Chapter 3: Series

Definition **Series and Partial Sums**

Let a_n be a complex sequence. Then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots$$

is the **series** whose nth term is a_n. We often notate this series as $\sum_n a_n$ or $\sum a_n$.

Furthermore, the expression

$$\sum_{n=1}^{m} a_n = a_1 + a_2 + \cdots + a_m$$

is the **partial sum** of the series $\sum a_n$.

If the following limit exists:

$$\lim_{m\to\infty} \sum_{n=1}^{m} a_n = \ell$$

then we say that $\sum a_n$ converges to ℓ. We write this limit as $\sum_{n=1}^{\infty} a_n$. Otherwise, we say $\sum a_n$ diverges.

Note

If a_n begins with $n = r$, then we write

$$\sum_{r}^{\infty} a_n = \lim_{m\to\infty} \sum_{n=r}^{m} a_n.$$

If a_n is a real sequence and $\sum_{n=1}^{m} a_n \to \infty$ as $n \to \infty$, then we say that $\sum a_n$ diverges to $\pm\infty$.

Lemma 17 **Arithmetic Properties of Series**

Let a_n be a sequence in $\mathbb{C}$.

(i) If $\sum a_n$ converges and $\sum b_n$ converges, then $\sum(a_n + b_n)$ converges and

$$\sum(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

(ii) If c is a nonzero complex number, then $\sum ca_n$ converges if and only if $\sum a_n$ converges. In this case,

$$c\sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} ca_n.$$

(iii) If $\sum a_n$ converges and $a_n = b_n$ for all but finitely many values of n, then $\sum a_n$ also converges.

(iv) If $\sum a_n$ converges, then so does any series obtained by bracketing terms together.

(v) If a_n and b_n are real sequences, then $\sum a_n$ and $\sum b_n$ are convergent if and only if $\sum(a_n + b_n)$ is convergent in $\mathbb{C}$.

Proof:

(i) Lemma 8 says that $\lim f(n) + \lim g(n) = \lim(f(n) + g(n))$.

The result follows from letting $f(n) = \sum_{n=1}^{\infty} a_n$ and $g(n) = \sum_{n=1}^{\infty} b_n$.

(ii) The result follows from Lemma 8 (ii).

(iii) Observe that $\sum_{n=1}^{m} a_n = \sum_{n=1}^{\infty} b_n$ for all sufficiently large n.

(iv) The partial sums of the bracketed series are a subsequence of the partial sums of $\sum a_n$.

(v) Use Lemma 9 on $\sum_{n=1}^{m} a_n$ and $\sum_{n=1}^{m} b_n$. ■

Example **Bracketing Terms of Series**

The series $\sum a_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ converges. Lemma 17 (iv) tells us we can bracket terms of this series together to get a new convergent series. Consider

$$\sum b_n = \left(1 + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{32}\right) + \cdots$$

The partial sum of $\sum a_n$ where $m = 1$ has no equivalent partial sum of $\sum b_n$. However, the partial sum at $m = 2$ does:

$$\sum_{n=1}^{2} a_n = 1 + \frac{1}{2} = \sum_{n=1}^{1} b_n$$

More generally, $\sum_{n=1}^{2m} a_n = \sum_{n=1}^{m} b_n$.

Thus, letting $f(n)$ be the sequence of partial sums of $\sum a_n$, the partial sums of the series $\sum b_n$ form a subsequence $f(2n)$. Since $f(n)$ converges, so too must $f(2n)$.

Note

The series $\sum(-1)^n$ does not converge, but

$$(-1 + 1) + (-1 + 1) + \cdots$$

does converge. So brackets cannot necessarily be removed while keeping convergence the same.

Theorem 18 **The General Principle of Convergence for Series**

Let a_n be a complex sequence. Then $\sum a_n$ converges if and only if, given $\varepsilon > 0$, there is some natural number n_0 such that for all $r, s \geq n_0$,

$$\left|\sum_r^s a_n\right| < \varepsilon.$$

Proof: Let $f(m) = \sum_{n=1}^m a_n$. Then

$$\left|\sum_{n=r}^m a_n - \sum_{n=s}^m a_n\right| = |f(r) - f(s)| = \left|\sum_r^s a_n\right|.$$

The result follows from applying Theorem 15 to $f(m)$. ∎

Example

The series $\sum \frac{1}{n}$ diverges because, for any natural r,

$$\sum_{2^r+1}^{2^{r+1}} \frac{1}{n} = \frac{1}{2^r+1} + \frac{1}{2^r+2} + \cdots + \frac{1}{2^{r+1}} > \underbrace{\frac{1}{2^{r+1}} + \cdots + \frac{1}{2^{r+1}}}_{2^r \text{ times}} = 2^r \cdot \frac{1}{2^{r+1}} = \frac{1}{2}.$$

The General Principle of Convergence does not hold, so the series diverges.

Lemma 19

Suppose that $\sum a_n$ converges. Then $a_n \to 0$ as $n \to \infty$.

Proof: If $\sum a_n$ converges, then

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty}\left(\sum_{n=1}^m a_n - \sum_{n=1}^{m-1} a_n\right) = \sum_{n=1}^\infty a_n - \sum_{n=1}^\infty a_n = 0.$$ ∎

Alternative Proof: By the General Principle of Convergence, given $\varepsilon > 0$, we have

$$\left|\sum_r^r a_n\right| = |a_n| < \varepsilon \quad \forall r \geq n_0 \text{ for some } n_0 \in \mathbb{N}.$$ ∎

Note

Lemma 19 can be used to prove that a series diverges. For example, $\sum \sqrt{n}$ diverges since $\sqrt{n}$ does not tend to zero as $n \to \infty$.

However, we know $\sum \frac{1}{n}$ diverges while $1/n \to 0$ as $n \to \infty$, so the converse to this lemma is false.

Lemma 20 **Geometric Progressions**

Let z be a complex number. Then $\sum_n z^n$ converges if $|z| < 1$ and diverges if $|z| \geq 1$.

Proof: If $|z| \geq 1$, then $|z|^n \geq 1$. Hence $|z|^n$ does not tend to zero as $n \to \infty$, and so we have divergence.

If $|z| < 1$, then we have

$$\sum_{n=0}^{m} z^n = \frac{1 - z^{m+1}}{1 - z} = \frac{1}{1 - z} - \frac{z^{m+1}}{1 - z}.$$

However, $z^{m+1} \to 0$ as $n \to \infty$ since $\lim |z|^{m+1} = 0$ if $|z| < 1$. Thus,

$$\sum_{n=0}^{m} z^n \longrightarrow \frac{1}{1 - z} \quad \text{as } m \longrightarrow \infty.$$

Therefore, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$. ■

Exercise

Suppose $S = 1 + z + z^2 + \cdots + z^m$. Derive the closed form

$$S = \frac{1 - z^{m-1}}{1 - z}$$

used in the above proof.

Definition **Absolute Convergence**

If a_n is a complex sequence, then we say $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

Theorem 21

Suppose a_n is a complex sequence. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges. In other words, absolute convergence implies convergence.

Proof: Suppose that $\sum |a_n|$ converges. Then, given $\varepsilon > 0$, there is some natural n_0 such that for all $r, s \geq n_0$,

$$\sum_r^s |a_n| < \varepsilon.$$

Moreover,

$$\left| \sum_r^s a_n \right| \leq \sum_r^s |a_n| < \varepsilon \quad \forall r, s \geq n_0.$$

Hence, the General Principle of Convergence implies that $\sum a_n$ converges. ■

Note

The converse of this theorem is false. Consider $\sum \frac{1}{n}$ and $\sum (-1)^n/n$. The former diverges, but we will show later that the latter converges.

Lemma 22

Suppose a_n is a real sequence where $a_n \geq 0$ for every n. Then $\sum a_n$ converges if and only if

$$\left\{ \sum_{n=1}^{m} : m \in \mathbb{N} \right\}, \quad \text{the set of all partial sums}$$

is bounded above.

Proof: The forward direction follows from Lemma 10, wherein $f(n)$ converging to a limit implies $\{f(n) : n \in \mathbb{N}\}$ is bounded.

The converse follows from Theorem 12. Take $f(m) = \sum_{n=1}^{n}$.

Since $f(m)$ is an increasing function of m and is bounded above, it therefore tends to a limit. In this case, $f(m) \leq \lim f(m)$ by Theorem 12. ■

Theorem 23 **The Comparison Test**

Suppose for some natural n_0 that for all $n \geq n_0$,

$$0 \leq a_n \leq b_n \quad \text{where } a_n, b_n \in \mathbb{R}.$$

(i) If $\sum b_n$ converges, then $\sum a_n$ converges.

(ii) If $\sum a_n$ diverges, then $\sum b_n$ diverges.

Proof: Suppose that $\sum b_n$ converges. Then

$$\sum_{n_0}^{m} a_n \leq \sum_{n_0}^{m} b_n \leq \sum_{n_0}^{\infty} b_n.$$

Therefore, the set

$$\left\{ \sum_{n_0}^{m} a_n : m \in \mathbb{N}, m \geq n_0 \right\}$$

is bounded above by $\sum_{n_0}^{\infty} b_n$. Hence,

$$\sum_{n=1}^{m} a_n \leq \sum_{n=1}^{n_0-1} + \sum_{n_0}^{\infty} b_n.$$

Therefore, $\sum_{n=1}^{\infty} a_n$ exists by Lemma 22. Part (ii) is left as an exercise. ■