Exercise

Suppose $f(n)$ and $g(n)$ are real-valued sequences. Show that

$$\underline{\lim}\, f(n) + \underline{\lim}\, g(n) \leq \underline{\lim}(f(n) + g(n)) \leq \overline{\lim}(f(n) + g(n)) \leq \overline{\lim}\, f(n) + \overline{\lim}\, g(n).$$

Continuing our discussion about the Principle of General Convergence, we first generalize it to complex numbers then use another method to prove the converse. We restate the theorem below.

Theorem 15 **The General Principle of Convergence**

Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence. Then $f(n)$ converges to some finite limit in $\mathbb{R}$ if and only if, given $\varepsilon > 0$, there is some natural number n_0 for which $r, s \geq n_0$ implies

$$|f(r) - f(s)| < \varepsilon.$$

The proof we gave last lecture does not hold in the complex plane, since we have not yet defined the concept of upper and lower limits for complex sequences. Nevertheless, we can use the same theorem over real sequences to prove the complex case.

Remark

Theorem 15 is true for complex sequences of the form $f : \mathbb{N} \to \mathbb{C}$.

Proof: Use Theorem 15 on the real and imaginary parts of $f(n)$, where

$$|\mathrm{Re}(f(r)) - \mathrm{Re}(f(s))| \leq |f(r) - f(s)|$$
$$|\mathrm{Im}(f(r)) - \mathrm{Im}(f(r))| \leq |f(r) - f(s)|.$$

Hence, if $f : \mathbb{N} \to \mathbb{C}$ is a Cauchy sequence, then $\mathrm{Re}(f(n))$ converges, so $f(n)$ converges. ∎

We now prove Theorem 15 without using upper or lower limits.

Proof: The forward direction stands as before.

In the converse direction, suppose $f : \mathbb{N} \to \mathbb{R}$ is a sequence and that given $\varepsilon > 0$, there is some number n_0 such that for all $r, s \geq n_0$,

$$|f(r) - f(s)| < \varepsilon.$$

Let $S = \{f(n) : n \in \mathbb{N}\} \subseteq \mathbb{R}$ be the set of all values of $f(n)$. Then S is bounded because $|f(n)| \leq B$ for every n if we take B to be

$$B = \max\{|f(n_0) + \varepsilon|, f(1), f(2), \ldots, f(n_0 - 1)\}.$$

Suppose S is a finite set. then there is some real number ℓ such that $f(n) = \ell$ for infinitely many n. Hence, $f(n) = \ell$ for some $n \geq n_0$.

Then, for all $r \geq n_0$, choose $s \geq n_0$ such that $s = \ell$. We obtain

$$|f(r) - f(s)| = |f(r) - \ell| < \varepsilon \quad \forall r \geq n_0.$$

Therefore, $f \to \ell$.

In the case where S is infinite, we have

$$-B \leq f(n) \leq B \quad \forall n \in \mathbb{N}.$$

We now use the technique of repeated bisection. Our goal is essentially to bisect the values of $f(n)$, choose the section that contains infinitely many values, and repeat. Inductively, we will show that $f(n)$ is a Cauchy sequence.

Construct a sequence

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq b_2 \leq b_1 \leq b_0$$

where $a_0 = -B$, and $b_0 = B$. Note that all values of S must lie between a_0 and b_0.

Now, select a_1 and b_1 such that $a_1 < b_1$ and

$$b_1 - a_1 = \frac{b_0 - a_0}{2} = \frac{2B}{2}.$$

We require there to exist infinitely many values of S between a_1 and b_1. For example, choose $a_1 = a_0$ and $b_1 = (a_0 + b_0)/2$ if possible; otherwise, choose $a_1 = (a_0 + b_0)/2$ and $b_1 = b_0$.

TODO: Insert a diagram here!

Inductively, choose a_n and b_n such that

$$a_{n-1} \leq a_n < b_n \leq b_{n-1}, \qquad b_n - a_n = \frac{2B}{2^n}$$

and there exists infinitely many members of S between a_n and b_n.

Then a_n forms an increasing bounded sequence. Thus, a_n tends to ℓ for some real number ℓ.

> **Note**
> Importantly, ℓ is not necessarily a rational number.
>
> *The rational numbers are not complete. When the Ancient Greeks discovered this, they started killing themselves.*

It follows that

$$b_n = a_n + \frac{B}{2^{n-1}} \longrightarrow \ell + 0 \quad \text{as } n \to \infty.$$

Hence, there is some $n_1 \in \mathbb{N}$ such that, for all $n \geq n$,

$$\ell - \varepsilon < a_n < b_n < \ell + \varepsilon.$$

Therefore, there are infinitely many elements of S between $\ell - \varepsilon$ and $\ell + \varepsilon$. Furthermore, there is a number $s_0 \geq n_0$ for which

$$\ell - \varepsilon < f(s_0) < \ell + \varepsilon \quad \iff \quad |f(s_0) - \ell| < \varepsilon.$$

Thus, if $r \geq n_0$, we have

$$|f(r) - \ell| \leq |f(r) - f(s_0)| + |f(s_0) - \ell| < \varepsilon + \varepsilon = 2\varepsilon.$$

Therefore, $f(r) \to \ell$ as $n \to \infty$. ∎

Subsequences

Definition **Subsequence**

Let $f : \mathbb{N} \to S$ be a sequence in a set S. A **subsequence** of f is a sequence $f(\mu(n))$ where

$$\mu : \mathbb{N} \longrightarrow \mathbb{N}, \quad f \circ \mu : \mathbb{N} \longrightarrow S$$

and μ is a strictly increasing function. That is, if $f(1), f(2), \ldots$ is the original sequence, then $f(\mu(1)), f(\mu(2)), \ldots$ is a subsequence.

In plain English, a subsequence is a subcollection of the terms in the same order.

If we have a sequence a_n, we may notate a subsequence as a_{n_1}, a_{n_2}, and so on.

Remark

If $f(n)$ is a sequence in $\mathbb{R}$ or $\mathbb{C}$, and $f(n) \to \ell$ as $n \to \infty$, then any subsequence $f(\mu(n))$ also converges to ℓ.

Proof: Given $\varepsilon > 0$, there is a natural number n_0 such that

$$|f(n) - \ell| < \varepsilon \quad \forall n \geq n_0.$$

So, if $\mu(n) \geq n_0$, then we also get $|f(\mu(n)) - \ell| < \varepsilon$ since μ is strictly increasing. ∎

Note

The sequence $f(n) = (-1)^n$ does not converge. However, if $\mu(n) = 2n$, then $f(\mu(n)) = 1$ for all n. Similarly, if $\mu(n) = 2n + 1$, then $f(\mu(n))$ is always -1.

The moral here is that a sequence that does not converge may have subsequences that do.

Proposition 16

Suppose $f(n)$ is a bounded real sequence. Then there exists a subsequence $f(\mu(n))$ that converges to $\overline{\lim}\, f(n)$.

Proof: Let $\Lambda = \overline{\lim}\, f(n)$. Then, by Theorem 14, there exists a number n_1 for which

$$\begin{aligned} f(n) &< \Lambda + 1 \quad \forall n \geq n_1 \\ f(n) &> \Lambda - 1 \quad \text{for infinitely many } n. \end{aligned}$$

Let $\mu(1)$ be the least value of n satisfying both conditions.

Inductively, there is an n_r such that

$$\begin{aligned} f(n) &< \Lambda + 1/r \quad \forall n \geq n_r \\ f(n) &> \Lambda - 1/r \quad \text{for infinitely many } n. \end{aligned}$$

Let $\mu(r)$ be the least integer satisfying both conditions with the added requirement that $\mu(r) > \mu(r-1)$.

Then we have a subsequence defined inductively, and $f(\mu(r)) \to \Lambda$ as $r \to \infty$. ∎

Exercise

Prove that there exists a subsequence $f(v(n))$ that converges to $\underline{\lim}\, f(n)$.

Note

If $\overline{\lim}\, f(n) = \infty$, i.e. if $f(n)$ is unbounded above, then we can let $\mu(r)$ be the smallest n such that $f(n) \geq r$ and $n > \mu(r-1)$. Then $f(\mu(r)) \to \infty$ as $n \to \infty$.

Note

Proposition 16 implies that every sequence of real numbers, bounded above or below, has a convergent subsequence. This is known as the **Bolzano-Weierstrass theorem**.