

Example 4

$$\lim_{n \rightarrow \infty} n^{1/n} = 1.$$

Proof: Write $n^{1/n}$ as $1 + \delta(n)$ where $\delta(n) \geq 0$. Then

$$\begin{aligned} n &= (1 + \delta(n))^n \\ &= 1 + n\delta(n) + \frac{n(n-1)\delta(n)^2}{2!} + \dots + \delta(n)^n \\ &\geq \frac{n(n-1)\delta(n)^2}{2!} \end{aligned}$$

Isolating $\delta(n)$, we find

$$\sqrt{\frac{2}{n-1}} \geq d(n) \geq 0.$$

Now $\sqrt{\frac{2}{n-1}} \rightarrow 0$ as $n \rightarrow \infty$ by Example 2. Then, by Lemma 11, $\delta(n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $n^{1/n} \rightarrow 1$ as $n \rightarrow \infty$. ■

Example 5

$\left(1 + \frac{1}{n}\right)^n$ tends to some limit e for which $2 \leq e \leq 3$.

Proof:

Using the binomial theorem, we get

$$\left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \dots + \left(\frac{1}{n}\right)^n. \quad (1)$$

The $(r+1)$ st term of this expansion is

$$\frac{1}{r! \cdot n^r} \cdot n(n-1)\dots(n-r+1) = \frac{1}{r!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{r-n}{n}\right). \quad (2)$$

The $(r+1)$ st term increases as n increases since each term in the product increases. As n increases, the number of terms in the expansion of $\left(1 + \frac{1}{n}\right)^n$, which is $n+1$, also increases. Hence, $\left(1 + \frac{1}{n}\right)^n$ increases as n increases.

To establish an upper bound on the sequence, Equation 2 tells us that the r th term of the expansion in Equation 1 is at most $1/r!$. Then,

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} \\ &= 3 - \frac{1}{2^{n-1}} < 3. \end{aligned}$$

So $\left(1 + \frac{1}{n}\right)^n$ is increasing and is bounded above. Therefore,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \text{ exists, and } \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \leq 3.$$

■

Definition 6

Let $f, g : \mathbb{N} \rightarrow \mathbb{R}$ be real sequences. Then we introduce the following notation.

- (i) $f = O(g)$ or $f(n) = O(g(n))$ if there is some constant $K \in \mathbb{R}$ and some $n_0 \in \mathbb{N}$ for which

$$|f(n)| < K|g(n)| \quad \forall n \geq n_0.$$

- (ii) $f = o(g)$ or $f(n) = o(g(n)) \iff f(n)/g(n) \rightarrow 0$ as $n \rightarrow \infty$.
- (iii) $f \sim g$ or $f(n) \sim g(n) \iff f(n)/g(n) \rightarrow 1$ as $n \rightarrow \infty$.

Example

Let $f(n) = 2n^3 + 3n^2 + 27$.

- (i) $f(n) = O(n^3)$ since $|f(n)| < 3n^3$ if $3n^2 + 27 \leq n^3$, which is true when $n \geq 5$.
- (ii) $f(n) = o(n^4)$.
- (iii) $f(n) \sim 2n^3 + 5n + 1$.

Example

- (i) $n = O(n^2)$, and $n = o(n^2)$, but $n \not\sim n^2$.
- (ii) $n + \sin(n) \sim n$.

Note

You should **never, ever** write $f(n) \rightarrow g(n)$ as $n \rightarrow \infty$, because we have no precise definition of what this means.

There is a special place in hell for people who write this. People shouldn't commit crimes either, but this is much worse. At least, frequently, crimes are meaningful.

Definition 7

If $f : \mathbb{N} \rightarrow \mathbb{R}$, then f is said to **tend to infinity** if, given any $K \in \mathbb{R}$, there is a natural number n_0 such that $f(n) \geq K$ for all $n \geq n_0$.

We write this as

$$f(n) \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \text{or} \quad \lim_{n \rightarrow \infty} f(n) = \infty.$$

Exercise 8

Write a similar definition for sequences tending to negative infinity.

Note

Saying that a sequence tends to infinity is different from saying it is unbounded.

For example, $f(n) = (-1)^n n^2$ is unbounded, but does not tend to infinity.

Lemma 13

- (i) An unbounded monotonic sequence tends to either $+\infty$ or $-\infty$.
- (ii) Let f be a sequence such that $f(n) > 0$ for all sufficiently large n . Then, as n approaches infinity,

$$f(n) \rightarrow 0 \iff 1/f(n) \rightarrow \infty.$$

Proof of 13.i: Suppose, without much loss of generality, that $f(n)$ is strictly increasing and unbounded. Then, given $K \in \mathbb{R}$, K is not an upper bound of the set

$$f(\mathbb{N}) = \{f(n) : n \in \mathbb{N}\}.$$

Thus, there is some $n_0 \in \mathbb{N}$ for which $K < f(n_0)$. For any $n \geq n_0$, it is still true that $K < f(n)$. ■

Proof of 13.ii: If $1/f(n) \rightarrow \infty$, then $1/f(n)$ is defined for all sufficiently large n . Then, given $\varepsilon > 0$, there is some $n_1 \in \mathbb{N}$ for which $1/\varepsilon < 1/f(n)$ for all $n \geq n_1$. So

$$0 < f(n) < \varepsilon \quad \forall n \geq n_1$$

and thus $f(n) \rightarrow \infty$ as $n \rightarrow \infty$.

If $f(n) \rightarrow 0$ as $n \rightarrow \infty$, then given $K \in \mathbb{R}$ with $K > 0$, there is some $n_2 \in \mathbb{N}$ for which $|f(n)| < 1/K$ for all $n \geq n_2$.

Then, since $f(n) > 0$, we can rewrite the inequality as

$$K < 1/f(n) \quad \forall n \geq n_2$$

proving that $1/f(n) \rightarrow \infty$ as $n \rightarrow \infty$. ■

Upper and Lower Limits of Real Sequences

Definition

Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be a sequence. If $f(\mathbb{N})$ is bounded above, we define the **upper limit** of $f(n)$ as n tends to infinity to be

$$\limsup_{r \rightarrow \infty} \{f(r), f(r+1), f(r+2), \dots\}.$$

If $f(\mathbb{N})$ is bounded below, we define the **lower limit** of $f(n)$ to be

$$\liminf_{r \rightarrow \infty} \{f(r), f(r+1), f(r+2), \dots\}.$$

We introduce the following notation for upper and lower limits respectively:

$$\limsup_{n \rightarrow \infty} f(n) = \overline{\lim}_{n \rightarrow \infty} f(n) \quad \liminf_{n \rightarrow \infty} f(n) = \underline{\lim}_{n \rightarrow \infty} f(n)$$

If $f(\mathbb{N})$ is not bounded above, we define $\overline{\lim}_{n \rightarrow \infty} f(n)$ to be $+\infty$.

Similarly, if $f(\mathbb{N})$ is not bounded below, we define $\underline{\lim}_{n \rightarrow \infty} f(n)$ to be $-\infty$.

Remark

The precise definitions of upper and lower limits above can be understood using properties of suprema and infima. Suppose $A \subseteq B \subseteq \mathbb{R}$ and B is bounded above. Recall that A must also be bounded above, and $\sup(A) \leq \sup(B)$.

Thus the expression in the definition of the upper limit of $f(n)$ is a decreasing function. That is,

$$\sup\{f(r), f(r+1), \dots\} \geq \sup\{f(r+1), f(r+2), \dots\}.$$

Therefore, this function either tends to a limit (Theorem 12) or to $-\infty$ (Lemma 13).

Similarly, the sequence in the definition of the lower limit is increasing, so its limit is either a real number ℓ or $+\infty$.

Example

(i) Let $f(n) = (-1)^n + 1/n$.

Then $\overline{\lim}_{n \rightarrow \infty} f(n) = 1$ since $1 + 1/r \rightarrow 1$ as $r \rightarrow \infty$. Similarly, $\underline{\lim}_{n \rightarrow \infty} f(n) = -1$.

(ii) Let $r : \mathbb{N} \rightarrow \mathbb{Q}$ be an enumeration of the rationals, i.e. a bijection. Then

$$\overline{\lim}_{n \rightarrow \infty} r(n) = +\infty \quad \underline{\lim}_{n \rightarrow \infty} r(n) = -\infty$$

$$\overline{\lim}_{n \rightarrow \infty} |r(n)| = +\infty \quad \underline{\lim}_{n \rightarrow \infty} |r(n)| = 0.$$

(iii) If $f(n) = n$, then $\overline{\lim}_{n \rightarrow \infty} f(n) = \infty = \underline{\lim}_{n \rightarrow \infty} f(n)$.