Lemma 11

(i) Let $f(n)$ and $g(n)$ be sequences in $\mathbb{R}$ with $f(n) \to \ell$ and $g(n) \to m$ as $n \to \infty$. Suppose that $f(n) \leq g(n)$ for all sufficiently large n. Then $\ell \leq m$.

(ii) Suppose that $f(n) \leq g(n) \leq h(n)$ for all sufficiently large n. Suppose further that $f(n) \to \ell$ and $h(n) \to \ell$ as $n \to \infty$. Then $g(n)$ converges, and $g(n) \to \ell$ as $n \to \infty$.

Proof of 11.i: We are given that, for some natural number n_0, $f(n) \leq g(n)$ for all $n \geq n_0$. Additionally, given $\varepsilon > 0$, there are also numbers n_1 and n_2 for which

$$|f(n) - \ell| < \varepsilon \ \forall n \geq n_1$$
$$|g(n) - m| < \varepsilon \ \forall n \geq n_2.$$

Hence, for all values of $n \geq \max\{n_0, n_1, n_2\}$, we have

$$\ell - \varepsilon < f(n) \leq g(n) < m + \varepsilon$$

Thus, $\ell - m < 2\varepsilon$.

If $\ell - m > 0$, we could choose $\varepsilon = (\ell - m)/2$, which would imply $\ell - m < \ell - m$, a contradiction. Therefore, $\ell - m \leq 0$, and so $\ell \leq m$. ∎

Corollary

If $f(n) \leq K$ for all sufficiently large n, and $f(n) \to \ell$ as $n \to \infty$, then $\ell \leq K$. (This is obtained from the above by setting $g(n) = K$ for all n.)

Note

Let $f(n) = 1 - 1/n$. Then $f(n) < 1$ for all n, but $\lim_{n\to\infty} f(n) = 1$. Here, the limit of $f(n)$ does not fall under the same strict inequality that the sequence itself does. More generally, $f(n) < K$ implies $f(n) \leq K$, meaning $\lim_{n\to\infty} f(n) \leq K$.

Proof of 11.ii: There is some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$f(n) \leq g(n) \leq h(n).$$

Given $\varepsilon > 0$, there are also natural numbers n_1 and n_2 such that

$$|f(n) - \ell| < \varepsilon \ \forall n \geq n_1$$
$$|h(n) - \ell| < \varepsilon \ \forall n \geq n_2.$$

Hence, if $n \geq \max\{n_1, n_2, n_3\}$, then

$$\ell - \varepsilon < f(n) \leq g(n) \leq h(n) < \ell + \varepsilon.$$

By definition, this means $g(n)$ converges to ℓ as $n \to \infty$. ∎

Example

$$0 < \frac{1}{n\sqrt{n}} \le \frac{1}{n}.$$

We know that $\lim_{n\to\infty} = 0$. Hence, $\lim_{n\to\infty} \frac{1}{n\sqrt{n}}$ exists and is equal to zero.

Definition

Let $f(n)$ be a sequence in $\mathbb{R}$.

- We say that $f(n)$ is **increasing** if $n_1 \le n_2 \Rightarrow f(n_1) \le f(n_2)$.
- We say that $f(n)$ is **decreasing** if $n_1 \le n_2 \Rightarrow f(n_1) \ge f(n_2)$.
- f is **strictly increasing** if $n_1 < n_2 \Rightarrow f(n_1) < f(n_2)$.
- f is **strictly decreasing** if $n_1 < n_2 \Rightarrow f(n_1) > f(n_2)$.
- f is (*strictly*) **monotonic** if f is (*strictly*) increasing or decreasing.

Theorem 12

Let $f(n)$ be a sequence in $\mathbb{R}$. Suppose that

(i) f is increasing
(ii) f is bounded above; that is, there is some $K \in \mathbb{R}$ such that $f(n) \le K$ for all n

Then f converges to some limit.

Proof: Let $F = \{f(n) : n \in \mathbb{N}\}$ be the set of all values in the sequence. Moreover, let $\ell = \sup(F)$, which exists because K is an upper bound of F. Then $f(n) \le \ell$ for all values of n.

Since ℓ is the least upper bound, $\ell - \varepsilon$ is not an upper bound of F for any $\varepsilon > 0$. Therefore, there exists a natural number n_0 for which $\ell - \varepsilon < f(n_o)$.

Since f is increasing, for every $n \ge n_0$, we have $f(n_0) \le f(n)$. It follows that

$$\ell - \varepsilon < f(n_0) \le f(n) \le \ell \le \ell + \varepsilon \ \forall n \ge n_0.$$

Equivalently, $|f(n) - \ell| < \varepsilon$. This proves the result. ■

Corollary

Suppose that $f(n)$ is a sequence in $\mathbb{R}$ which is decreasing and bounded below. Then $f(n)$ tends to a limit ℓ as $n \to \infty$.

Proof: If $f(n)$ is decreasing, then $-f(n)$ is increasing. Additionally, if $f(n)$ is bounded below, then $-f(n)$ is bounded above. Theorem 12 implies that $-f(n)$ converges to some limit ℓ. Hence, $f(n) \to \ell$ as $n \to \infty$. ■

Exercise

Alternatively, use an analogous proof to that of Theorem 12.

Example

Suppose k is a real number where $0 < k < 1$. Then $k^n \to 0$ as $n \to \infty$.

Proof: The sequence k^n is decreasing, and $k^n > 0$ for every n. Hence, $k^n \to \ell$ where $\ell \geq 0$.

Suppose $k^n \to \ell$. Then $k \cdot k^n \to k \cdot \ell$. But $k \cdot k^n = k^{n+1}$, and this is the same sequence. Hence $k^{n+1} \to \ell$, and $k \cdot \ell = \ell$. Since $k \neq 1$, it follows that $\ell = 0$. ∎

Example

If q is a positive integer, then $\frac{1}{n^{1/q}} \to 0$ as $n \to \infty$.

Proof: $f(n) = n^{-1/q}$ is a decreasing function of n, and $n^{-1/q} > 0$. So, by Theorem 12, the sequence $n^{-1/q}$ tends to a limit $\ell \geq 0$ as $n \to \infty$. Now,

$$\begin{aligned} \frac{1}{n^{1/q}} \to \ell \quad &\implies \quad \frac{1}{n} \to \ell^q \\ &\implies \quad \ell^q = 0 \quad \implies \quad \ell = 0. \end{aligned}$$

Similarly, if p is a natural number, then $\left(\frac{1}{n^{1/q}}\right)^p \to 0$ as $n \to \infty$. ∎

Example

Suppose that $k > 0$. Then $k^{1/n} \to 1$ as $n \to \infty$.

Proof: If $k > 1$, then the sequence $k^{1/n}$ is decreasing, and $k^{1/n} \geq 1$. Theorem 12 tells us that $k^{1/n} \to \ell$ for some $\ell \geq 1$.

Suppose that $\ell = 1 + \delta$ for some $\delta \geq 0$. Then for all n,

$$\begin{aligned} k^{1/n} &\geq \ell = 1 + \delta \\ k &\geq (1 + \delta)^n \\ k &= 1 + n\delta + n(n-1)\delta^2 + \cdots + \delta^n \\ k &\geq 1 + n\delta. \end{aligned}$$

Therefore,

$$\frac{k-1}{n} \geq \delta.$$

The left-hand sequence tends to zero, so δ must equal 0, meaning $\ell = 0$. ∎