MATH 117 Notes Jan. 15 2026 (Draft)

Let $A \subseteq \mathbb{R}$, $A \neq \varnothing$. Then $\operatorname{glb}(A)$, the **infimum** of of A, exists when A is bounded below. If $\operatorname{glb}(A) \leq a\ \forall a \in A$, then $\operatorname{glb}(A)$ is a lower bound.

If $\delta > 0$, then $\delta + \operatorname{glb}(A)$ is not a lower bound of A, i.e. $\operatorname{glb}(A)$ is the **greatest lower bound**.

Note that if $A \neq \varnothing$ is bounded, then $\operatorname{glb}(A) \leq a \leq \operatorname{lub}(A)\ \forall a \in A$.

Distance

Definition. The **distance** $d(a, b)$ between $a, b \in \mathbb{R}$ is defined as

$$d(a, b) = |a - b|.$$

Properties of Distance

- $d(a, b) \geq 0$
- $d(a, b) = 0$ if and only if $a = b$
- $d(a, b) = d(b, a)$
- $d(a, b) + d(b, c) \geq d(a, c)$; known as the triangle inequality

The Complex Numbers $\mathbb{C}$

As a set, $\mathbb{C} = \mathbb{R} \times \mathbb{R}$. A complex number is an ordered pair (x, y) of real numbers.

We define complex and addition and multiplication.

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Note that

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$
$$(x_1, 0) \cdot (x_2, 0) = (x_1 x_2, 0)$$

thus the collection $\{(x, 0) \in \mathbb{C} : x \in \mathbb{R}\}$ is a copy of $\mathbb{R}$.

Therefore, we identify $(x, 0)$ with x. Then $\mathbb{R} \subseteq \mathbb{C}$. In this case, $\mathbb{R}$ s a subfield of $\mathbb{C}$.

Let $i = (0, 1)$. Then $i^2 = (-1, 0)$, and hence $i^2 = -1$. Finally,

$$\begin{aligned}(x, y) &= (x, 0) + (0, 1) \cdot (y, 0) \\ &= x + iy.\end{aligned}$$

For some more notation, $z = x + iy$ where $z \in \mathbb{C}$, $x, y \in \mathbb{R}$. We say that $x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z)$.

We define $\overline{z} := x - iy$ to be the **complex conjugate** of z.

Complex Conjugates

- $\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
- $z + \overline{z} = 2\operatorname{Re}(z)$
- $z \cdot \overline{z} = 2\operatorname{Im}(z)$

Thus conjugation is a bijection from $\mathbb{C} \longrightarrow \mathbb{C}$, where $z \mapsto \overline{z}$, that preserves addition and multiplication. Note that this bijection is of order 2; that is, $\overline{\overline{z}} = z$.

Exercise. Show that there does not exist such a map on $\mathbb{R}$.

Absolute Value

Definition. The **modulus** or **absolute value** $|z|$ of $z = x + iy$ is defined as $\sqrt{x^2 + y^2}$.

Some properties of this definition include:
- $|z|^2 = z \cdot \overline{z}$
- $|z| = |\overline{z}|$
- $|z| > 0$ unless $z = 0$
- $|z_1 z_2| = |z_1| \cdot |z_2|$
- $\text{Re}(z) \leq |z|$

The Triangle Inequality

Lemma 5. $|z_1 + z_2| \leq |z_1| + |z_2|$. We prove this as follows:

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2) \cdot \left(\overline{z_1 + z_2}\right) \\ &= z_1 \overline{z_1} + z_1 \overline{z_2} + \overline{z_1} z_2 + z_2 \overline{z_2} \\ &= |z_1|^2 + |z_2|^2 + 2|z_1\|z_2| \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

$$\therefore |z_1 + z_2| \leq |z_1| + |z_2|.$$

Corollary 1. $|a - c| \leq |\underbrace{a - b}_{z_1}| + |\underbrace{b - c}_{z_2}| \ \forall a, b, c \in \mathbb{C}$.

Define $d(a, b) = |a - b|$. Then $d(a, b) \leq d(a, b) + d(b, c)$.

Corollary 2. $|a| - |b| \leq |a - b|$. Let $a = z_1 + z_2$ and $b = z_2$ for this result to follow.

Geometry of Complex Numbers

If $z = x + iy$ is a complex number, observe that

$$z = |z| \left(\frac{x}{|z|} + \frac{iy}{|z|} \right).$$

Then $\frac{x}{|z|} = \cos(\theta)$, $\frac{y}{|z|} = \sin(\theta)$, where $\theta = \text{Arg}(z)$. *TODO: insert a diagram!*

Theorem 6. The Cauchy-Schwartz Inequality

In Russia, this is called something else... well, everything is called something else there.

Let $a_1, ..., a_n$ and $b_1, ..., b_n$ be complex numbers. Then

$$\left(\sum_{i=1}^{n} a_i \cdot \overline{b_i} \right)^2 \leq \sum_{i=1}^{n} |a_i|^2 \cdot \sum_{i=1}^{n} |b_i|^2.$$

Proof. For any $\lambda, \mu \in \mathbb{C}$, we know:

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} |\lambda a_i + \mu b_i|^2 \\ &= \sum_{i=1}^{n} (\lambda a_i + \mu b_i) \cdot \left(\overline{\lambda} \cdot \overline{a_i} + \overline{\mu} \cdot \overline{b_i} \right) \\ &= |\lambda^2| \sum_{i=1}^{n} |a_i|^2 + \lambda \overline{\mu} \sum_{i=1}^{n} a \cdot \overline{b_i} + \overline{\lambda} \mu \sum_{i=1}^{n} \overline{a_i} \cdot b - |\mu|^2 \sum_{i=1}^{n} |b_i|^2 \end{aligned}$$

Then, we choose λ and μ.

$$\lambda = \sum_{i=1}^{n} |b_i|^2 = \overline{\lambda} \text{ } (\lambda \text{ is real})$$
$$\mu = -\sum_{i=1}^{n} a_i \cdot \overline{b_i}$$

Substituting yields:

$$0 \leq \lambda \left(\lambda \sum_{i=1}^{n} |a_i|^2 - 2\mu\overline{\mu} + |\mu|^2 \right)$$
$$= \lambda \left(\lambda \sum_{i=1}^{n} |a_i|^2 - |\mu|^2 \right)$$

We know that $\lambda \geq 0$. If $\lambda = 0$, then $b_i = 0 \ \forall i$, and in this case, the inequality we seek to prove holds. Otherwise, if $\lambda > 0$, we multiply both sides of the inequality by λ^{-1} to obtain the following:

$$|\mu|^2 \leq \lambda \sum_{i=1}^{n} |a_i|^2$$

With one final substitution,

$$|\sum_{i=1}^{n} a_i \cdot \overline{b_i}|^2 \leq \sum_{i=1}^{n} |a_i|^2 \cdot \sum_{i=1}^{n} |b_i|^2. \ \blacksquare$$

Notes

Consider $a_1, ..., a_n$ and $b_1, ..., b_n$ to be real numbers. Then,

$$|\sum_{i=1}^{n} a_i \cdot b_i|^2 \leq \sum_{i=1}^{n} |a_i|^n \cdot \sum_{i=1}^{n} |b_i|^2.$$

Now look at $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times).

If $\hat{a} \in \mathbb{R}^n$, then $\hat{a}$ is an ordered n-tuple $(a_1, ..., a_n)$. Define $\|\hat{a}\|$ to be the **norm** of a where

$$\|\hat{a}\| = +\sqrt{\sum_{i=1}^{n} (a_i)^2}.$$

Define $\hat{a} \cdot \hat{b} = \sum_{i=1}^{n} a_i \cdot b_i$. Then $\|\hat{a}\| = +\sqrt{\hat{a} \cdot \hat{a}}$.

The Cauchy-Schwartz inequality says $\left(\hat{a} \cdot \hat{b}\right)^2 \leq \|\hat{a}\|^2 \cdot \|\hat{b}\|^2$, and this can be used to define *angle*.

Chapter II: Sequences

Definition. A **sequence** in a set S is a function $f : \mathbb{N} \to S$, i.e. a rule associating $f(n) \in S$ with each $n \in \mathbb{N}$. Usually for us, S is $\mathbb{R}$ or $\mathbb{C}$. For example,

$$f(n) = \frac{1}{n}, \ f(n) = +\sqrt{n}, \ f(n) = \left(1 + \frac{1}{n}\right)^n$$

$$f(n) = \text{the number of digits in the } n\text{th prime}$$

We often say $f(n)$ is a sequence in S. For other notation, we use a_n or x_n instead of $f(n)$.

It won't matter if f is not defined for finitely many values of n. For example,

$$f(n) = \frac{1}{(n-1)(n-5)}$$

is not defined for $n = 1$ or $n = 5$. We often take limits of these sequences as $n \to \infty$.

Convergence

Definition. Let $f(n)$ be a sequence in $\mathbb{R}$ or $\mathbb{C}$. Then the sequence $f(n)$ **converges** to a limit $\ell \in \mathbb{R}$ (or $\mathbb{C}$) if, given $\varepsilon \in \mathbb{R}$ where $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|f(n) - \ell| < \varepsilon \ \forall n \geq n_0$.

We write that $f(n) \to \ell$ as $n \to \infty$. We may also write $\lim\limits_{n\to\infty} f(n) = \ell$.

Notes about sequences

1. Sequences are **not** series.
2. ∞ is not an element of $\mathbb{R}$ or $\mathbb{C}$, but tending to infinity ($n \to \infty$) is permitted.
3. ε is standard notation for an arbitrarily small number.
4. ε must be given before n_0 is given, and usually n_0 depends upon ε.

Examples

1. $f(n) = 1/n$; then $f(n) \to 0$ as $n \to \infty$ from the corollary to theorem 2.
2. $f(n) = c \ \forall n$, a constant sequence;
   Then $f(n) \to c$ as $n \to \infty$ since $|f(n) - c| = 0 < \varepsilon \ \forall n$.
3. $f(n) = \frac{2}{n^2}$. Given $\varepsilon > 0$, $|f(n) - 0| < \varepsilon$ provided that $n > \sqrt{\frac{2}{\varepsilon}}$. So take $n_0 = \left\lfloor \frac{2}{\varepsilon} \right\rceil + 1$.