

**Example 4**

$$\lim_{n \rightarrow \infty} n^{1/n} = 1.$$

*Proof:* Write  $n^{1/n}$  as  $1 + \delta(n)$  where  $\delta(n) \geq 0$ . Then

$$\begin{aligned} n &= (1 + \delta(n))^n \\ &= 1 + n\delta(n) + \frac{n(n-1)\delta(n)^2}{2!} + \cdots + \delta(n)^n \\ &\geq \frac{n(n-1)\delta(n)^2}{2!} \end{aligned}$$

Isolating  $\delta(n)$ , we find

$$\sqrt{\frac{2}{n-1}} \geq \delta(n) \geq 0.$$

Now  $\sqrt{\frac{2}{n-1}} \rightarrow 0$  as  $n \rightarrow \infty$  by Example 2. Then, by Lemma 11,  $\delta(n) \rightarrow 0$  as  $n \rightarrow \infty$ .

Therefore,  $n^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ . ■

**Example 5**

$\left(1 + \frac{1}{n}\right)^n$  tends to some limit  $e$  for which  $2 \leq e \leq 3$ .

*Proof:*

Using the binomial theorem, we get

$$\left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \cdots + \left(\frac{1}{n}\right)^n. \quad (1)$$

The  $(r+1)$ st term of this expansion is

$$\frac{1}{r! \cdot n^r} \cdot n(n-1)\cdots(n-r+1) = \frac{1}{r!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{r-n}{n}\right). \quad (2)$$

The  $(r+1)$ st term increases as  $n$  increases since each term in the product increases. As  $n$  increases, the number of terms in the expansion of  $\left(1 + \frac{1}{n}\right)^n$ , which is  $n+1$ , also increases. Hence,  $\left(1 + \frac{1}{n}\right)^n$  increases as  $n$  increases.

To establish an upper bound on the sequence, Equation 2 tells us that the  $r$ th term of the expansion in Equation 1 is at most  $1/r!$ . Then,

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \\ &= 3 - \frac{1}{2^{n-1}} < 3. \end{aligned}$$

So  $\left(1 + \frac{1}{n}\right)^n$  is increasing and is bounded above. Therefore,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \text{ exists, and } \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \leq 3. \quad \blacksquare$$

### Definition 6

Let  $f, g : \mathbb{N} \rightarrow \mathbb{R}$  be real sequences. Then we introduce the following notation.

(i)  $f = O(g)$  or  $f(n) = O(g(n))$  if there is some constant  $K \in \mathbb{R}$  and some  $n_0 \in \mathbb{N}$  for which

$$|f(n)| < K|g(n)| \quad \forall n \geq n_0.$$

(ii)  $f = o(g)$  or  $f(n) = o(g(n)) \iff f(n)/g(n) \rightarrow 0$  as  $n \rightarrow \infty$ .

(iii)  $f \sim g$  or  $f(n) \sim g(n) \iff f(n)/g(n) \rightarrow 1$  as  $n \rightarrow \infty$ .

### Example

Let  $f(n) = 2n^3 + 3n^2 + 27$ .

(i)  $f(n) = O(n^3)$  since  $|f(n)| < 3n^3$  if  $3n^2 + 27 \leq n^3$ , which is true when  $n \geq 5$ .

(ii)  $f(n) = o(n^4)$ .

(iii)  $f(n) \sim 2n^3 + 5n + 1$ .

### Example

(i)  $n = O(n^2)$ , and  $n = o(n^2)$ , but  $n \neq n^2$ .

(ii)  $n + \sin(n) \sim n$ .

### Note

You should **never, ever** write  $f(n) \rightarrow g(n)$  as  $n \rightarrow \infty$ , because we have no precise definition of what this means.

*There is a special place in hell for people who write this. People shouldn't commit crimes either, but this is much worse. At least, frequently, crimes are meaningful.*

### Definition 7

If  $f : \mathbb{N} \rightarrow \mathbb{R}$ , then  $f$  is said to **tend to infinity** if, given any  $K \in \mathbb{R}$ , there is a natural number  $n_0$  such that  $f(n) \geq K$  for all  $n \geq n_0$ .

We write this as

$$f(n) \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \text{or} \quad \lim_{n \rightarrow \infty} f(n) = \infty.$$

### Exercise 8

Write a similar definition for sequences tending to negative infinity.

### Note

Saying that a sequence tends to infinity is different from saying it is unbounded. For example,  $f(n) = (-1)^n n^2$  is unbounded, but does not tend to infinity.

### Lemma 13

- (i) An unbounded monotonic sequence tends to either  $+\infty$  or  $-\infty$ .
- (ii) Let  $f$  be a sequence such that  $f(n) > 0$  for all sufficiently large  $n$ . Then, as  $n$  approaches infinity,

$$f(n) \rightarrow 0 \iff 1/f(n) \rightarrow \infty.$$

*Proof of 13.i:* Suppose, without much loss of generality, that  $f(n)$  is strictly increasing and unbounded. Then, given  $K \in \mathbb{R}$ ,  $K$  is not an upper bound of the set

$$f(\mathbb{N}) = \{f(n) : n \in \mathbb{N}\}.$$

Thus, there is some  $n_0 \in \mathbb{N}$  for which  $K < f(n_0)$ . For any  $n \geq n_0$ , it is still true that  $K < f(n)$ . ■

*Proof of 13.ii:* If  $1/f(n) \rightarrow \infty$ , then  $1/f(n)$  is defined for all sufficiently large  $n$ . Then, given  $\varepsilon > 0$ , there is some  $n_1 \in \mathbb{N}$  for which  $1/\varepsilon < 1/f(n)$  for all  $n \geq n_1$ . So

$$0 < f(n) < \varepsilon \quad \forall n \geq n_1$$

and thus  $f(n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

If  $f(n) \rightarrow 0$  as  $n \rightarrow \infty$ , then given  $K \in \mathbb{R}$  with  $K > 0$ , there is some  $n_2 \in \mathbb{N}$  for which  $|f(n)| < 1/K$  for all  $n \geq n_2$ .

Then, since  $f(n) > 0$ , we can rewrite the inequality as

$$K < 1/f(n) \quad \forall n \geq n_2$$

proving that  $1/f(n) \rightarrow \infty$  as  $n \rightarrow \infty$ . ■

## Upper and Lower Limits of Real Sequences

### Definition

Let  $f : \mathbb{N} \rightarrow \mathbb{R}$  be a sequence. If  $f(\mathbb{N})$  is bounded above, we define the **upper limit** of  $f(n)$  as  $n$  tends to infinity to be

$$\limsup_{r \rightarrow \infty} \{f(r), f(r+1), f(r+2), \dots\}.$$

If  $f(\mathbb{N})$  is bounded below, we define the **lower limit** of  $f(n)$  to be

$$\liminf_{n \rightarrow \infty} \{f(n), f(n+1), f(n+2), \dots\}.$$

We introduce the following notation for upper and lower limits respectively:

$$\limsup_{n \rightarrow \infty} f(n) = \overline{\lim}_{n \rightarrow \infty} f(n) \quad \liminf_{n \rightarrow \infty} f(n) = \underline{\lim}_{n \rightarrow \infty} f(n)$$

If  $f(\mathbb{N})$  is not bounded above, we define  $\overline{\lim}_{n \rightarrow \infty} f(n)$  to be  $+\infty$ .

Similarly, if  $f(\mathbb{N})$  is not bounded below, we define  $\underline{\lim}_{n \rightarrow \infty} f(n)$  to be  $-\infty$ .

### Remark

The precise definitions of upper and lower limits above can be understood using properties of suprema and infima. Suppose  $A \subseteq B \subseteq \mathbb{R}$  and  $B$  is bounded above. Recall that  $A$  must also be bounded above, and  $\sup(A) \leq \sup(B)$ .

Thus the expression in the definition of the upper limit of  $f(n)$  is a decreasing function. That is,

$$\sup\{f(n), f(n+1), \dots\} \geq \sup\{f(n+1), f(n+2), \dots\}.$$

Therefore, this function either tends to a limit (Theorem 12) or to  $-\infty$  (Lemma 13).

Similarly, the sequence in the definition of the lower limit is increasing, so its limit is either a real number  $\ell$  or  $+\infty$ .

### Example

(i) Let  $f(n) = (-1)^n + 1/n$ .

Then  $\overline{\lim}_{n \rightarrow \infty} f(n) = 1$  since  $1 + 1/n \rightarrow 1$  as  $n \rightarrow \infty$ . Similarly,  $\underline{\lim}_{n \rightarrow \infty} f(n) = -1$ .

(ii) Let  $r : \mathbb{N} \rightarrow \mathbb{Q}$  be an enumeration of the rationals, i.e. a bijection. Then

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} r(n) &= +\infty & \underline{\lim}_{n \rightarrow \infty} r(n) &= -\infty \\ \overline{\lim}_{n \rightarrow \infty} |r(n)| &= +\infty & \underline{\lim}_{n \rightarrow \infty} |r(n)| &= 0. \end{aligned}$$

(iii) If  $f(n) = n$ , then  $\overline{\lim}_{n \rightarrow \infty} f(n) = \infty = \underline{\lim}_{n \rightarrow \infty} f(n)$ .