Reference Equations Catalog for IMPACT Precipitation Code Validation

IMPACT Project Literature Collection Task 3.0.0

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Contents

1 Energy Dissipation Parameterization (Fang et al. 2010)

1.1 Normalized Atmospheric Column Mass

From Fang et al. (2010), Equation (1):

$$y = \frac{2}{E_{\text{mono}}} (\rho H)^{0.7} \left(6 \times 10^{-6}\right)^{-0.7} \tag{1}$$

where:

- y = normalized atmospheric column mass (dimensionless)
- E_{mono} = incident monoenergetic electron energy (keV)
- ρ = atmospheric mass density (g cm^{-3})
- H = atmospheric scale height (cm)
- 6×10^{-6} = reference density for normalization (g cm^{-3})
- 0.7 = empirical exponent from curve fitting

Validity Range: 100 eV $\leq E_{\text{mono}} \leq$ 1 MeV
Units: E_{mono} in keV, ρ in g cm^{-3}, H in cm

Physical Interpretation: This equation normalizes the atmospheric column mass to account for the energy-dependent penetration depth of precipitating electrons. Higher energy electrons penetrate deeper, and this normalization allows a unified parameterization across the full energy range.

Source: Fang et al. (2010), *Geophysical Research Letters*, 37, L22106, doi:10.1029/2010GL045406, Equation (1), Page L22106-2

Implementation Location: IMPACT_MATLAB/calc_Edissipation.m, line 33

1.2 Energy Dissipation Rate

From Fang et al. (2010), Equation (4):

$$f(y) = C_1 y^{C_2} \exp\left(-C_3 y^{C_4}\right) + C_5 y^{C_6} \exp\left(-C_7 y^{C_8}\right) \tag{2}$$

where:

- f = normalized energy dissipation rate (dimensionless)
- y = normalized atmospheric column mass from Equation (1)
- C_i $(i = 1, \ldots, 8)$ = energy-dependent coefficients

Physical Interpretation: The double exponential form captures the characteristic energy deposition profile, with the first term representing the primary ionization peak and the second term accounting for secondary ionization processes. The coefficients C_i are determined by the electron energy.

Source: Fang et al. (2010), *Geophysical Research Letters*, 37, L22106, doi:10.1029/2010GL045406, Equation (4), Page L22106-3

1.3 Coefficient Energy Dependence

From Fang et al. (2010), Equation (5):

$$C_i(E) = \exp\left(\sum_{j=0}^{3} P_{ij} \left[\ln(E)\right]^j\right) \tag{3}$$

where:

- C_i = coefficient for energy dissipation parameterization ($i = 1, \ldots, 8$)
- E = electron energy (keV)
- P_{ij} = polynomial coefficients from Table 1
- $\ln(E)$ = natural logarithm of energy

Implementation Location: IMPACT_MATLAB/calc_Edissipation.m, lines 35-43

1.4 Pij Polynomial Coefficients

From Fang et al. (2010), Table 1:

Table 1: Parameterization Coefficients P_{ij} for Isotropically Incident Monoenergetic 100 eV to 1 MeV Electrons

$i \backslash j$	$j = 0$	$j = 1$	$j = 2$	$j = 3$
1	1.24616	1.45903	-0.242269	0.0595459
2	2.23976	-4.22918$\times10^{-7}$	0.0136458	0.00253332
3	1.41754	0.144597	0.0170433	0.000639717
4	0.248775	-0.150890	6.30894$\times10^{-9}$	0.00123707
5	-0.465119	-0.105081	-0.0895701	0.0122450
6	0.386019	0.00175430	-0.000742960	0.000460881
7	-0.645454	0.000849555	-0.0428502	-0.00299302
8	0.948930	0.197385	-0.00250603	-0.00206938

Storage Location: IMPACT_MATLAB/coeff_fang10.mat (MATLAB binary file)

Implementation Location: IMPACT_MATLAB/sav_fang10_coeff.m

Source: Fang et al. (2010), *Geophysical Research Letters*, 37, L22106, doi:10.1029/2010GL045406, Table 1, Page L22106-4

2 Ionization Rate Formula (Fang et al. 2010)

2.1 Total Ionization Rate

From Fang et al. (2010), Equation (2) and surrounding text:

$$q_{\text{tot}} = \frac{Q_{\text{mono}} f}{D^* H} \tag{4}$$

where:

- q_{tot} = total ionization production rate (cm^{-3} s^{-1})
- Q_{mono} = incident monoenergetic electron energy flux (keV cm^{-2} s^{-1})
- f = normalized energy dissipation rate from Equation (2)
- D^* = mean energy loss per ion pair = 0.035 keV (35 eV)
- H = atmospheric scale height (cm)

Units: Q_{mono} in keV cm^{-2} s^{-1}, H in cm, q_{tot} in cm^{-3} s^{-1}

Physical Interpretation: This equation converts the energy dissipation profile into actual ionization rates by dividing by the mean energy required to produce one ion-electron pair. The 35 eV value is a well-established "rule of thumb" for high-energy electrons in Earth's atmosphere.

Source: Fang et al. (2010), *Geophysical Research Letters*, 37, L22106, doi:10.1029/2010GL045406, Section 2, Page L22106-2

Critical Note: The constant $D^* = 0.035$ keV is derived from laboratory measurements (Rees 1989) and represents the "mean energy loss per ion pair production." However, Fang et al. (2010) note that this value "is accurate for precipitating high-energy electrons but not for low-energy particles."

Implementation Location: IMPACT_MATLAB/calc_ionization.m, line 35

2.2 Ionization Constant Source

The constant 0.035 keV (35 eV) has the following provenance:

$$D^* = 35 \text{ eV/ion pair} = 0.035 \text{ keV/ion pair} \tag{5}$$

Literature Source: Rees, M. H. (1989), *Physics and Chemistry of the Upper Atmosphere*, Cambridge University Press, Cambridge.

Validation: Laboratory measurements of electron ionization efficiency in atmospheric gases.

Type: Physical constant - This represents a fundamental ionization efficiency for high-energy electrons in Earth's atmosphere.

3 Bounce Period Formula (Dipole Field Theory)

3.1 Relativistic Bounce Period

From standard magnetospheric physics (Roederer 1970; Schulz and Lanzerotti 1974):

$$T_b = 4\frac{R_E}{c}\frac{L}{\gamma\beta}\int_0^{\alpha_{\text{eq}}} \frac{\cos\alpha\, d\alpha}{\sqrt{1 - \frac{B_{\text{eq}}}{B_m(\alpha)}\sin^2\alpha}} \tag{6}$$

where:

- T_b = bounce period (seconds)
- R_E = Earth radius = 6.371×10^6 m
- c = speed of light = 2.998×10^8 m/s
- L = magnetic shell parameter
- γ = Lorentz factor = $(1 - \beta^2)^{-1/2}$
- $\beta = v/c$ (ratio of particle speed to speed of light)
- α_{eq} = equatorial pitch angle
- B_{eq} = equatorial magnetic field strength
- $B_m(\alpha)$ = magnetic field strength at mirror point for pitch angle α

Physical Interpretation: This exact formula gives the complete relativistic bounce period for a charged particle trapped in Earth's dipole magnetic field. The integral accounts for the variation of magnetic field strength along the particle's bounce trajectory.

Source: Roederer, J. G. (1970), *Dynamics of Geomagnetically Trapped Radiation*, Springer-Verlag, Berlin.

3.2 Momentum Calculation

From bounce_time_arr.m, line 38:

$$pc = \sqrt{\left(\frac{E}{mc^2} + 1\right)^2 - 1} \cdot mc^2 \tag{7}$$

where:

- pc = momentum times speed of light (MeV)
- E = kinetic energy (MeV)
- mc^2 = rest mass energy (0.511 MeV for electrons, 938 MeV for protons)

Units: E in MeV, pc in MeV

3.3 T_pa Polynomial Approximation

From bounce_time_arr.m, lines 46-47:

$$T_{\text{pa}} = 1.38 + 0.055 \sin^{1/3} \alpha - 0.32 \sin^{1/2} \alpha - 0.037 \sin^{2/3} \alpha - 0.394 \sin \alpha + 0.056 \sin^{4/3} \alpha \quad (8)$$

where:

- T_{pa} = pitch angle scaling factor for bounce period (dimensionless)
- α = equatorial pitch angle (radians)

Coefficients:

Term	Coefficient	Power of sin(α)
Constant	1.38	$\sin^0(\alpha)$
Term 1	0.055	$\sin^{1/3}(\alpha)$
Term 2	-0.32	$\sin^{1/2}(\alpha)$
Term 3	-0.037	$\sin^{2/3}(\alpha)$
Term 4	-0.394	$\sin^1(\alpha)$
Term 5	0.056	$\sin^{4/3}(\alpha)$

Physical Interpretation: This polynomial approximation provides a computationally efficient way to evaluate the pitch angle dependence of the bounce period integral. The exact integral in Equation (6) can be approximated by $T_b \propto T_{\text{pa}}(\alpha_{\text{eq}})$.

Critical Open Question: The specific coefficients (1.38, 0.055, -0.32, -0.037, -0.394, 0.056) have not been definitively traced to a primary literature source. While the polynomial form is consistent with standard dipole theory (Roederer 1970), the exact coefficients require further investigation.

Type: Empirical/Algorithmic - Numerical approximation to the exact bounce period integral.

Implementation Location: IMPACT_MATLAB/bounce_time_arr.m, lines 46-47

3.4 Complete Bounce Period Calculation

From bounce_time_arr.m, line 50:

$$T_b = 4LR_E \frac{mc^2}{pc} \frac{T_{\text{pa}}}{c} \frac{1}{86400} \quad (9)$$

where:

- T_b = bounce period in days (final output units)
- All other terms as defined above
- Factor of 86400 converts seconds to days

Alternative Formulation: The code actually outputs in days, but the fundamental bounce period formula can be expressed as:

$$T_b(\text{seconds}) = 4LR_E \frac{mc^2}{pc} T_{\text{pa}} \frac{1}{c} \tag{10}$$

Implementation Location: IMPACT_MATLAB/bounce_time_arr.m, line 50

3.5 Particle Mass Dependence

The bounce period depends on particle rest mass through the momentum calculation:

$$m_e c^2 = 0.511 \text{ MeV (electrons)}, \quad m_p c^2 = 938.272 \text{ MeV (protons)} \tag{11}$$

Implementation Location: IMPACT_MATLAB/bounce_time_arr.m, lines 26-28

4 Mirror Altitude Formula (Dipole Field)

4.1 Dipole Magnetic Field Ratio

From dipole_mirror_altitude.m, line 14:

$$\frac{B}{B_{\text{eq}}} = \frac{\cos^6 \lambda}{\sqrt{1 + 3\sin^2 \lambda}} \tag{12}$$

where:

- B = magnetic field strength at latitude λ
- B_{eq} = equatorial magnetic field strength
- λ = magnetic latitude (radians)

Physical Interpretation: This equation describes how the magnetic field strength varies with latitude in an ideal dipole field. The field is strongest at the poles and weakest at the equator.

Source: Standard dipole field theory, e.g., Roederer (1970), *Dynamics of Geomagnetically Trapped Radiation*, Springer-Verlag, Berlin.

Implementation Location: IMPACT_MATLAB/dipole_mirror_altitude.m, line 14

4.2 Mirror Point Altitude

From dipole_mirror_altitude.m, line 27:

$$r = LR_E \cos^2 \lambda \tag{13}$$

where:

- r = radial distance from Earth's center (km)
- L = magnetic shell parameter
- R_E = Earth radius = 6371 km
- λ = magnetic latitude (radians)

Resulting Altitude:

$$h = r - R_E = LR_E \cos^2 \lambda - R_E \tag{14}$$

where h is altitude above Earth's surface (km).

Physical Interpretation: This equation gives the radial position of a magnetic field line at a given latitude. Particles with a specific equatorial pitch angle will mirror at the latitude where the magnetic field strength equals their mirror field strength.

Implementation Location: IMPACT_MATLAB/dipole_mirror_altitude.m, line 27

4.3 Loss Cone Angle

From loss cone theory:

$$\sin^2 \alpha_{\rm LC} = \frac{B_{\rm eq}}{B_m} \tag{15}$$

where:

- $\alpha_{\rm LC}$ = loss cone angle at the equator
- $B_{\rm eq}$ = equatorial magnetic field strength
- B_m = magnetic field strength at the mirror point

Physical Interpretation: Particles with equatorial pitch angles less than $\alpha_{\rm LC}$ will have mirror points below the atmosphere and will be lost through atmospheric collisions. Typical loss cone angles are 5-10 degrees depending on L-shell.

Atmospheric Boundary: Precipitation occurs when mirror altitude ¡ 1000 km.

Source: Roederer, J. G. (1970), *Dynamics of Geomagnetically Trapped Radiation*, Springer-Verlag, Berlin.

5 Summary of Critical Constants

Type Classification:

- **Physical**: Fundamental physical constant with units
- **Empirical**: Fitted parameter from experimental/observational data
- **Normalization**: Scaling factor for dimensionless parameterization
- **Algorithmic**: Numerical approximation or computational choice

Table 2: Constant Traceability Matrix for IMPACT Precipitation Code

Constant	Value	Code Location	Literature Source	Equation	Type
D^*	0.035 keV	calc_ionization.m:35	Rees (1989)	(4)	Physical
ρ_{ref}	6×10^{-6} g cm^{-3}	calc_Edissipation.m:33	Fang et al. (2010)	(1)	Normalization
Exponent	0.7	calc_Edissipation.m:33	Fang et al. (2010)	(1)	Empirical
P_{11}	1.24616	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{12}	1.45903	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{13}	-0.242269	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{14}	0.0595459	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{21}	2.23976	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{22}	-4.22918$\times 10^{-7}$	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{23}	0.0136458	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{24}	0.00253332	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{31}	1.41754	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{32}	0.144597	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{33}	0.0170433	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{34}	0.000639717	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{41}	0.248775	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{42}	-0.150890	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{43}	6.30894$\times 10^{-9}$	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{44}	0.00123707	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{51}	-0.465119	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{52}	-0.105081	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{53}	-0.0895701	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{54}	0.0122450	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{61}	0.386019	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{62}	0.00175430	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{63}	-0.000742960	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{64}	0.000460881	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{71}	-0.645454	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{72}	0.000849555	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{73}	-0.0428502	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{74}	-0.00299302	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{81}	0.948930	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{82}	0.197385	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{83}	-0.00250603	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
P_{84}	-0.00206938	coeff_fang10.mat	Fang et al. (2010)	Table 1	Empirical
$T_{\mathrm{pa},0}$	1.38	bounce_time_arr.m:46	*UNIDENTIFIED*	(8)	Empirical/Algorith
$T_{\mathrm{pa},1}$	0.055	bounce_time_arr.m:46	*UNIDENTIFIED*	(8)	Empirical/Algorith
$T_{\mathrm{pa},2}$	-0.32	bounce_time_arr.m:46	*UNIDENTIFIED*	(8)	Empirical/Algorith
$T_{\mathrm{pa},3}$	-0.037	bounce_time_arr.m:46	*UNIDENTIFIED*	(8)	Empirical/Algorith
$T_{\mathrm{pa},4}$	-0.394	bounce_time_arr.m:46	*UNIDENTIFIED*	(8)	Empirical/Algorith
$T_{\mathrm{pa},5}$	0.056	bounce_time_arr.m:46	*UNIDENTIFIED*	(8)	Empirical/Algorith
$m_e c^2$	0.511 MeV	bounce_time_arr.m:26	Physical constant	(7)	Physical
$m_p c^2$	938 MeV	bounce_time_arr.m:28	Physical constant	(7)	Physical
R_E	6371 km	bounce_time_arr.m:41	Physical constant	(9)	Physical
c	2.998×10^8 m/s	bounce_time_arr.m:42	Physical constant	(6)	Physical

6 References

- Fang, X., C. E. Randall, D. Lummerzheim, W. Wang, G. Lu, S. C. Solomon, and R. A. Frahm (2010), Parameterization of monoenergetic electron impact ionization, *Geophysical Research Letters*, 37, L22106, doi:10.1029/2010GL045406.

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