A Finite Element Primer *

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Contents

*This is a summary of finite element theory for a diffusion problem in one dimension. It provides the mathematical foundation for Chapter 1 of our reference book *Finite Elements and Fast Iterative Solvers with Applications in Incompressible Fluid Dynamics*, see http://www.oup.co.uk/isbn/0-19-852868-X

1. A Model Diffusion Problem

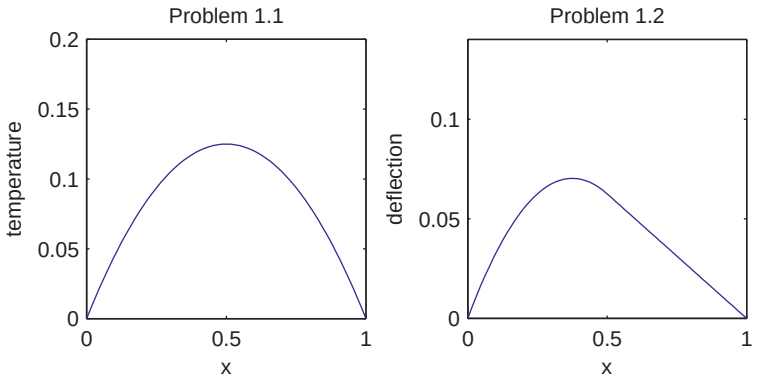
The problem we consider herein is a two point boundary value problem. A formal statement is: given a real function $f \in C^0(0,1)$ (see Definition x.2 below), we seek a function $u \in C^2(0,1) \cap C^0[0,1]$ (see below) satisfying

$$\left.\begin{aligned} -\frac{d^2u}{dx^2} &= f \quad \text{for} \quad 0 < x < 1 \\ u(0) &= 0; \quad u(1) = 0. \end{aligned}\right\} \qquad (D)$$

The term $-\frac{d^2u}{dx^2}$ represents "diffusion", and f is called the "source" term. A sufficiently smooth function u satisfying (D) is called a "strong" (or "classical") solution.

Problem 1.1 $(f = 1)$ This is a model for the temperature in a wire with the ends kept in ice. There is a current flowing in the wire which generates heat. Solving (D) gives the parabolic "hump"

$$u(x) = \frac{1}{2}(x - x^2).$$



Some basic definitions will need to be added if our statement of (D) is to make sense.

Definition x.1 (Domain)
A **domain** is a bounded open set; for example, $\Omega = (0,1)$, which identifes where a differential equation is defined. The "closure" of the set, denoted $\overline{\Omega}$, includes all the points on the boundary of the domain; for example, $\overline{\Omega} = [0,1]$.

Definition x.2 (Continuous function)
A real **function** f is mapping which assigns a unique real number to every point in a domain: $f : \Omega \to \mathbb{R}$.

- $C^0(\Omega)$ is the set of all continuous functions defined on Ω.
- $C^k(\Omega)$ is the set of all continuous functions whose kth derivatives are also continuous over Ω.
- $C^0(\overline{\Omega})$ is the set of all functions $u \in C^0(\Omega)$ such that u can be extended to a continuous function on $\overline{\Omega}$.

The standard way of categorizing spaces of functions is to use the notion of a "norm". This is made explicit in the following definition.

Definition x.3 (Normed vector space)
A **normed vector space** V, has (or more formally is "equipped with") a mapping $\|\cdot\| : V \to \mathbb{R}$ which satisfies four axioms:

① $\|u\| \geq 0 \qquad \forall u \in V;$ (where $\forall$ means "for all")

② $\|u\| = 0 \iff u = 0;$ (where $\iff$ means "if and only if")

③ $\|\alpha u\| = |\alpha|\|u\|, \quad \forall \alpha \in \mathbb{R}$ and $\forall u \in V;$

④ $\|u + v\| \leq \|u\| + \|v\| \quad \forall u, v \in V.$

Note that, if the second axiom is relaxed to the weaker condition $\|u\| = 0 \Leftarrow u = 0$ then V is only equipped with a **semi–norm**. A normed vector space that is "complete" (see Definition x.10 below) is called a **Banach Space**.

Two examples of normed vector spaces are given below.

Example x.3.1 Suppose that $V = \mathbb{R}^2$, that is, all vectors $\mathbf{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$.
Valid norms are

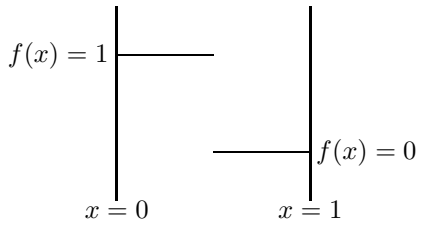
$$\begin{aligned} \|\mathbf{u}\|_1 &= |u_x| + |u_y|; && \ell_1 \text{ norm} \\ \|\mathbf{u}\|_2 &= (u_x^2 + u_y^2)^{1/2}; && \ell_2 \text{ norm} \qquad \heartsuit \\ \|\mathbf{u}\|_\infty &= \max\{|u_x|, |u_y|\}. && \ell_\infty \text{ norm} \end{aligned}$$

Example x.3.2 Suppose that $V = C^0(\Omega)$. A valid norm is

$$\|u\| = \max_{x \in \overline{\Omega}} |u(x)|. \qquad L_\infty \text{ norm.} \qquad \heartsuit$$

Returning to (D), the source function $f(x)$ may well be "rough", $f \notin C^0(\Omega)$. An example is given below.

Problem 1.2 ($f(x) = 1 - H(1/2)$; where $H(x)$ is the "unit step" function)



This is a model for the deflection of a simply supported elastic beam subject to a discontinuous load. Solving the differential equation over the two intervals and imposing continuity of the solution and the first derivative at the interface point $x = 1/2$ gives the "generalized" solution shown in the figure on page 1:

$$u(x) = \begin{cases} -\frac{x^2}{2} + \frac{3}{8}x & 0 \le x < \frac{1}{2} \\ -\frac{x}{8} + \frac{1}{8} & \frac{1}{2} \le x \le 1. \end{cases}$$

Solving (D) means finding two functions:

$$\begin{aligned} \text{first,} \quad & v \text{ such that} \quad \frac{dv}{dx} = -f, \\ \text{second,} \quad & u \text{ such that} \quad \frac{du}{dx} = v. \end{aligned}$$

We will see that an appropriate starting point for constructing function spaces for v (and hence u) is the space of square integrable functions.

Definition x.4 (Square integrable function)
$L_2(\Omega)$ is the vector space of square integrable functions defined on Ω :

$$u \in L_2(\Omega) \text{ if and only if } \int_\Omega u^2 < \infty.$$

Functions that are not continuous in $[0, 1]$ may still be square integrable. We give two examples below.

Example x.4.1 Consider $f = x^{-1/4}$.

$$\int_0^1 f^2 \, dx = \int_0^1 x^{-1/2} \, dx = 2$$

hence $\int_0^1 f^2 < \infty$ so that $f \in L_2(\Omega)$. □

Example x.4.2 Consider $f = \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}$.

$$\begin{aligned} \int_0^1 f^2\,dx &= \int_0^{1/2} f^2\,dx + \int_{1/2}^1 f^2\,dx \\ &= \underbrace{\int_0^{1/2} 0\,dx}_{0} + \underbrace{\int_{1/2}^1 dx}_{1/2} \end{aligned}$$

hence $\int_0^1 f^2 < \infty$ so that $f \in L_2(\Omega)$. □

$L_2(\Omega)$ is a Banach space.

Example x.3.3 Suppose that $V = L_2(\Omega)$ with $\Omega = (0,1)$. A valid norm is

$$\|u\| = \left(\int_0^1 u^2\,dx\right)^{1/2}. \qquad L_2 \text{ norm} \qquad \heartsuit$$

$L_2(\Omega)$ is pretty special—it is also equipped with an inner product.

Definition x.5 (Inner product space)
An **inner product** space V, has a mapping $(\cdot,\cdot) : V \times V \to \mathbb{R}$ which satisfies four axioms:

❶ $(u,w) = (w,u) \qquad \forall u, w \in V;$

❷ $(u,u) \ge 0 \qquad \forall u \in V;$

❸ $(u,u) = 0 \iff u = 0;$

❹ $(\alpha u + \beta v, w) = \alpha(u,w) + \beta(v,w) \qquad \forall \alpha, \beta \in \mathbb{R}; \quad \forall u, v, w \in V.$

Example x.5.1 Suppose that $V = \mathbb{R}^2$. A valid inner product is given by

$$(\mathbf{u}, \mathbf{w}) = u_x w_x + u_y w_y \quad = \mathbf{u} \cdot \mathbf{w} \qquad \heartsuit$$

Example x.5.2 Suppose that $V = L_2(\Omega)$, with $\Omega = (0,1)$.
A valid inner product is given by

$$(u,w) = \int_0^1 uw. \qquad \heartsuit$$

A complete inner product space like $L_2(\Omega)$ is called a **Hilbert Space**.

Note that an inner product space is also a normed space. There is a "natural" (or "energy") norm

$$\|u\| = (u,u)^{\frac{1}{2}}.$$

Inner products and norms are related by the Cauchy-Schwarz inequality.

Definition x.6 (Cauchy-Schwarz inequality)

$$|(u,v)| \le \|u\|\,\|v\| \qquad \forall u,v \in V. \qquad\qquad (C\text{–}S)$$

Example x.6.1 Suppose that $V = \mathbb{R}^2$. We have the discrete version of C–S:

$$\mathbf{u}\cdot\mathbf{w} \le |\mathbf{u}\cdot\mathbf{w}| \le (u_x^2+u_y^2)^{1/2}(w_x^2+w_y^2)^{1/2}. \qquad \heartsuit$$

Example x.6.2 Suppose that $V = L_2(\Omega)$, with $\Omega = (0,1)$. We have

$$\int_0^1 uw \le \left|\int_0^1 uw\right| \le \left(\int_0^1 u^2\right)^{1/2}\left(\int_0^1 w^2\right)^{1/2}. \qquad \heartsuit$$

Returning to Problem 1.2, we now address the question of where (in which function space) do we look for the generalized solution u when the function f is square integrable but *not* continuous? The answer to this is "in a Sobolev space".

Definition x.7 (Sobolev space)
For a positive index k, the Sobolev space $H^k(0,1)$ is the set of functions $v : (0,1) \to \mathbb{R}$ such that v and all derivatives up to and including k are square integrable:

$$u \in H^k(0,1) \iff \int_0^1 u^2 < \infty,\ \int_0^1 \left(\frac{du}{dx}\right)^2 < \infty,\ \ldots,\ \int_0^1 \left(\frac{d^k u}{dx^k}\right)^2 < \infty.$$

Note that $H^k(0,1)$ defines a Hilbert space with inner product

$$(u,w)_k = \int_0^1 uw + \int_0^1 \left(\frac{du}{dx}\right)\left(\frac{dw}{dx}\right) + \ldots + \int_0^1 \left(\frac{d^k u}{dx^k}\right)\left(\frac{d^k w}{dx^k}\right)$$

and norm

$$\|u\|_k = \left(\int_0^1 u^2 + \int_0^1 \left(\frac{du}{dx}\right)^2 + \ldots + \int_0^1 \left(\frac{d^k u}{dx^k}\right)^2\right)^{1/2}.$$

Returning to Problem 1.2, the appropriate solution space turns out to be

$$H_0^1(0,1) = \underbrace{\left\{ u \in L_2(0,1),\ \frac{du}{dx} \in L_2(0,1) \right\}}_{u \in H^1(0,1)};\ \underbrace{u(0) = 0,\ u(1) = 0}_{\text{essential b.c.'s}}\Bigg\}.$$

This is a big surprise:

- There are no second derivatives in the definition of H_0^1!
- First derivatives need not be continuous!

To help understand where $H_0^1(0,1)$ comes from, it is useful to reformulate (D) as a minimization problem. Specifically, given $f \in L_2(0,1)$, we look for a "minimizing function" $u \in H_0^1(0,1)$ satisfying

$$F(u) \le F(v), \qquad \forall v \in H_0^1(0,1), \qquad (M)$$

where $F : H_0^1(0,1) \to \mathbb{R}$ is the so-called "energy functional"

$$F(v) = \frac{1}{2} \underbrace{\int_0^1 \left(\frac{dv}{dx}\right)^2}_{*} - \underbrace{\int_0^1 fv}_{**},$$

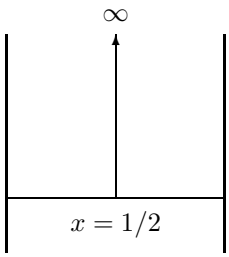
and $H_0^1(0,1)$ is the associated "minimizing set". Note that both right hand side terms are finite,

$$(*) \quad v \in H^1(0,1) \ \Rightarrow \ \int_0^1 \left(\frac{dv}{dx}\right)^2 < \infty$$

$$(**) \quad \int_0^1 fv \ \le \ \underbrace{\left(\int_0^1 f^2\right)^{1/2}}_{f \in L_2(\Omega)} \underbrace{\left(\int_0^1 v^2\right)^{1/2}}_{v \in H^1(0,1)} \quad \text{using } C\text{–}S \ \Rightarrow \int_0^1 fv < \infty.$$

Note also that the definition of (M) and the construction $(**)$ suggests that even rougher load data f may be allowable.

Problem 1.3 ($f(x) = \delta(1/2)$; where $\delta(x)$ is the "Dirac function")



This gives a model for the deflection of a simply supported elastic beam subject to a point load. The solution is called a "fundamental solution" or a Green's function. In this case, since $v \in H^1(0,1) \Rightarrow v \in C^0(0,1)$, we have that $\int_0^1 fv = v\,(1/2) < \infty$, so that (M) is well defined. (Note that this statement is only true for domains in $\mathbb{R}^1$—in higher dimensions the Dirac delta function is not admissible as load data.)

Returning to (M), we can compute u by solving the following "variational formulation" : given $f \in L_2(0,1)$ find $u \in H_0^1(0,1)$ such that

$$\int_0^1 \frac{du}{dx}\frac{dv}{dx} = \int_0^1 fv \qquad \forall v \in H_0^1(0,1). \qquad (V)$$

A solution to (V) (or, equivalently a solution to (M)) is called a "weak" solution. The relationship between (D), (M) and (V) is explored in the following three theorems.

Theorem 1.1 $((D) \Rightarrow (V))$
If u solves (D) then it solves (V).

Proof. Let u satisfy (D). Since continuous functions are square integrable then $u \in L_2(0,1)$ and $\frac{du}{dx} \in L_2(0,1)$. Furthermore since $u(0) = 0 = u(1)$ from the statement of (D), we have that $u \in H_0^1(0,1)$.

To show (V), let $v \in H_0^1(0,1)$, multiply (D) by v and integrate over Ω:

$$-\int_0^1 \frac{d^2u}{dx^2}\,v = \int_0^1 fv.$$

Using integrating by parts gives

$$-\int_0^1 \frac{d^2u}{dx^2}\,v = \int_0^1 \frac{du}{dx}\frac{dv}{dx} - \left[\frac{du}{dx}\,v\right]_0^1,$$

where

$$\left[\frac{du}{dx}\,v\right]_0^1 = \frac{du}{dx}(1)v(1) - \frac{du}{dx}(0)v(0),$$

and since $v \in H_0^1(0,1)$ we have $v(0) = v(1) = 0$ so that the boundary term is zero. Thus we have that u satisfies

$$\int_0^1 \frac{du}{dx}\frac{dv}{dx} = \int_0^1 fv \quad \forall v \in H_0^1(0,1)$$

as required. □

The above proof shows us how to "construct" a weak formulation from a classical formulation. We now use the properties of an inner product in Definition x.5 to show that (V) has a unique solution.

Theorem 1.2 *A solution to* (V) *is unique.*

Proof. Let $V = H_0^1(0,1)$ and assume that there are two weak solutions $u_1(x) \in V$, $u_2(x) \in V$ such that

$$\begin{aligned}\left(\frac{du_1}{dx}, \frac{dv}{dx}\right) &= (f,v) \quad \forall v \in V, \qquad (a,b) = \int_0^1 ab;\\ \left(\frac{du_2}{dx}, \frac{dv}{dx}\right) &= (f,v) \quad \forall v \in V.\end{aligned}$$

Subtracting

$$\left(\frac{du_1}{dx}, \frac{dv}{dx}\right) - \left(\frac{du_2}{dx}, \frac{dv}{dx}\right) = 0 \quad \forall v \in V,$$

and then using ❹ gives

$$\left(\frac{du_1}{dx} - \frac{du_2}{dx}, \frac{dv}{dx}\right) = 0 \quad \forall v \in V.$$

We now define $w = u_1 - u_2$, so that

$$\left(\frac{dw}{dx}, \frac{dv}{dx}\right) \quad \forall v \in V. \qquad (\ddagger)$$

Our aim is to show that $w = 0$ in $(0,1)$ (so that $u_1 = u_2$.) To do this we note that $w \in V$ (by the definition of a vector space) and set $v = w$ in $(\ddagger)$. Using ❸ then gives

$$\left(\frac{dw}{dx}, \frac{dw}{dx}\right) = 0 \Rightarrow \frac{dw}{dx} = 0. \qquad (*)$$

From which we might deduce that w is constant. Finally, if we use the fact that functions in V are continuous over $[0,1]$, and are zero at the end points, we can see that $w = 0$ as required. □

The "hole" in the above argument is that two square integrable functions which are identical in $[0,1]$ except at a finite set of points are **equivalent** to each other. (They cannot be distinguished from each other in the sense of taking their L_2 norm.) Thus, a more precise statement of $(*)$ is that $dw/dx = 0$ "almost everwhere". Thus a more rigorous way of establishing uniqueness is to use the famous Poincaré–Friedrich inequality.

Lemma 1.3 (Poincaré–Friedrich)
If $w \in H_0^1(0,1)$ *then*

$$\int_0^1 w^2 \leq \int_0^1 \left(\frac{dw}{dx}\right)^2. \qquad (P\text{–}F)$$

Proof. See below. □

Thus, starting from $(*)$ and using P–F gives

$$\underbrace{\int_0^1 w^2}_{\geq 0} \leq \int_0^1 \left(\frac{dw}{dx}\right)^2 = 0$$

and we deduce that $w = 0$ almost everywhere in $(0,1)$, so that there is a unique solution to (V) in the L_2 sense. □

Proof. **(of P–F)**
Suppose $w \in H_0^1(0,1)$, then

$$w(x) = w(0) + \int_0^x \frac{dw}{dx}(\xi)\, d\xi.$$

Thus, since $w(0) = 0$ we have

$$\begin{aligned} w^2 &= \left|\int_0^x \frac{dw}{dx}\right|^2 \\ &\leq \left(\int_0^x 1^2\right)\left(\int_0^x \left(\frac{dw}{dx}\right)^2\right) \qquad \text{using } C\text{–}S \\ &\leq \underbrace{\left(\int_0^1 1^2\right)}_{=1}\left(\int_0^1 \left(\frac{dw}{dx}\right)^2\right) \qquad \text{because } x \leq 1. \end{aligned}$$

Hence $w^2(x) \leq \int_0^1 \left(\frac{dw}{dx}\right)^2$, and integrating over (0,1) gives

$$\int_0^1 w^2 \leq \int_0^1 \underbrace{\left\{\int_0^1 \left(\frac{dw}{dx}\right)^2\right\}}_{\in R^+} dx \quad = \quad \left\{\int_0^1 \left(\frac{dw}{dx}\right)^2\right\} \underbrace{\int_0^1 dx}_{=1}$$

as required. □

Theorem 1.4 ($(V) \Leftrightarrow (M)$)
If u solves (V) then u solves (M) and vice versa.

Proof.

(I) $(V) \Rightarrow (M)$
Let $u \in H_0^1(0,1)$ be the solution of (V), that is,

$$\left(\frac{du}{dx}, \frac{dv}{dx}\right) = (f, v) \qquad \forall v \in H_0^1(0,1).$$

Suppose $v \in H_0^1(0,1)$, and define $w = v - u \in H_0^1(0,1)$, then using the symmetry ❶ and linearity ❹ of the inner product gives

$$\begin{aligned}
F(v) &= F(u+w) \\
&= \frac{1}{2}\left(\frac{d}{dx}(u+w), \frac{d}{dx}(u+w)\right) - (f, u+w) \\
&= \frac{1}{2}\left(\frac{du}{dx}, \frac{du}{dx}\right) - (f,u) + \frac{1}{2}\left(\frac{du}{dx}, \frac{dw}{dx}\right) \\
&\quad + \frac{1}{2}\underbrace{\left(\frac{dw}{dx}, \frac{du}{dx}\right)}_{\frac{1}{2}\left(\frac{du}{dx}, \frac{dw}{dx}\right)} - (f,w) + \frac{1}{2}\left(\frac{dw}{dx}, \frac{dw}{dx}\right) \\
&= \frac{1}{2}\left(\frac{du}{dx}, \frac{du}{dx}\right) - (f,u) + \frac{1}{2}\left(\frac{dw}{dx}, \frac{dw}{dx}\right) + \underbrace{\left(\frac{du}{dx}, \frac{dw}{dx}\right) - (f,w)}_{=0} \\
&= F(u) + \frac{1}{2}\left(\frac{dw}{dx}, \frac{dw}{dx}\right).
\end{aligned}$$

Finally, using ❷, we have that $F(v) \geq F(u)$ as required. □

(II) $(V) \Leftarrow (M)$.
Let $u \in H_0^1(0,1)$ be the solution of (M), that is

$$F(u) \leq F(v) \qquad \forall v \in H_0^1(0,1).$$

Thus, given a function $v \in H_0^1(0,1)$ and $\varepsilon \in \mathbb{R}$, $u + \varepsilon v \in H_0^1(0,1)$, so that $F(u) \leq F(u+\varepsilon v)$. We now define $g(\varepsilon) := F(u+\varepsilon v)$. This function is is minimised when $\varepsilon = 0$, so that we have that $\left.\frac{dg}{d\varepsilon}\right|_{\varepsilon=0} = 0$. Now,

$$\begin{aligned}
g(\varepsilon) &= \frac{1}{2}\left(\frac{d}{dx}(u+\varepsilon v), \frac{d}{dx}(u+\varepsilon v)\right) - (f, u+\varepsilon v) \\
&= \frac{1}{2}\varepsilon^2\left(\frac{dv}{dx}, \frac{dv}{dx}\right) + \varepsilon\left(\frac{du}{dx}, \frac{dv}{dx}\right) - \varepsilon(f,v) \\
&\qquad + \frac{1}{2}\left(\frac{du}{dx}, \frac{du}{dx}\right) - (f,u), \\
\text{and so,} \quad \frac{dg}{d\varepsilon} &= \varepsilon\left(\frac{dv}{dx}, \frac{dv}{dx}\right) + \left(\frac{du}{dx}, \frac{dv}{dx}\right) - (f,v).
\end{aligned}$$

Finally, setting $\left.\frac{dg}{d\varepsilon}\right|_{\varepsilon=0} = 0$ we see that u solves (V).

□

In summary, we have that $(D) \Rightarrow (V) \Leftrightarrow (M)$. The converse implication $(D) \Leftarrow (V)$ is not true unless $u \in H_0^1(0,1)$ is smooth enough to ensure that $u \in C^2(0,1)$. In this special case, we have that $(D) \Leftrightarrow (V)$.

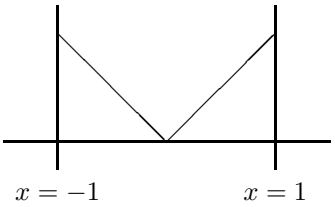
Returning to (M) and the space $H_0^1(0,1)$, a very important property of $L_2(\Omega)$ is the concept of a "weak derivative".

Definition x.8 (Weak derivative)
$u \in L_2(\Omega)$ possesses a weak derivative $\partial u \in L_2(\Omega)$ satisfying

$$(\phi, \partial u) = -\left(\frac{d\phi}{dx}, u\right) \qquad \forall \phi \in C_0^\infty(\Omega), \qquad (W\text{–}D)$$

where $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions which are zero outside Ω.

Example x.8.1 Consider the function $u = |x|$ with $\Omega = (-1,1)$.



This function is not differentiable in the classical sense. However, starting from the right hand side of W–D, and integrating by parts gives

$$\begin{aligned}
-\left(\frac{d\phi}{dx}, u\right) &= -\int_{-1}^{1} |x| \frac{d\phi}{dx} \\
&= -\int_{-1}^{0} (-x) \frac{d\phi}{dx} - \int_{0}^{1} x \frac{d\phi}{dx} \\
&= \int_{-1}^{0} \frac{d}{dx}(-x)\,\phi + \int_{0}^{1} \frac{d}{dx}(x)\,\phi \\
&\qquad - \underbrace{\Big[(-x)\phi\Big]_{-1}^{0}}_{0\phi(0)-1\phi(-1)} - \underbrace{\Big[(x)\phi\Big]_{0}^{1}}_{1\phi(1)-0\phi(0)} \\
&= \int_{-1}^{0} \phi \left\{\frac{d}{dx}(-x)\right\} + \int_{0}^{1} \phi \left\{\frac{d}{dx}(x)\right\} = (\phi, \partial u).
\end{aligned}$$

Thus the weak derivative is a step function,

$$\partial u = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases} .$$

Note that the value of ∂u is not defined at the origin. □

Example x.8.2 Consider the step function $u(x) = H(1/2)$. Constructing the weak derivative using W–D gives

$$-\left(\frac{d\phi}{dx}, u\right) = -\underbrace{\int_0^{\frac{1}{2}} (0)\frac{d\phi}{dx}}_{=0} - \int_{\frac{1}{2}}^1 (1)\frac{d\phi}{dx}.$$

Integrating by parts then gives

$$-\left(\frac{d\phi}{dx}, u\right) = \underbrace{\int_{\frac{1}{2}}^1 \phi\frac{d}{dx}(1)}_{0} - \underbrace{\Big[1\phi\Big]_{1/2}^1}_{\phi(1)-\phi(1/2)} .$$

Finally, since $\phi(1) = 0$, we conclude that

$$-\left(\frac{d\phi}{dx}, u\right) = \phi(\frac{1}{2}) = (\phi, \partial u)\,.$$

Thus, if we relax the requirement that $\partial u \in L_2(\Omega)$, we see that the weak derivative of a step function is a delta function. □

2. Galerkin Approximation

We now introduce a finite dimensional subspace $V_k \subset H_0^1(0,1)$. This is associated with a set of basis functions

$$V_k = \text{ span}\,\{\phi_1(x), \phi_2(x), \ldots, \phi_k(x)\}$$

so that every element of V_k, say u_k, can be uniquely written as

$$u_k = \sum_{j=1}^{k} \alpha_j \phi_j, \quad \alpha_j \in \mathbb{R}.$$

To compute the Galerkin approximation, we pose the variational problem (V) over V_k. That is, we seek $u_k \in V_k$ such that

$$\left(\frac{du_k}{dx}, \frac{dv_k}{dx}\right) = (f, v_k) \qquad \forall v_k \in V_k. \qquad (V_h)$$

Equivalently, since $\{\phi_i\}_{i=1}^k$ are a basis set, we have that

$$\begin{aligned}
\left(\frac{du_k}{dx}, \frac{d\phi_i}{dx}\right) &= (f, \phi_i), \qquad i = 1, 2, \ldots, k \\
\left(\frac{d}{dx}(\sum_{j=1}^{k} \alpha_j \phi_j), \frac{d\phi_i}{dx}\right) &= (f, \phi_i), \\
\sum_{j=1}^{k} \alpha_j \left(\frac{d\phi_j}{dx}, \frac{d\phi_i}{dx}\right) &= (f, \phi_i).
\end{aligned}$$

This can be written in matrix form as

$$A\mathbf{x} = \mathbf{f} \qquad (V_h')$$

$$\begin{aligned}
\text{with} \quad A_{ij} &= \left(\tfrac{d\phi_j}{dx}, \tfrac{d\phi_i}{dx}\right), \quad i, j = 1, \ldots, k; \\
x_j &= \alpha_j, \qquad j = 1, \ldots, k; \\
\text{and} \quad f_i &= (f, \phi_i), \qquad i = 1, \ldots, k.
\end{aligned}$$

(V_h') is called the Galerkin system, A is called the "stiffness matrix", $\mathbf{f}$ is the "load vector" and $u_k = \sum_{j=1}^k \alpha_j \phi_j$ is the "Galerkin solution".

Theorem 2.1 *The stiffness matrix is symmetric and positive definite.*

Proof. Symmetry follows from ❶. To establish positive definiteness we consider the quadratic form and use ❹:

$$\begin{aligned}
\mathbf{x}^T A\mathbf{x} &= \textstyle\sum_{j=1}^k \sum_{i=1}^k \alpha_j A_{ji} \alpha_i \\
&= \textstyle\sum_{j=1}^k \sum_{i=1}^k \alpha_j \left(\tfrac{d\phi_j}{dx}, \tfrac{d\phi_i}{dx}\right) \alpha_i \\
&= \left(\textstyle\sum_{j=1}^k \alpha_j \tfrac{d\phi_j}{dx}, \sum_{i=1}^k \alpha_i \tfrac{d\phi_i}{dx}\right) \\
&= \left(\tfrac{du_k}{dx}, \tfrac{du_k}{dx}\right).
\end{aligned}$$

Thus from ❷ we see that A is at least semi-definite. Definiteness follows from the fact that $\mathbf{x}^T A\mathbf{x} = 0$ if and only if $du_k/dx = 0$. But since $u_k \in H_0^1(0,1)$ then $du_k/dx = 0$ implies that $u_k = 0$. Finally, since $\{\phi_i\}_{i=1}^k$ are a basis set, we have that $u_k = 0$ implies that $\mathbf{x} = \mathbf{0}$. □

Theorem 2.1 implies that A is nonsingular. This means that the solution $\mathbf{x}$ (and hence u_k) exists and is unique.

An alternative approach, the so-called Rayleigh–Ritz method, is obtained by posing the minimization problem (M) over the finite dimensional subspace V_k. That is, we seek $u_k \in V_k$ such that

$$F(u_k) \le F(v_k) \qquad \forall v_k \in V_k. \qquad (M_h)$$

Doing this leads to the matrix system (V_h') so that the Ritz solution and the Galerkin solution are one and the same.

The beauty of Galerkin's method is the "best approximation" property.

Theorem 2.2 (Best approximation)
If u is the solution of (V) and u_k is the Galerkin solution, then

$$\|\frac{d}{dx}(u-u_k)\| \leq \|\frac{d}{dx}(u-v_k)\| \qquad \forall v_k \in V_k. \qquad (B\text{–}A)$$

Proof. The functions u_k and u satisfy the following

$$u_k \in V_k; \quad \left(\frac{du_k}{dx}, \frac{dv_k}{dx}\right) = (f, v_k) \quad \forall v_k \in V_k$$

$$u \in V; \qquad \left(\frac{du}{dx}, \frac{dv}{dx}\right) = (f, v) \qquad \forall v \in V.$$

But since $V_k \subset V$ we have that

$$\left(\frac{du}{dx}, \frac{dv_k}{dx}\right) = (f, v_k) \qquad \forall v_k \in V_k.$$

Subtracting equations and using ❹ gives

$$\begin{aligned} \left(\frac{du}{dx}, \frac{dv_k}{dx}\right) - \left(\frac{du_k}{dx}, \frac{dv_k}{dx}\right) &= 0 \qquad \forall v_k \in V_k \\ \left(\frac{du}{dx} - \frac{du_k}{dx}, \frac{dv_k}{dx}\right) &= 0 \\ \left(\frac{d}{dx}(u-u_k), \frac{dv_k}{dx}\right) &= 0 \qquad \forall v_k \in V_k \qquad (G\text{–}O) \end{aligned}$$

This means that the error $u - u_k$ is "orthogonal" to the subspace V_k—a property known as **Galerkin orthogonality**. To establish the best approximation property we start with the left hand side of B–A and use Galerkin

orthogonality as follows:

$$\begin{aligned}
\|\frac{d}{dx}(u-u_k)\|^2 &= \left(\frac{d}{dx}(u-u_k), \frac{d}{dx}(u-u_k)\right) \\
&= \left(\frac{d}{dx}(u-u_k), \frac{du}{dx}\right) - \underbrace{\left(\frac{d}{dx}(u-u_k), \frac{du_k}{dx}\right)}_{=0 \quad G\text{–}O} \qquad u_k \in V_k \\
&= \left(\frac{d}{dx}(u-u_k), \frac{du}{dx}\right) - \overbrace{\left(\frac{d}{dx}(u-u_k), \frac{dv_k}{dx}\right)} \qquad v_k \in V_k \\
&= \left(\frac{d}{dx}(u-u_k), \frac{d}{dx}(u-v_k)\right) \\
&\leq \|\frac{d}{dx}(u-u_k)\| \, \|\frac{d}{dx}(u-v_k)\|. \qquad \text{using } C\text{–}S
\end{aligned}$$

Hence, dividing by $\|\frac{d}{dx}(u-u_k)\| > 0$[1]

$$\|\frac{d}{dx}(u-u_k)\| \leq \|\frac{d}{dx}(u-v_k)\| \quad \forall v_k \in V_k,$$

as required. □

An important observation here is that we have a natural norm to measure errors—which is inherited from the minimimization problem (M).

Example x.3.4 Suppose $V = H_0^1(0,1)$. A valid norm is

$$\|v\|_E = \left(\frac{dv}{dx}, \frac{dv}{dx}\right)^{1/2} = \|\frac{dv}{dx}\|.$$

This is called the **energy norm**. ♡

A technical issue that arises here is that the best approximation property does not automatically imply that the Galerkin method converges in the sense that

$$\|u-u_k\|_E \to 0 \quad \text{as} \quad k \to \infty.$$

For convergence, we really need to introduce the concept of a "complete" space that was postponed earlier.

Definition x.9 (Cauchy sequence)
A sequence $(v^{(k)}) \in V$ is called a **Cauchy sequence** in a normed space V if for any $\varepsilon > 0$, there exists a positive integer $k_0(\varepsilon)$ such that

$$\|v^{(\ell)} - v^{(m)}\|_V < \varepsilon \quad \forall \ell, m \geq k_0.$$

[1] If $\|\frac{d}{dx}(u-u_k)\| = 0$ then $B\text{–}A$ holds trivially.

A Cauchy sequence is convergent, so the only issue is whether the limit of the sequence is in the "correct space". This motivates the following definition.

Definition x.10 (Complete space)
A normed space V is **complete** if it contains the limits of all Cauchy sequences in V. That is, if $(v^{(k)})$ is a Cauchy sequence in V, then there exists $\xi \in V$ such that

$$\lim_{k\to\infty} \|v^{(k)} - \xi\|_V = 0.$$

We write this as $\lim_{k\to\infty} v^{(k)} = \xi$.

Example x.10.1 The space $H_0^1(0,1)$ is complete with respect to the energy norm $\|\cdot\|_E$. The upshot is that the Galerkin approximation is guaranteed to converge to the weak solution in the limit $k \to \infty$. ♡

We now introduce a simple-minded Galerkin approximation based on "global" polynomials. That is, we choose

$$V_k = \text{span}\left\{1, x, x^2, \ldots, x^{k-1}\right\}.$$

To ensure that $V_k \subset V$, the function $u_k = \sum_j \alpha_j x^{j-1}$ must satisfy two conditions:

(I) $u_k \in H^1(0,1)$;

(II) $u_k(0) = 0 = u_k(1)$.

The first condition is no problem, $u_k \in C^\infty(0,1)$! To satisfy the second condition we need to modify the basis set to the following,

$$V_k^* = \text{span}\left\{x(x-1), x^2(x-1), x^3(x-1)\ldots, x^k(x-1)\right\}.$$

Problem 2.1 $(f = 1)$
Consider

$$V_2^* = \text{span}\{\underbrace{x(x-1)}_{\phi_1}, \underbrace{x^2(x-1)}_{\phi_2}\}$$

Then constructing the matrix system (V_h'), we have that

$$\begin{aligned}
A_{11} &= \int_0^1 \left(\frac{d\phi_1}{dx}\right)^2 = \int_0^1 \left(\frac{d}{dx}(x^2-x)\right)^2 = \frac{1}{3} \\
A_{12} &= \int_0^1 \frac{d\phi_2}{dx}\frac{d\phi_1}{dx} = \int_0^1 \frac{d}{dx}(x^2-x)\,\frac{d}{dx}(x^3-x^2) = \frac{1}{6} = A_{21} \\
A_{22} &= \int_0^1 \left(\frac{d\phi_2}{dx}\right)^2 = \int_0^1 \left(\frac{d}{dx}(x^3-x^2)\right)^2 = \frac{2}{15} \\
f_1 &= \int_0^1 \phi_1 = \int_0^1 x^2 - x = -\frac{1}{6} \\
f_2 &= \int_0^1 \phi_2 = \int_0^1 x^3 - x^2 = -\frac{1}{12}.
\end{aligned}$$

This gives

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{15} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} \\ -\frac{1}{12} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}.$$

So the Galerkin solution is

$$u_2(x) = -\frac{1}{2}\phi_1 + 0\phi_2 = \frac{1}{2}(x - x^2) = u(x).$$

The fact that the Galerkin approximation agrees with the exact solution is to be expected given the best approximation property and noting that $u \in V_2^*$.

Problem 2.2 ($f(x) = H(1/2)$; where $H(x)$ is the "unit step" function)
Consider V_2^* as above. In this case,

$$\begin{aligned}
f_1 &= \int_{\frac{1}{2}}^1 \phi_1 = -\frac{1}{12} \\
f_2 &= \int_{\frac{1}{2}}^1 \phi_2 = \int_0^1 x^3 - x^2 = -\frac{5}{192}.
\end{aligned}$$

This gives the Galerkin system

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{15} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{12} \\ -\frac{5}{192} \end{pmatrix}.$$

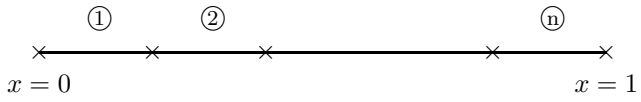
Note that the Galerkin solution is not exact in this case.

The big problem with global polynomial approximation is that the Galerkin matrix becomes increasingly ill conditioned as k is increased. Computationally, it behaves like a Hilbert matrix and so reliable computation for $k > 10$ is not possible. For this reason, **piecewise polynomial** basis functions are used in practice instead of global polynomial functions.

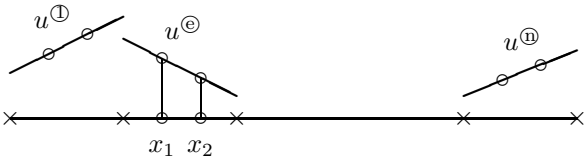
3. Finite Element Galerkin Approximation

A piecewise polynomial approximation space can be constructed in four steps.

Step(i) Subdivision of $\overline{\Omega}$ into "elements".
For $\overline{\Omega} = [0, 1]$ the elements are intervals as illustrated below.



Step (ii) Piecewise approximation of u using a low-order polynomial (e.g. linear):



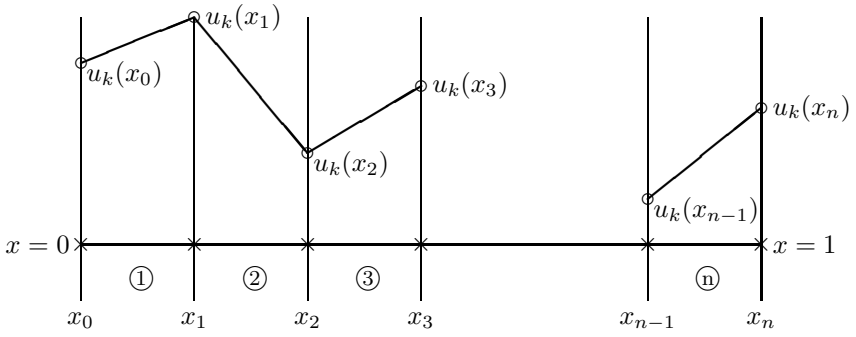
In general, a linear function $u^{ⓔ}(x)$ is defined by its values at two distinct points $x_1 \neq x_2$

$$u^{ⓔ}(x) = \underbrace{\frac{(x - x_2)}{(x_1 - x_2)}}_{\ell_1^{ⓔ}(x)} u^{ⓔ}(x_1) + \underbrace{\frac{(x - x_1)}{(x_2 - x_1)}}_{\ell_2^{ⓔ}(x)} u^{ⓔ}(x_2).$$

$\ell_i(x)$ are called "nodal" basis functions and satisfy the interpolation conditions

$$\ell_1^{ⓔ}(x) = \begin{cases} \text{linear over } ⓔ \\ 1 \quad \text{if } x = x_1 \\ 0 \quad \text{if } x = x_2 \end{cases} ; \qquad \ell_2^{ⓔ}(x) = \begin{cases} \text{linear over } ⓔ \\ 1 \quad \text{if } x = x_2 \\ 0 \quad \text{if } x = x_1 \end{cases} .$$

Step (iii) Satisfaction of the smoothness requirement (I).
This is done by carefully positioning the nodes, so that x_1 and x_2 are at the end points of the interval.

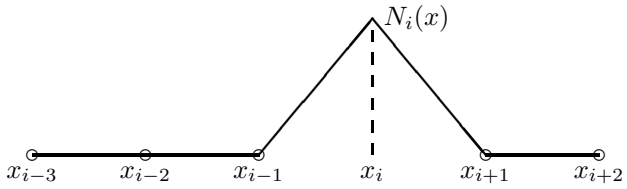


Concatenating the element functions $u^{\textcircled{e}}(x)$ gives a global function $u_k(x)$

$$u_k(x) = \bigwedge_{\textcircled{e}=1}^{n} u^{\textcircled{e}}(x).$$

Thus $u_k(x)$ is defined by $k = n-1$ internal values, and the two boundary values $u_k(x_0) = u_k(0)$ and $u_k(x_n) = u_k(1)$. We can then define $N_i(x)$, the so called "global" basis function, so that

$$N_i(x) = \begin{cases} \text{linear over } [0,1] \\ 1 \text{ if } x = x_i \\ 0 \text{ if } x = x_j \ (j \neq i) \end{cases},$$



and write

$$u_k(x) = \alpha_0 N_0(x) + \alpha_1 N_1(x) + \ldots + \alpha_n N_n(x).$$

Note that $u_k(x_i) = \alpha_i$, so that the "unknowns" are the function values at the nodes.

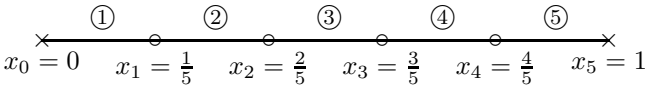
Step (iv) Satisfaction of the essential boundary condition requirement (II). This is easy—we simply remove the basis functions $N_0(x)$ and $N_n(x)$ from the basis set. The modified Galerkin approximation is

$$u_k^*(x) = \alpha_1 N_1(x) + \ldots + \alpha_{n-1} N_{n-1}(x),$$

and is associated with the approximation space

$$V_{n-1}^* = \text{span}\{N_1(x), \ldots, N_{n-1}(x)\}\,.$$

Problem 3.1 $(f(x) = 1)$
Consider five equal length elements



so that

$$V_6 = \text{span}\{N_i(x)\}_{i=0}^{5}\,.$$

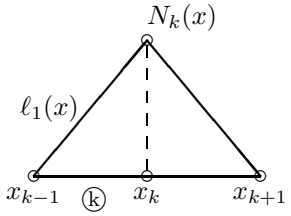
For computational convenience the Galerkin system coefficients

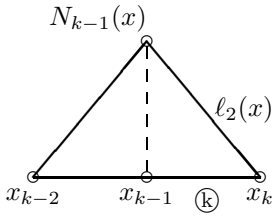
$$A_{ij} = \int_0^1 \frac{dN_j}{dx}\,\frac{dN_i}{dx}\,, \quad b_i = \int_0^1 N_i\,,$$

may be computed element–by–element. That is,

$$A_{ij} = \sum_{\text{k}=1}^{5} \underbrace{\int_{x_{k-1}}^{x_k} \frac{dN_j}{dx}\,\frac{dN_i}{dx}}_{\int_{x_{k-1}}^{x_k} \frac{d\ell_s}{dx}\,\frac{d\ell_t}{dx}}\,, \quad b_i = \sum_{\text{k}=1}^{5} \underbrace{\int_{x_{k-1}}^{x_k} N_i}_{\int_{x_{k-1}}^{x_k} \ell_t},$$

where s and t are local indices referring to the associated element basis functions illustrated below.





In particular, in element ⓚthere are two nodal basis functions

$$\ell_1(x) = \frac{(x - x_{k-1})}{(x_k - x_{k-1})}, \qquad \ell_2(x) = \frac{(x - x_k)}{(x_{k-1} - x_k)},$$

so that

$$\frac{d\ell_1}{dx} = \frac{1}{h}, \qquad \frac{d\ell_2}{dx} = -\frac{1}{h},$$

with $h = x_k - x_{k-1} = 1/5$. This generates a 2×2 "element contribution" matrix

$$A^{ⓚ} = \begin{bmatrix} \int_{x_{k-1}}^{x_k} (\frac{d\ell_1}{dx})^2 & \int_{x_{k-1}}^{x_k} (\frac{d\ell_2}{dx})(\frac{d\ell_1}{dx}) \\ \int_{x_{k-1}}^{x_k} (\frac{d\ell_1}{dx})(\frac{d\ell_2}{dx}) & \int_{x_{k-1}}^{x_k} (\frac{d\ell_2}{dx})^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{h} & -\frac{1}{h} \\ -\frac{1}{h} & \frac{1}{h} \end{bmatrix}$$

and a (2×1) "element contribution" vector

$$b^{ⓚ} = \begin{bmatrix} \int_{x_{k-1}}^{x_k} \ell_1 \\ \int_{x_{k-1}}^{x_k} \ell_2 \end{bmatrix} = \begin{bmatrix} \frac{h}{2} \\ \frac{h}{2} \end{bmatrix}.$$

Summing over the element contributions gives the stiffness matrix and the load vector,

$$A = \underbrace{\begin{bmatrix} \frac{1}{h} & -\frac{1}{h} & 0 & 0 & 0 & 0 \\ -\frac{1}{h} & \frac{1}{h} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A^{①}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{h} & -\frac{1}{h} & 0 & 0 & 0 \\ 0 & -\frac{1}{h} & \frac{1}{h} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A^{②}} +$$

$$\dots + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{h} & -\frac{1}{h} \\ 0 & 0 & 0 & 0 & -\frac{1}{h} & \frac{1}{h} \end{bmatrix}}_{A^{⑤}}$$

$$b = \begin{bmatrix} \frac{h}{2} \\ \frac{h}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{h}{2} \\ \frac{h}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{h}{2} \\ \frac{h}{2} \end{bmatrix}.$$

Thus, the "assembled " Galerkin system is

$$\underbrace{\begin{bmatrix} \frac{1}{h} & -\frac{1}{h} & 0 & 0 & 0 & 0 \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & 0 & 0 & 0 \\ 0 & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & 0 & 0 \\ 0 & 0 & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & 0 \\ 0 & 0 & 0 & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ 0 & 0 & 0 & 0 & -\frac{1}{h} & \frac{1}{h} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} \frac{h}{2} \\ h \\ h \\ h \\ h \\ \frac{h}{2} \end{bmatrix}}_{b}.$$

Note that this system is singular (and inconsistent!). All the columns sum to zero, so that $A\mathbf{1} = \mathbf{0}$. This problem arises because we have not yet imposed the essential boundary conditions. To do this we simply need to remove $N_0(x)$ and $N_5(x)$ from the basis set, that is, delete the first and last row and column from the system. Doing this gives the nonsingular reduced system

$$\underbrace{\begin{bmatrix} \frac{2}{h} & -\frac{1}{h} & 0 & 0 \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & 0 \\ 0 & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ 0 & 0 & -\frac{1}{h} & \frac{2}{h} \end{bmatrix}}_{A^*} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}}_{x^*} = \underbrace{\begin{bmatrix} h \\ h \\ h \\ h \end{bmatrix}}_{b^*}.$$

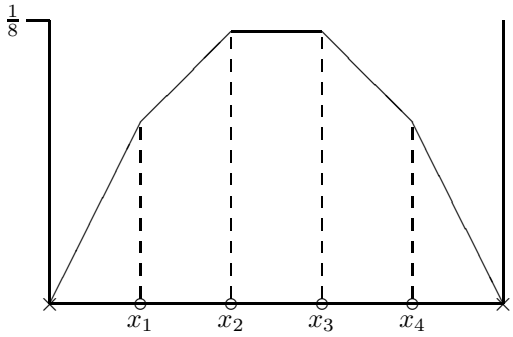
Setting $h = 1/5$, and solving gives

$$\alpha_1 = \alpha_4 = 0.08; \quad \alpha_2 = \alpha_3 = 0.12;$$

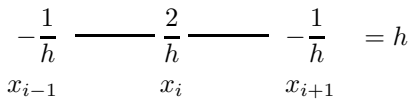
so the Galerkin finite element solution is

$$u_4^*(x) = 0.08N_1(x) + 0.12N_2(x) + 0.12N_3(x) + 0.08N_4(x).$$

It is illustrated below. Note that $u_4^*(x_i) = u(x_i)$ so the finite element solution is exact at the nodes! (It is not exact at any point between the nodes though.)



Note also that the generic equation



corresponds to a centered finite difference approximation to $-\frac{d^2u}{dx^2} = 1$.

To complete the discussion we would like to show that the finite element solution converges to the weak solution in the limit $h \to 0$.

Theorem 3.1 *(Convergence in the energy norm)*
If u is the solution of (V) and u_k is the finite element solution based on linear approximation then

$$\|u - u_k\|_E \leq h\|f\|$$

where h is the length of the longest element in the subdivision (which does not have to be uniform.)

Proof. We now formally introduce the linear interpolant, $u^* \in V_k$ of the exact solution, so that

$$u(x_i) = u^*(x_i) \qquad i = 0, 1, 2, \ldots, n\,.$$

Note that we cannot assume that $u_k = u^*$ in general. Introducing $e(x) = u(x) - u^*(x)$, we see that $e \in V_k$ and that

$$e(x_i) = 0, \qquad i = 0, 1, 2, \ldots, n\,.$$

We can now bound the element interpolation error using standard tools from approximation theory

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} \left(\frac{de}{dx}\right)^2 &\leq (x_i - x_{i-1})^2 \int_{x_{i-1}}^{x_i} \left(\frac{d^2e}{dx^2}\right)^2 \\
&= (x_i - x_{i-1})^2 \int_{x_{i-1}}^{x_i} \left(\frac{d^2u}{dx^2}\right)^2 \\
&\leq h^2 \int_{x_{i-1}}^{x_i} \left(\frac{d^2u}{dx^2}\right)^2
\end{aligned}$$

where $h = \max_i |x_i - x_{i-1}|$. Summing over the intervals then gives the estimate

$$\int_0^1 \left(\frac{de}{dx}\right)^2 \leq h^2 \int_0^1 \left(\frac{d^2u}{dx^2}\right)^2 .$$

Finally, using B–A

$$\|\frac{d}{dx}(u-u_k)\| \le \|\frac{d}{dx}(u-u^*)\| \le h\|\frac{d^2u}{dx^2}\| = h\|f\| < \infty.$$

Thus $\lim_{h\to 0} u_k = u$ in the energy norm. □

To get an error estimate in L_2 we use a very clever "duality argument".

Theorem 3.2 *(Aubin–Nitsche)*
If u is the solution of (V) and u_k is the finite element solution based on linear approximation then

$$\|u-u_k\| \le h^2\|f\|\,.$$

Proof. Let w be the solution of the dual problem

$$-\frac{d^2w}{dx^2} = u-u_k \quad x\in(0,1); \qquad w(0)=0=w(1).$$

Then we have

$$\begin{aligned}
\|u-u_k\|^2 &= (u-u_k, u-u_k)\\
&= (u-u_k, -\frac{d^2w}{dx^2})\\
&= (\frac{d}{dx}(u-u_k), \frac{dw}{dx}) \qquad (\text{since } w(0)=w(1)=0)\\
&= (\frac{d}{dx}(u-u_k), \frac{dw}{dx}) - (\frac{d}{dx}(u-u_k), \frac{dw^*}{dx}), \quad w^*\in V_k, \qquad G\text{–}O
\end{aligned}$$

where w^* is the interpolant of w in V_k. Hence

$$\begin{aligned}
\|u-u_k\|^2 &= \left(\frac{d}{dx}(u-u_k), \frac{d}{dx}(w-w^*)\right)\\
&\le \|\frac{d}{dx}(u-u_k)\| \underbrace{\|\frac{d}{dx}(w-w^*)\|}_{} \qquad C\text{–}S\\
&\le h\,\|\frac{d^2w}{dx^2}\| = h\,\|u-u_k\|.
\end{aligned}$$

Hence, assuming that $\|u-u_k\| > 0$, we have that

$$\|u-u_k\| \le h\,\|\frac{d}{dx}(u-u_k)\| \le h^2\|f\|,$$

as required. □

A more complete discussion of these issues can be found in Chapters 11 and 14 of the following reference book.

- ENDRE SÜLI & DAVID MAYERS, *An Introduction to Numerical Analysis*, Cambridge University Press, 2003.