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Graphical characterization of positive definite non symmetric quasi-Cartan matrices

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ABSTRACT

It is known that each positive definite quasi-Cartan matrix A is $\mathbb{Z}$-equivalent to a Cartan matrix A_Δ called Dynkin type of A, the matrix A_Δ is uniquely determined up to conjugation by permutation matrices. However, in most of the cases, it is not possible to determine the Dynkin type of a given connected quasi-Cartan matrix by simple inspection. In this paper, we give a graph theoretical characterization of non-symmetric connected quasi-Cartan matrices. For this purpose, a special assemblage of blocks is introduced. This result complements the approach proposed by Barot (1999, 2001), for $\mathbb{A}_n$, $\mathbb{D}_n$ and $\mathbb{E}_m$ with $m = 6, 7, 8$.

1. Introduction and results

Quasi-Cartan matrices are present in many areas of mathematics. The motivation is based on the classical theory of complex semi-simple Lie algebras, (see [12]). These algebras can be characterized by a base of the root system from which a Cartan matrix is obtained. A *symmetrizer* of a matrix A is an integer diagonal matrix D with positive diagonal entries such that DA is symmetric. If A has a symmetrizer D then A is called *symmetrizable* (D is not unique). Following [6], by a *quasi-Cartan* matrix of size $n \geq 2$ we mean a square $n \times n$ matrix $A = [A_{ij}] \in \mathbb{M}_n(\mathbb{Z})$ with integer coefficients A_{ij} such that A is symmetrizable and $A_{ii} = 2$, for all i. The set of all quasi Cartan matrices $A \in \mathbb{M}_n(\mathbb{Z})$ is denoted by $\mathbf{qC}$. We say that a matrix $A \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ is *positive definite*, if the symmetric matrix $DA \in \mathbb{M}_n(\mathbb{Z}) \subset \mathbb{M}_n(\mathbb{R})$ is positive definite, for some symmetrizer D. The set of all positive definite quasi-Cartan matrices $A \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ is denoted by $\mathbf{qC}^+$. We note that a matrix $A \in \mathbf{qC}^+$ is a Cartan matrix if $A_{ij} \leq 0$ for all pairs i, j with $i \neq j$. The quasi-Cartan matrices $A, A' \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ are defined to be $\mathbb{Z}$-*equivalent* (we denote it by $A \sim A'$) if there exists a $\mathbb{Z}$-invertible matrix $E \in \mathbb{M}_n(\mathbb{Z})$ and symmetrizers $D, D' \in \mathbb{M}_n(\mathbb{Z})$ such that $D'A' = E^t(DA)E$ and D' is conjugate to D by a permutation matrix. For general purposes, it will be convenient to switch to a more graphical language.

Following [7], by a *mixed graph* we mean the triple $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that consists of a set $\mathcal{V} \neq \emptyset$ of vertices, a set $\mathcal{E}$ of edges (undirected) and a set $\mathcal{A}$ of arrows. In this paper, a *bigraph* B is a mixed graph G together with a function $\omega : \mathcal{E} \cup \mathcal{A} \to \mathbb{Z}$ that assigns to every $e \in \mathcal{E} \cup \mathcal{A}$ an integer number called the *weight* of e. A vertex $v \in \mathcal{V}$ is a *source* (respectively *sink*) vertex if for all $a_{ij} \in \mathcal{A}$ the vertex i (respectively j) is equal to v and $e_{vj}, e_{iv} \notin \mathcal{E}$.

Definition 1.1. To each quasi-Cartan matrix $A \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$, with $n \geq 2$, we associate its bigraph $B_A = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \omega)$, with n vertices, as follows:

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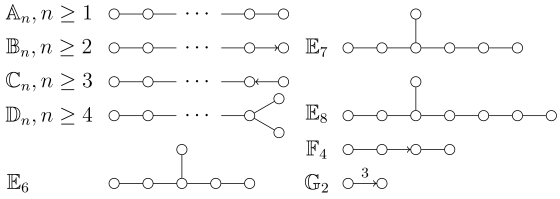


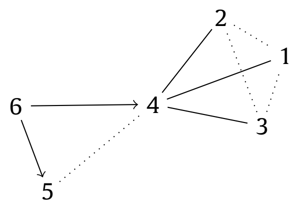
Fig. 1. Dynkin diagrams.

- $\mathcal{V} = \{1, 2, \dots, n\}$
- $\mathcal{E} = \{e_{ij} \mid i, j \in \mathcal{V}$ with $i \neq j$ and $|A_{ij}| = |A_{ji}| \neq 0\}$
- $\mathcal{A} = \{a_{ij} \mid i, j \in \mathcal{V}$ with $i \neq j$ and $|A_{ij}| < |A_{ji}|\}$
- for each $e \in \mathcal{E} \cup \mathcal{A}$, we set $\omega(e) = A_{ji}$, where $|A_{ij}| \leq |A_{ji}|$.

Notice that from Definition 1.1 the sets $\mathcal{E}$ and $\mathcal{A}$ are disjoint. A *path* P in B_A from vertex v_1 to vertex v_r is a subgraph $P = v_1 v_2 \dots v_r$ induced in B_A by the set of vertices $v_i \in \mathcal{V}$ where for all i, $1 \leq i \leq r$ the vertices v_i are pairwise distinct and there exists $e \in \mathcal{E} \cup \mathcal{A}$ between the vertices v_i and v_{i+1}. We say that B_A is *connected* if there exists a path from v_i to v_j for all $v_i, v_j \in \mathcal{V}$ [7]. A *chordless cycle* is a connected induced sub-bigraph such that every vertex is adjacent with exactly two vertices. A bigraph B_A satisfies the *chordless cycle condition* if every induced chordless cycle of B_A has an odd number of dotted connections (edges or arrows). Every bigraph B_A associated to a quasi-Cartan matrix A can be represented as a diagram of dots (vertices in B_A), lines and arrows (solid and dotted). All edges and arrows are represented as follows: if $e_{ij} \in \mathcal{E}$ then e_{ij} is indicated by a dotted line with weight ω, $i \overset{\omega}{\cdots\cdots} j$ if $\omega(e_{ij}) > 0$, and solid $i \overset{\omega}{\text{——}} j$ if $\omega(e_{ij}) < 0$. Similarly for $a_{ij} \in \mathcal{A}$, a_{ij} is indicated by a dotted arrow $i \overset{\omega}{\cdots\!\!>} j$ if $\omega(a_{ij}) > 0$, and solid $i \xrightarrow{\omega} j$ if $\omega(a_{ij}) < 0$. We denote by $\Phi(A)$ the *frame* of a quasi-Cartan matrix A, that is, the graph obtained from B_A by turning all broken edges and broken arrows into solid ones [3]. A frame $\Phi(A)$ is called *positive* if A is a positive definite matrix. Throughout this paper, all the solid (dotted) arrows are considered with $\omega = -2$ ($\omega = 2$) unless otherwise indicated, and no distinction is made between the bigraph B_A and its diagram.

Example 1.2. A quasi-Cartan matrix and its associated bigraph.

$$\begin{pmatrix} 2 & 1 & 1 & -1 & 0 & 0 \\ 1 & 2 & 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$



If A is a Cartan matrix, the bigraph B_A is actually a bigraph with $\omega(e) < 0$ for all $e \in \mathcal{E} \cup \mathcal{A}$, moreover if A is connected (i.e. B_A is connected) then B_A is known as *Dynkin diagram* (see Fig. 1). From now on, we will only consider connected matrices.

If $A' \in \mathbb{M}_n(\mathbb{Z})$ is a connected quasi-Cartan in $\mathbf{qC}^+$, and A_Δ is a Cartan matrix such that $A' \sim A_\Delta$, then Δ will be referred to be the *Dynkin type* of B_A, that is, the Dynkin diagram associated to A_Δ . The existence of the Cartan matrix A_Δ such that $A' \sim A_\Delta$ will be proved in the Section 2, see also [13], a proof for the symmetric case is given in [8]. It follows that two connected matrices in $\mathbf{qC}^+$ with the same Dynkin type are $\mathbb{Z}$-equivalent; therefore, it is important to have a simple characterization of positive definite connected quasi-Cartan matrices. For this purpose we study in the following paragraph some graphical and combinatorial aspects for the various parameters characterizing the Dynkin types of positive definite connected quasi-Cartan matrices.

Let X and Y be disjoint sets of vertices. We denote by $F[X, Y]$ the non-separable bigraph obtained by joining each pair of vertices x, y with $x \in X$ and $y \in Y$ by a solid edge, and all other pairs of vertices by a dotted edge; such bigraph is called an $\mathbb{A}$-*block*, see [2], [1]. If v is a vertex in $F[X, Y]$ and $|X \cup Y| \geq 2$, we denote by $\overrightarrow{F_v}[X, Y]$ ($\overleftarrow{F_v}[X, Y]$) and we call $\mathbb{B}$-block ($\mathbb{C}$-block) to the bigraph obtained from $F[X, Y]$ after substituting every solid or dotted edge over v by a solid or dotted arrow pointing to (coming out of) the vertex v. The vertex v is the sink (source) vertex of $\overrightarrow{F_v}[X, Y]$ ($\overleftarrow{F_v}[X, Y]$). In both cases, we call to vertex v a *distinguished vertex*. (See Fig. 2.)

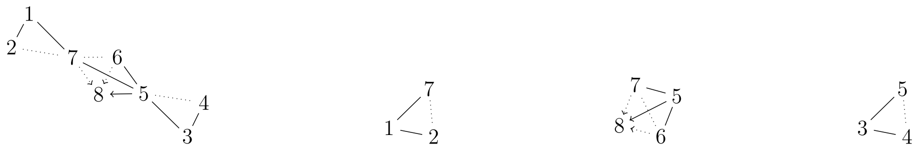
Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A}, \omega)$, $G' = (\mathcal{V}', \mathcal{E}', \mathcal{A}', \omega')$. Then, we define the sum of G and G' by $G \oplus G' = (\mathcal{V} \cup \mathcal{V}', \mathcal{E}'', \mathcal{A}'', \omega'')$ where:

$$\omega''(e) = \begin{cases} \omega(e), & \text{if } e \in (\mathcal{E} \setminus \mathcal{E}') \cup (\mathcal{A} \setminus \mathcal{A}') \\ \omega'(e), & \text{if } e \in (\mathcal{E}' \setminus \mathcal{E}) \cup (\mathcal{A}' \setminus \mathcal{A}) \\ \omega'(e) + \omega(e), & \text{if } e \in (\mathcal{E} \cap \mathcal{E}') \cup (\mathcal{A} \cap \mathcal{A}') \end{cases}$$

$\mathcal{E}'' = (\mathcal{E} \cup \mathcal{E}') \setminus \{e \in \mathcal{E} \cap \mathcal{E}' \mid \omega'(e) + \omega(e) = 0\}$ and $\mathcal{A}'' = (\mathcal{A} \cup \mathcal{A}') \setminus \{e \in \mathcal{A} \cap \mathcal{A}' \mid \omega'(e) + \omega(e) = 0\}$.

(a) $\overleftarrow{F_v}$ [X,Y]. (b) $\overrightarrow{F_v}$ [X,Y].

Fig. 2. $\mathbb{C}$-block and $\mathbb{B}$-block.



(a) G. (b) A_1. (c) B_1. (d) A_2.

Fig. 3. Decomposition of the $\mathbb{B}$-assemblage G into blocks.

Remark 1.3. The operation $\oplus$ is commutative and associative.

The assemblage of $\mathbb{A}$-blocks was introduced by Barot in [2]. Abarca and Rivera [1] reinterpreted this concept to define the $\mathbb{A}$-block tree. Now we present a general class of assemblages in terms of the bigraphs sum defined earlier.

Let $G_1, G_2, \ldots, G_n$ be bigraphs, then the *assemblage* of the bigraphs G_i, $i \in \{1, \ldots, n\}$, is defined as follows:

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_n.$$

Let $G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$ be the connected bigraph composed by the assemblage of the $\mathbb{A}$-blocks $G_1, G_2, \ldots, G_{n-1}$ and the $\mathbb{B}$-block G_n, such that the biconnected components of G are exactly $G_1, G_2, \ldots, G_n$, every separating vertex is the intersection of exactly two blocks and the distinguished vertex is not a separating vertex. We will call the resulting bigraph, $\mathbb{B}$-*assemblage* (see Fig. 3). Under the assumptions made earlier, we define the $\mathbb{C}$-*assemblage*, by replacing $\mathbb{B}$-block with $\mathbb{C}$-block.

The following theorem is the main result of this paper.

Theorem 1.4. *Let $A \in$ **qC**. Then A is positive definite and has Dynkin type $\mathbb{B}_n$ ($\mathbb{C}_n$) if and only if B_A is a $\mathbb{B}$-assemblage ($\mathbb{C}$-assemblage).*

2. $\mathbb{Z}$-equivalent quasi-Cartan matrices

Each positive definite quasi-Cartan matrix $A \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ gives rise to the integral quadratic form q_{DA} given by $q_{DA}(x) = \frac{1}{2}x^t DAx$ on $\mathbb{Z}^n$, where D is a fixed symmetrizer of A. If $D = diag(d_1, \ldots, d_n)$, then the set $R(q_{DA}) = \bigcup_{i=1}^{N} q_{DA}^{-1}(d_i)$ is finite and the set:

$$P(q_{DA}) = \{x \in R(q_{DA}) \mid x_i \geq 0 \text{ for all } i = 1, \ldots, n\},$$

is known as the set of positive roots of q_{DA} (see [4]). Given an integer σ, we denote by $E_{sr}^{\sigma} = I + \sigma e_s e_r^t$ where I is the identity matrix of size $n \times n$ and e_j the jth basic vector of the group $\mathbb{Z}^n$.

Lemma 2.1. *If $A \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ and D a symmetrizer of A, then:*

$$D^{-1}(E_{rs}^{-A_{sr}})D = E_{rs}^{-A_{rs}}.$$

Proof. Let $B = D^{-1}(E_{rs}^{-A_{sr}})D$. Then $(B)_{rs} = -d_r^{-1}d_sA_{sr}$ and $-A_{sr}d_s = -A_{rs}d_r$, yields $-A_{sr}d_sd_r^{-1} = -A_{rs}$. $\square$

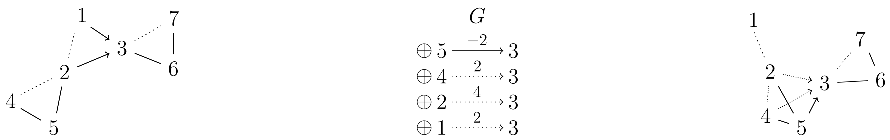
Remark 2.2. For any matrix $A \in \mathbf{qC}^+ \subseteq \mathbb{M}_n(\mathbb{Z})$, the equality $A' = E_{rs}^{-A_{rs}}AE_{sr}^{-A_{sr}}$ holds if and only if $A' = D^{-1}(E_{sr}^{-A_{rs}})^tDAE_{sr}^{-A_{sr}}$. To simplify the notation, we set

$$T_{sr}^{\sigma}(A) := D^{-1}(E_{sr}^{\sigma})^tDAE_{sr}^{\sigma},$$

in particular, if $\sigma = -A_{sr}$ then

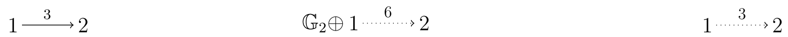
$$T_{sr}^{-A_{sr}}(A) = E_{rs}^{-A_{rs}}AE_{sr}^{-A_{sr}}.$$

Following Barot-de la Peña [5] and von Höhne [9] (see also Ovsienko [13]), the operator $A \to T_{sr}^{\sigma}(A)$ is called an *inflation* if $\sigma < 0$ and *deflation* if $\sigma > 0$. In general, $A \to T_{sr}^{\sigma}(A)$ is called a *flation* on A. It is easy to check that the operator $A \to T_{sr}^{\sigma}(A)$



(a) G. (b) Theorem 3.1. (c) $T_{23}^{-}(G)$.

Fig. 4. Applying an inflation over the bigraph G.



(a) $\mathbb{G}_2$. (b) $T_{12}^{-1}(\mathbb{G}_2)$. (c) $T_{21}^{-3}(\mathbb{G}_2)$.

Fig. 5. Applying all the possible flations over the bigraph $\mathbb{G}_2$.

carries any matrix $A \in \mathbf{qC}^+$ to a matrix in $\mathbf{qC}^+$. If $A' = T_{s_1r_1}^{-A_{s_1r_1}}(A)$, then the composition of the flation operators $T_{s_1r_1}^{-A_{s_1r_1}}$ and $T_{s_2r_2}^{-A'_{s_2r_2}}$ is denoted by $T = T_{s_2r_2}^{-A'_{s_2r_2}} \circ T_{s_1r_1}^{-A_{s_1r_1}}$ and defined by

$$T(A) = E_{r_2s_2}^{-A'_{r_2s_2}} E_{r_1s_1}^{-A_{r_1s_1}} A E_{s_1r_1}^{-A_{s_1r_1}} E_{s_2r_2}^{-A'_{s_2r_2}}.$$

We include a proof based on Theorem 8.26, [4] for convenience to the reader (see also [13] and [8]).

Theorem 2.3. *For each positive definite connected quasi-Cartan matrix A there exist a composite flation operator T and a Cartan matrix A_Δ of the form $A_\Delta = T(A)$ such that $A \sim A_\Delta$.*

Proof. If there exists a pair s, r, with $r \neq s$ and $A_{sr} > 0$, we set $q := q_{DA}$, $A' = T_{sr}^{-A_{sr}}(A)$ and $q' = q_{DA'}$. Then $T_{sr}^{A_{sr}}$ carries bijectively the finite set $R(q)$ to the finite set $R(q')$, and carries injectively the finite set $P(q)$ into $P(q')$. Since $e_r = T_{sr}^{A_{sr}}(e_r - A_{sr}e_s) \in P(q')$ but $e_r - A_{sr}e_s \notin P(q)$, there are less elements in $P(q)$ than in $P(q')$. Continuing this procedure, we obtain a Cartan matrix $A_\Delta = T(A)$, where T is the composition of the flations T_{sr}. Here we use the fact that at each iteration the cardinality of the finite set $P(q')$ is bounded by the cardinality of $R(q)$, but increases in each step. It follows that the procedure terminates and ends up with a Cartan matrix A_Δ. □

Pérez, Abarca, and Rivera [14] showed that the inflation algorithm runs in $O(n^6)$ and proposed an algorithm to decide whether an admissible quasi-Cartan matrix is positive definite and compute the Dynkin type in just $O(n^3)$.

3. Operations over bigraphs associated to quasi-Cartan matrices

Let A be a non symmetric quasi-Cartan matrix. The main aim of this section is to study how the inflation operators act directly over the bigraph associated to A by terms of the sum of bigraphs. Let $b_{ri} = -A_{rs}A_{si}$ and $b_{ir} = -A_{is}A_{sr}$ for all $i \neq r$ and $G_{\{i,r\}}^{b_i}$ the bigraph associated to an edge or arrow between the vertices i and r where $b_i = \{b_{ir}, b_{ri}\}$. If $b_{ir} = b_{ri} \neq 0$ then $G_{\{i,r\}}^{b_i} := i \overset{b_{ri}}{\text{—}} r$ (respectively $i \overset{b_{ri}}{\cdots} r$) if $b_{ir} < 0$ ($b_{ir} > 0$). If $|b_{ri}| > |b_{ir}|$, $G_{\{i,r\}}^{b_i} := i \xrightarrow{b_{ri}} r$ ($i \overset{b_{ri}}{\dashrightarrow} r$) if $b_{ir} < 0$ ($b_{ir} > 0$), similarly if $|b_{ir}| > |b_{ri}|$, $G_{\{i,r\}}^{b_i} := r \xrightarrow{b_{ir}} i$ ($r \overset{b_{ir}}{\dashrightarrow} i$) if $b_{ir} < 0$ ($b_{ir} > 0$).

Let $A \in \mathbf{qC}$ then each flation could be graphically interpreted as follows:

Theorem 3.1. *Assume that $A \in \mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ and B_A is its associated bigraph. Then*

$$T_{sr}^{-A_{sr}}(B_A) = B_A \oplus \bigoplus_{i \neq r, b_{ir} \neq 0} G_{\{i,r\}}^{b_i}.$$

Let G the bigraph in Fig. 4(a). Note that by Theorem 3.1, $T_{23}^{-}(G) = G \oplus G_{\{1,3\}}^{\{1,2\}} \oplus G_{\{2,3\}}^{\{2,4\}} \oplus G_{\{4,3\}}^{\{1,2\}} \oplus G_{\{5,3\}}^{\{-1,-2\}}$. (See Fig. 4.)

Proof. By computing $T_{sr}^{-A_{sr}}(A) = A'$ we have:

$$A'_{ij} = \begin{cases} A_{ij} & \text{if } i \neq r, j \neq r \\ A_{rj} - A_{rs}A_{sj} & \text{if } i = r, j \neq r \\ A_{ir} - A_{sr}A_{is} & \text{if } i \neq r, j = r \end{cases}$$

the rest comes directly from the sum of graphs properties. □

In the following sections we will address the problem to determine the Dynkin type of every connected, positive definite, non symmetric quasi-Cartan matrix. We will discuss the cases $\mathbb{B}_n$, $\mathbb{C}_n$ and $\mathbb{F}_4$ in detail. The case $\mathbb{G}_2$ is very simple (see Fig. 5).

(a) G. (b) $T_{45}^{-A_{45}}(G)$.

Fig. 6. Effects of the flations over $\mathbb{B}$-blocks.

4. Bigraphs of Dynkin type $\mathbb{B}_n$

In this section we use flation operations to characterize the Dynkin type $\mathbb{B}_n$. This characterization is written in the language of sum of bigraphs and the flations described before. Let $G - x$ denote the bigraph obtained from G by removing a vertex x and all edges e_{ix} and arrows a_{ix}, a_{xi}.

4.1. Flation properties

Lemma 4.1. *Let $G = \overrightarrow{F_v}[X, Y]$ be a $\mathbb{B}$-block. Then for all flation $T_{sr}^{-A_{sr}}$, $s, r \in X \cup Y$:*

(*a*) $T_{sr}^{-A_{sr}} G = (G - r) \oplus s \frac{\quad}{} r$ *if* $s, r \notin \{v\}$ *and* $A_{sr} > 0$, $T_{sr}^{-A_{sr}} G = (G - r) \oplus s \cdots r$ *if* $s, r \notin \{v\}$ *and* $A_{sr} < 0$.
(*b*) $T_{sr}^{-A_{sr}} G = (G - r) \oplus s \longrightarrow r$ *if* $r = v$ *and* $A_{sr} > 0$, $T_{sr}^{-A_{sr}} G = (G - r) \oplus s \cdots\!\!> r$ *if* $r = v$ *and* $A_{sr} < 0$.
(*c*) $T_{sr}^{-A_{sr}} G = \overrightarrow{F_v}[X \cup \{r\}, Y \setminus \{r\}]$ *if* $s = v$.

Proof.

(*a*) Assume that $s \neq v$ and $r \neq v$. Without loss of generality we can assume that $v, r \in Y$ and $A_{sr} > 0$. By applying $T_{sr}^{-A_{sr}}$ we obtain:

$$\begin{aligned} T_{sr}^{-A_{sr}}(G) &= G \oplus \bigoplus_{\substack{i \neq r \\ b_{ir} \neq 0}} G_{\{i,r\}}^{b_i}, \\ &= \overrightarrow{F_v}[X, Y] \oplus \bigoplus_{\substack{i \neq s,v \\ b_{ir} > 0}} i \cdots r \oplus \bigoplus_{\substack{i \neq s,v \\ b_{ir} < 0}} i \frac{\quad}{} r \oplus r \longrightarrow v \oplus G_{\{s,r\}}^{b_s}, \\ &= \overrightarrow{F_v}[X, Y] \oplus \bigoplus_{i \in X} i \cdots r \oplus \bigoplus_{i \in Y \setminus \{r,s,v\}} i \frac{\quad}{} r \oplus r \longrightarrow v \oplus s \overset{2}{\frac{\quad}{}} r, \\ &= \overrightarrow{F_v}[X, Y \setminus \{r\}] \oplus s \cdots r \oplus s \overset{2}{\frac{\quad}{}} r, \\ T_{sr}^{-A_{sr}}(G) &= (\overrightarrow{F_v}[X, Y] - r) \oplus s \frac{\quad}{} r. \end{aligned}$$

(*b*) If $r = v$ and assume that $A_{sr} > 0$, then by applying the same procedure we have:

$$T_{sr}^{-A_{sr}}(G) = \overrightarrow{F_v}[X, Y \setminus \{r\}] \oplus s \cdots\!\!> r \oplus G_{\{s,r\}}^{b_s}, \tag{4.1}$$

from which the result follows. (See Fig. 6.)

(*c*) If $s = v$ for all edge or arrow between the vertices i and r with weight ω there is an edge or arrow between the same vertices with weight -2ω. Therefore the result follows. □

Lemma 4.2. *Let $G = \overrightarrow{F_v} \oplus F$ be a $\mathbb{B}$-*assemblage, *where $\overrightarrow{F_v}$ is a $\mathbb{B}$-block, F is an $\mathbb{A}$-block and s is the separation vertex, then:*

- *if* $r \neq v$, $T_{sr}^{-A_{sr}}(G) = (\overrightarrow{F_v} - r) \oplus F'$ *where* $F' - r = F$,
- *if* $r = v$, $T_{sr}^{-A_{sr}}(G) = \overrightarrow{F'_v} \oplus F''$ *where* $\overrightarrow{F'_v} - r = F$, $F'' = \overrightarrow{F_v} - r$.

Proof. Assume that $\overrightarrow{F_v} = \overrightarrow{F_v}[X, Y]$, $F = F[X', Y']$, $r \in X \cup Y - \{v\}$, and $s \in X \cap X'$, then, if $A_{sr} < 0$:

$$\begin{aligned} T_{sr}^{-A_{sr}}(G) &= \overrightarrow{F_v}[X, Y] \oplus F[X', Y'] \oplus \bigoplus_{i \neq r, b_{ir} \neq 0} G_{\{i,r\}}^{b_i}, \\ &= \overrightarrow{F_v}[X, Y] \oplus F[X', Y'] \oplus \bigoplus_{i \neq r, i \in X \cup Y} G_{\{i,r\}}^{b_i} \oplus \bigoplus_{j \in X' \cup Y'} G_{\{j,r\}}^{b_j}, \end{aligned}$$

$$= (\overrightarrow{F_v}[X,Y]-r) \oplus \bigoplus_{j\in X'\cup Y'} G^{b_j}_{\{j,r\}} \oplus F[X',Y'],$$

$$= (\overrightarrow{F_v}[X,Y]-r) \oplus F[X'\cup\{r\},Y']$$

from which it can be chosen $F[X'\cup\{r\},Y']$ as F', note that if $A_{sr} > 0$ then $F' = F[X', Y'\cup\{r\}]$. If $r = v$ by choosing $F'' = \overrightarrow{F_v}[X,Y]-r$ and $\overrightarrow{F'_v} = F[X'\cup\{r\},Y']$ the result follows. □

4.2. Proof of the main result for $\mathbb{B}_n$

First we show that the existence of a $\mathbb{B}$-assemblage is invariant under flation operators, i.e. that if G is a $\mathbb{B}$-assemblage then $T^{\sigma}_{sr}G$ is also a $\mathbb{B}$-assemblage.

Lemma 4.3. *If A is a quasi-Cartan matrix such that B_A is a $\mathbb{B}$-assemblage, then $T^{-A_{sr}}_{sr}(B_A)$ is a $\mathbb{B}$-assemblage.*

Proof. Since B_A is a $\mathbb{B}$-assemblage, we set:

$$B_A = G \oplus F[X',Y'] \oplus \overrightarrow{F_v}[X,Y],$$

where $\overrightarrow{F_v}[X,Y]$ is the $\mathbb{B}$-block in B_A, $F[X',Y']$ is an $\mathbb{A}$-block, G and $G\oplus F[X',Y']$ are $\mathbb{A}$-assemblages, and $F[X',Y'] \oplus \overrightarrow{F_v}[X,Y]$ is a $\mathbb{B}$-assemblage. If $s, r \in \mathcal{V}(G\oplus F[X',Y'])$ and $s \notin X\cup Y$, then we have:

$$T^{-A_{sr}}_{sr}(B_A) = T^{-A_{sr}}_{sr}(G\oplus F[X',Y']) \oplus \overrightarrow{F_v}[X,Y].$$

In view of the results of [2] and [1], $T^{-A_{sr}}_{sr}(G\oplus F[X',Y'])$ is also an $\mathbb{A}$-assemblage. Since s is not a separation vertex, then $T^{-A_{sr}}_{sr}(B_A)$ is a $\mathbb{B}$-assemblage.

If $s, r \in (X\cup Y)$ and $s \in X'\cup Y'$ is a separation vertex, then:

$$T^{-A_{sr}}_{sr}(B_A) = G \oplus T^{-A_{sr}}_{sr}(F[X',Y'] \oplus \overrightarrow{F_v}[X,Y]).$$

Without loss of generality we can assume that $s \in X'$ and $r \in Y$. In view of Lemma 4.2, if $A_{sr} < 0$ then

$$T^{-A_{sr}}_{sr}(B_A) = G \oplus (F[X'\cup\{r\},Y'] \oplus (\overrightarrow{F_v}[X,Y]-r)),$$

and therefore $T^{-A_{sr}}_{sr}(B_A)$ is a $\mathbb{B}$-assemblage. Similarly, if $s, r \in (X'\cup Y')$ and $s \in X\cup Y$ is a separation vertex, then the assumptions $s \in X$ and $v \in Y'$ yield.

$$T^{-A_{sr}}_{sr}(B_A) = G \oplus ((F[X',Y']-r) \oplus \overrightarrow{F_v}[X\cup\{r\},Y]).$$

If $s, r \in X\cup Y$ and s, r are not separation vertices, by applying Lemma 4.1 the result follows, that is, the bigraph $T^{-A_{sr}}_{sr}(B_A)$ is a $\mathbb{B}$-assemblage. □

Theorem 4.4. *Assume that A is a quasi-Cartan matrix such that B_A is a $\mathbb{B}$-assemblage and T is a composition of flations, then $T(B_A)$ is a $\mathbb{B}$-assemblage.*

Proof. This result follows easily by induction over the number of flations in composition T. We proved the base case, and the inductive step, in Lemma 4.3. □

Corollary 4.5. *Assume that A is a quasi-Cartan matrix with Dynkin type $\mathbb{B}_n$, then B_A is a $\mathbb{B}$-assemblage.*

Proof. If B_A is a bigraph with Dynkin type $\Delta = \mathbb{B}_n$, then by Theorem 2.3 (see also [4], [8] and [13]) there exists a composition of flations T such that $B_A = T(\Delta)$. Since Δ is a $\mathbb{B}$-assemblage itself, then by Theorem 4.4, B_A is a $\mathbb{B}$-assemblage. □

We have one implication of Theorem 4.8. For the converse we start proving that every $\mathbb{B}$-*block* has Dynkin type $\mathbb{B}_n$.

Lemma 4.6. *Every $\mathbb{B}$-block G has Dynkin type $\mathbb{B}_n$.*

Proof. Assume that $G = \overrightarrow{F_v}[X,Y]$, $v \in Y$, and $X \neq \emptyset$, $X = \{x_1,\dots,x_k\}$. By applying $T^{-1}_{x_2x_1}$ over G we get:

$$T^{-1}_{x_2x_1}(G) = \overrightarrow{F_v}[X\setminus\{x_1\},Y] \oplus x_1 \text{---} x_2\,,$$

this result can be generalized by induction applying successively the operators $T^{-1}_{x_ix_{i-1}}$ to G as follows:

$$T^{-1}_{x_kx_{k-1}}\dots T^{-1}_{x_2x_1}(G) = \overrightarrow{F_v}[\{x_k\},Y] \oplus \bigoplus_{i\,\in\,[2\dots k]} x_{i-1}\text{---}x_i\,.$$

If $|Y| > 1$ and considering $Y = \{y_1, \ldots, y_{k'}, v\}$ using an analogous argument, $T^{-1}_{y_1y_2} \ldots T^{-1}_{y_{k'-1}y_{k'}} T^{-1}_{y_{k'}v} T^{-1}_{x_kx_{k-1}} \ldots T^{-1}_{x_2x_1}(G)$,

$$= F[\{x_k\}, \{y_1\}] \oplus \bigoplus_{i\,\in\,[2\ldots k],} x_{i-1} \text{---} x_i \oplus \bigoplus_{j\,\in\,[2\ldots k'],} y_{i-1} \text{---} y_i \oplus y_{k'} \longrightarrow v\,.$$

If $X = \emptyset$ applying directly $T^{-1}_{y_1y_2} \ldots T^{-1}_{y_{k'-1}y_{k'}} T^{-1}_{y_{k'}v}$ over G the result follows, that is, $\overrightarrow{F_v}[X, Y]$ has Dynkin type $\mathbb{B}_n$. $\square$

Theorem 4.7. *Let G be a $\mathbb{B}-$assemblage, then there exists a sequence of flations with composition T so that $T(G)$ is a $\mathbb{B}$-block.*

Proof. Let

$$G = \left[\bigoplus_{i\in[2\ldots k'],} F[X_i, Y_i] \right] \oplus \overrightarrow{F_v}[X, Y],$$

following [1] there exist X', Y' and a sequence of inflations with composition T so that $F[X', Y'] = T(\bigoplus i \in [2\ldots k'], F[X_i, Y_i])$. Note that if $s \in Y \cup Y'$ is the separation vertex between $F[X', Y']$ and $\overrightarrow{F_v}[X, Y]$, by the procedure in [[1], Lemma 3.6] this composition T always avoids s, then :

$$T(G) = F[X', Y'] \oplus \overrightarrow{F_v[X, Y]},$$

in view of Lemma 4.2:

$$T^1_{sx'}(G) = \overrightarrow{F_v}[X, Y \cup \{x'\}] \oplus F[X' \setminus \{x'\}, Y'].$$

By applying this procedure for all x' in $X' \cup Y'$ respectively, the $\mathbb{B}$-assemblage can be embedded into one single $\mathbb{B}$-*block*. The rest of the proof could be easily deduced from Lemma 4.6. $\square$

Kasjan and Simson in [10] proposed a result related with Theorem 4.7 in which they affirmed that if a connected positive definite quasi-Cartan matrix A such that B_A has a sink vertex, then it can be reduced by inflations to a Dynkin diagram of type $\mathbb{B}_n$.

The following theorem is a direct consequence of all the results presented above and is the main result of this section.

Theorem 4.8. *A bigraph G has Dynkin type $\mathbb{B}_n$ if and only if it is a $\mathbb{B}$-assemblage.*

5. Bigraphs of Dynkin type $\mathbb{C}_n$

Lemma 5.1. *Let A be a positive quasi-Cartan matrix such that A^t is a positive quasi-Cartan matrix, and*

$$A' = T^{-A_{sr}}_{sr}(A),$$

then, the matrix $(A')^t$ has the same Dynkin type as A^t.

Proof.

$$\begin{aligned}
(A')^t &= (E^{-A_{rs}}_{rs} A E^{-A_{sr}}_{sr})^t\,,\\
&= (E^{-A_{sr}}_{sr})^t A^t (E^{-A_{rs}}_{rs})^t\,,\\
&= E^{-A_{sr}}_{rs} A^t E^{-A_{rs}}_{sr}\,,\\
&= E^{-A^t_{rs}}_{rs} A^t E^{-A^t_{sr}}_{sr}\,,\\
&= T^{-A^t_{sr}}_{sr}(A^t),
\end{aligned}$$

then the matrix $(A')^t$ has the same Dynkin type as A^t. $\square$

Lemma 5.2. *If A and A^t are positive quasi-Cartan matrices and $\{T^{\sigma_k}_{ij}\}$ is a sequence of flations with composition T such that:*

$$A' = T(A)\,, \tag{5.1}$$

then, the matrix $(A')^t$ has the same Dynkin type as A^t.

Proof. The proof can be realized by applying induction over the number of elements in the sequence $\{T^{\sigma_k}_{ij}\}$. The base case and the inductive step were obtained in Lemma 5.1. $\square$

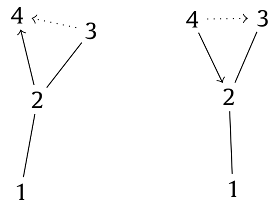
Theorem 5.3. *Any positive quasi-Cartan matrix A has Dynkin type $\mathbb{B}_n$ if and only if its transpose A^t has Dynkin type $\mathbb{C}_n$.*

Proof. Since $A^t_{\Delta} = A_{\Delta'}$ where $\Delta = \mathbb{B}_n$, $\Delta' = \mathbb{C}_n$, and by Lemma 5.2 the result follows. $\square$

Let $A \in \mathbf{qC}$ and B_A be a $\mathbb{B}$-assemblage and v the distinguished vertex. Then $(A^t)_{ij} = A_{ji} = A_{ij}$ if $i, j \neq v$, $|(A^t)_{vj}| = |A_{jv}| = 1$ and $|(A^t)_{jv}| = |A_{vj}| = 2$. From which B_{A^t} remains the same as B_A, but, with the arrows in an opposite direction.

Example 5.4. A matrix $A \in \mathbf{qC}^+$ of Dynkin type $\mathbb{B}_n$, its associated bigraph B_A, and B_{A^t}.

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & -2 & 2 & 2 \end{pmatrix}$$



The following theorem is a direct consequence of the results presented before and it is the main theorem of this section.

Theorem 5.5. *Any bigraph G has Dynkin type $\mathbb{C}_n$ if and only if it is a $\mathbb{C}$-assemblage.*

6. Bigraphs of Dynkin type $\mathbb{F}_4$

In [11] Kasjan and Simson studied a slightly different approach of the $\mathbb{Z}$-equivalence of quasi-Cartan matrices in $\mathbb{M}_4(\mathbb{Z})$. A connected positive definite quasi-Cartan matrix A (with symmetrizer D) has Dynkin type $\mathbb{F}_4$ if and only if:

(i) $D = P^t D_{\Delta} P$ with P a permutation matrix and $D_{\Delta} = diag(1, 1, 2, 2)$.
(ii) $\mathbf{specc}_{G_{\Delta}} = \mathbf{specc}_{G_{\Delta'}}$ where $\mathbf{specc}_{G_{\Delta'}}$ is the spectrum of the upper triangular matrix $G_{\Delta'} = (g_{ij})$ given by $g_{ii} = \frac{1}{2}(DA)_{ii}$ and $g_{ij} = (DA)_{ij}$ for all $i < j$.

In this section we present a complete description of bigraphs of Dynkin type $\mathbb{F}_4$ in terms of the $\mathbb{Z}$-equivalence defined in the beginning of this article. This characterization is constructive and allows to determine if a bigraph is of Dynkin type $\mathbb{F}_4$ by simple inspection.

Lemma 6.1. *Let G be a bigraph such that $G = \mathbb{F}_4$, then, for each sequence of flations with composition T, $T(G)$ is an extension by a vertex of a bigraph of Dynkin type $\mathbb{B}_3$ or $\mathbb{C}_3$.*

Proof. Let A' be the quasi-Cartan matrix associated to $T(G)$. Since A' is a positive connected quasi-Cartan matrix of Dynkin type $\mathbb{F}_4$, every connected principal submatrix S of A' with $S \in \mathbb{M}_3(\mathbb{Z})$ has to be a positive quasi-Cartan matrix of Dynkin type $\mathbb{A}_3$, $\mathbb{B}_3$ or $\mathbb{C}_3$. To prove that there exists a connected principal submatrix S with Dynkin type $\mathbb{B}_3$ or $\mathbb{C}_3$, suppose that every connected principal submatrix S of A' with $S \in \mathbb{M}_3(\mathbb{Z})$ has Dynkin type $\mathbb{A}_3$. Since A' is a non symmetric matrix, there exist two vertices r, v of $T(G)$ such that $\max\{|A_{rv}|, |A_{vr}|\} \neq 1$ and the connected principal submatrix containing the correspondent rows and columns of r and v clearly have not Dynkin type $\mathbb{A}$. Therefore A' necessarily has a connected principal submatrix of Dynkin type $\mathbb{B}_3$ or $\mathbb{C}_3$ and the result follows. $\square$

Lemma 6.2. *Any bigraph B_A has Dynkin type $\mathbb{F}_4$ if and only if is an extension of a bigraph G by a vertex y such that:*

- *G is a bigraph of Dynkin type $\mathbb{B}_3$ or $\mathbb{C}_3$,*
- *if v is the distinguished vertex of G, then there is an edge between v and y,*
- *if there exists a connection e between y and some non distinguished vertex x, then e is an arrow such that it is pointing to (coming out from) y if v is a sink (source) vertex of G,*
- *the bigraph B_A satisfies the chordless cycle condition.*

Proof. Let A be the quasi-Cartan matrix associated to B_A. It is not difficult to check that if $\Phi(A)$ is one of the following bigraphs and B_A accomplishes the cycle condition then it has Dynkin type $\mathbb{F}_4$. For this aim the reader can use the results obtained in (Theorem 5.1, [11]).

Now, to prove that the frames in Fig. 7 are all the frames associated to the bigraphs with Dynkin type $\mathbb{F}_4$, suppose that x_1, x_2 are the non distinguished vertices in G. If we add an arrow between v and y then we have one of the following non

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$$

(a)

$$\begin{tikzcd} 2 \arrow[d, dash] & 3 \arrow[d, dash] \arrow[dl] \\ 1 & 4 \arrow[l] \end{tikzcd}$$

(b)

$$\begin{tikzcd} 2 \arrow[d, dash] & 3 \arrow[d, dash] \\ 1 \arrow[r] \arrow[ur] & 4 \end{tikzcd}$$

(c)

$$\begin{tikzcd} 2 \arrow[d, dash] \arrow[r] & 3 \arrow[d, dash] \\ 1 \arrow[r] & 4 \end{tikzcd}$$

(d)

$$\begin{tikzcd} 2 \arrow[d, dash] \arrow[r] & 3 \arrow[d, dash] \\ 1 \arrow[r] \arrow[ur] & 4 \end{tikzcd}$$

(e)

$$\begin{tikzcd} 2 \arrow[d, dash] \arrow[r] \arrow[dr] & 3 \arrow[d, dash] \\ 1 \arrow[r] \arrow[ur] & 4 \end{tikzcd}$$

(f)

Fig. 7. Frames of the bigraphs of Dynkin type $\mathbb{F}_4$.

positive subgraphs:

$$x_2 \longrightarrow v \longrightarrow y \quad x_2 \longrightarrow v \longleftarrow y \quad x_2 \longleftarrow v \longrightarrow y \quad x_2 \longleftarrow v \longleftarrow y$$

If we add at least one edge e_{yx_i} and the edge e_{vy} then we have one of the non positive subgraphs:

$$\begin{tikzcd} & y \arrow[d, dash] \\ x_i \arrow[ur, dash] \arrow[r] & v \end{tikzcd} \quad \begin{tikzcd} & y \arrow[d, dash] \\ x_i \arrow[ur, dash] & v \arrow[l] \end{tikzcd} \quad \begin{tikzcd} x_1 \arrow[r, dash] \arrow[d, dash] & y \arrow[d, dash] \\ x_2 & v \arrow[l] \end{tikzcd} \quad \begin{tikzcd} x_1 \arrow[r, dash] \arrow[d, dash] & y \arrow[d, dash] \\ x_2 \arrow[r] & v \end{tikzcd}$$

If we add at least one of the edges e_{yx_i} or at least one of the arrows a_{yx_i} then we have the following possibilities for the resulting bigraph:

- it has Dynkin type $\mathbb{B}_4$ or $\mathbb{C}_4$,
- it has one of the non positive subgraph described before or is one of the following non positive bigraphs:

$$y \longrightarrow x_1 \longrightarrow x_2 \longrightarrow v \qquad y \longrightarrow x_1 \longrightarrow x_2 \longleftarrow v \qquad \square$$

7. Concluding remarks

1. The main result of the paper, described in Theorem 1.4, completes the results of Barot [2,3] obtained for the types $\mathbb{A}_n$, $\mathbb{D}_n$, and $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.
2. The notion of the assemblage of $\mathbb{A}$-blocks we use in the paper was introduced by Barot in [2]. Abarca and Rivera [1] reinterpreted this concept to define the $\mathbb{A}$-block tree. By applying it they obtained a new construction which fully characterizes the Dynkin type $\mathbb{D}_n$. They also presented a polynomial algorithm to determine the Dynkin type of an integer symmetric matrix if it has one.
3. Positive and non-negative quasi-Cartan matrices in $\mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ are recently studied by Kasjan and Simpson [10,11] by means of so called edge-bipartite graphs. A relation between quasi-Cartan matrices in $\mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ (equivalently, $\mathbb{Z}$-symmetrizable Cartan matrices $C \in \mathbb{M}_n(\mathbb{Z})$ and Cox-regular edge-bipartite graphs with n vertices are studied by Simson, [15]).
4. It would be interesting to have a result analogous to Theorem 1.4 for connected non- negative quasi-Cartan matrices in $\mathbf{qC} \subseteq \mathbb{M}_n(\mathbb{Z})$ of rank $n-1$, where $n \geq 2$.

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References

[1] Mario Abarca, Daniel Rivera, Graph theoretical and algorithmic characterizations of positive definite symmetric quasi-Cartan matrices, Fund. Inform. 149 (3) (2016) 241–261.
[2] Michael Barot, A characterization of positive unit forms, Bol. Soc. Mat. Mexicana 5 (1999) 87–94.
[3] Michael Barot, A characterization of positive unit forms, part II, Bol. Soc. Mat. Mexicana 7 (2001) 13–22.
[4] Michael Barot, Introduction to the Representation Theory of Algebras, Springer International Publishing Switzerland, 2015. http://dx.doi.org/10.1007/978-3-319-11475-0.
[5] Michael Barot, Jose Antonio de la Peña, The Dynkin type of a non-negative unit form, Expo. Math. 17 (4) (1999) 339–348.
[6] Michael Barot, Daniel Rivera, Generalized Serre relations for Lie algebras associated with positive unit forms, J. Pure Appl. Algebra 211 (2007) 360–373.
[7] Guillaume Fertin, Hafedh Mohamed-Babou, Irena Rusu, Path-driven orientation of mixed graphs, Discrete Appl. Math. (116) (2015) 271–278.
[8] Peter Gabriel, Andrei V. Roiter, Representations of Finite-Dimensional Algebras, in: Encyclopaedia of Mathematical Sciences, vol. 73, Springer, 1997.
[9] Hans Joachim Von Höhne, On weakly positive unit forms, Comment. Math. Helv. 63 (1988) 312–336.
[10] Stanisław Kasjan, Daniel Simson, Algorithms for isotropy groups of Cox-regular edge-bipartite graphs, Fund. Inform. 139 (2015) 249–275.
[11] Stanisław Kasjan, Daniel Simson, Mesh algorithms for Coxeter spectral classification of Cox-regular edge-bipartite graphs with loops,II. application to Coxeter spectral analysis, Fund. Inform. 139 (2015) 185–209.

[12] Anthony W. Knapp, Lie Groups Beyond an Introduction, second ed., in: Progress in Mathematics, vol. 140, Birkhäuser, 2002.
[13] Sergei. A. Ovsienko, Integer weakly positive forms, Inst. Mat. Akad. Nauk USSR (1978) 3–17 Schurian Matrix Problems and Quadratic Forms (Russian), Preprint 78.25.
[14] Claudia Pérez, Mario Abarca, Daniel Rivera, Cubic algorithm to compute the Dynkin type of a positive definite quasi-Cartan matrix, Fund. Inform. 158 (2018) 369–384.
[15] Daniel Simson, Symbolic algorithms computing Gram congruences in the Coxeter spectral classification of edge-bipartite graphs, II. Isotropy mini-groups, Fund. Inform. 145 (2016) 49–90.