# MATH 117 Notes Jan. 13 2026 (Draft)

## Elementary properties of $\mathbb{R}$

- $\exists\ 0, 1 \in \mathbb{R}$
- $1 > 0$
- $1 + 1 + \cdots + 1 > 0$, and so on

We denote $1 + 1 + \cdots + 1$ ($n$ times) by $n$. So we have a copy of $\mathbb{N}$ in $\mathbb{R}$, whence we also have a copy of $\mathbb{Z}$ in $\mathbb{R}$.

$\exists\ m^{-1} \in \mathbb{R}$ ($m \neq 0$), and $m \cdot n^{-1}$ corresponds to $m/n \in \mathbb{Q}$. So there is a copy of $\mathbb{Q}$ in $\mathbb{R}$.

$\mathbb{R} \setminus \mathbb{Q} = \{x \in \mathbb{R} : x \notin \mathbb{Q}\}$ is called the set of **irrational numbers**.

We will show that $\exists\ \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ such that $\left(\sqrt{2}\right)^2 = 2$.

Note that $1 - \sqrt{2}$ is irrational, as the sum of a rational and an irrational number is irrational, but $\left(1 - \sqrt{2}\right) + \sqrt{2} \in \mathbb{Q}$. Thus, the irrational numbers are *not* closed under addition.

The set $\mathbb{Q}$ is countable, but the set $\mathbb{R}$ is uncountable.

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \subseteq \mathbb{H}$ (where $\mathbb{H}$ is the set of quaternions).

## Theorems of $\mathbb{R}$

**Theorem 2.** *Archimedes' Axiom.* Let $a \in \mathbb{R}$. Then $\exists n \in \mathbb{Z}$ st. $n > a$.

*Proof.* Suppose not. Then $a \geq n\ \forall n \in \mathbb{Z}$, i.e. $\mathbb{Z}$ is bounded above by $a$.
So there exists a least upper bound $b \in \mathbb{R}$ of the set $\mathbb{Z}$ (by axiom C). Hence $b - 1$ is not an upper bound of $\mathbb{Z}$, and so $\exists n_0 \in \mathbb{Z}$ with $b - 1 < n_0 \leq b$.
Adding 1 to the inequality gives $b < n_0 + 1$. But $n_0 + 1 \in \mathbb{Z}$, a contradiction. This proves the result.

Thus there is no "$\infty$" $\in \mathbb{R}$; there is no biggest real number.

**Corollary.** Given any $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$, $\exists N \in \mathbb{Z}$ st. if $n \in \mathbb{Z}$ and $n > N$, then $\varepsilon > \frac{1}{n} > 0$.

*Proof.* There exists $N \in \mathbb{Z}$ st. $N > \frac{1}{\varepsilon}$ (by Thm. 2). If $n > N$, then $n > \frac{1}{\varepsilon}$.
Therefore $\varepsilon > \frac{1}{\varepsilon}$ (by axiom B4).

**Lemma 3.** $\exists \sqrt{2} \in \mathbb{R}$, i.e. $\exists x \in \mathbb{R}$ st. $x^2 = 2$, and $1 < x < 2$.

*Proof.* Let $A = \{a \in \mathbb{R} : a^2 < 2\}$. Now $1 \in A$, so $A \neq \varnothing$. If $a \in A$, then $a < 2$ since $2^2 = 4 > 2$.
Hence $A$ is a non-empty set, and 2 is an upper bound.
Let $x = \operatorname{lub}(A)$. This exists by axiom C, and $2 > x > 1$.

1. Suppose that $x^2 < 2$.
   Let $\alpha = 2 - x^2 > 0$.
   Suppose that $0 < \delta < 1$. Then,
   $$(x + \delta)^2 - x^2 = \delta(2x + \delta) < \delta(2 \cdot 2 + 1) = 5\delta.$$
   Choose $\delta$ st. $0 < \delta < \frac{\alpha}{5}$. Then,
   $$(x + \delta)^2 - x^2 < \alpha \Longrightarrow (x + \delta)^2 < \alpha + x^2 \Longrightarrow (x + \delta)^2 < 2$$
   i.e. $x + \delta \in A$.
   Hence $x$ is not an upper bound for $A$ since $x \ngeq x + \delta$, a contradiction.

2. Suppose that $x^2 > 2$.
   Let $\beta = x^2 - 2 > 0$.
   Suppose that $0 < \delta < 1$. Then,
   $$x^2 - (x-\delta)^2 = \delta(2x - \delta) < \delta(2x) < 4\delta.$$
   Choose $\delta$ st. $0 < \delta < \frac{\beta}{4}$. Then,
   $$x^2 - (x-\delta)^2 < \beta \implies 2 < (x-\delta)^2.$$
   If $a \in A$, then $a^2 < 2 < (x-\delta)^2$. Therefore, $a < x - \delta$ (note that $x - \delta > 0$).
   Therefore $x - \delta$ is an upper bound for $A$. Hence $x$ is not the lowest upper bound of $A$, a contradiction.

We therefore conclude that $x^2 = 2$.

**Theorem 4.** If $a, b \in \mathbb{R}$, and $a < b$, then there exist rational and irrational numbers strictly between $a$ and $b$.

*Proof.* Without loss of generality, assume that $0 < a < b$.

Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < b - a$ ($\because$ corollary to Thm. 2)
Now consider the set $\left\{n \cdot \left(\frac{1}{q}\right) : n \in \mathbb{N}\right\}$. Thm. 2 implies that this set is unbounded.
Let $n_0$ be the largest non-negative integer such that $\frac{n_0}{q} \leq a$. Then,

$$a < \frac{n_0 + 1}{q} \leq a + \frac{1}{q} < a + (b - a) = b$$

and so we have constructed a rational number $\alpha = \frac{n_0+1}{q}$ with $a < \alpha < b$.

Now, we know that $0 < \frac{\sqrt{2}}{2} < 1$. The number $a + (b-a)\frac{\sqrt{2}}{2}$ lies between $a$ and $b$ and is irrational.
This completes the proof.

## More on bounds

**Remark.** If $A \subseteq \mathbb{R}$, $A \neq \varnothing$, and $A$ is bounded above, $A$ has a least upper bound. As an exercise, show that this least upper bound is unique.

If $A$ has a maximum, i.e. $\exists M \in A$ st. $M \geq a\ \forall a \in A$, then $M = \operatorname{lub}(A)$.

### Lower bounds

Suppose $\varnothing \neq A \subseteq \mathbb{R}$ and that $A$ is bounded below, i.e. $\exists b \in \mathbb{R}$ st. $b \leq a\ \forall a \in A$.
Then $-b \geq -a\ \forall a \in A$.

Define $-A = \{-a : a \in A\}$.
Then $-b$ is an upper bound for $-A$, and so $\operatorname{lub}(-A)$ exists.
Define the **greatest lower bound** of $A$ to be $\operatorname{lub}(-A)$.

We notate this as $\operatorname{glb}(A)$, the **infimum** of $A$, $\inf(A)$, or $\inf_{a \in A}(a)$.