Traffic Rules

For convenience let's rename the inputs. Let the maximum car speed be V km/h, the maximum speed allowed at the sign be W km/h and the overall road length L km.

Let $v(t)$ be the car speed at the time t and

$$\ell(t) = \int\limits_0^t v(\tau)\, d\tau$$

be the distance that the car has covered at that moment. Out task is to organize car movement so that the time t_* such that $\ell(t_*) = L$ is minimal.

There are following restrictions on the car movement.

1. The starting speed is zero: $v(0) = 0$.

2. The maximum car speed cannot surpass V: $v(t) \leqslant V$.

3. The speed at the sign doesn't surpass W, that is, when $\ell(t) = d$ the speed $v(t) \leqslant W$. Let t_d be the time of reaching the sign.

4. The car can't accelerate and decelerate faster than a km/h^2: $|v'(t)| \leqslant a$.

The solution splits into a variety of cases which we group based on whether we decelerate at the sign and whether we reach the maximum speed.

1 We don't decelerate at the sign

First we'll consider the solution itself for this case and then when it occurs.

Since we don't decelerate at the sign in this case, the optimal car movement is as follows. We accelerate with the maximum acceleration a until the maximum speed V is reached and then proceed with the speed V until the road ends.

$$v(t) = \begin{cases} at, & 0 \leqslant t \leqslant t_V, \\ V, & t > t_V. \end{cases}$$

It might happen that the road ends before we reach the maximum speed though.

Given the optimal car movement, the time to reach the maximum speed

$$t_V = \frac{V}{a}$$

and the distance that the car covers during this time is

$$L_V = \frac{V^2}{2a}.$$

If $L < L_V$, that is when $\boxed{2aL < V^2}$, then the road ends before we reach the full speed and

$$L = \ell(t_*) = \frac{at_*^2}{2}, \quad \text{thus} \quad \boxed{t_* = \sqrt{\tfrac{2L}{a}}}.$$

In the opposite case $\boxed{2aL \geqslant V^2}$ we have

$$L = \ell(t_*) = \int\limits_0^{t_V} a\tau \, d\tau + \int\limits_{t_V}^{t_*} V \, d\tau = \frac{at_V^2}{2} + V(t_* - t_V)\,.$$

Here we express

$$t_* = \frac{1}{V} \cdot \left(L - \frac{at_V^2}{2} \right) + t_V = \boxed{\tfrac{L}{V} + \tfrac{V}{2a}}.$$

When does the car not decelerate at the sign? First, if $W \geqslant V$, clearly we don't have to decelerate. Otherwise we would cover distance

$$L_W = \frac{W^2}{2a}$$

in order to reach speed W and if $d \leqslant L_W$, that is, $2ad \leqslant W^2$, we would not accelerate enough to violate the speed restriction at the sign. To summarize, we don't decelerate at the sign when

$$\boxed{W \geqslant V \quad \text{or} \quad 2ad \leqslant W^2}.$$

2 We have to decelerate at the sign

In this more complicated case the optimal car movement looks as follows. First we accelerate to speed V, drive with that speed, then decelerate to speed W at the sign. After the sign we again accelerate to speed V and drive until the road ends. Again, we might not hit speed V before the sign as well as after the sign.

Let's first find the time t_d to reach the sign.

2.1 Maximum speed reached before the sign

If we hit the maximum speed before reaching the sign, the optimal speed

$$v(t) = \begin{cases} at, & 0 \leqslant t \leqslant t_V, \\ V, & t_V < t \leqslant t_s, \\ W - a(t - t_d), & t_s < t \leqslant t_d, \end{cases}$$

where t_s is the time we start decelerating.

We have already found the time t_V and the distance L_V of accelerating to speed V.

$$t_V = \frac{V}{a}, \quad L_V = \frac{V^2}{2a}.$$

Since we expect continuity of $v(t)$ in the point t_s, we have

$$W - a(t_s - t_d) = V, \quad \text{and the deceleration time} \qquad t_d - t_s = \frac{V - W}{a}.$$

The deceleration distance is

$$L_{VW} = \int_{t_s}^{t_d} v(\tau)\, d\tau = \frac{V^2 - W^2}{2a}.$$

From this we get the following expression for the distance to the sign.

$$d = \ell(t_d) = L_V + (t_s - t_V) \cdot V + L_{VW} = \frac{2V^2 - W^2}{2a} + (t_d - t_V - (t_d - t_s)) \cdot V = \frac{2V^2 - W^2}{2a} + \left(t_d - \frac{2V - W}{a}\right).$$

Therefore the time to reach the sign in this case is

$$\boxed{t_d = \frac{2V-W}{a} + \frac{1}{V} \cdot \left(d - \frac{2V^2-W^2}{2a}\right)}.$$

The formula is effective when the speed V is reached, that is, when $L_V + L_{VW} \leqslant d$, or

$$\boxed{2V^2 - W^2 \leqslant 2ad}.$$

2.2 Maximum speed not reached before the sign

If we don't hit the maximum speed before reaching the sign, the optimal speed

$$v(t) = \begin{cases} at, & 0 \leqslant t \leqslant t_X, \\ W - a(t - t_d), & t_X < t \leqslant t_d, \end{cases}$$

where t_X is the moment when the acceleration stops and we start decelerating before the sign. Let's get the expression for the distance to the sign.

$$d = \ell(t_d) = \int_0^{t_X} a\tau\, d\tau + \int_{t_X}^{t_d} (W - a(\tau - t_d))\, d\tau.$$

The first integral equals $\frac{at_X^2}{2}$. We change the variable in the second integral: $u = W - a(\tau - t_d)$, $d\tau = -\frac{du}{a}$.

$$\int_{t_X}^{t_d} (W - a(\tau - t_d))\, d\tau = \int_{W-a(t_X-t_d)}^{W} u \cdot \left(-\frac{du}{a}\right).$$

The point t_X is defined to satisfy equality $W - a(t_X - t_d) = at_X$, which we'll use in the integration limit.

$$\int\limits_{t_X}^{t_d} (W - a(\tau - t_d))\, d\tau = -\frac{1}{a} \cdot \int\limits_{at_X}^{W} u\, du = \frac{1}{a} \cdot \int\limits_{W}^{at_X} u\, du = \frac{(a^2 t_X^2 - W^2)}{2a} = \frac{at_X^2}{2} - \frac{W^2}{2a}.$$

In total for the $\ell(t_d)$ we have

$$d = \ell(t_d) = at_X^2 - \frac{W^2}{2a}.$$

From this we can express the time t_X.

$$t_X = \sqrt{\frac{d}{a} + \frac{W^2}{2a^2}} = \frac{\sqrt{4ad + 2W^2}}{2a}.$$

From the condition on t_X we can find already t_d.

$$t_d = 2\left(t_X - \frac{W}{2a}\right) = \boxed{\frac{\sqrt{4ad+2W^2} - W}{a}}.$$

From now on we consider the time t_d to reach the sign a known quantity. We can concentrate on finding the rest of the time needed to reach to destination.

2.3 Maximum speed reached after the sign

If the car reaches the maximum speed after the sign before finishing the ride, the speed function looks as follows.

$$v(t) = \begin{cases} W + a(t - t_d), & t_d < t \leqslant t_{WV}, \\ V, & t_{WV} < t \leqslant t_*, \end{cases}$$

where t_{WV} is the time the car reaches the maximum speed V.

The time to accelerate from the speed W to V and the distance covered during the acceleration is clearly the same as the time and distance for deceleration.

$$t_{WV} - t_d = \frac{V - W}{a}, \quad L_{WV} = \frac{V^2 - W^2}{2a}.$$

Since the overall distance from the sign to the end is $L - d$, we have the following condition for the case: $L_{WV} \leqslant L - d$, that is, $\boxed{V^2 - W^2 \leqslant 2a(L - d)}$.

We find t_* in this case by expanding the definition $\ell(t_*) = L$.

$$L_{WV} + V(t_* - t_{WV}) + d = L,$$

that is,

$$\boxed{t_* = t_d + \frac{V-W}{a} + \frac{1}{V} \cdot \left(L - d - \frac{V^2 - W^2}{2a}\right)}.$$

2.4 Maximum speed not reached after the sign

In this case the optimal movement speed is

$$v(t) = W + a(t - t_d) \quad \text{for } t_d < t \leqslant t_*.$$

From this we get the following expression for the distance covered after the sign:

$$\int\limits_{t_d}^{t_*} (W + a(\tau - t_d))\, d\tau.$$

We know that this distance equals $L - d$ though. Let's change the integration variable with substitution $u = W + a(\tau - t_d)$, $d\tau = \frac{1}{a}\, du$.

$$L - d = \int\limits_{W}^{W-a(t_*-t_d)} \frac{u\, du}{a} = \frac{1}{2a} \cdot \left((W + a(t_* - t_d))^2 - W^2\right).$$

From this we get the formula for t_* in the final case:

$$\boxed{t_* = t_d + \frac{\sqrt{W^2+2a(L-d)}-W}{a}}.$$