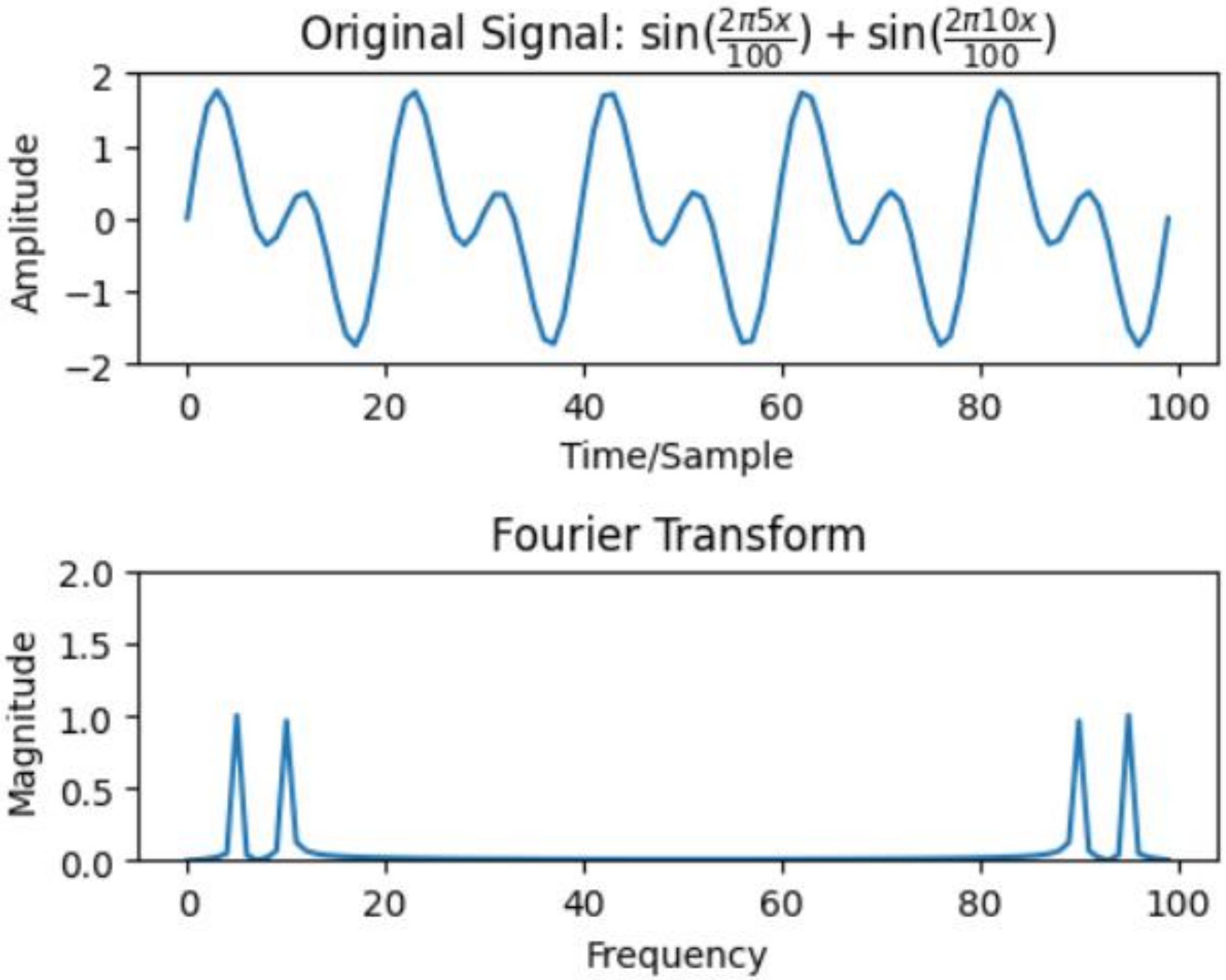

Original Signal: $2\sin(\frac{2\pi 5x}{100})$
Amplitude
2
1
0
−1
−2
0
20
40
60
80
100
Time/Sample
Fourier Transform
Magnitude
2.0
1.5
1.0
0.5
0.0
0
20
40
60
80
100
Frequency

Examples



Original Signal: $\sin(\frac{2\pi 10x}{100})$
Amplitude
2
1
0
−1
−2
0
20
40
60
80
100
Time/Sample
Fourier Transform
Magnitude
2.0
1.5
1.0
0.5
0.0
0
20
40
60
80
100
Frequency

Examples



Original Signal: $\sin(\frac{2\pi 5x}{100}) + \sin(\frac{2\pi 10x}{100})$
Amplitude
2
1
0
−1
−2
0
20
40
60
80
100
Time/Sample
Fourier Transform
Magnitude
2.0
1.5
1.0
0.5
0.0
0
20
40
60
80
100
Frequency

Fourier Basis Vectors

DFT is a **basis transform** –

We are moving from the ***standard basis*** *to the* ***Fourier basis***

$$(f(0), f(1), f(2), \dots, f(N-1)) \xrightarrow[Fourier]{} (F(0), F(1), F(2), \dots, f(N-1))$$

Spatial domain

(Standard basis)

→

Frequency domain

(Fourier basis)

Standard Basis Vectors

$$
\begin{aligned}
f &= \big(f(0), f(1), \dots, f(N-1)\big) = \\
&f(0) \cdot (1, 0, \dots, 0) + \\
&f(1) \cdot (0, 1, \dots, 0) + \\
&\dots + \\
&f(N-1) \cdot (0, 0, \dots, 1)
\end{aligned}
$$

$(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$, …, $(0, 0, \dots, 1)$ are the standard basis vectors

The DFT Matrix

$$F(\omega) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x \omega}{N}} \leftrightarrow \vec{F} = M_{N\times N}\vec{f}$$

$$\begin{pmatrix} \gamma^0 & \gamma^0 & \gamma^0 & & \gamma^0 \\ \gamma^0 & \gamma^1 & \gamma^2 & \cdots & \gamma^{N-1} \\ \gamma^0 & \gamma^2 & \gamma^4 & & \gamma^{2(N-1)} \\ & \vdots & & & \vdots \\ \gamma^0 & \gamma^{N-1} & \gamma^{2(N-1)} & \cdots & \gamma^{(N-1)^2} \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(N-1) \end{pmatrix} = \begin{pmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ \boxed{F(N-1)} \end{pmatrix}$$

Where $\gamma = e^{-\frac{2\pi i}{N}}$ is the Fourier basis

The DFT Matrix

$$F(N-1) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x (N-1)}{N}}$$

The DFT Matrix

$$F(N-1) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x(N-1)}{N}}$$

$$= \sum_{x=0}^{N-1} f(x) \left(e^{-\frac{2\pi i}{N}}\right)^{x(N-1)}$$

The DFT Matrix

$$F(N-1) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x(N-1)}{N}}$$

$$= \sum_{x=0}^{N-1} f(x) \boxed{\left(e^{-\frac{2\pi i}{N}}\right)}^{x(N-1)} \longrightarrow \boxed{\omega = e^{-\frac{2\pi i}{N}}}$$

The DFT Matrix

$$F(N-1) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x(N-1)}{N}}$$

$$= \sum_{x=0}^{N-1} f(x) \left(e^{-\frac{2\pi i}{N}}\right)^{x(N-1)} = \sum_{x=0}^{N-1} f(x)\, \omega^{x(N-1)}$$

The DFT Matrix

$$F(N-1) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x(N-1)}{N}}$$

$$= \sum_{x=0}^{N-1} f(x) \left(e^{-\frac{2\pi i}{N}}\right)^{x(N-1)} = \sum_{x=0}^{N-1} f(x)\, \omega^{x(N-1)}$$

$$= f(0)\cdot\omega^{0} + f(1)\cdot\omega^{N-1} + f(2)\cdot\omega^{2(N-1)} + \cdots + f(N-1)\cdot\omega^{(N-1)^2}$$

The DFT Matrix

$$F(\omega) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x \omega}{N}} \leftrightarrow \vec{F} = M_{N\times N}\vec{f}$$

$$\frac{1}{N}\begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \cdots & \omega^{N-1} \\ \omega^0 & \omega^2 & \omega^4 & & \omega^{2(N-1)} \\ & \vdots & & & \vdots \\ \omega^0 & \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(N-1) \end{pmatrix} = \begin{pmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ F(N-1) \end{pmatrix}$$

Where $\omega = e^{-\frac{2\pi i}{N}}$ is the Fourier basis

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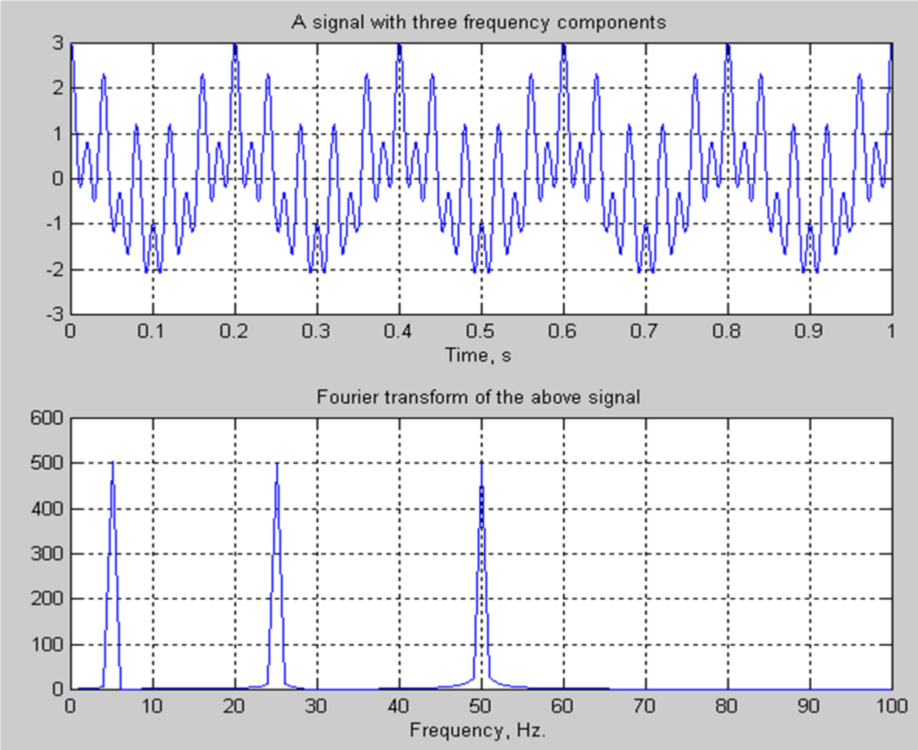
Stationary vs. Non-Stationary

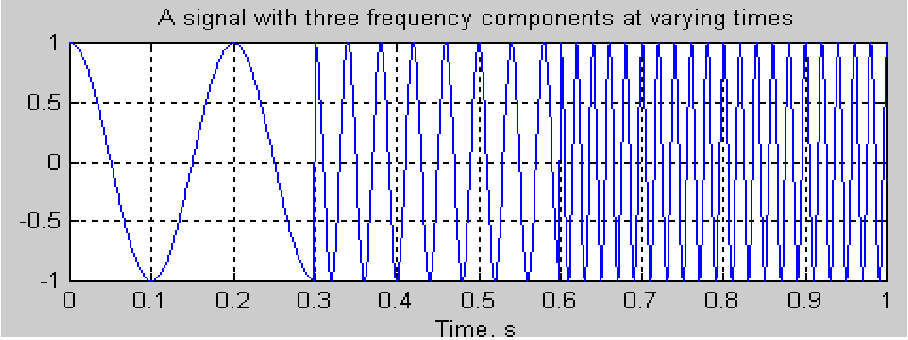
Stationary:

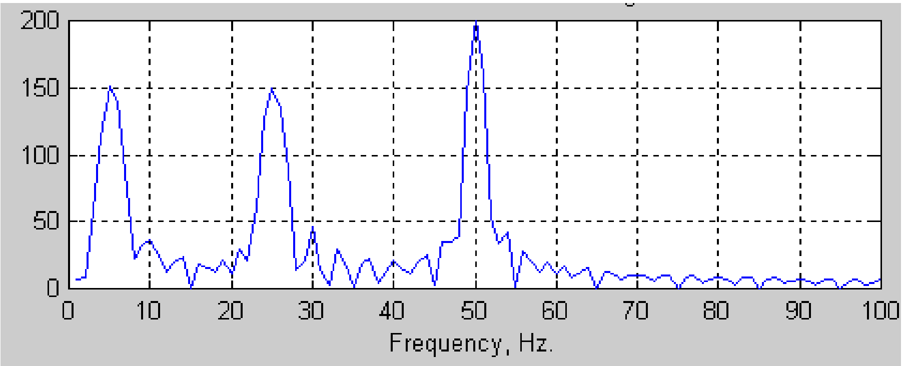
$\cos(2\pi \cdot 5t) + \cos(2\pi \cdot 25t) + \cos(2\pi \cdot 50t)$

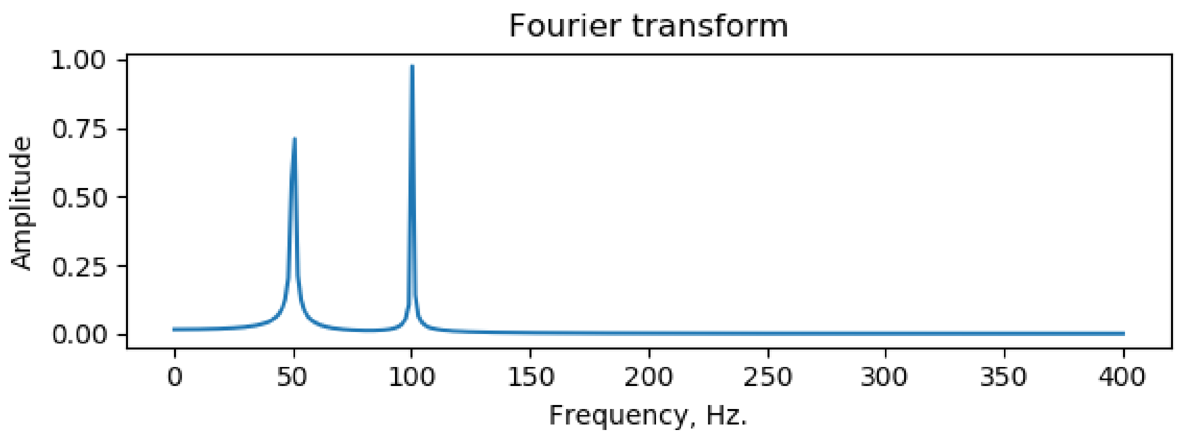


Non-stationary

$\cos(2\pi \cdot 5t)$ ***then*** $\cos(2\pi \cdot 25t)$ ***then*** $\cos(2\pi \cdot 50t)$







Fourier provides **localization in the frequency** domain but **no localization in the time** domain.

Introduction

Periodic Functions

Back to Fourier

Non-Stationary Signals

Sound

STFT

Spectrograms

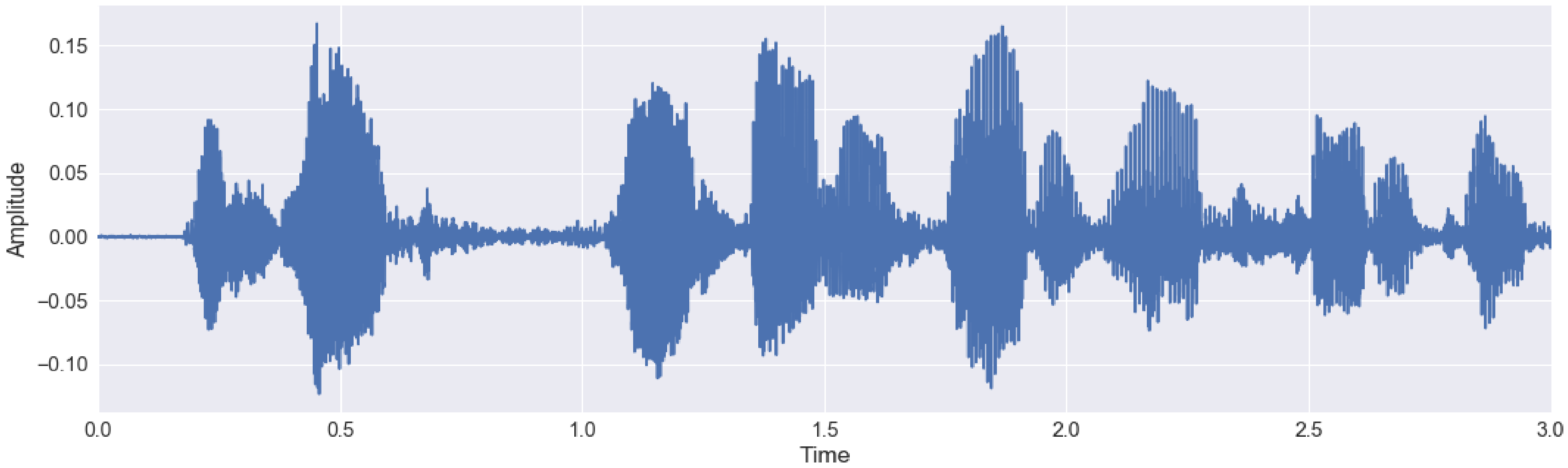
Sound

- Sound is a vibration that typically propagates as an audible wave of pressure, through a transmission medium such as a gas, liquid or solid. (Wikipedia)

- We represent sound as a signal

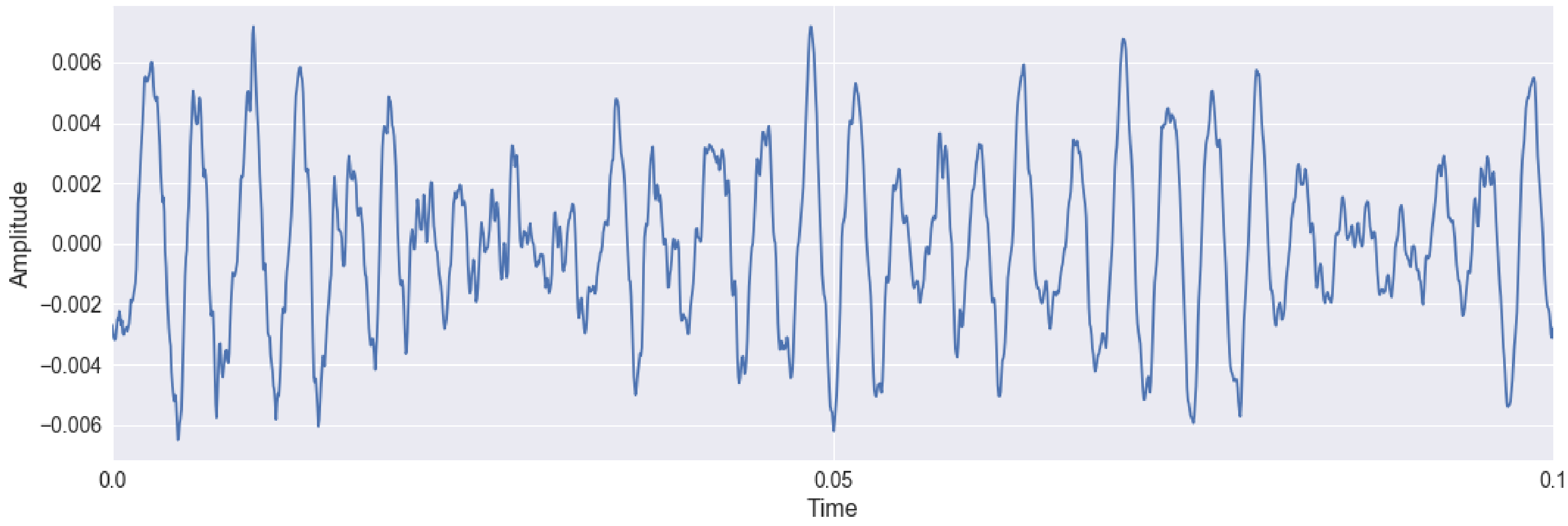
Sound - Waveform

A non-stationary 1D signal



Sound - Waveform

Zoom in..



Sound VS Images

Images and sound representations are not very different:

- Quality:
  - Image quality depends on resolution, number of pixels in the image
  - Sound quality depends on sampling rate (samples/second)
- Memory:
  - Images are represented using 8 bits per pixel (values between $[0, 255]$)
  - In sound the bit depth (bits/sample) is defined as the amplitude resolution
    - Telephone, AM radio – 8 bit: $[-127, +127]$
    - Audio CDs – 16 bit: $[-32,768, +32,767]$

Periodic Functions

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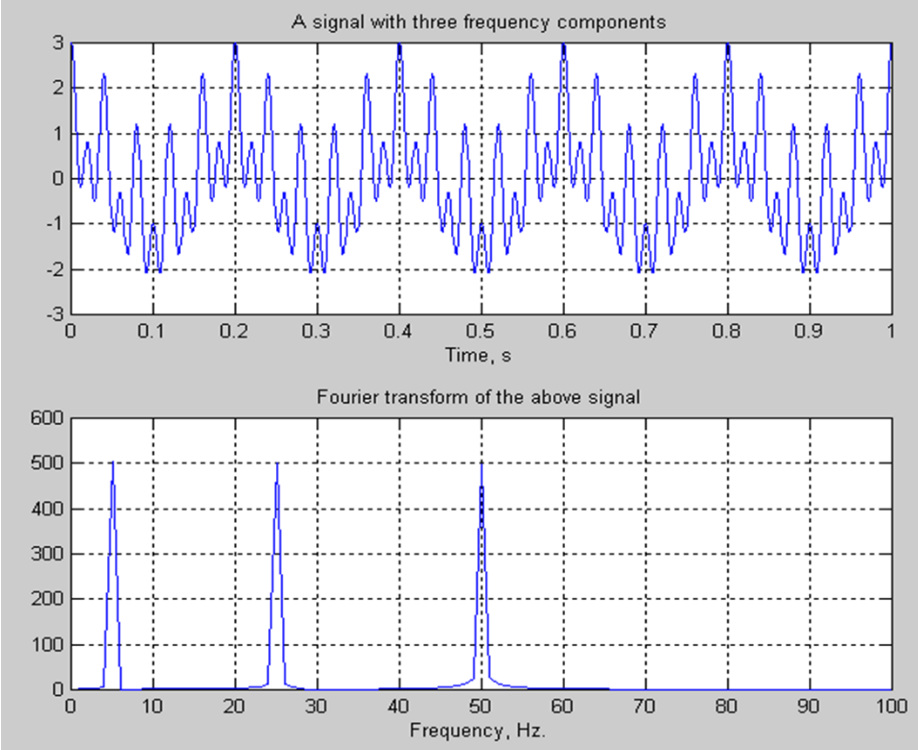
Stationary vs. Non-Stationary

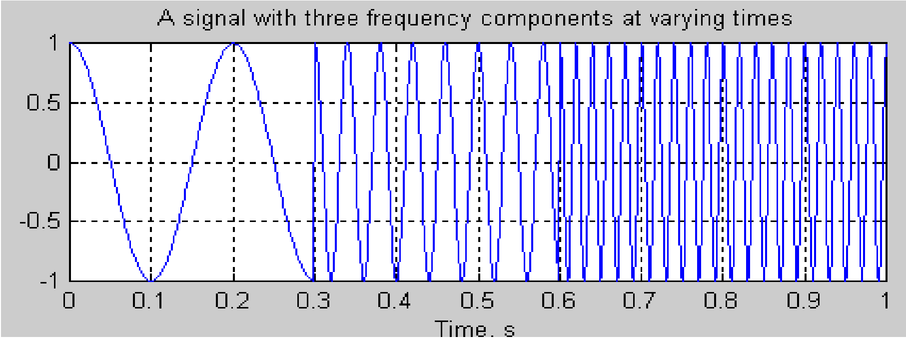
Stationary:

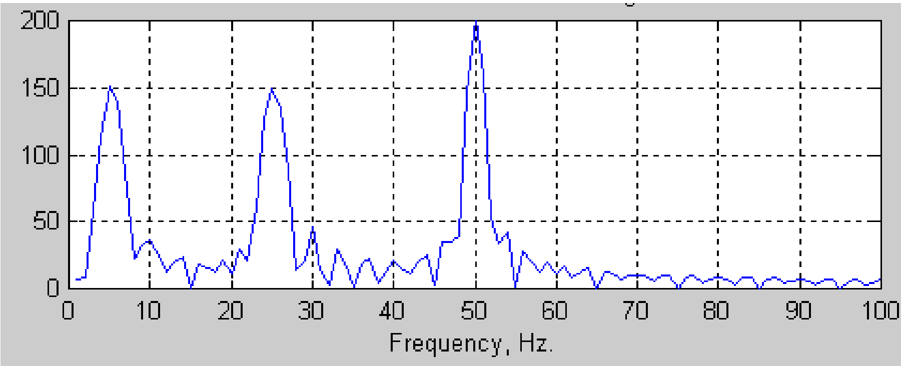
$\cos(2\pi \cdot 5t) + \cos(2\pi \cdot 25t) + \cos(2\pi \cdot 50t)$

Non-stationary

$\cos(2\pi \cdot 5t)$ ***then*** $\cos(2\pi \cdot 25t)$ ***then*** $\cos(2\pi \cdot 50t)$



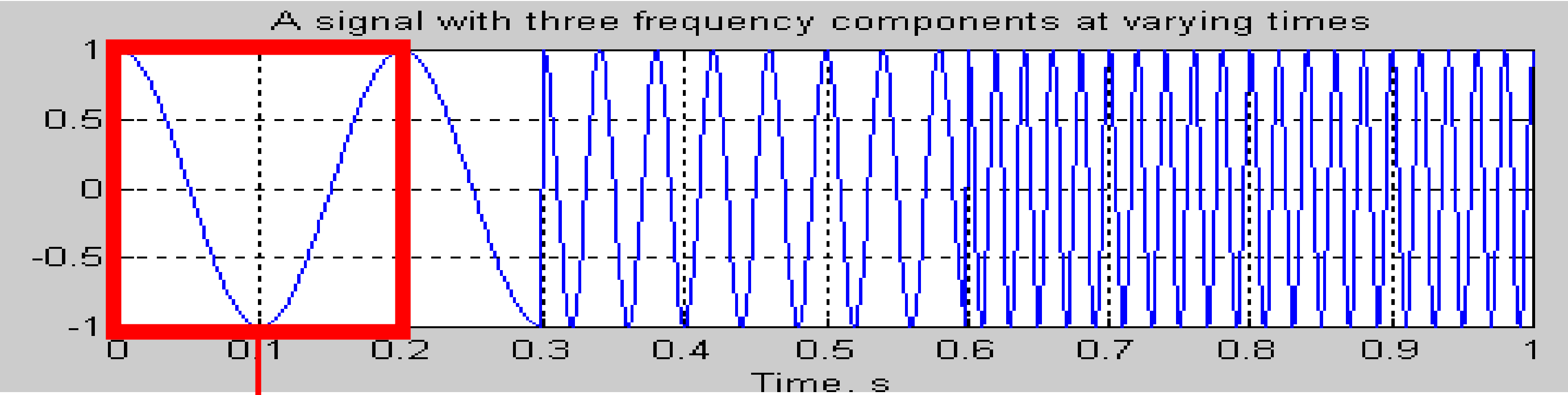




Short-Time Fourier Transform

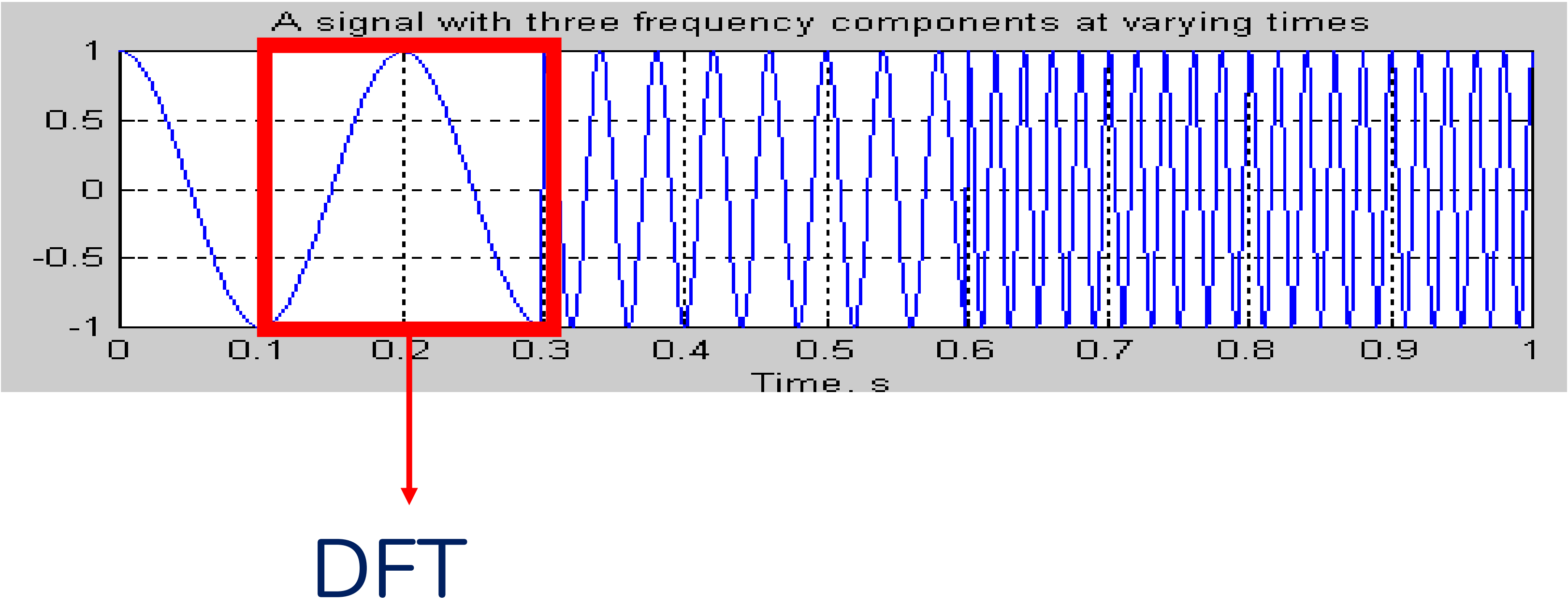
- When applying DFT on non-stationary signals, **localization in time is lost**
- To avoid this, we would like to apply DFT independently on **short-time segments**
- We assume the segments are short enough to be **considered stationary**
- This is called the **Short-Time Fourier Transform (STFT)**

Short-Time Fourier Transform

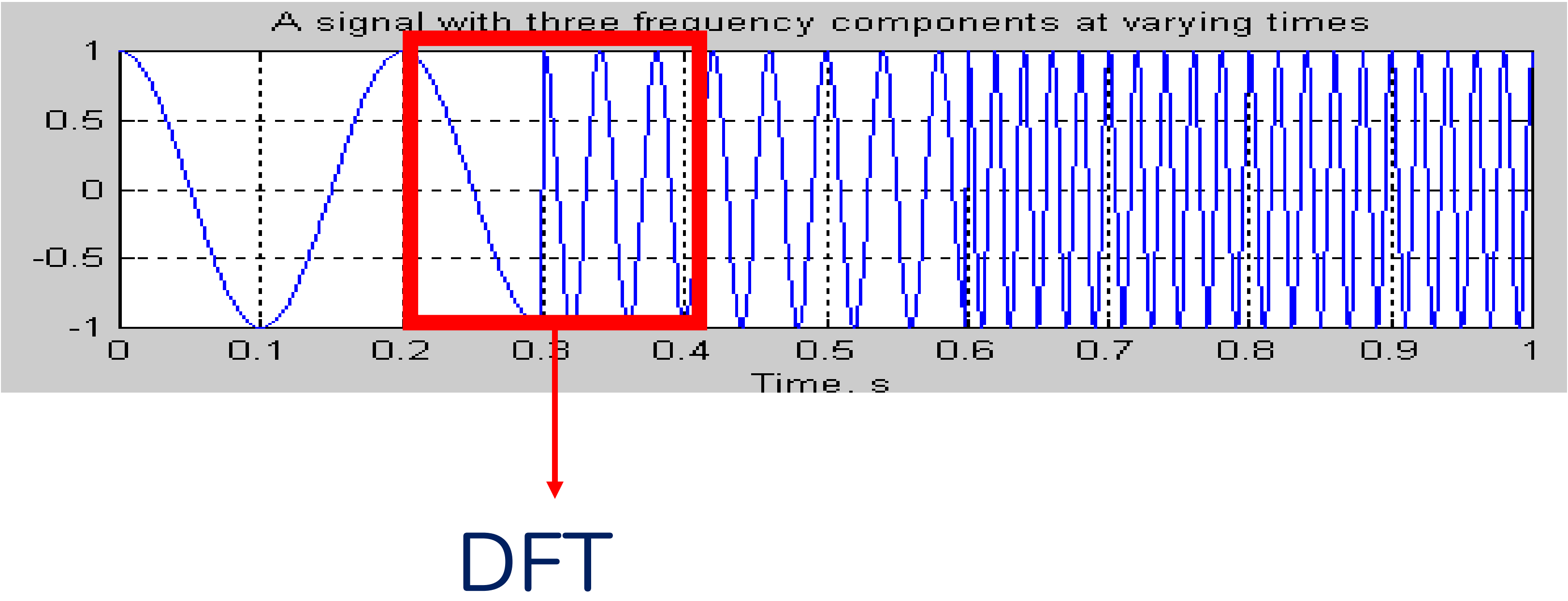


DFT

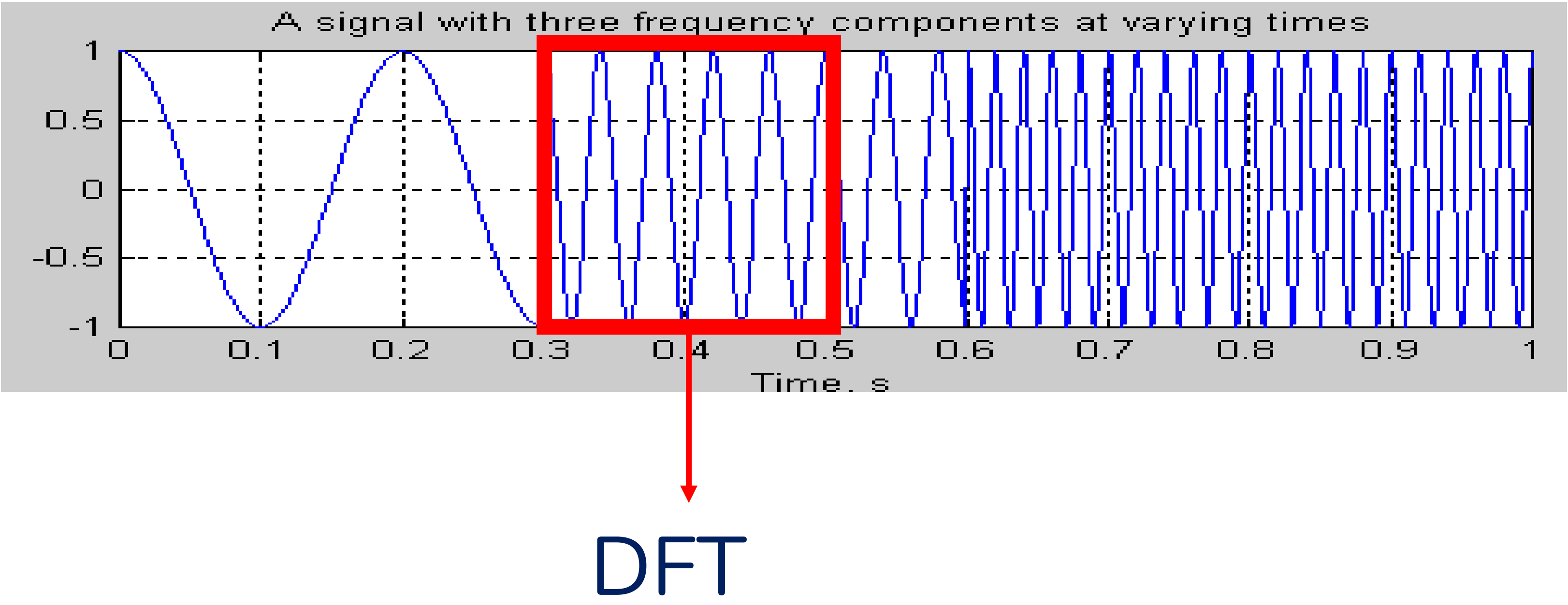
Short-Time Fourier Transform



Short-Time Fourier Transform



Short-Time Fourier Transform



STFT - Steps

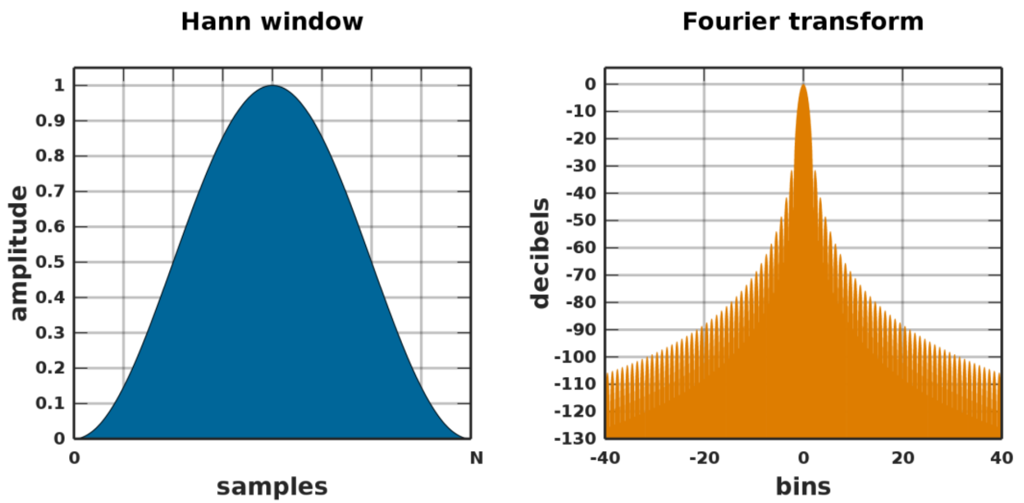
1. Choose a window of finite length
2. Place the window on top of signal at $t = 0$
3. Truncate the signal using this window
4. Compute DFT of the truncated signal, save results
5. Incrementally slide the window to the right
6. Go back to step 3, until window reaches the end of the signal

Windows

- **Shape**: rectangular, Gaussian, triangle.
- **Length** (W) – the size of the window
  - Longer: better frequency resolution
  - Shorter: better time resolution
- **Shift** (L) – how much does the window move every time
  - Smaller: smoother results
  - Larger: less computation

Windows

- Window overlap: $W - L$
- Why do we want overlap?
  - To avoid unnatural discontinuities between the segments
- This is why we generally don't use rectangular window shape



STFT

DFT: $F(\omega) = \sum_{x=0}^{N-1} f(x)\, e^{-\frac{2\pi i x \omega}{N}}$

IDFT: $f(x) = \frac{1}{N} \sum_{\omega=0}^{N-1} F(\omega)\, e^{\frac{2\pi i x \omega}{N}}$

STFT for time t:

$$F(\omega, t) = \sum_{x=-\infty}^{\infty} f(x)\, W(t-x) e^{-\frac{2\pi i x \omega}{N}}$$

ISTFT: (K is a normalization constant)

$$f(x) = K \sum_{p=-\infty}^{\infty} \sum_{u=0}^{N-1} F(u, pL) e^{\frac{2\pi i u x}{N}}$$

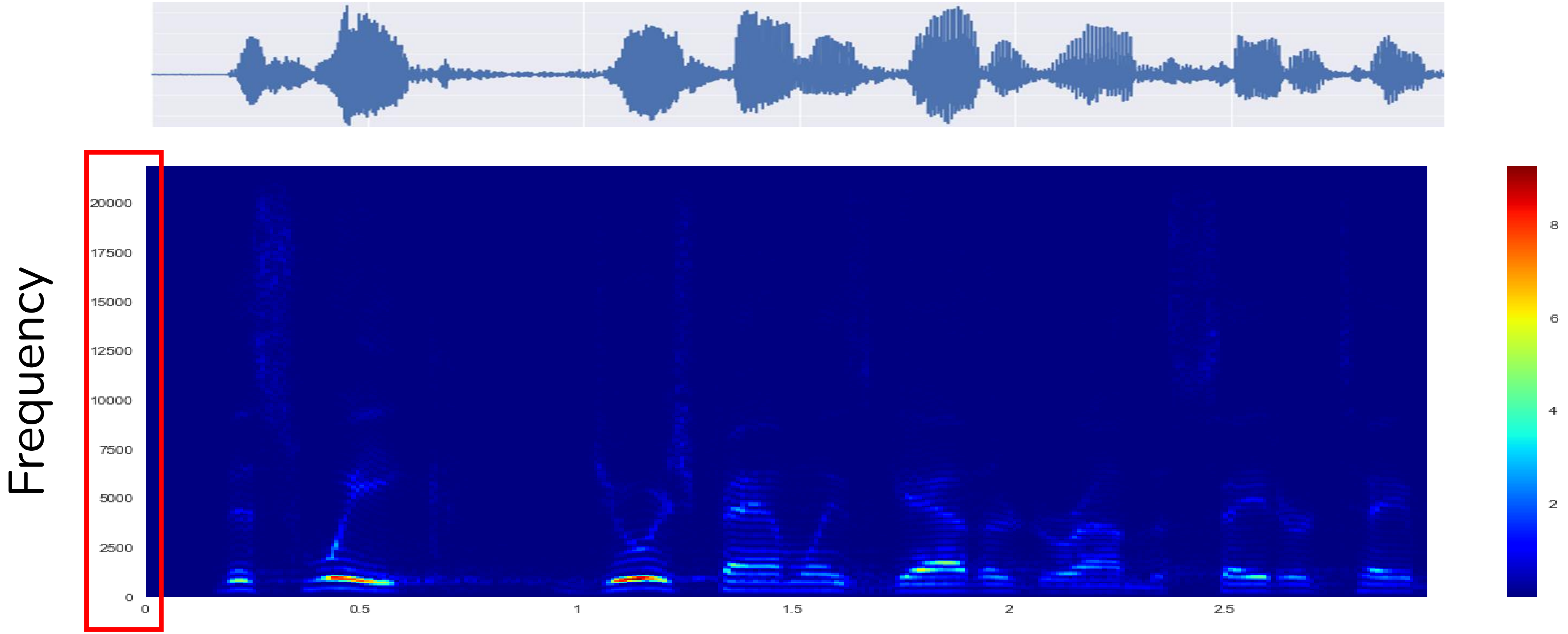
Back to Fourier
Non-Stationary Signals
Sound
STFT

Spectrograms

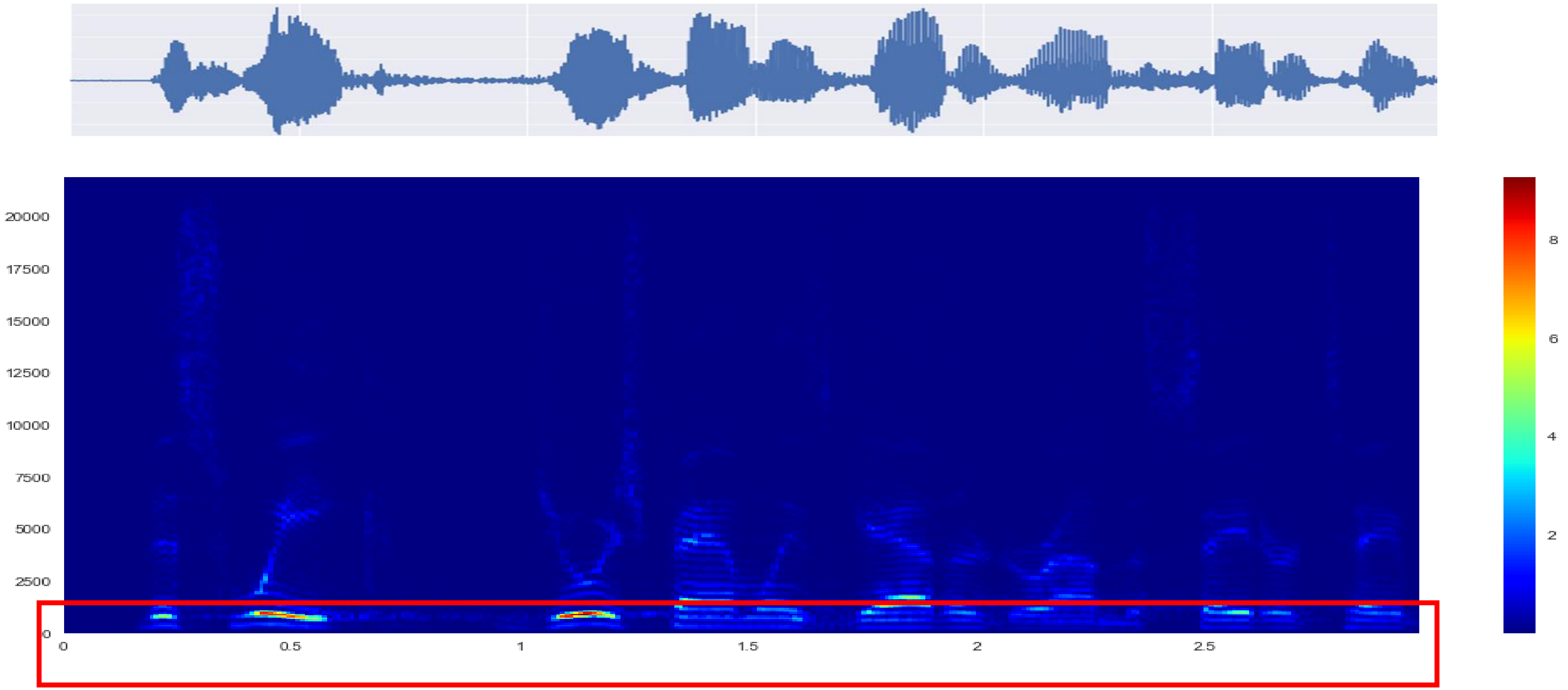
Spectrograms

- Visual representation of the spectrum of frequencies of a signal as it varies with time. (Wikipedia)
- 3D representation –
  – X-axis represents time
  – Y-axis represents frequency
  – Colors represent amplitude\magnitude
- Helps us present the localization in time we achieved from STFT

Spectrograms

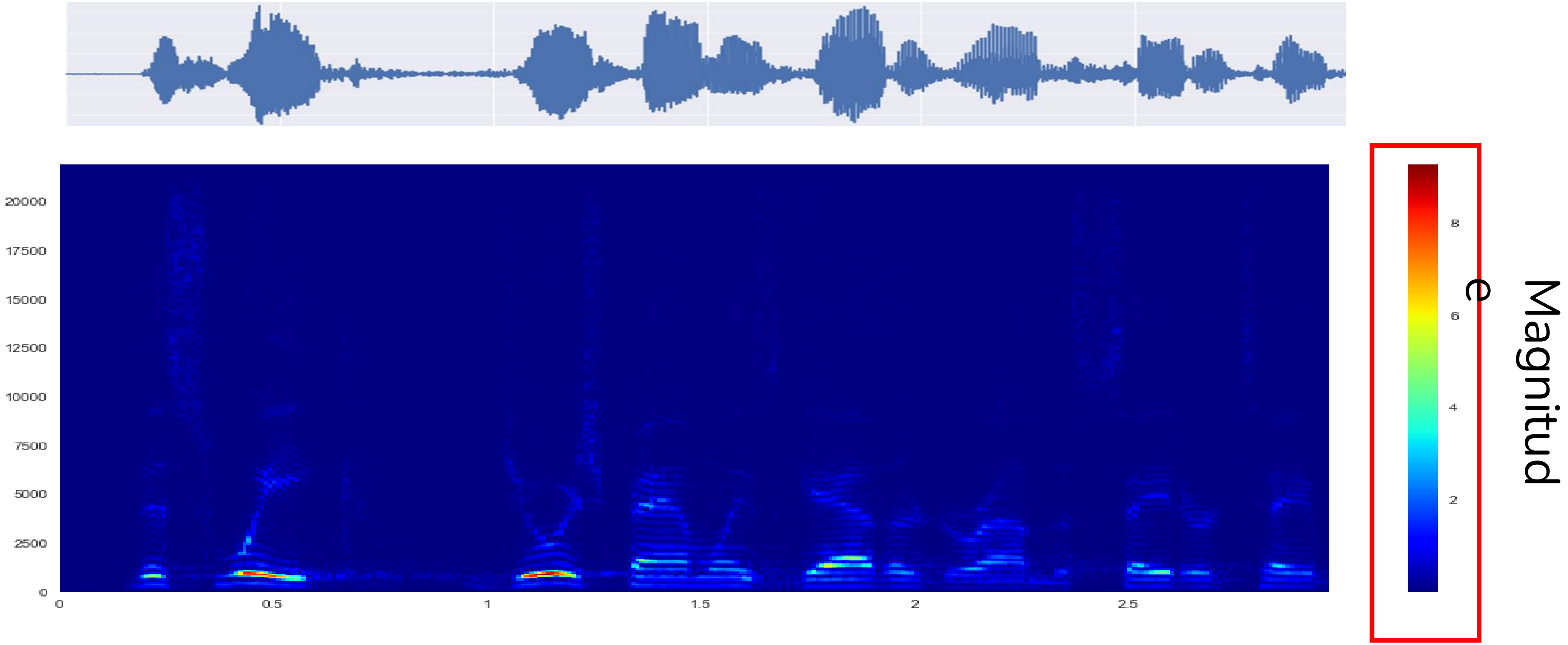


Spectrograms

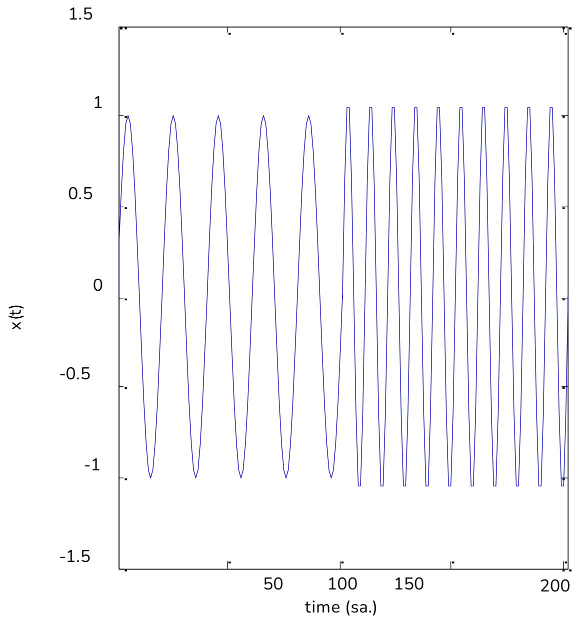


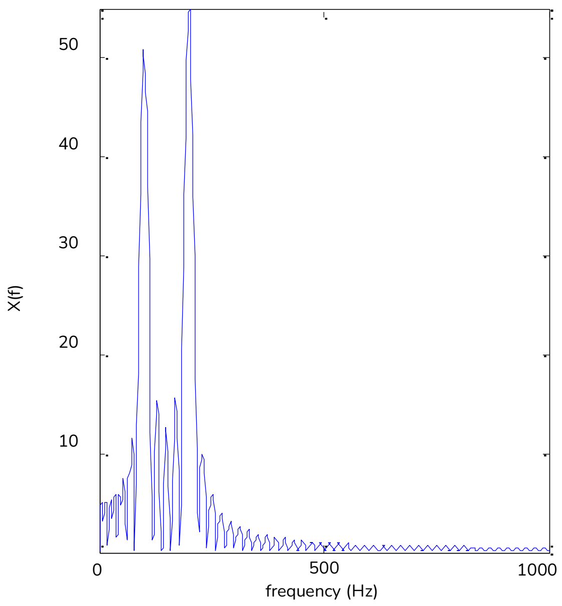
Time

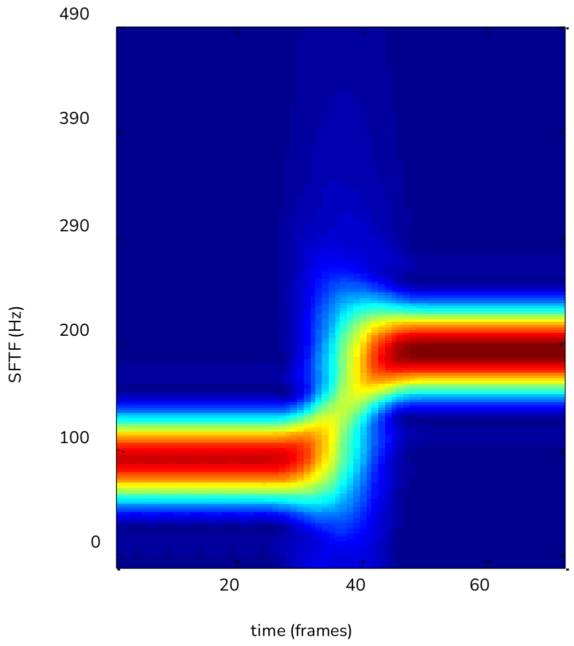
Spectrograms



Spectrograms







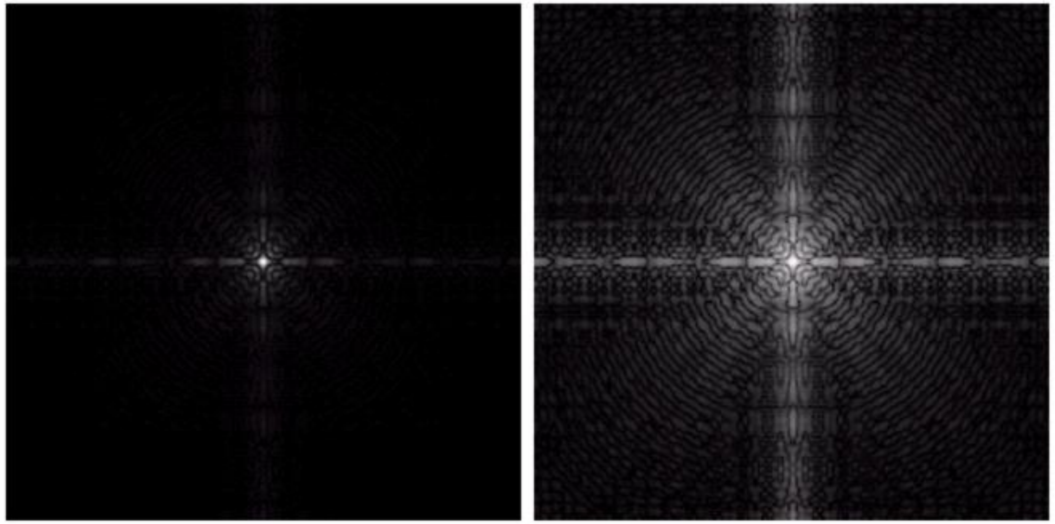
Spectrograms – Log Compression

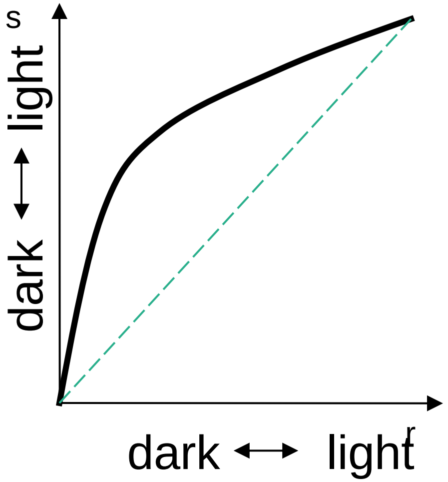
- Sometimes visualization isn’t clear due to the output range of Fourier
  - Can be very large values!
- We want to compress the range

Remember this slide?

- Log Transform

$$s = c * \log(1 + r)$$



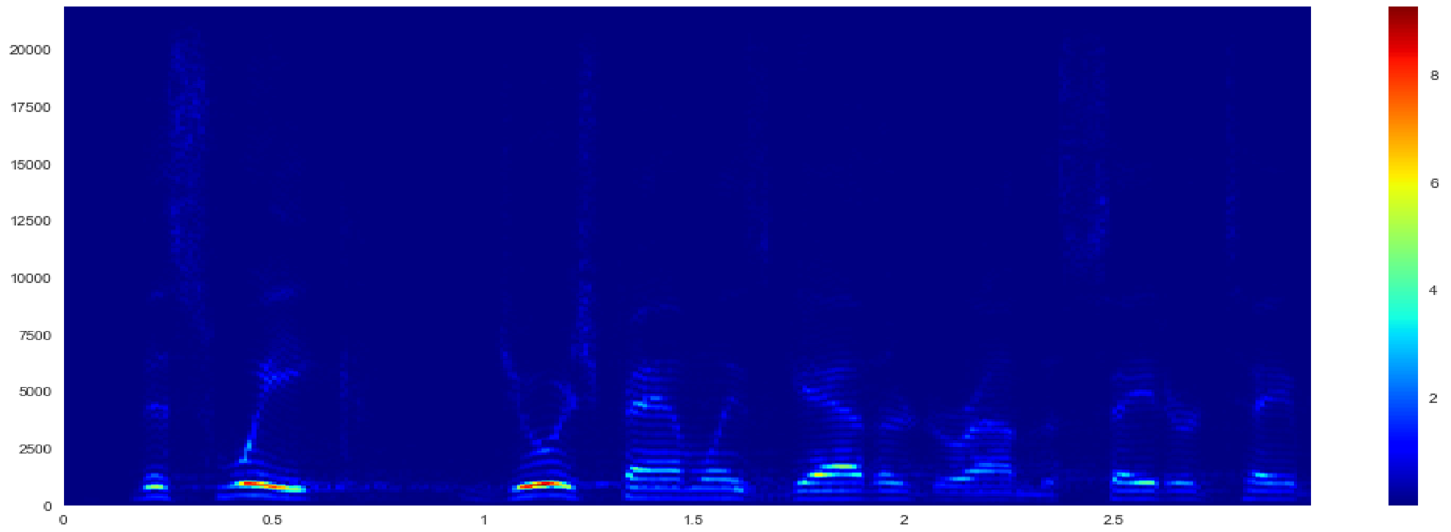


Spectrograms – Log Compression

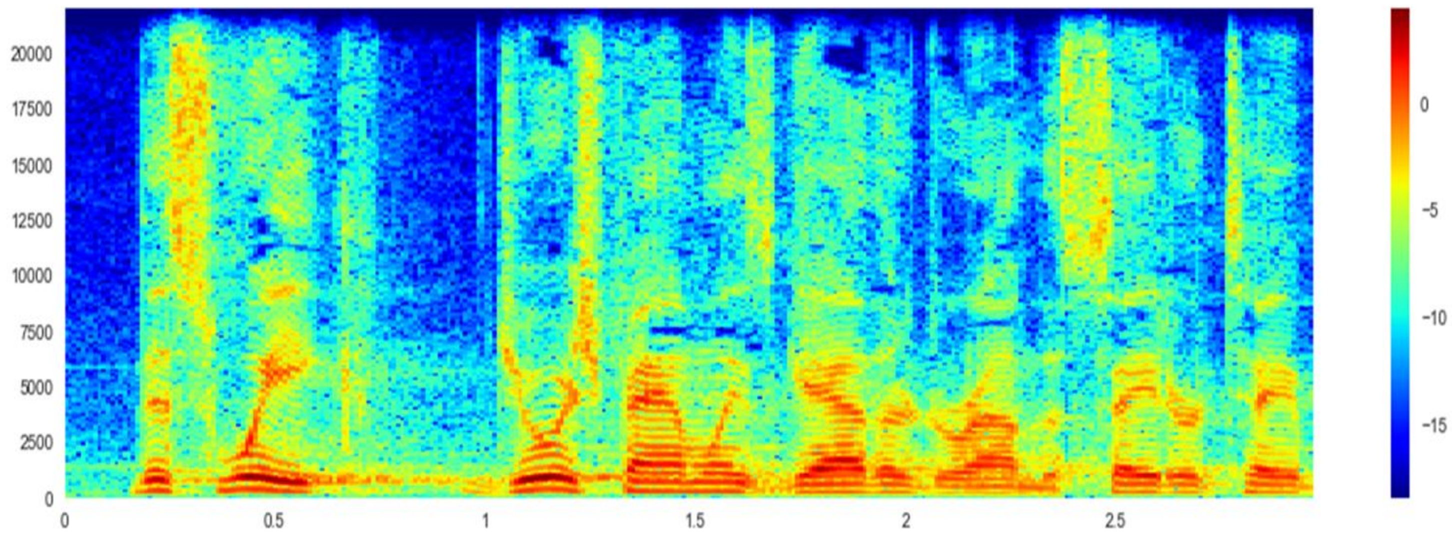
- Sometimes visualization isn't clear due to the output range of Fourier
  - – Can be very large values!
- We want to compress the range
- Log transform enables us to compress the dynamic range of the values
- The steps –
  - Compute $\log(|F(u)| + 1)$
  - Scale to full grey-level range

Spectrograms – Log Compression

Original



Log compressed



Next week:

2D Fourier Transform and Images