Exercise 1

Given a field S, prove:

(i) $0 \cdot a = 0 \ \forall a \in S$

(ii) $-(-a) = a \ \forall a \in S$

(iii) $(-1) \cdot a = -a \ \forall a \in S$

(iv) $(-a)(-b) = ab \ \forall a, b \in S$

Exercise 2

Given an ordered field S, prove that if $a \in S$, then $a^{-1} > 0$.

Exercise 3

If $A \subseteq \mathbb{R}$, A is non-empty, and A is bounded above, then A has a least upper bound. Show that this least upper bound is unique.

Exercise 4

Complex conjugation is a bijection from $\mathbb{C} \to \mathbb{C}$, where $z \mapsto \overline{z}$, that preserves addition and multiplication. This bijection is second-order, meaning $\overline{\overline{z}} = z$.

Show that there does not exist such a map on $\mathbb{R}$.

Exercise 5

Suppose that $f(n)$ is a sequence in $\mathbb{R}$ which is decreasing and bounded below. Then $f(n)$ tends to a limit ℓ as $n \to \infty$.

Prove this statement using a proof analogous to that of Theorem 12.

Exercise 6

Write a definition for sequences tending to negative infinity.

Exercise 7

Prove the converse to the Important Remark. That is, suppose $f(n)$ is a real sequence where $f(n) \to \ell$ as $n \to \ell$. Prove $\overline{\lim} f(n) = \underline{\lim} f(n) = \ell$.

Exercise 8

Prove Theorem 14.ii.

Let $f(n)$ be a real sequence. Suppose that $\overline{\lim} f(n) = \Lambda$ and $\underline{\lim} f(n) = \lambda$ for some real numbers Λ and λ. Given $\varepsilon > 0$, prove there is some natural number $n_1 \in \mathbb{N}$ such that, for infinitely many $n \geq n_1$, $f(n) > \lambda - \varepsilon$ and $\lambda + \varepsilon < f(n)$.

Exercise 9

Complete the argument that $\lambda = \lambda'$ in the proof of Theorem 14.iii.

Exercise 10

Suppose $f(n)$ and $g(n)$ are real-valued sequences. Show that

$$\underline{\lim}\, f(n) + \underline{\lim}\, g(n) \leq \underline{\lim}(f(n) + g(n)) \leq \overline{\lim}(f(n) + g(n)) \leq \overline{\lim}\, f(n) + \overline{\lim}\, g(n).$$

Exercise 11

Suppose $f(n)$ is a bounded real sequence. Prove that there exists a subsequence $f(v(n))$ that converges to $\underline{\lim}\, f(n)$.