Important Remark
Suppose $f(n)$ is a real sequence where $\overline{\lim}\, f(n) = \underline{\lim}\, f(n) = \ell$. Then $f(n) \to \ell$ as $n \to \ell$.

Proof: Given $\varepsilon > 0$, there is some natural number r_1 such that, for every $r \geq r_1$,

$$\ell - \varepsilon < \sup\{f(r), f(r+1), \ldots\} < \ell + \varepsilon.$$

Additionally, there is another number r_2 such that, for all $r \geq r_2$,

$$\ell - \varepsilon < \inf\{f(r), f(r+1), \ldots\} < \ell + \varepsilon.$$

Therefore, for any $r \geq \max\{r_1, r_2\}$, we have

$$\ell - \varepsilon < \inf\{f(r), f(r+1), \ldots\} \leq f(r) \leq \sup\{f(r), f(r+1), \ldots\} < \ell < \varepsilon.$$

By definition, $|f(r) - \ell| < \varepsilon$, and so $f(r) \to \ell$ as $n \to \infty$. ∎

Exercise
Prove the converse.

General Properties of Upper and Lower Limits

(i) $\underline{\lim}\, f(n) \leq \overline{\lim}\, f(n)$.

This is because $\inf\{f(n) : n \geq r\} \leq \sup\{f(n) : n \geq r\}$ for any r. Then, we can apply Lemma 11.

(ii) $\overline{\lim}\, f(n) \leq \sup f(\mathbb{N})$ if the supremum exists;

$\underline{\lim}\, f(n) \geq \inf f(\mathbb{N})$ if the infimum exists.

(iii) If $f(n) \leq g(n)$ for all sufficiently large n, then

$$\overline{\lim}\, f(n) \leq \overline{\lim}\, g(n), \qquad \underline{\lim}\, f(n) \leq \underline{\lim}\, g(n).$$

Theorem 14
Let $f(n)$ be a real sequence. Suppose that $\overline{\lim}\, f(n) = \Lambda$ and $\underline{\lim}\, f(n) = \lambda$ for some real numbers Λ and λ. Then,

(i) Given $\varepsilon > 0$, there is some natural number $n_0 \in \mathbb{N}$ such that, for infinitely many $n \geq n_0$, $f(n) > \Lambda + \varepsilon$ and $\Lambda - \varepsilon < f(n)$.

(ii) Given $\varepsilon > 0$, there is some natural number $n_1 \in \mathbb{N}$ such that, for infinitely many $n \geq n_1$, $f(n) > \lambda - \varepsilon$ and $\lambda + \varepsilon < f(n)$.

(iii) If Λ' and λ' have the same properties respectively, then $\Lambda = \Lambda'$ and $\lambda = \lambda'$.

Proof of 14.i: Given $\varepsilon > 0$, there is some $R \in \mathbb{N}$ such that, for every $r \geq R$, we have

$$|\sup\{f(r), f(r+1), \ldots\} - \Lambda| < \varepsilon.$$

Equivalently,

$$\Lambda - \varepsilon < \sup\{f(r), f(r+1), \ldots\} < \Lambda + \varepsilon.$$

So for all $r \geq R$, we know $\Lambda - \varepsilon$ is not an upper bound of the set $\{f(r), f(r+1), \ldots\}$. Thus, $\Lambda - \varepsilon < f(r')$ for some $r' \geq r \geq R$, i.e. for infinitely many r'.

Moreover, for all $r \geq R$, we have $f(r) < \Lambda + \varepsilon$. Then the required result follows. ∎

Exercise
The proof of Theorem 14.ii is similar. Write such a proof.

Proof of 14.iii: Suppose Λ' and λ' have properties (i) and (ii) respectively.

Then, given $\varepsilon > 0$, we know $f(n) < \Lambda + \varepsilon$ for sufficiently large values of n.

Also, $\Lambda' - \varepsilon < f(n)$ for infinitely many n.

Choose an n satisfying both of these conditions. Then,

$$\Lambda' - \varepsilon < f(n) < \Lambda + \varepsilon \quad \Longrightarrow \quad \Lambda' - \Lambda = 2\varepsilon.$$

Since ε is arbitrarily small, it follows that $\Lambda' - \Lambda \leq 0$. Similarly, $\Lambda - \Lambda \leq 0$, so $\Lambda = \Lambda'$. A similar argument shows that $\lambda = \lambda'$. ∎

Exercise
Write the argument for $\lambda = \lambda'$ mentioned above.

Corollary
If $f(n)$ is a bounded sequence, then $\Lambda = \lambda$ if and only if $f(n)$ tends to a limit ℓ where $\ell = \Lambda = \lambda$.

Proof: Suppose $\Lambda = \lambda$. Then given $\varepsilon > 0$, for all sufficiently large n, we have

$$\begin{aligned} &\lambda - \varepsilon < f(n) < \Lambda + \varepsilon \\ \Longrightarrow\ &\ell - \varepsilon < f(n) < \lambda + \varepsilon. \end{aligned}$$

Hence, $f(n) \to \ell$ as $n \to \infty$.

Conversely, if $f(n)$ tends to a limit ℓ, then ℓ satisfies properties (i) and (ii) of Theorem 14, so $\Lambda = \lambda = \ell$. ∎

Theorem 15 **The General Principle of Convergence**

Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence. Then $f(n)$ converges to some finite limit in $\mathbb{R}$ if and only if, given $\varepsilon > 0$, there is some natural number n_0 for which $r, s \geq n_0$ implies

$$|f(r) - f(s)| < \varepsilon.$$

Proof: Suppose $f(n) \to \ell$ for some real limit ℓ. Then, given $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$|f(n) - \ell| < \varepsilon/2.$$

If r and s are at least n_0, then

$$\begin{aligned} |f(r) - f(s)| &= |f(r) - \ell + \ell - f(s)| \\ &\leq |f(r) - \ell| + |f(s) - \ell| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Conversely, suppose that given $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}$ where for all $r, s \geq n_0$,

$$|f(r) - f(s)| < \varepsilon.$$

Then,

$$f(n_0) - \varepsilon < f(n) < f(n_0) + \varepsilon \ \forall n \geq n_0.$$

Hence, $f(n)$ is bounded, and we have

$$|f(n)| \leq \max\{|f(n_0) + \varepsilon|, |f(1)|, |f(2)|, \dots, |f(n_0 - 1)|\}.$$

Thus,

$$\begin{gathered} f(n_0) - \varepsilon \leq \underline{\lim}\, f(n) \leq \overline{\lim}\, f(n) \leq f(n_0) + \varepsilon \\ 0 \leq \overline{\lim}\, f(n) - \underline{\lim}\, f(n) \leq 2\varepsilon. \end{gathered}$$

Therefore, $\overline{\lim}\, f(n) = \underline{\lim}\, f(n)$, and $f(n)$ tends to a limit by the corollary to Theorem 14. ■

Remark

Sequences having the property of the theorem are called **Cauchy Sequences**.