Exercise 1

Prove:

(i) Any field is a vector space over itself.

(ii) For any field $\mathbb{F}$, the set $\mathbb{F}^n$ is a vector space over $\mathbb{F}$.

(iii) $\mathbb{C}$ is a vector space over $\mathbb{R}$.

Exercise 2

Define multiplication on $\mathbb{F}^S$ as follows:

$$(f \cdot g)(s) = f(s) \cdot g(s)\ \forall s \in S.$$

Determine whether $\mathbb{F}^S$ is a field, and justify your answer.

Exercise 3

Suppose W is a subset of $\mathbb{F}^4$ where

$$W = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4\}.$$

Show that W is a subspace of $\mathbb{F}^4$.

Exercise 4

Prove that the set of *continuous* real functions is a subspace of the set of all real functions.

Exercise 5

Prove that the intersection of an arbitrary number of subspaces, not just finitely many, is also a subspace.

Exercise 6

Let S be the set defined as

$$S \coloneqq \left\{\begin{bmatrix} a & a \\ a & b \end{bmatrix} : a, b \in \mathbb{R}\right\}.$$

(i) Prove that S is a subspace of $M_{2\times 2}(\mathbb{F})$.

(ii) Give a basis for S.

Exercise 7

Consider $S = \left\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\right\}$.

(i) Show that S is a subspace of $\mathbb{R}^3$.

(ii) Find a basis for S.

Exercise 8

Show that $\beta = \{1\}$ is a basis for the vector space $\mathbb{C}(\mathbb{C})$, then deduce that the dimension of this space is 1.

Exercise 9

Prove that $\beta_m \coloneqq \left\{1, x, x^2, \dots, x^m\right\}$ is a basis for $P_m(\mathbb{F})$.

Exercise 10

Let $S_{3\times 3}$ be the set of all 3×3 real symmetric matrices. Then, we define a basis:

$$\beta_{3\times 3} = \left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$$

Verify that $\beta_{3\times 3}$ is a basis by showing that it spans $S_{3\times 3}$ while also being linearly independent.

Exercise 11

(i) Prove S, the set of $n \times n$ real diagram matrices, is a subspace of $M_{n\times n}(\mathbb{R})$.

(ii) Show that β is a basis for S, where

$$\beta = \left\{E^{ii} : 1 \le i \le n\right\} = \left\{E^{11}, E^{22}, \dots, E^{nn}\right\}.$$

Exercise 12

Suppose S is a subspace of $\mathbb{R}^5$ with a basis β such that

$$S = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 + x_3 + x_5 = 0,\ x_2 = x_4\right\},$$
$$\beta = \{(0, 1, 0, 1, 0), (1, 0, -1, 0, 0), (0, 0, -1, 0, 1)\}.$$

We can extend β to a basis for $\mathbb{R}^5$. One such basis is

$$\gamma = \beta \cup \{(0, 0, 0, 1, 0), (0, 0, 1, 0, 0)\}.$$

Prove γ is a basis for $\mathbb{R}^5$.