Lemma 7

Suppose that $f(n)$ is a sequence in $\mathbb{R}$ or $\mathbb{C}$. Suppose further that $f(n) \to \ell_1$ and $f(n) \to \ell_2$ as $n \to \infty$. Then $\ell_1 = \ell_2$.

Proof: If $\ell_1 \neq \ell_2$, then $|\ell_1 - \ell_2| > 0$. Let $\varepsilon = \frac{1}{2}|\ell_1 - \ell_2|$. Then

$$\exists n_1 \in \mathbb{N} \text{ st. } |f(n) - \ell_1| < \varepsilon \ \forall n \geq n_1,$$
$$\exists n_2 \in \mathbb{N} \text{ st. } |f(n) - \ell_2| < \varepsilon \ \forall n \geq n_2.$$

Choose $n = \max(n_1, n_2)$. Then, using the triangle inequality,

$$|\ell_1 - \ell_2| \leq |\ell_1 - f(n)| + |\ell_2 - f(n)| < 2\varepsilon.$$

However, $|\ell_1 - \ell_2| = 2\varepsilon$, so this is a contradiction. Hence $\ell_1 = \ell_2$. ∎

Definition **Sequence Divergence**

We say that a sequence $f(n)$ **diverges** if and only if $f(n)$ does not converge.

Lemma 8

Suppose that $f(n)$ and $g(n)$ are sequences in $\mathbb{R}$ or $\mathbb{C}$. If $f(n) \to \ell$ and $g(n) \to m$ as $n \to \infty$, then:

(i) $f(n) + g(n) \to \ell + m$.

(ii) $f(n) \cdot g(n) \to \ell \cdot m$.

(iii) If $m \neq 0$, then $f(n)/g(n)$ is defined for all but finitely many $n \in N$, and $f(n)/g(n)$ converges to ℓ/m.

Proof of 8.i: Given $\varepsilon > 0$,

$$\exists n_1 \in \mathbb{N} \text{ st. } |f(n) - \ell| < \varepsilon/2 \ \forall n \geq n_1,$$
$$\exists n_2 \in \mathbb{N} \text{ st. } |g(n) - m| < \varepsilon/2 \ \forall n \geq n_2.$$

Then, for all $n \geq \max(n_1, n_2)$,

$$\begin{aligned} |f(n) + g(n) - \ell - m| &\leq |f(n) - \ell| + |g(n) - m| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore, $f(n) + g(n) \to \ell + m$ as $n \to \infty$ by definition. ∎

Remark

If $f(n)$ can be shown to converge to a limit ℓ given $\varepsilon' > 0$ where ε' takes the form $\varepsilon' = k \cdot \varepsilon$ for another $\varepsilon > 0$ and a constant k, then $f(n)$ also satisfies the convergence definition given ε. If we have:

$$\exists n_0 \in \mathbb{N} \text{ st. } |f(n) - \ell| < \varepsilon' = k \cdot \varepsilon \ \forall n \geq n_0$$

then we simply choose $\varepsilon' = \varepsilon/k$ for the required result. This technique is useful when constructing proofs like the above.

Proof of 8.ii: Observe that

$$\begin{aligned}|f(n)g(n) - \ell m| &= |f(n)g(n) - f(n)m + f(n)m - \ell m| \\ &\leq |f(n)| \cdot |g(n) - m| + |m| \cdot |f(n) - \ell|.\end{aligned}$$

Given $\varepsilon > 0$, we also know

$$\begin{aligned}&\exists n_1 \in \mathbb{N} \text{ st. } |f(n) - \ell| < \min(\varepsilon, 1) \; \forall n \geq n_1, \\ &\exists n_2 \in \mathbb{N} \text{ st. } |g(n) - m| < \varepsilon \qquad\quad \forall n \geq n_2.\end{aligned}$$

Hence, if $n \geq n_1$ then $|f(n)| < |\ell| + 1$, and

$$|f(n)g(n) - \ell m| < \varepsilon(|\ell| + 1 + |m|) \; \forall n \geq \max\big(n_1, n_2\big).$$

therefore, $f(n) \cdot g(n) \to \ell \cdot m$ as $n \to \infty$. ∎

Proof of 8.iii: For this part, it will be sufficient to show that $1/g(n) \to 1/m$ and then apply 8.ii above. We first show that $1/g(n)$ exists, i.e. is defined for all sufficiently large n.

Now since $g(n) \to m$, there is a sufficiently small $\varepsilon > 0$ such that

$$\exists n_0 \in \mathbb{N} \text{ st. } |g(n) - m| < \varepsilon < |m|/2 \; \forall n \geq n_0.$$

Then,

$$\begin{aligned}|m| &\leq |g(n)| + |m - g(n)| \\ &< |g(n)| + |m|/2 \\ \implies |m|/2 &< |g(n)| \; \forall n \geq n_0.\end{aligned}$$

Hence for every n greater than n_0, $g(n)$ is not zero (it is positive) and so $1/g(n)$ exists.

For a more general $\varepsilon > 0$, $g(n)$ satisfies the definition of sequence convergence tending to m for some sufficiently large $n_1 \in \mathbb{N}$. Combined with the previous inequalities, it follows that

$$\left|\frac{1}{m} - \frac{1}{g(n)}\right| = \left|\frac{g(n) - m}{m \cdot g(n)}\right| < \frac{2\varepsilon}{|m|^2} \qquad \forall n \geq \max\big(n_0, n_1\big).$$

Therefore, $1/g(n) \to 1/m$ as $n \to \infty$. ∎

Lemma 9

Suppose that $f(n) = x(n) + iy(n)$ is a sequence in $\mathbb{C}$ where $x(n)$ and $y(n)$ are both sequences in $\mathbb{R}$. Then $f(n) \to a + ib$, for some real numbers a and b, if and only if $x(n) \to a$ and $y(n) \to b$.

Proof: In the reverse direction, the result immediately follows from Lemma 8.i applied to complex sequences.

We handle the forward direction. Suppose $f(n) \to a + ib$. Then, given $\varepsilon > 0$,

$$\exists n_0 \in \mathbb{N} \text{ st. } |x(n) + iy(n) - a - ib| < \varepsilon \ \forall n \geq n_0.$$

We know that for every complex number z, its real and imaginary components satisfy the following conditions:

$$|\mathrm{Re}(z)| \leq |z|$$
$$|\mathrm{Im}(z)| \leq |z|$$

Applying these properties to the first equation, we see that

$$\exists n_0 \in \mathbb{N} \text{ st. } |x(n) - a| < |[x(n) - a] + i[y(n) - b]| < \varepsilon$$
$$|y(n) - b| < |[x(n) - a] + i[y(n) - b]| < \varepsilon \ \forall n \geq n_0.$$

Therefore, $x(n) \to a$ and $y(n) \to b$ as $n \to \infty$. ■

Lemma 10

If $f(n)$ is a sequence in $\mathbb{R}$ or $\mathbb{C}$ and $f(n) \to \ell$ as $n \to \infty$, then $f(n)$ is **bounded** by a real number k. Specifically,

$$|f(n)| \leq k \ \forall n \in \mathbb{N}.$$

Equivalently, $f(\mathbb{N}) = \{f(n) : n \in \mathbb{N}\}$ is a bounded set.

Proof: Using the definition of sequence convergence,

$$\exists n_0 \in \mathbb{N} \text{ st. } |f(n) - \ell| < 1 \ \forall n \geq n_0.$$

Hence, for every $n \geq n_0$, it holds that $|f(n)| < |\ell| + 1$.

Now choose $k = \max\{|f(1)|, |f(2)|, \ldots, |f(n_0 - 1)|, |\ell| + 1\}$. Then $f(n) \leq k$ for all n. ■

Remark

In the proof above, the sequence terms from 1 to $n_0 - 1$ are the finitely many terms which lie outside of a small region around ℓ. In this case, these terms differ from ℓ by more than 1, but in general, these special terms always form a finite set.

Note

The converse to Lemma 10 is false; that is, not every bounded sequence converges to a limit. For example, $f(n) = (-1)^n \in \mathbb{R}$ is bounded, since for every nth term, $|f(n)| \leq 1$. However, $f(n)$ does not converge, instead oscillating between 1 and -1 forever.