

More on the Comparison Test

Example

Consider the series

$$\sum_{n \geq 2} \frac{1}{n(\log n)^\alpha}.$$

This series converges if $\alpha > 1$ and diverges if $\alpha \leq 1$.

Proof: Let $a_n = 1/n(\log n)^\alpha$. Then

$$2^n a_{2^n} = \frac{2^n}{2^n (\log 2^n)^\alpha} = \frac{1}{(\log 2^n)^\alpha} = \frac{1}{(\log 2)^\alpha n^\alpha}.$$

But $\sum \frac{1}{(\log 2)^\alpha n^\alpha}$ converges if and only if $\sum \frac{1}{n^\alpha}$ converges, so we are done. ■

Remark

The series $\sum \frac{1}{n \log n}$ diverges very slowly. In fact,

$$\sum_3^{120,000} \frac{1}{n \log n} < 8.$$

Furthermore, the series

$$\sum \frac{1}{n \log n (\log \log n)^\alpha}$$

converges if $\alpha > 1$ and diverges, even more slowly, if $\alpha \leq 1$.

To generalize,

$$\sum \frac{1}{n \log n \cdot (\log \log n) \cdot \dots \cdot (\log \log \dots \log n)^\alpha}$$

can be made to diverge arbitrarily slowly. This is called the **logarithmic scale** of convergent and divergent series, and it is useful for comparison tests.

Definition

Power Series

Suppose that a_n is a complex sequence. The series $\sum a_n z^n$ is called the **power series** with $\{a_n\}$ as coefficients.

- (i) In the trivial case, $\sum a_n z^n$ converges when $z = 0$.
- (ii) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then f is a function defined as

$$f : \left\{ z : \sum a_n z^n \text{ converges} \right\} \longrightarrow \mathbb{C}.$$

Lemma 27

Suppose that $\sum a_n z^n$ converges for some complex number z_0 . Then for any z such that $|z| < |z_0|$, the power series converges absolutely.

Proof: Observe the following equality:

$$|a_n z^n| = |a_n (z_0)^n| \cdot \left| \frac{z}{z_0} \right|^n$$

We know $\sum a_n (z_0)^n$ converges, and so $a_n (z_0)^n \rightarrow 0$ as $n \rightarrow \infty$. Thus, given some ε where $0 < \varepsilon < 1$, there is some natural number n_0 such that for every $n \geq n_0$,

$$|a_n (z_0)^n| < \varepsilon < 1 \implies |a_n (z_0)^n| < \left| \frac{z}{z_0} \right|^n < 1.$$

Hence $\sum |z/z_0|^n$ converges. The Comparison Test implies that $\sum |a_n z^n|$ converges also. ■

Definition

Radius of Convergence

The **radius of convergence** R of the power series $\sum a_n z^n$ is defined by

$$R = \sup \left\{ |z| : \sum a_n z^n \text{ converges} \right\}.$$

We say that R is infinite if the above set is unbounded.

Theorem 28

The power series $\sum a_n z^n$ with radius of convergence R converges absolutely for all z with $|z| < R$. It diverges if $|z| > R$.

Proof: If R is infinite, then given $z \in \mathbb{C}$, there exists $z_0 \in \mathbb{C}$ such that $|z_0| > |z|$ and $\sum a_n (z_0)^n$ converges, since the set $\{|z| : \sum a_n z^n \text{ converges}\}$ is unbounded. The result now follows from Lemma 27. Moreover, the series converges absolutely.

Otherwise, R is a positive real number. Suppose that z is a complex number for which $|z| < R$.

From the definition of R , there is another complex number z_0 where $|z| < |z_0| < R$ such that $\sum a_n (z_0)^n$ converges. Lemma 27 implies that $\sum a_n z^n$ converges absolutely.

If $|z| > R$, then it follows from the definition of R that $\sum a_n z^n$ diverges. ■

Theorem 29

Let R be the radius of convergence of $\sum a_n z^n$. Then

$$1/R = \overline{\lim} |a_n|^{1/n}.$$

By convention, if R is infinite, then $1/R = 0$.

Proof: Consider $\overline{\lim} |a_n z^n|^{1/n} = |z| \cdot \overline{\lim} |a_n|^{1/n} = |z| \cdot \Lambda$.

By the Cauchy n th Root Test, the series $\sum a_n z^n$ converges if $|z| \cdot \Lambda < 1$ and diverges if $|z| \cdot \Lambda > 1$.

Equivalently, $|z| < 1/\Lambda$ implies convergence, and $|z| > 1/\Lambda$ implies divergence. ■

Example 1

Consider $\sum n^n z^n$. This series converges at the origin, where $z = 0$, and nowhere else. If $|z| > 0$, then $n|z| > 1$ for all sufficiently large n , so $(n|z|)^n$ does not tend to zero as $n \rightarrow \infty$. Hence, $R = 0$.

Example 2

Consider $\sum z^n$. On the circle of convergence, where $z = 1$, the series diverges as $|z|^n = 1$, meaning $|z|^n$ does not tend to zero as $n \rightarrow \infty$. Hence, $R = 1$.

Example 3

Consider $\sum \frac{z^n}{n!}$. Here, $R = \infty$.

Example 4

Let's power down this series a bit...

Consider $\sum \frac{z^n}{n}$. Using Theorem 29, observe that $\left(\frac{1}{n}\right)^{1/n} \rightarrow 1$ as $n \rightarrow \infty$. Hence $R = 1$.

Alternatively, observe that the series diverges at $z = 1$ but converges at $z = -1$.

Example 5

Consider $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$. Here, $R = 1$ since, for example, $\lim (1/n^2)^{1/n} = 1$.

Note that if $|z| = 1$, then $|z^n/n^2| = 1/n^2$. Since $\sum \frac{1}{n^2}$ converges, it follows that $\sum \frac{z^n}{n^2}$ converges absolutely for $|z| = 1$.

Note

It is possible to construct power series with arbitrary behavior on the circle of convergence. For example, we could construct a series diverging at countably infinitely many points on this circle.

Theorem 30

Alternating Series Test

Let a_n be a nonnegative real sequence such that

- (i) $a_n \rightarrow 0$ as $n \rightarrow \infty$, and
- (ii) a_n is a decreasing sequence.

Then $\sum (-1)^n a_n$ converges.

Proof: Let $s_m = \sum_{n=0}^m (-1)^n a_n$. We wish to show that s_m tends to a real limit.

Consider

$$s_{2r} - s_{2r-2} = \sum_{n=2r-2}^{2r} a_n = a_{2r} - a_{2r-1} \leq 0.$$

Hence, s_{2r} is a decreasing sequence. Expanding s_{2r} yields

$$s_{2r} = a_{2r} - a_{2r-1} + a_{2r-2} - \cdots + a_0.$$

Since $a_{2r} - a_{2r-1} \geq 0$, applying similarly to the other terms, we see $s_{2r} \geq a_{2r} \geq 0$. Hence s_{2r} is a decreasing sequence that is bounded below.

Therefore, s_{2r} tends to a real limit ℓ as $r \rightarrow \infty$.

Now, we have $s_{2r} - s_{2r-1} = a_{2r}$, and this tends to zero by assumption. Therefore,

$$(s_{2r-1} - s_{2r}) + s_{2r} \longrightarrow 0 + \ell.$$

Since $s_{2r-1} \rightarrow \ell$, it follows that $s_m \rightarrow \ell$ as $m \rightarrow \infty$. ■