MATH 108A Notes Jan 6. 2026 (Draft)

Introduction

Linear Algebra is the study of linear equations and linear maps between linear spaces.

A **linear equation** is an equation of the form

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0$$

where each c_i is a **scalar** and each x_i is a **variable**.

A **linear map** is a function $f : X \to Y$

Fields

We are familiar with $\mathbb{R}$, the set of real numbers, which may be defined as

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{R} \setminus \mathbb{Q}$$

Consider $\mathbb{C}$, the set of complex numbers. The set is defined as

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} \text{ where } i = \sqrt{-1}.$$

We can add and multiply complex numbers like so:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$\begin{aligned}(a + bi) \cdot (c + di) &= ac + adi + bci - bd \\ &= ac - bd + (ad + bc)i\end{aligned}$$

The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ are examples of **fields**.

Properties of $\mathbb{C}$

1. **Commutativity of $+$ and $\cdot$**
   $\forall \alpha, \beta \in \mathbb{C}, \alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$
2. **Associativity of $+$ and $\cdot$**
   $\forall \alpha, \beta, \gamma \in \mathbb{C}, (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$
3. **Existence of the additive identity**
   $\exists!\ 0 \in \mathbb{C}$ such that $\forall \alpha \in \mathbb{C}, \alpha + 0 = 0 + \alpha = \alpha$
4. **Existence of the multiplicative identity**
   $\exists!\ 1 \in \mathbb{C}$ such that $\forall \alpha \in \mathbb{C}, \alpha \cdot 1 = 1 \cdot \alpha = \alpha$
5. **Existence of the additive inverse**
   $\forall \alpha \in \mathbb{C}, \exists!\ \beta \in \mathbb{C}$ such that $\alpha + \beta = \beta + \alpha = 0$. We notate β as $-\alpha$.
6. **Existence of the multiplicative inverse**
   $\forall \alpha \in \mathbb{C}$ where $\alpha \neq 0, \exists!\ \beta \in \mathbb{C}$ such that $\alpha\beta = \beta\alpha = 1$ We notate β as $\frac{1}{\alpha}$.
7. **Distributivity**
   $\forall \alpha, \beta, \gamma \in \mathbb{C}, \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ and $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$

Ex. Consider $2 + 3i \in \mathbb{C}$. The additive inverse is $-2 - 3i$, and the multiplicative inverse can be found like so:

$$\frac{1}{2 + 3i} \cdot \frac{2 - 3i}{2 - 3i} = \frac{2}{13} - \frac{3i}{13}$$

Ex. Prove that $\alpha\beta = \beta\alpha$ for every pair $\alpha, \beta \in \mathbb{C}$.

Proof. Let $a, \beta \in \mathbb{C}$. Then $\alpha = a + bi$ and $\beta = c + di$ for some $a, b, c, d \in \mathbb{R}$.

Expanding the expression $\alpha\beta$, we get:

$$\begin{aligned}\alpha\beta &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ca + dai + cbi + dbi^2 \\ &= (c + di)(a + bi) \\ &= \beta\alpha. \blacksquare\end{aligned}$$

Examples of Fields

We notate a general field with the symbol $\mathbb{F}$. Importantly, elements of a field $\mathbb{F}$ are **scalars**. Some notable fields include:

- $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$
- $\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$
- $\mathbb{R}^n = \{(x_1, x_2, ..., x_n) \mid x_i \in \mathbb{R} \text{ for } i = 1, 2, ..., n\}$
- $\mathbb{C}^2 = \{(\zeta_1, \zeta_2) \mid \zeta_1, \zeta_2 \in \mathbb{C}\} = \{(a + bi, c + di) \mid a, b, c, d \in \mathbb{R}\}$
- $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}\}$

More generally,

$$\mathbb{F}^n = \{\underbrace{(x_1, x_2, ..., x_n)}_{n\text{-tuple}} \mid x_i \in \mathbb{F} \text{ for } i = 1, 2, ..., n\} \text{ is a field.}$$

In this course, we will restrict ourselves to using $\mathbb{R}$ and $\mathbb{C}$.

Operations in Fields

x_i is the ith coordinate of the n-tuple $(x_1, x_2, ..., x_n)$.

Given a field $\mathbb{F}^n$ and $x_i, y_i \in \mathbb{F}^n$ where $i = 1, 2, ..., n$,

$$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n).$$

Ex. Observe the addition of two tuples in $\mathbb{R}^3$:

$$(2, 3, 4) + (-1, 9, -6) = (2 - 1, 3 + 9, 4 - 6) = (1, 12, -2)$$

It follows that the additive identity of $\mathbb{F}^n$ is an n-tuple $(0, 0, ..., 0)$ where 0 is the additive identity of $\mathbb{F}$.

Fields also support scalar multiplication. Given a scalar $\lambda \in \mathbb{F}$ and a tuple $x \in \mathbb{F}^n$,

$$\lambda x = \lambda(x_1, x_2, ..., x_n) = (\lambda x_1, \lambda x_2, ..., \lambda x_n).$$

Binary Operations

An operation $\circ$ on a set S is said to be a **binary** if for every pair $a, b \in S$, it holds that $a \circ b \in S$. That is, $\circ$ is a function where

$$\circ : S \times S \to S.$$

We say that S is **closed** under $\circ$.

Examples of Binary Operations

- Addition on $\mathbb{R}$, i.e. $f(a, b) = a + b$
- Multiplication on $\mathbb{R}$, i.e. $f(a, b) = a \cdot b$

- Addition and multiplication on $\mathbb{Q}$
- Addition on $M_{2\times 2}(\mathbb{R})$, or real 2-by-2 matrices

Multiplication is *not* a binary option on the irrationals $\mathbb{R} \setminus \mathbb{Q}$. For instance, $\sqrt{2} \cdot \sqrt{2} \notin \mathbb{R} \setminus \mathbb{Q}$.

Vector Spaces

A **vector space** V over a field $\mathbb{F}$ is a set with two operations, namely addition $(+)$ and scalar multiplication $(\cdot)$, for which the following properties hold:

1. **Commutativity**
   $u + v = v + u \ \forall u, v \in V$
2. **Associativity**
   $(u + v) + w = u + (v + w) \ \forall u, v, w \in V$
3. **Existence of the additive identity**
   $\exists\ 0 \in V$ such that $v + 0 = 0 + v \ \forall v \in V$
4. **Existence of the additive inverse**
   $\forall u \in V \ \exists v \in V$ such that $u + v = 0$. We notate v as $-u$.
5. **Existence of the multiplicative identity**
   $1 \cdot v = v \cdot 1 = v \ \forall v \in V$ where $1 \in \mathbb{F}$
6. **Distributivity**
   $(ab)v = a(bv) \ \forall a, b \in \mathbb{F} \ \forall v \in V$
7. *unnamed*
   $a(u + v) = au + av \ \forall a \in \mathbb{F} \ \forall u, v \in V$
8. *unnamed*
   $(a + b)v = av + bv \ \forall a, b \in \mathbb{F} \ \forall v \in V$

Note that addition is a binary operation, but scalar multiplication is not.

Examples of Vector Spaces

- $\mathbb{R}$ is a vector space over itself
- $\mathbb{C}$ is a vector space over itself
- $\mathbb{C}$ is a vector space over $\mathbb{R}$
- $\mathbb{R}^n$ is a vector space over $\mathbb{R}$
- $\mathbb{C}^n$ is a vector space over $\mathbb{C}$
- $\mathbb{F}^n$ is a vector space over $\mathbb{F}$ for any field $\mathbb{F}$

Ex. Is $\mathbb{R}$ a vector space over $\mathbb{R}^2$?

> **No**. For instance, consider a vector $x \in \mathbb{R}$ and a the multiplicative identity $(1, 1) \in \mathbb{R}^2$. We would expect $(1, 1) \cdot x = x$, but in fact $(1, 1) \cdot x = (x, x) \notin \mathbb{R}$.

Ex. Is the empty set a vector space?

> **No**, the empty set is missing identities (0 and 1). Therefore, a vector space can never be empty. The simplest vector space is $\{0\}$ over $\mathbb{R}$.