Chapter IV: Continuity

In the past, functions were called continuous if one could draw them smoothly. However, we can cook up very nasty functions which are continuous but can't be drawn at all.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a real function, and let a be a real number. Then tends to a limit ℓ as $x \to a$ if, given $\varepsilon > 0$. there exists $\delta > 0$ such that, for every value of x where $0 < |x - a| < \delta$, we have $|f(x) - \ell| < \varepsilon$.

We write $f(x) \to \ell$ as $x \to a$ or $\lim_{x \to a} f(x) = \ell$.

Note

The value of $f(a)$ as $x \to a$ is irrelevant to the definition of the limit.

Example 1

Suppose that $f(x) = c$ for some real constant c at all x. Then $\lim_{x \to a} f(x) = c$.

Example 2

Suppose that $f(x) = x$. Then $\lim_{x \to a} f(x) = a$.

Example 3

Suppose f is defined as

$$f(x) = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Then $\lim_{x \to 0} f(x) = 1$, even though $f(0) = 0$.

Example 4

Suppose f is defined as

$$f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then $\lim_{x \to 0} f(x) = 0$, since $|f(x)| \leq |x|$ for all x.

Example 5

Suppose f is defined as

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then $\lim_{x \to 0} f(x)$ does not exist.

Example 6

Suppose f is defined as

$$f(x) = \begin{cases} 0 \text{ if } x \in \mathbb{Q} \\ 1 \text{ if } x \notin \mathbb{Q} \end{cases}$$

Then $\lim_{x \to a} f(x)$ does not exist for any value of a.

Note

(i) In the definition of what it means to say $\lim_{x \to a} f(x) = \ell$, we could use $k\varepsilon$, or $|f(x) - \ell| < k\varepsilon$, for some fixed $k > 0$.

(ii) The same definition holds if f is defined on $\{ x : 0 < |x - a| < r \}$ for some r.

Lemma 36

If $f(x) \to \ell_1$ and $f(x) \to \ell_2$ as $x \to a$, then $\ell_1 = \ell_2$.

Proof: If $\ell_1 \neq \ell_2$, let $\varepsilon = \frac{1}{2}|\ell_1 - \ell_2|$. Then

$$\begin{aligned} \exists \delta_1 > 0 \quad \text{st.} \quad 0 < |x - a| < \delta_1 &\implies |f(x) - \ell_1| \\ \exists \delta_2 > 0 \quad \text{st.} \quad 0 < |x - a| < \delta_2 &\implies |f(x) - \ell_2|. \end{aligned}$$

Choose x such that $|x - a| < \min\{\delta_1, \delta_2\}$. It follows that

$$|\ell_1 - \ell_2| \leq |f(x) - \ell_1| + |f(x) - \ell_2| < 2\varepsilon = |\ell_1 - \ell_2|.$$

This is a contradiction. Hence $\ell_1 = \ell_2$ as asserted. ■

Lemma 37

Suppose that as $x \to a$, $f(x) \to \ell$ and $g(x) \to m$.

(i) $f(x) + g(x) \to \ell + m$ as $x \to a$.

(ii) $f(x) \cdot g(x) \to \ell m$ as $x \to a$.

(iii) $f(x)/g(x) \to \ell/m$ as $x \to a$ provided $m \neq 0$.

Proof of (ii): Given $\varepsilon > 0$, we have

$$\exists \delta_1 > 0 \quad \text{st.} \quad 0 < |x - a| < \delta_1 \implies |f(x) - \ell| < \min\{\varepsilon, 1\}$$
$$\exists \delta_2 > 0 \quad \text{st.} \quad 0 < |x - a| < \delta_2 \implies |f(x) - \ell| < \varepsilon.$$

Then if $0 < |x - a| < \min\{\delta_1, \delta_2\}$ we have

$$\begin{aligned} |f(x) \cdot g(x) - \ell m| &= |f(x) \cdot g(x) - f(x)m + f(x)m - \ell m| \\ &\leq |f(x)||g(x) - m| + |m||f(x) - \ell| \\ &\leq (|\ell| + 1 + |m|)\varepsilon. \end{aligned}$$

Henc $f(x) \cdot g(x) \to \ell m$ as $x \to a$, as required. ■

Exercise

Parts (i) and (iii) of Lemma 37 can be proved in a similar way to Lemma 8. Complete these proofs.

Corollary

By taking $f(x) = x^n$, we see that $x^n \to a^n$ as $x \to a$, whence $\sum_0^m c_r x^r = \sum_0^m c_r a^r$.

Definition — **Function Continuity**

A real function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at a real number a if $\lim_{x \to a} f(x)$ exists and is equal to $f(a)$. That is, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

We say that f is **continuous** if f is continuous at every real a.

We sometimes call functions continuous if they are continuous at every point at which they are defined.

Theorem 38 — **General Principle of Convergence for Functions**

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $f(x)$ tends to a limit as $x \to a$ if and only if given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \text{and} \quad 0 < |y - a| < \delta \quad \implies \quad |f(x) - f(y)| < \varepsilon.$$

Proof: Suppose $f(x) \to \ell$ as $x \to a$. Then given $\varepsilon > 0$,

$$\begin{aligned} \exists \delta > 0 \quad \text{st.} \quad & 0 < |x - a| < \delta \implies |f(x) - \ell| < \varepsilon/2 \\ & 0 < |y - a| < \delta \implies |f(y) - \ell| < \varepsilon/2. \end{aligned}$$

Using the triangle inequality, we get

$$|f(x) - f(y)| = |f(x) - \ell| + |f(y) - \ell| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Conversely, suppose that given $\varepsilon > 0$, there is a $\delta > 0$ that satisfies the convergence condition.

Consider the sequence $x_n = a + 1/n$. Then $\lim x_n = a$. Hence, for all $r, s \geq 1/\delta$,

$$0 < |x_r - a| < \delta \quad \text{and} \quad 0 < |x_s - a| < \delta.$$

It follows that $|f(x_r) - f(x_s)| < \varepsilon$. The General Principle of Convergence applied to the sequence $f(x_n)$ implies that $f(x_n)$ tends to a real limit ℓ as $n \to \infty$.

Therefore, there is a natural number n_0 such that for all $n \geq n_0$, $|f(x_n) - \ell| < \varepsilon$.

Now, suppose x satisfies $0 < |x - a| < \delta$. Moreover, choose $n \geq \max\{n_0, 1/\delta\}$. It follows that

$$|f(x_n) - f(x)| < \varepsilon \quad \text{and} \quad |f(x_n) - \ell| < \varepsilon.$$

By the triangle inequality,

$$|f(x) - \ell| < 2\varepsilon$$

and by definition, $f(x) \to \ell$ as $x \to a$. ∎

Note

If $f(x) \to \ell$ as $x \to a$, and $x_n \to a$ as $n \to \infty$, but $x_n \neq a$ for all sufficiently large values of n, the sequence $f(x_n)$ tends to ℓ as $n \to \infty$.

Proof: Given $\varepsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - \ell| < \varepsilon$. Noting that δ is arbitrarily small, there is a natural number n_0 such that for every $n \geq n_0$, we have $|x_n - a| < \delta$ and $x_n \neq a$.

Hence for all $n \geq n_0$, we get $|f(x_n) - \ell| < \varepsilon$, and so $f(x_n) \to \ell$ as $n \to \infty$. ∎