

Theorem

Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent subset of a vector space $V(\mathbb{F})$, and suppose $v \in V$ where v is also a vector in S .

Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

Proof: Suppose $S \cup \{v\}$ is linearly dependent. Then there is some nonzero constant c for which

$$\begin{aligned} c_1 v_1 + c_2 v_2 + \dots + c_n v_n + cv &= \mathbf{0} \\ \implies c_1 v_1 + c_2 v_2 + \dots + c_n v_n &= -cv \quad c_i \in \mathbb{F} \forall i \in [n]. \end{aligned}$$

Then, since $c \neq 0$, we multiply both sides by c^{-1} to get

$$v = c^{-1}(c_1 v_1 + c_2 v_2 + \dots + c_n v_n).$$

Thus, $v \in \text{span}(S)$.

Conversely, suppose $v \in \text{span}(S)$. Then v can be expressed as a linear combination of vectors in S , like so:

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n \quad c_i \in \mathbb{F} \forall i \in [n].$$

Adding $-v$ to both sides, we obtain

$$\mathbf{0} = -1v + c_1 v_1 + c_2 v_2 + \dots + c_n v_n.$$

Thus, there exists a nontrivial combination of vectors in $S \cup \{v\}$ that equals the zero vector. By definition, this set is linearly dependent. ■

Theorem

Let $S = \{v_i\}_{i=1}^n$ be a linearly dependent subset of a vector space $V(\mathbb{F})$. Then there exists some index $k \in [n]$ for which

$$v_k \in \text{span}\{v_1, v_2, \dots, v_{k-1}\}.$$

Furthermore, $\text{span}(S) = \text{span}(S \setminus \{v_k\})$.

Proof: Given that $S = \{v_1, \dots, v_n\}$ is linearly dependent, there are some $c_i \in \mathbb{F}$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = \mathbf{0}.$$

Choose $k \in [n]$ to be the largest number for which $c_k \neq 0$. We then have

$$c_1 v_1 + \dots + c_k v_k + \dots + c_n v_n = \mathbf{0}.$$

Adding $-c_k v_k$ to both sides and multiplying by c_k^{-1} , we get

$$\begin{aligned} v_k &= c_k^{-1}(c_1 v_1 + \dots + c_{k-1} v_{k-1} + 0v_k + 0v_{k+1} + \dots + 0v_n) \\ &= c_k^{-1}(c_1 v_1 + \dots + c_{k-1} v_{k-1}). \end{aligned}$$

Therefore, $v_k \in \text{span}\{v_1, v_2, \dots, v_{k-1}\}$.

We now show that $\text{span}(S) = \text{span}(S \setminus \{v_k\})$. Let $v \in \text{span}(S \setminus \{v_k\})$. Then

$$\begin{aligned} v &= \sum_{i=1}^{k-1} c_i v_i + \sum_{i=k+1}^n c_i v_i \\ &= \sum_{i=1}^{k-1} c_i v_i + 0v_k + \sum_{i=k+1}^n c_i v_i \quad v_i \in S, c_i \in \mathbb{F} \forall i \in [n]. \end{aligned}$$

Thus, $v \in \text{span}(S)$, which implies $\text{span}(S \setminus \{v_k\}) \subseteq \text{span}(S)$.

Conversely, let $v \in \text{span}(S)$. Then

$$v = \sum_{i=1}^n c_i v_i = \sum_{i=1}^{k-1} c_i v_i + c_k v_k + \sum_{i=k+1}^n c_i v_i \quad v_i \in S, c_i \in \mathbb{F} \forall i \in [n].$$

We know that v_k , and thereby $c_k v_k$, exist in $\text{span}\{v_1, v_2, \dots, v_{k-1}\}$. This implies

$$c_k v_k = d_1 v_1 + d_2 v_2 + \dots + d_{k-1} v_{k-1} \quad d_i \in \mathbb{F} \forall i \in [n].$$

Substituting this expression back into the expansion of v , we get

$$v = \sum_{i=1}^{k-1} (c_i + d_i) v_i + \sum_{i=k+1}^n c_i v_i.$$

Thus, v can be written as a linear combination of vectors in $S \setminus \{v_k\}$. This implies $v \in \text{span}(S \setminus \{v_k\})$, meaning $\text{span}(S) \subseteq \text{span}(S \setminus \{v_k\})$.

It follows that $\text{span}(S) = \text{span}(S \setminus \{v_k\})$. ■

Remark

This theorem tells us that deleting a vector from a linearly dependent set does not change its span.

Theorem

The Replacement Theorem

Let $V(\mathbb{F})$ be a vector space generated by a finite set S for which $|S| = n < \infty$. Furthermore, let L be a linearly independent subset of V for which $|L| = m < \infty$.

- (i) $m \leq n$, and $|S| \leq |L|$.
- (ii) There is a subset $H \subseteq S$ containing exactly $n - m$ vectors such that $L \cup H$ generates V .

Proof: Let $S = \{x_i\}_{i=1}^n \subseteq V(\mathbb{F})$ such that $\text{span}(S) = V$. We proceed by mathematical induction on m .

If $m = 0$, then $m \leq n$ for every natural number n . Moreover, $m = 0$ implies L is the empty set, so taking $H = S$ gives the desired result.

Now, assume that the result holds for some integer $m > 0$. We show that the result also holds for $m + 1$.

Let $L = \{v_1, v_2, \dots, v_{m+1}\}$ be a linearly independent subset of V containing $m + 1$ elements. By a previous result, any finite subset of a linearly independent set is also linearly independent. Thus, the set $\{v_1, v_2, \dots, v_m\} \subseteq L$ is linearly independent.

By the inductive hypothesis, $m \leq n$, and there exists a subset $\{x_1, x_2, \dots, x_{n-m}\} \subseteq S$ such that $\{v_1, v_2, \dots, v_m\} \cup \{x_1, x_2, \dots, x_{n-m}\}$ generates V .

Now, we know $v_{m+1} \in V$, so it can be written as a linear combination of vectors in the above union. That is,

$$v_{m+1} = c_1 v_1 + \dots + c_m v_m + d_1 x_1 + \dots + d_{n-m} x_{n-m}$$

$$c_i \in \mathbb{F} \forall i \in [m], \quad d_i \in \mathbb{F} \forall i \in [n - m].$$

Note that $n - m > 0$, since otherwise $n - m = 0$ would imply v_{m+1} is a linear combination of vectors in $\{v_1, v_2, \dots, v_m\} \subseteq L$. That would be a contradiction, since L is linearly independent. Thus, $n > m$, and $n \geq m + 1$.

Moreover, some d_i must be nonzero, otherwise we would reach the same contradiction. Take $d_1 \neq 0$. From our expansion of v_{m+1} , we get

$$x_1 = -d_1^{-1}(c_1 v_1 + \dots + c_m v_m + d_2 x_2 + \dots + d_{n-m} x_{n-m} - v_{m+1}).$$

Let $H = \{x_2, x_3, \dots, x_{n-m}\}$. Note that $|H| = n - m - 1 = n - (m + 1)$. Then x_1 is a linear combination of vectors in L and H . In other words, $x_1 \in \text{span}(L \cup H)$.

All vectors v_1, \dots, v_m exist in L , so they are also elements of $\text{span}(L \cup H)$. Similarly, since x_2, \dots, x_{n-m} are all in H , they are vectors in $\text{span}(L \cup H)$. We just showed that $x_1 \in \text{span}(L \cup H)$. These facts combined give $\{v_1, \dots, v_m, x_1, \dots, x_{n-m}\} \subseteq \text{span}(L \cup H)$.

It follows that $\text{span}\{v_1, \dots, v_m, x_1, \dots, x_{n-m}\} \subseteq \text{span}(L \cup H)$. However, we also know the set $\{v_1, \dots, v_m, x_1, \dots, x_{n-m}\}$ generates V . Therefore, $\text{span}(L \cup H) = V$. The conclusion then holds for $m + 1$. ■

Definition

Basis for a Vector Space

A **basis** for a vector space $V(\mathbb{F})$ is a linearly independent subset of V that generates V . We usually represent bases with the symbol β .

Example

The set $\{(1, 2), (3, 5)\}$ is a basis for \mathbb{R}^2 .

Example

The basis for the zero vector space $\{\mathbf{0}\}$ is the empty set.

Example

One basis for \mathbb{F}^n is a set of n -tuples

$$\{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, \dots, 0, 1)\} = \{e_1, e_2, \dots, e_n\}.$$

This is the **standard basis** for \mathbb{F}^n .