

## More on the Comparison Test

### Example

Consider the series

$$\sum_{n \geq 2} \frac{1}{n(\log n)^\alpha}.$$

This series converges if  $\alpha > 1$  and diverges if  $\alpha \leq 1$ .

*Proof:* Let  $a_n = 1/n(\log n)^\alpha$ . Then

$$2^n a_{2^n} = \frac{2^n}{2^n (\log 2^n)^\alpha} = \frac{1}{(\log 2^n)^\alpha} = \frac{1}{(\log 2)^\alpha n^\alpha}.$$

But  $\sum \frac{1}{(\log 2)^\alpha n^\alpha}$  converges if and only if  $\sum \frac{1}{n^\alpha}$  converges, so we are done. ■

### Remark

The series  $\sum \frac{1}{n \log n}$  diverges very slowly. In fact,

$$\sum_3^{120,000} \frac{1}{n \log n} < 8.$$

Furthermore, the series

$$\sum \frac{1}{n \log n (\log \log n)^\alpha}$$

converges if  $\alpha > 1$  and diverges, even more slowly, if  $\alpha \leq 1$ .

To generalize,

$$\sum \frac{1}{n \log n \cdot (\log \log n) \cdot \dots \cdot (\log \log \dots \log n)^\alpha}$$

can be made to diverge arbitrarily slowly. This is called the **logarithmic scale** of convergent and divergent series, and it is useful for comparison tests.

### Definition

### Power Series

Suppose that  $a_n$  is a complex sequence. The series  $\sum a_n z^n$  is called the **power series** with  $\{a_n\}$  as coefficients.

- (i) In the trivial case,  $\sum a_n z^n$  converges when  $z = 0$ .
- (ii) Let  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ . Then  $f$  is a function defined as

$$f : \left\{ z : \sum a_n z^n \text{ converges} \right\} \rightarrow \mathbb{C}.$$

### Lemma 27

Suppose that  $\sum a_n z^n$  converges for some complex number  $z_0$ . Then for any  $z$  such that  $|z| < |z_0|$ , the power series converges absolutely.

*Proof:* Observe the following equality:

$$|a_n z^n| = |a_n(z_0)^n| \cdot \left| \frac{z}{z_0} \right|^n$$

We know  $\sum a_n(z_0)^n$  converges, and so  $a_n(z_0)^n \rightarrow 0$  as  $n \rightarrow \infty$ . Thus, given some  $\varepsilon$  where  $0 < \varepsilon < 1$ , there is some natural number  $n_0$  such that for every  $n \geq n_0$ ,

$$|a_n(z_0)^n| < \varepsilon < 1 \implies |a_n(z_0)^n| < \left| \frac{z}{z_0} \right|^n < 1.$$

Hence  $\sum |z/z_0|^n$  converges. The Comparison Test implies that  $\sum |a_n z^n|$  converges also. ■

### Definition

### Radius of Convergence

The **radius of convergence**  $R$  of the power series  $\sum a_n z^n$  is defined by

$$R = \sup \{ |z| : \sum a_n z^n \text{ converges} \}.$$

We say that  $R$  is infinite if the above set is unbounded.

### Theorem 28

The power series  $\sum a_n z^n$  with radius of convergence  $R$  converges absolutely for all  $z$  with  $|z| < R$ . It diverges if  $|z| > R$ .

*Proof:* If  $R$  is infinite, then given  $z \in \mathbb{C}$ , there exists  $z_0 \in \mathbb{C}$  such that  $|z_0| > |z|$  and  $\sum a_n(z_0)^n$  converges, since the set  $\{|z| : \sum a_n z^n \text{ converges}\}$  is unbounded. The result now follows from Lemma 27. Moreover, the series converges absolutely.

Otherwise,  $R$  is a positive real number. Suppose that  $z$  is a complex number for which  $|z| < R$ .

From the definition of  $R$ , there is another complex number  $z_0$  where  $|z| < |z_0| < R$  such that  $\sum a_n(z_0)^n$  converges. Lemma 27 implies that  $\sum a_n z^n$  converges absolutely.

If  $|z| > R$ , then it follows from the definition of  $R$  that  $\sum a_n z^n$  diverges. ■

### Theorem 29

Let  $R$  be the radius of convergence of  $\sum a_n z^n$ . Then

$$1/R = \overline{\lim} |a_n|^{1/n}.$$

By convention, if  $R$  is infinite, then  $1/R = 0$ .

*Proof:* Consider  $\overline{\lim} |a_n z^n|^{1/n} = |z| \cdot \overline{\lim} |a_n|^{1/n} = |z| \cdot \Lambda$ .

By the Cauchy  $n$ th Root Test, the series  $\sum a_n z^n$  converges if  $|z| \cdot \Lambda < 1$  and diverges if  $|z| \cdot \Lambda > 1$ .

Equivalently,  $|z| < 1/\Lambda$  implies convergence, and  $|z| > 1/\Lambda$  implies divergence. ■

### Example 1

Consider  $\sum n^n z^n$ . This series converges at the origin, where  $z = 0$ , and nowhere else. If  $|z| > 0$ , then  $n|z| > 1$  for all sufficiently large  $n$ , so  $(n|z|)^n$  does not tend to zero as  $n \rightarrow \infty$ . Hence,  $R = 0$ .

### Example 2

Consider  $\sum z^n$ . On the circle of convergence, where  $z = 1$ , the series diverges as  $|z|^n = 1$ , meaning  $|z|^n$  does not tend to zero as  $n \rightarrow \infty$ . Hence,  $R = 1$ .

### Example 3

Consider  $\sum \frac{z^n}{n!}$ . Here,  $R = \infty$ .

### Example 4

*Let's power down this series a bit...*

Consider  $\sum \frac{z^n}{n}$ . Using Theorem 29, observe that  $\left(\frac{1}{n}\right)^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ . Hence  $R = 1$ .

Alternatively, observe that the series diverges at  $z = 1$  but converges at  $z = -1$ .

### Example 5

Consider  $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ . Here,  $R = 1$  since, for example,  $\lim(1/n^2)^{1/n} = 1$ .

Note that if  $|z| = 1$ , then  $|z^n/n^2| = 1/n^2$ . Since  $\sum 1/n^2$  converges, it follows that  $\sum \frac{z^n}{n^2}$  converges absolutely for  $|z| = 1$ .

### Note

It is possible to construct power series with arbitrary behavior on the circle of convergence. For example, we could construct a series diverging at countably infinitely many points on this circle.

**Theorem 30****Alternating Series Test**

Let  $a_n$  be a nonnegative real sequence such that

- (i)  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ , and
- (ii)  $a_n$  is a decreasing sequence.

Then  $\sum (-1)^n a_n$  converges.

*Proof:* Let  $s_m = \sum_{n=0}^m (-1)^n a_n$ . We wish to show that  $s_m$  tends to a real limit.

Consider

$$s_{2r} - s_{2r-2} = \sum_{2r-2}^{2r} a_n = a_{2r} - a_{2r-1} \leq 0.$$

Hence,  $s_{2r}$  is a decreasing sequence. Expanding  $s_{2r}$  yields

$$s_{2r} = a_{2r} - a_{2r-1} + a_{2r-2} + \cdots + a_0.$$

Since  $a_{2r} - a_{2r-1} \geq 0$ , applying similarly to the other terms, we see  $s_{2r} \geq a_{2r} \geq 0$ . Hence  $s_{2r}$  is a decreasing sequence that is bounded below.

Therefore,  $s_{2r}$  tends to a real limit  $\ell$  as  $r \rightarrow \infty$ .

Now, we have  $s_{2r} - s_{2r-1} = a_{2r}$ , and this tends to zero by assumption. Therefore,

$$(s_{2r-1} - s_{2r}) + s_{2r} \longrightarrow 0 + \ell.$$

Since  $s_{2r-1} \rightarrow \ell$ , it follows that  $s_m \rightarrow \ell$  as  $m \rightarrow \infty$ . ■