World Religion Populations

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Abstract

There are several different models that examine population dynamics using differential equations, including predator-prey models and SIR models. We modify these models to examine population dynamics and interactions between populations of different religions. Cases involving two religions, three religions, and the introduction of a new religion are examined. Solution existence of the model is proven, and equilibrium conditions are examined in each case.

1 Background/Movitation

There are over 4,000 religions in the world today. All of these belief systems are distinct, having their own customs, cultures, rituals, and teachings. One natural consequence arising from this context is that much like overall population numbers, the population numbers of each religion are constantly in a state of change. We are interested in modeling this phenomenon. Unlike the population at large however, each religion's respective population is affected by more than just births and deaths.

1.1 The General Phenomenon

In order to understand the phenomenon that is religion growth rates, we need to think about what factors affect the population count of any given religion at a particular time.

Just like in the general population, religious populations will be heavily impacted by birth and death rates. However, for each religion, we also need to think about some other things, such as the following:

- How many people stay in a religion for their entire lives?
- Does a given religion convert many people over a given period of time?
- Do other religions 'steal' members of the religion through conversion processes of their own?
- How many people are becoming irreligious/agnostic/atheist from this religion?
- Are the birth or death rates in this religion significantly different than in other religions (or the general population)?

All of these questions are important to think about. We will of course be making some simplifying assumptions, but it is important to first understand the theoretical framework of the general phenomenon.

1.2 A (Brief) Literature Review

While this is a relatively novel modeling endeavor, there are a few studies that have considered the phenomenon. A 2011 study entitled "A Simple Differential Equation System for the Description of Competition among Religions", focuses on the competition between two religious groups and one irreligious group. This study uses a modified predator/prey model to examine different growth dynamics, finding that as long as there remains in each religion a core group of strong followers, the predator/prey model captures the phenomenon well. This paper also notes that for a system of three religions, there often exists an equilibrium, although the birth/death/conversion rates necessary to achieve that equilibrium can become unrealistic depending on initial conditions [4]. Another study supported these conclusions, while noting that one driving force behind growth (or decline) in church populations was "through contact between religious enthusiasts and unbelievers"[2].

There have also been a handful of studies focused more on real-world applications. One paper looked at the application of these models to The Church of Jesus Christ of Latter-Day Saints, finding that the equilibrium population may be somewhere around 30 million members. The modeling in this approach was slightly different, focused less on interactions between religions and more on a solitary differential equation for one religion[1].

There is already a decent foundation of work done regarding differential modeling of religious population numbers. We intend to take these findings and use them, while also modifying our approach slightly to focus more on interactions between religions and less on overall population growth like many numerical studies have.

2 Modeling

To begin our mathematical study of the subject at hand, we will start in a general setting, considering arbitrary religions A, B, and C.

It is important to stop here and note that although we are referring to religions, the mathematical construct for these population changes can be extended in generality to "Non-Religions" such as populations of atheists, agnostics, or others. The reasoning behind this will become clear as we continue.

2.1 The Base Case: Religions A and B

To understand the motivation behind our governing equations, let's consider the base case where there are only two religions to think about, A and B. Let's first consider the population change in A for any given time period t.

With $P_A(t)$ referring to the population of A at time t, we have that:

$$P_A' = (\text{birth rate of } A - \text{death rate of } A) + (\text{conversions to } A - \text{conversion rate from } A)$$

Next, note that in this universe we have only two religions, A and B. We can therefore simply our equation to:

$$P_A' = (\text{birth rate } A - \text{death rate of } A) + (\text{conv. rate to } A - \text{conv. rate to } B) \cdot (\text{A-B Interaction})$$

We consider the interaction between religions A and B as the product of the two populations, $P_A(t)P_B(t)$. Finally, since the birth rate and death rate for each religion can be simplified to one number, we can write the final equations as:

$$\begin{aligned} P'_A(t) &= P_A(t)(n_A + (C_{AB} - C_{BA})P_B(t)), \\ P'_B(t) &= P_B(t)(n_B + (C_{BA} - C_{AB})P_A(t)). \end{aligned} \tag{1}$$

With:

$$\begin{aligned} n_A &= \text{birth rate A - death rate A}, \\ C_{AB} &= \text{conversion rate to A from B}. \end{aligned} \tag{2}$$

2.2 Extension to the Nth Religion

Expanding our universe past 2 religions is fairly trivial. For N religions, we have that for Religion i,

$$P'_i(t) = P_i(t)\left(n_i + \sum_{j=1}^{N}(C_{ij} - C_{ji})P_j(t)\right) \tag{3}$$

The only difference here is that we are summing over all $N-1$ interaction effects given by expanding our universe ($N-1$ instead of N because when $j = i$ we have $(C_{ii} - C_{ii})P_i(t) = 0$, because a religion does not meaningfully self-interact in this model).

We can also define a matrix C containing the conversion rates:

$$C = \begin{bmatrix} 0 & C_{AB} & C_{AC} & ... \\ C_{BA} & 0 & C_{BC} & ... \\ C_{CA} & C_{CB} & 0 & ... \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

The diagonal is all zeros for the reason stated in the previous paragraph–if $C_{ii} \neq 0$, it would make no difference for the model.

2.2.1 Existence and Uniqueness

Like all differential equations we have to question if a solution exists. Furthermore, if a solution exists, is it unique. Luckily we can use the Picard-Lindelöf theorem to answer this question.

Theorem 1. *Let $D \subset \mathbb{R}^N \times R$ be open. The N dimensional religion growth initial value problem defined by*

$$\boldsymbol{P}'(t) = \boldsymbol{F}(\boldsymbol{P}(t), t), \tag{5}$$

$$\boldsymbol{P}(t_0) = \boldsymbol{P}_0, \tag{6}$$

where $\boldsymbol{F} : D \to \mathbb{R}^N$ and $(\boldsymbol{P}_0, t_0) \in D$ has a unique solution on some $I \subset D$.

Proof. It is easy to see that the function is continuous by the fact that $\boldsymbol{F}$ is differentiable. In fact $\boldsymbol{F} \in C^\infty(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^n)$. Using this, by the integral mean value theorem we know that $\boldsymbol{F}$ is uniformly Lipschitz and by the Picard-Lindelöf theorem we know that there exists some interval $I \subset D$ such that the solution is unique. □

A nice consequence of this theorem is because we have that $\boldsymbol{F}$ is uniformly Lipschitz then we also know that there is continuous dependence among initial conditions. While this is a good start it is hardly enough to analyze our ODE, expecially when we would like to consider long term effects. Questions like, will a solution ever not exist? Does our solution blow up in finite time? Still need to be answered before we begin analyzing numerical solutions and trust the results (provided our numerical methods are stable enough to be reliable). The question becomes, does a global solution exist?

One property that our system has is that it is "close" to linear. Because the way the sum is written P_i is never multiplied by itself. This characteristic is what allows us to say that a Global solution exists.

Theorem 2. *Because of the close-to-linear property we described above the N dimensional initial value problem eq.* (5) *has a global solution on* $[t_0, \infty)$. *The proof of this is described in [3].*

Now that we can expect our results to make sense we can start analyze solutions to our system. We begin by identifying the assumptions we have made thus far.

2.3 Simplifying Assumptions

It is important to point out a few simplifying assumptions made in this model.

- We are assuming that all rates remain constant for each time period
- We are counting each baby born into a religion as a member of that religion.
- We are assuming that the net conversion rates between two religions are generally proportional to the amount that members of the religions interact.

Note that in this model we are claiming that beliefs such as agnosticism, atheism, and other "non-religions" are functionally equivalent to religions in our model. This is so that we can model the population growth of such a belief as they will "act like" a religion in a mathematical sense. It is left to the reader to decide the philosophical truth of these assumptions.

2.4 Plan for Analysis

2.4.1 Our Universe

We have shown that we can generalize this model to as many religions as we want. In order to keep our analysis tenable, we will focus on a system of 5 or 6 religions. With the way we have defined our system it is useful for (at least) two things. The first is to see how religions are likely to change in the future.The second is to see if we can accurately model what may have happened in the past. We know that if we consider the present as the Nth time period, we can solve a boundary value problem to see what a religion's population dynamics likely evolved from in the past.

2.4.2 Things to Explore

In order to analyze these governing equations, we will want to see how sensitive they are to very slight changes in initial conditions. We will also want to see how our system behaves as we slowly add in more religions. We will most likely start by exploring our base case (A and B) before seeking to generalize our findings to a full system. Finally, we will want to explore the stability of our system and see if this tells us anything about the general relationship between religions.

3 Results and Analysis

3.1 Case 1: Two Religions

There are many ways to parametrize a two religion system, but they are not always interesting. Many parametrizations lead to one religion going to zero, and the other exponentially growing to infinity. One interesting balance is if one religion has a positive n value (meaning it has more births than deaths), while the other religion has a negative n (more deaths than births) but converts members from the other religion while not losing any. Figure 1 is parametrized by

$$\mathbf{P}_0 = (100, 50), \quad \mathbf{n} = (0.1, -0.1), \quad C = \begin{bmatrix} 0 & 0 \\ 0.0005 & 0 \end{bmatrix}. \tag{7}$$

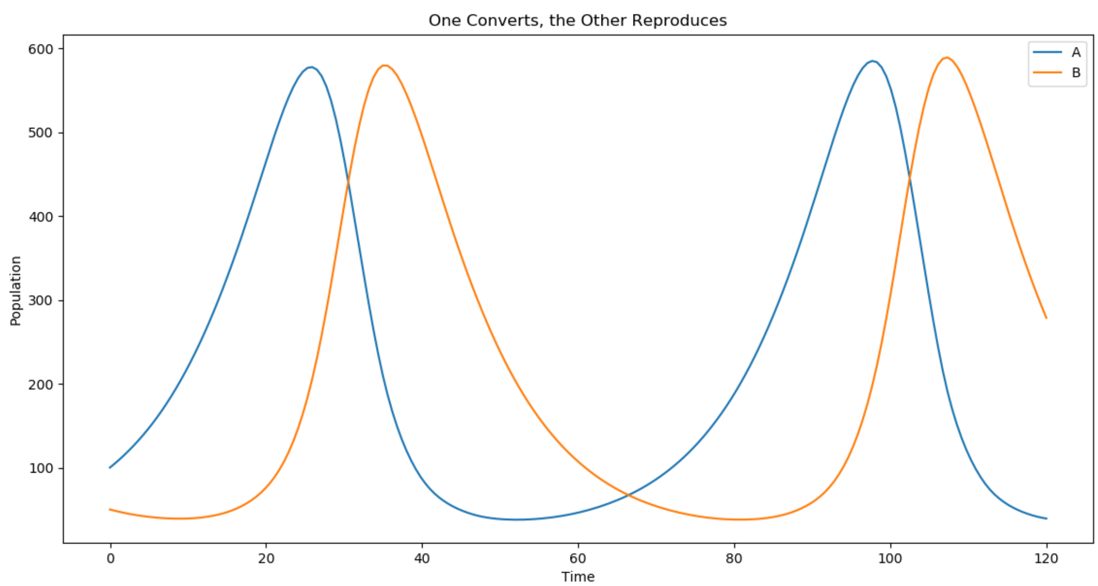

Figure 1: ‘Balanced’ System with Two Religions

The behavior is very interesting, with repetitive behavior being shown, as P_A grows due to its birth rate, then P_B grows due to conversion which also pushes P_A down, then P_B decreases due to its death rate, allowing P_A to grow and the cycle repeats.

3.2 Case 2: Three Religions with Different Birth and Conversion Rates

Different religions have different birth rates and conversion rates, so we tried to narrow in on these features to see how they interacted.

We first began by modeling how the different birth rates affected each religion. Since without conversion this would be simple exponential growth, a conversion factor was added to each religion.

A larger birth rate was added to the first group, whereas the other two were modeled after less populous groups. This trend can be seen in society, with different sub groups having different birth rates and people going from one group to another. The parametrization for fig. 2 became

$$\mathbf{P}_0 = (10, 10, 10), \quad \mathbf{n} = (0.2, -0.01, -0.02), \quad C = \begin{bmatrix} 0 & 0.001 & 0.001 \\ 0.01 & 0 & 0.01 \\ 0.01 & 0.01 & 0 \end{bmatrix}. \tag{8}$$

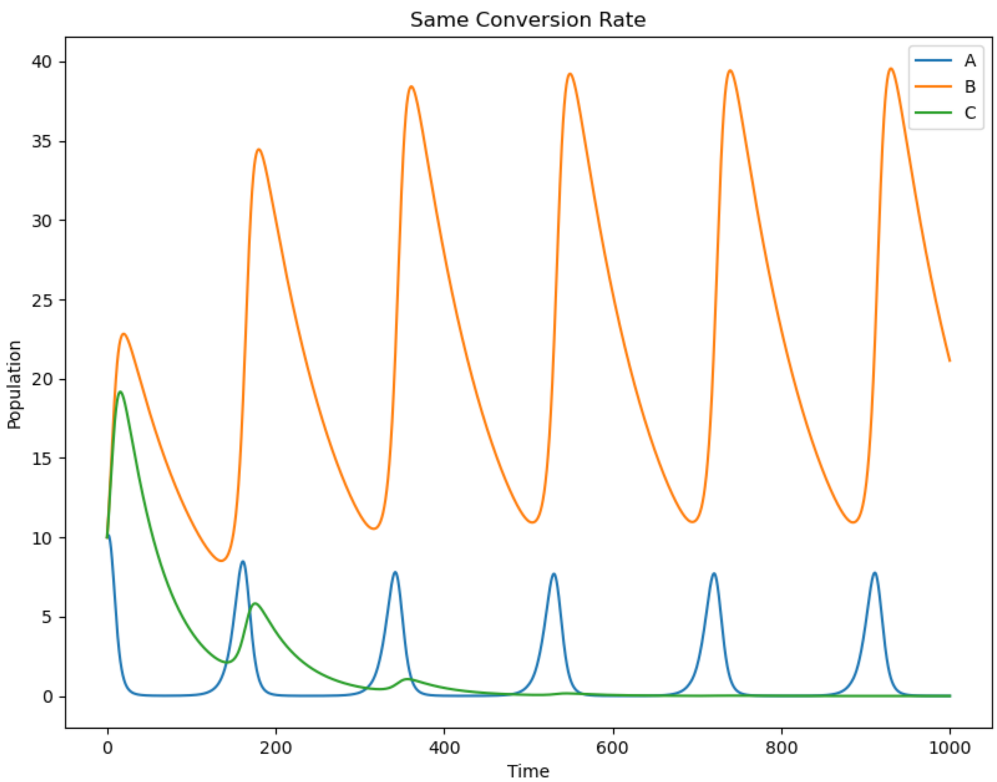

Figure 2: Same Conversion Rate

As evidenced by the graph, a small change in the death rate has a large impact on the population in the long run. Religion A is the religious group with positive population growth, and even though it adds 20 percent of its population to itself every generation, it quickly dies down as many members convert to the other two religions. However, due to the negative birth rate found in the other two, those religions quickly diminish. Later, religion A is able to grow without as many people to convert them away. The interesting part here comes between religions B and C. The difference between the two birth rates is only 1% of the population; however, this causes one group to almost die out while the other thrives. Although it is technically nonzero, it never spikes up, never growing above the bottom of the graph.

Using the similar parameters as above with a slight change in the conversion from B to C, we have

$$\mathbf{P}_0 = (10, 10, 10), \quad \mathbf{n} = (0.2, -0.01, -0.02), \quad C = \begin{bmatrix} 0 & .001 & .001 \\ 0.01 & 0 & 0.01 \\ 0.01 & 0.011 & 0 \end{bmatrix}. \tag{9}$$

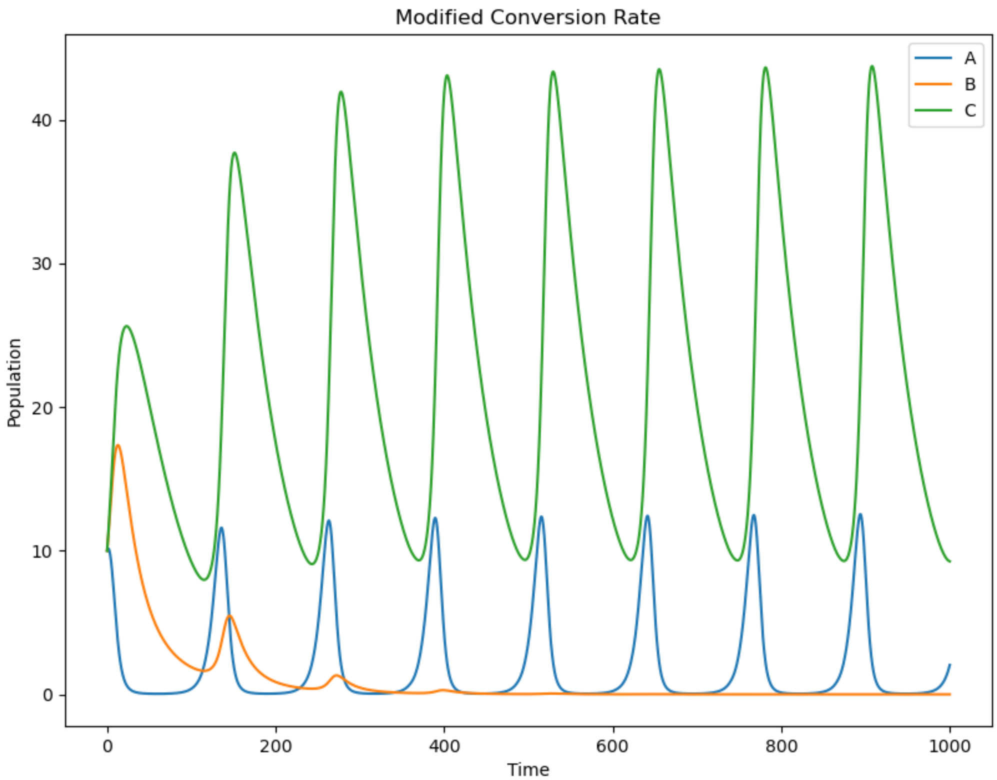

Figure 3: Modified Conversion Rate

We find fig. 3 to have a very different shape. Religion B has been replaced by C as the largest religion for most of the time by very slightly changing the conversion from B to C. Even though it was changed by just 1%, this was a large enough change B from being the largest to it almost going extinct over time. Time was run for 1000 time steps, which represents 1000 generations which may seem unrealistic, but it still useful to see convergence over time.

3.3 Case 3: Variable C Matrix and New Religion

One of our simplifying assumptions was that the conversion rates represented in the C matrix remained constant. However, we wanted to see what behavior would emerge if we actively went against that assumption. Specifically, we wanted to use a C matrix that is dependent on time that would help model the emergence of a new religion, where any new converts to that religion remained faithful to said religion until a certain amount of time passed. Our first attempt at modeling this was to have a piecewise matrix-valued function $C(t)$, where in this case we have

$$C(t) = \begin{bmatrix} 0 & 0.1 & 0.001 & 0 \\ 0 & 0 & 0 & 0 \\ 0.01 & 0.001 & 0 & 0 \\ 0.01 & 0.01 & 0.01 & 0 \end{bmatrix} \quad \text{if } t < 2; \qquad \begin{bmatrix} 0 & 0.1 & 0.001 & 0.01 \\ 0 & 0 & 0 & 0.01 \\ 0.01 & 0.001 & 0 & 0.01 \\ 0.01 & 0.01 & 0.01 & 0 \end{bmatrix} \quad \text{if } t \geq 2. \tag{10}$$

Our initial conditions were $\mathbf{P}_0 = (100, 100, 100, 1)$, and we had $\mathbf{n} = (0, 0.5, 0.01, 0)$. Here, religion D represents the new religion introduced, and the different conversion and birth/death rates for the other religions are only meant to be varied enough as to make this case somewhat interesting. Figure 4 shows the results:

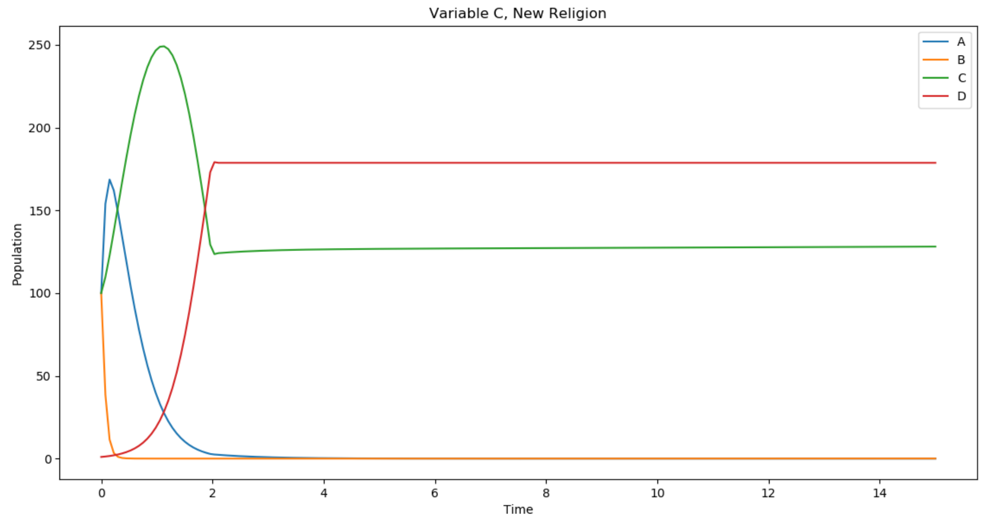

Figure 4: Behavior of model with a variable C matrix, and with religion D representing a newly introduced religion

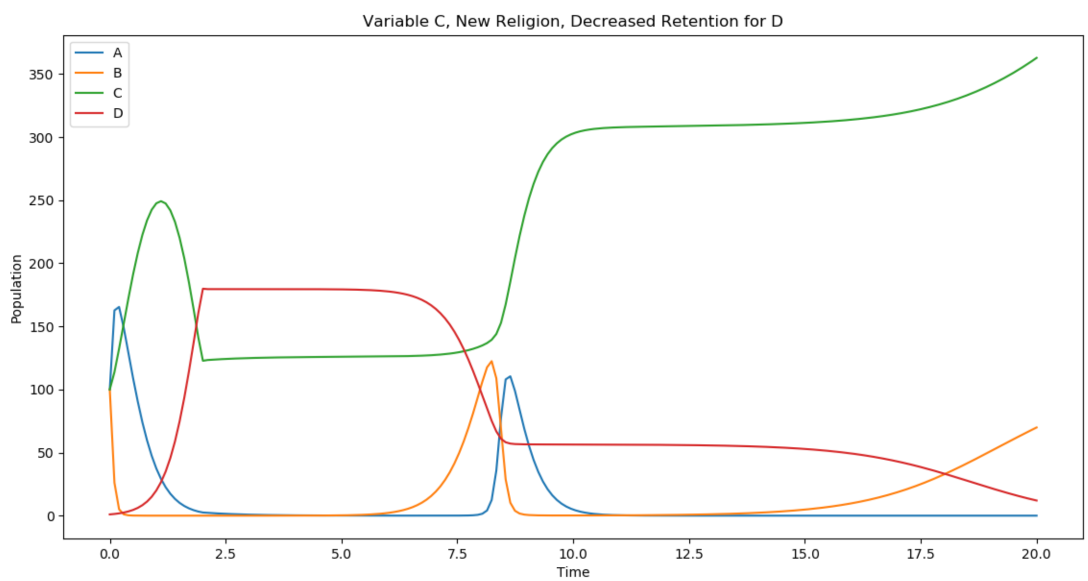

Figure 5: Modification of the model to decrease D's retention, which allows B to grow until it collapses again

This first attempt at modeling a variable C matrix is a good proof of concept, but the value of $C(t)$ is nearly symmetric for $t \geq 2$, especially for the relative populations of religions at that point in time, which means that the solution is nearly steady. A more interesting modification of our function would be for nearly-extinct religion B to have a higher rate of converting people from religion D (i.e. change $C_{BD}(t \geq 2)$ to be 0.02 instead of 0.01).

This difference in conversion rates gives quite a different behavior of the system, as shown in fig. 5.

The end behavior looks as though it is about to blow up due to the exponential increase in population, but there is very interesting behavior between $t = 7.5$ and $t = 10$. The increase in P_B is enough for the very large $n_B = 0.5$ to drastically increase P_B even further. However, religion A converts a large portion of religion B, rapidly decreasing P_B, and then much of religion A converts to religion C. Then, for $t = 10$ to $t = 15$ the system is nearly steady, with religion C dominating, D faring well, and A and B nearly extinct. The behavior past this seems to be dominated by the positive birth/death rates of religions B and C, and seem to be on the point of exponential explosion.

3.4 Case 4: Death Rate Interactions and Religious Wars

To model religious wars, we added an additional interaction matrix D that represents conflict between two religious groups. This interaction matrix is shown below.

$$D = \begin{bmatrix} 0 & D_{AB} & D_{AC} & \dots \\ D_{BA} & 0 & D_{BC} & \dots \\ D_{CA} & D_{CB} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{11}$$

These interaction terms are multiplied with the populations of other religions and summed, as shown in the equation below. They represent the rate at which interaction between religion A and religion B results in death in each population, These terms can be thought of as a way to modify a religions death rate in response to other religions.

$$P_i'(t) = P_i(t)\left(n_i + \sum_{j=1}^{N} D_{ij}P_j(t) + \sum_{j=1}^{N}(C_{ij} - C_{ji})P_j(t)\right) \tag{12}$$

In fig. 6, a war between two religious groups is modeled. Both religions start with the same population and the same growth rates. For simplicity, we assume there is no conversion between the two groups. However, the first religion is the more violent religion while the second religion

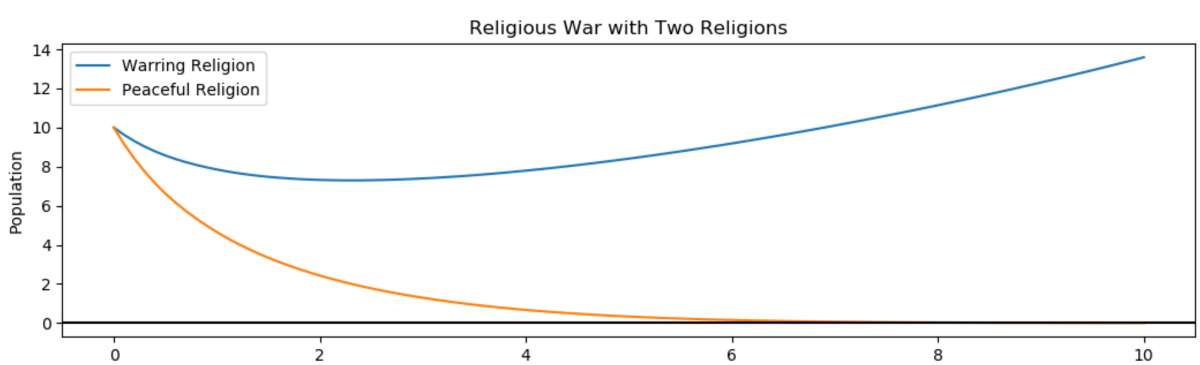

Figure 6: Base Case Religious War

is peaceful, resulting in a higher death rate for the peaceful religion than for the warring religion. The warring religion first decreases while fighting the second religion, then grows exponentially. The peaceful religion is decreased to near extinction.

$$\mathbf{P}_0 = (10, 10), \quad \mathbf{n} = (0.1, 0.1), \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -0.05 \\ -0.1 & 0 \end{bmatrix} \tag{13}$$

In fig. 7, a second warring religion is added. Warring religions decrease each other's populations at a high rate, and otherwise are identical to the warring religion from figure 6. The warring religions decrease each other's population, and the peaceful religion initially thrives. However, as time goes on, a single warring religion gains the largest population. Parameters are written below.

$$\mathbf{P}_0 = (1, 0.9, 1), \quad \mathbf{n} = (0.1, 0.1, 0.1), \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & -0.2 & -0.05 \\ -0.2 & 0 & -0.05 \\ -0.1 & -0.1 & 0 \end{bmatrix}. \tag{14}$$

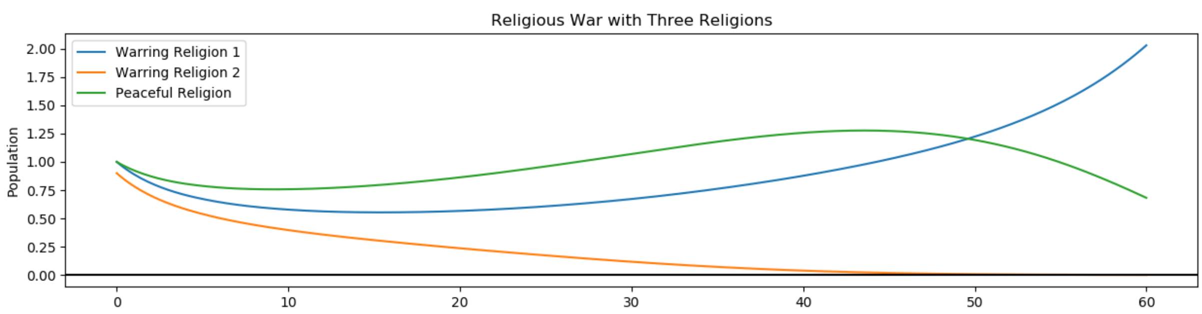

Figure 7: Second Case Religious War

4 Conclusion

There are a few key takeaways from these cases that give us insight into the general phenomenon of religious population dynamics. These are:

- The case with two religions shows that by 'balancing' death rates and conversion rates, the populations can fall into population cycles in tandem with each other.
- Subtle changes to model parameters can dramatically affect the outcome of the model; this modeling approach is relatively unstable.
- Each case had a different result, with about half of the cases displaying a periodic solution and half of the cases displaying near asymptotic stability.

These are interesting results. We suggest continued research in this area, focusing on actual population numbers over time. An interesting challenge for a future researcher would be an attempt to 'back-test' this approach, by trying to accurately recreate past (recorded) population numbers using this model.

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