Corollary

Let $V(\mathbb{F})$ be a vector space where $\dim(V) = n$.

(i) Any finite generating set for V contains at least n vectors, and any such set containing exactly n vectors is a basis for V.

(ii) Any linearly independent subset of V consisting of exactly n vectors is a basis for V.

(iii) Every linearly independent subset of V can be extended to a basis for V.

Proof of (i): Let S be a subset of V such that $\operatorname{span}(S) = V$. By a previous result, some subset $H \subseteq S$ is a basis for V.

Then $|H| = n$ by definition. Since $|H| \leq |S|$, it follows that $|S| \geq n$.

Conversely, suppose $|S| = n$. By the same result mentioned prior, choose $H = S$. Then S is a basis for V. ∎

Proof of (ii): Let L be a linearly independent subset of V for which $|L| = n$. Also, let β be a basis for V, meaning $|\beta| = n$.

By the Replacement Theorem, there is a subset $H \subseteq \beta$, where $|H| = n - n = 0$, such that $\operatorname{span}(L \cup H) = V$. Then H must be the the empty set, so $L \cup H = L$.

Therefore, $\operatorname{span}(L) = V$, and L forms a basis for V. ∎

Proof of (iii): Let L be a linearly independent subset of V where $|L| = m$.

By the Replacement Theorem, there is a subset $H \subseteq S$, where $|H| = n - m$, such that $\operatorname{span}(L \cup H) = V$. Since L and H are disjoint sets, $|L \cup H| = m + (n - m) = n$.

Therefore, by part (i), the set $L \cup H$ forms a basis for V. ∎

Example

In a previous exercise, it was shown that

$$\gamma = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

spans the set of real 2×2 matrices. Since $\dim\big(M_{2\times 2}(\mathbb{R})\big) = 4$, then γ must also be a basis for the set.

Example

The set $\beta = \{(1,1,0), (0,1,1), (1,0,1)\}$ is a linearly independent subset of $\mathbb{R}^3$. We know the dimension of $\mathbb{R}^3$ is 3, so β must also form a basis.

Dimensions of Subspaces

Theorem

If S is a subspace of a finite-dimensional vector space $V(\mathbb{F})$, then

(i) S is finite-dimensional, and $\dim(S) \leq \dim(V)$.

(ii) If $\dim(S) = \dim(V)$, then $S = V$.

Proof of (i): Let $\dim(V) = n$. If S is the zero subspace, then $\dim(S) = 0 \leq n$.

Now, let $S \neq \{0\}$ be a subspace of V. Then there exists some nonzero $v_1 \in S$. It follows that $\{v_1\}$ is a linearly independent subset of S.

Continue choosing vectors $v_1, v_2, \ldots, v_k$ from S such that $\{v_1, v_2, \ldots, v_k\}$ remains linearly independent. This process stops at some $k \leq n$, since the cardinality of linearly independent subset of V is at most $\dim(V)$.

It follows that adding more vectors from S to $\{v_1, v_2, \ldots, v_n\}$ would make the set linearly dependent. Therefore, this set generates S, meaning it is a basis for S.

Since there are k vectors in this basis, $\dim(S) = k \leq n$. ∎

Proof of (ii): Suppose $\dim(S) = \dim(V)$. Then S has a basis β such that $|\beta| = n$. Hence, β is a linearly independent subset of V whose cardinality is exactly $\dim(V)$. By the corollary, β is a basis for V. Therefore, $S = \operatorname{span}(\beta) = V$. ∎

Example

In a previous example, we showed that β is a basis for S where

$$S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\},$$
$$\beta = \{(1, -1, 0), (1, 0, -1)\}.$$

It follows that $\dim(S) = 2$, which is less than $\dim(\mathbb{R}^3) = 3$.

Example

The set of all $n \times n$ real square matrices is

$$M_{n\times n}(\mathbb{R}) = \left\{E^{ij} : 1 \leq i, j \leq n\right\}.$$

The subset of $n \times n$ symmetric matrices, $S_{n\times n}$, is a finite-dimensional subspace of the vector space $M_{n\times n}(\mathbb{R})$. Below are two bases for $S_{2\times 2}$ and $S_{3\times 3}$.

$$\beta_{2\times 2} = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

$$\beta_{3\times 3} = \left\{\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\1 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}\right\}$$

We can also represent $B_{3\times 3}$ like so:

$$\beta_{3\times 3} = \left\{E^{11}, E^{12} + E^{21}, E^{13} + E^{31}, E^{23} + E^{32}, E^{33}\right\}$$

The basis $\beta_{3\times 3}$ has a cardinality of six, hence $\dim(S_{3\times 3}) = 6$.

To understand the dimension of a subspace of symmetric matrices, we proceed with a counting argument for the cardinality of $\beta_{3\times 3}$. A symmetric 3×3 matrix has three distinct entries in the first row, so we represent them with the basis matrices E^{11}, $E^{12} + E^{21}$, and $E^{13} + E^{31}$. There are three more entries in the second row, but the first one is already covered by $E^{12} + E^{21}$, so we add two matrices instead of three. This pattern continues in the third row, for which we only add one basis matrix.

In general,

$$|\beta_{n\times n}| = n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}.$$

Exercise

Verify that $\beta_{3\times 3}$ is a basis by showing that it spans $S_{3\times 3}$ while also being linearly independent.

Example

Let S be the set of $n \times n$ real diagram matrices. That is,

$$S = \left\{A \in M_{n\times n}(\mathbb{R}) : A_{ij} = 0 \ \forall i \neq j\right\} \subseteq M_{n\times n}(\mathbb{R})$$

Below, we construct a basis for S:

$$\beta = \left\{E^{ii} : 1 \leq i \leq n\right\} = \left\{E^{11}, E^{22}, \ldots, E^{nn}\right\}$$

It follows that $\dim(S) = |\beta| = n$.

Exercise

(i) Prove S is a subspace of $M_{n\times n}(\mathbb{R})$.

(ii) Show that β is a basis for S.

Example

Let S be a subset of $\mathbb{R}^5$ defined as

$$S = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 + x_3 + x_5 = 0,\ x_2 = x_4\right\}.$$

We will construct a basis for S, prove it is indeed a basis, then determine $\dim(S)$.

First, we describe the basis:

$$\beta = \{(0,1,0,1,0),(1,0,-1,0,0),(0,0,-1,0,1)\}$$

To show β is linearly independent, consider the linear combination

$$a(0,1,0,1,0) + b(1,0,-1,0,0) + c(0,0,-1,0,1) = \mathbf{0}.$$

Simplifying, we get

$$(b,a,-b-c,a,c) = \mathbf{0} \quad \Longleftrightarrow \quad a = b = c = 0$$

Therefore, β is linearly independent.

Next, we show β spans S. Take any vector $(x_1,\dots,x_5) \in S$. From the conditions, we know $x_1 + x_3 + x_5 = 0$ and $x_2 = x_4$. We rewrite the first condition as $x_3 = -x_1 - x_5$ to isolate x_3.

We claim that our chosen vector can be represented as

$$(x_1,\dots,x_5) = a(0,1,0,1,0) + b(1,0,-1,0,0) + c(0,0,-1,0,1)$$
$$\text{where } a = x_2 = x_4,\ b = x_1,\ c = x_5.$$

Simplifying the right side, we get

$$(x_1,\dots,x_5) = (x_1, x_2, -x_1 - x_5, x_2, x_5)$$

and this equality follows directly from the set conditions. Thus, β spans S, and therefore β is a basis for S.

We can now deduce that $\dim(S) = |\beta| = 3$.

Exercise

Since S is a subspace of $\mathbb{R}^5$, we can extend β to a basis for $\mathbb{R}^5$. One such basis is

$$\gamma = \beta \cup \{(0,0,0,1,0),(0,0,1,0,0)\}.$$

Prove γ is a basis for $\mathbb{R}^5$.