Examples of the Comparison Test

Example 1

- The series $\sum \frac{1}{\sqrt{n}}$ diverges, since $\frac{1}{\sqrt{n}} < \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges.
- The series $\sum \frac{1}{\sqrt{n(n+1)}}$ diverges, since $\frac{1}{\sqrt{n(n+1)}} < \frac{1}{n+1}$, and $\sum \frac{1}{n+1}$ diverges.

Example 2

Consider the series

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \cdots$$

The nth root of this series is always at most $(1/2)^n$. Since $\sum \left(\frac{1}{2}\right)^n$ converges, the Comparison Test implies the series converges as well.

Example 3

Suppose that a_n and b_n are real sequences whose terms are strictly positive. Suppose further that, for every $n \geq n_0$,

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}.$$

Then if $\sum b_n$ converges, so does $\sum a_n$.

Proof: Observe that

$$\frac{a_{n_0+1}}{a_{n_0}} \cdot \frac{a_{n_0+2}}{a_{n_0+1}} \cdot \cdots \cdot \frac{a_{n+1}}{a_n} \leq \frac{b_{n_0+1}}{b_{n_0}} \cdot \frac{b_{n_0+2}}{b_{n_0+1}} \cdot \cdots \cdot \frac{b_{n+1}}{b_n}.$$

Most of these terms cancel, leaving the following inequality:

$$a_{n+1} \leq \frac{a_{n_0}}{b_{n_0}} \cdot b_{n+1}$$

Since a_{n_0}/b_{n_0} is a constant, $\sum \frac{a_{n_0}}{b_{n_0}} b_{n+1}$ converges. Therefore, $\sum a_{n+1}$ also converges by the Comparison Test. ∎

Example 4

If a_n/b_n tends to some nonzero limit ℓ where $a_n > 0$ and $b_n > 0$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof: Let $\varepsilon = \ell/2 > 0$. Then there is some natural number n_0 such that for every $n \geq n_0$,

$$\ell - \varepsilon < a_n/b_n < \ell = \varepsilon.$$

Substituting ε and rearranging the inequality, we get

$$\ell/2 \cdot b_n < a_n < 3\ell/2 \cdot b_n.$$

The result now follows by the Comparison Test. ∎

Theorem 24 **The Cauchy nth Root Test**

Suppose that a_n is a real sequence where $a_n \geq 0$.

(i) If $\overline{\lim}(a_n)^{1/n} < 1$, then $\sum a_n$ converges.

(ii) If $\overline{\lim}(a_n)^{1/n} > 1$, then $\sum a_n$ diverges.

Proof:

(i) Suppose that $\overline{\lim}(a_n)^{1/n} < 1$. Select a k such that $\overline{\lim}(a_n)^{1/n} < k < 1$.

Then there exists a number n_0 such that for all $n \geq n_0$,

$$(a_n)^{1/n} < k \quad \text{(Theorem 14)}.$$

Hence, $0 \leq a_n \leq k^n$ for every $n \geq n_0$. But $\sum k^n$ converges, so by the Comparison Test, $\sum a_n$ converges also.

(ii) Suppose $\overline{\lim}(a_n)^{1/n} > 1$. Then $(a_n)^{1/n} > 1$ and thus $a_n > 1$ for infinitely many n, so a_n does not tend to zero. Hence, $\sum a_n$ does not converge. ∎

Example

Let z be a complex number. Applying the Root Test to $z^n/n!$ yields

$$\overline{\lim_{n\to\infty}} |z^n/n!|^{1/n} = \overline{\lim_{n\to\infty}} \frac{|z|}{(n!)^{1/n}} = 0.$$

Therefore, $\sum \frac{z^n}{n!}$ converges because it converges absolutely.

Theorem 25 **The d'Alembert Ratio Test**

Suppose that a_n is a real sequence where $a_n \geq 0$.

(i) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and is less than 1, then $\sum a_n$ converges.

(ii) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and is greater than 1, then $\sum a_n$ diverges.

Proof:

(i) Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$. Choose k such that $\lim_{n\to\infty} a_{n+1} < k < 1$.

Then there is a number n_0 such that

$$\frac{a_{n+1}}{a_n} < k \quad \forall n \geq n_0.$$

Hence, if $r \geq 1$, we get

$$a_{n_0+r} < a_{n_0} \cdot k^r.$$

Since $\sum a_{n_0} k^r$ converges, we see $\sum_{n_0} a_n$ converges by the Comparison Test.

(ii) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\frac{a_{n+1}}{a_n} > 1 \quad \Longrightarrow \quad a_{n+1} > a_n \quad \Longrightarrow \quad a_{n+1} > a_n > \cdots > a_{n_0} > 0.$$

Hence a_n does not tend to zero as $n \to \infty$, so $\sum a_n$ diverges. ■

Refined Version of the Ratio Test

(i) If $\lim a_{n+1}/a_n < 1$, then $\sum a_n$ converges.

(ii) if $a_{n+1}/a_n > 1$ for all sufficiently large n, then $\sum a_n$ diverges.

Both criteria follow from the same proof above.

Example 1

Consider the series $\sum \frac{1}{n!}$. Applying the Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{1}{n} \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0.$$

Therefore, $\sum \frac{1}{n!}$ converges.

Example 2

Let z be a complex number, and consider $\sum \left|\frac{z^n}{n!}\right|$. It follows that

$$\frac{a_{n+1}}{a_n} = \left|\frac{z^{n+1}}{n+1}\right| \cdot \left|\frac{n!}{z^n}\right| = \frac{|z|}{n+1}.$$

This expression tends to zero as $n \to \infty$, so $\sum \left|\frac{z^n}{n!}\right|$ converges. Hence $\sum \frac{z^n}{n!}$ also converges.

We make the following definitions:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
$$\sin(z) = \sum_{n}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$
$$\cos(z) = \sum_{n}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

All of these series converge by the d'Alembert Ratio Test for any value of z. It follows that these definitions are complex functions.

Example 3

Consider $\sum \frac{1}{n}$. Then the ratio of subsequent terms tends to 1 as $n \to \infty$, so the Ratio Test tells us nothing in this case.

Now, consider $\sum \frac{1}{n^2}$. The ratio of subsequent terms again tends to 1, but this series converges while $\sum \frac{1}{n}$ does not. The Ratio Test cannot show this.

Observe that

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

Hence,

$$\sum_{2}^{m} \frac{1}{n^2} < 1 - \frac{1}{m} < 1.$$

Here, the partial sums are bounded. Therefore, $\sum \frac{1}{n^2}$ converges.

Theorem 26 **The Condensation Test**

Suppose that a_n is a real decreasing sequence where $a_n \geq 0$. Then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges.

Proof: Observe that we have

$$\frac{2^{n+1} a_{2^{n+1}}}{2} \leq \sum_{r=2^n+1}^{2^{n+1}} a_r \leq 2^n a_{2^n}.$$

If $\sum a_n$ converges, then

$$\frac{1}{2} \sum_{n}^{N} 2^n a_{2^n} \leq \sum_{r=2}^{2^N} a_r \leq \sum_{r=2}^{\infty} a_r$$

hence $\sum 2^n a_{2^n}$ also converges.

Conversely, if $\sum 2^n a_{2^n}$ converges, then for some N,

$$\sum_{r=2}^{m} a_r \leq \sum_{r=2}^{N+1} a_r \leq \sum_{n=0}^{N} 2^n a_{2^n} \leq \sum_{n=0}^{\infty} 2^n a_{2^n}.$$

Because the partial sums of a_n are bounded and $a_n \geq 0$, the series $\sum a_n$ converges.

TODO: Clarify this! A lot! ■

Example

The series $\sum \frac{1}{n^\alpha}$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$. Observe that

$$a_n = \frac{1}{n^\alpha} \implies 2^n a_{2^n} = \frac{2^n}{(2^n)^\alpha} = \frac{1}{(2^{\alpha-1})^n}.$$

The convergence of $\sum(\frac{1}{2^{\alpha-1}})^n$ is equivalent to $\frac{1}{2^{\alpha-1}} < 1 \Leftrightarrow \alpha > 1$.