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Cubic Algorithm to Compute the Dynkin Type of a Positive Definite Quasi-Cartan Matrix

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Abstract. *Inflations algorithm* is a procedure that appears implicitly in Ovsienko's classical proof for the classification of positive definite integral quadratic forms. The best known upper asymptotic bound for its time complexity is an exponential one. In this paper we show that this bound can be tightened to $\mathrm{O}\left(n^6\right)$ for the naive implementation. Also, we propose a new approach to show how to decide whether an admissible quasi-Cartan matrix is positive definite and compute the Dynkin type in just $\mathrm{O}\left(n^3\right)$ operations taking an integer matrix as input.

Keywords: quasi-Cartan matrix, Dynkin diagram, integral quadratic form, root system, inflations algorithm.

1. Introduction

Quasi-Cartan matrices are used in many branches of mathematics. The notion of quasi-Cartan matrices was introduced by Barot, Geiss and Zelevinsky in [1]. A *quasi-Cartan matrix* is an integer square matrix $\boldsymbol{A} \in \boldsymbol{M_n}(\mathbf{Z})$ with diagonal entries $(\boldsymbol{A})_{ii} = 2$ such that there exists a diagonal matrix

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$\boldsymbol{D}$ with positive integer diagonal entries for which $\boldsymbol{D}\,\boldsymbol{A}$ is symmetric. Following [2], we call such matrix $\boldsymbol{D}$ a *symmetrizer* of $\boldsymbol{A}$. A quasi-Cartan matrix $\boldsymbol{A}$ is a *Cartan matrix* if $(\boldsymbol{A})_{i\,j} \leq 0$ for all distinct $i, j \in \{1, \ldots, n\}$. Following [3] we denote by $\mathbf{qC_n}$ the set of all $n \times n$ quasi-Cartan matrices in $\boldsymbol{M_n}(\mathbf{Z})$. Complex semisimple Lie algebras can be characterized from the Cartan matrices by the well-known Serre's Theorem. In 2007 [4], Barot and Rivera proposed a generalization of this Theorem giving an explicit representation for any simply laced semisimple Lie algebra in terms of a positive definite symmetric quasi-Cartan matrix. In general two complex semisimple Lie algebras are isomorphic if and only if their asociated symmetric quasi-Cartan matrices are equivalents, i.e. they have the same Dynkin type. It follows from results of Ovsienko [5] proved in 1978, that every positive definite quasi-Cartan matrix is equivalent to a Cartan matrix, by applying an implicit algorithm that would be called "Inflations Algorithm". In 2016 Abarca and Rivera [3] proposed a polynomial algorithm $\mathrm{O}\left(n^5 \log^2 n\right)$ based on decomposition of graphs into biconnected components, to compute the Dynkin type of a positive definite symmetric quasi-Cartan matrix. Makuracki et al [6] use the inflation algorithm to classify positive connected Cox-regular edge-bipartite graphs. In this paper we present two fundamental ideas. We give a simplified proof of the well-know theorem of Ovsienko on the classification of integral quadratics forms, to demonstrate that the Inflations Algorithm has polynomial time complexity at most $\mathrm{O}\left(n^6\right)$. Also we propose an algorithm to decide if an admissible quasi-Cartan matrix is positive definite and compute its Dynkin type in only $\mathrm{O}\left(n^3\right)$ operations, taking an integer matrix as input.

1.1. Bigraphs

We say that an integer matrix $\boldsymbol{A} \in \boldsymbol{M_n}(\mathbf{Z})$ is locally symmetrizable if $A_{ii} = 2$ and the following conditions are met for all $i \neq j$:

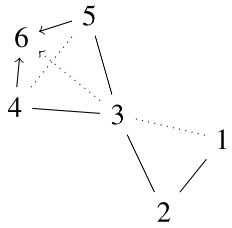
- $\mathrm{sgn}\left((\boldsymbol{A})_{i\,j}\right) = \mathrm{sgn}\left((\boldsymbol{A})_{j\,i}\right)$, where $\mathrm{sgn}\,(x) \in \{-1, 0, 1\}$ denotes the sign of a real number x;
- if $(\boldsymbol{A})_{i\,j} \neq (\boldsymbol{A})_{j\,i}$ then either $\frac{(\boldsymbol{A})_{i\,j}}{(\boldsymbol{A})_{ji}} \in \mathbf{Z}$ or $\frac{(\boldsymbol{A})_{ji}}{(\boldsymbol{A})_{i\,j}} \in \mathbf{Z}$.

Each quasi-Cartan matrix $\boldsymbol{A} \in \mathbf{qC_n}$ can be represented by an edge-bipartite edge-labelled mixed graph with n vertices, denoted here as $\mathbf{bigr}\,(\boldsymbol{A})$, see Figure 1.1. We call it briefly *bigraph*. This is a kind of graph that consists in a set $\mathcal{V} = \{1, \ldots, n\}$ of vertices, a set $\mathcal{E} \subseteq \mathcal{V}_2$ of edges (undirected), a set $\mathcal{A} \subseteq V \times V$ of arrows and a function $\omega : \mathcal{E} \cup \mathcal{A} \to \mathbb{Z}$, $\omega(e_{ij}) = \max\left(\left|(\boldsymbol{A})_{i\,j}\right|, \left|(\boldsymbol{A})_{j\,i}\right|\right)$ that defines the weight of $e_{ij} \in \mathcal{E} \cup \mathcal{A}$, where $\mathcal{V}_2 = \{\{i, j\} \mid i, j \in \mathcal{V} \text{ and } i \neq j\}$, and $e_{ij} = \{i, j\}$ if $(\boldsymbol{A})_{i\,j} = (\boldsymbol{A})_{j\,i}$, $e_{ij} = (i, j)$ if $\left|(A)_{i\,j}\right| < \left|(A)_{j\,i}\right|$. All edges and arrows are represented as follows:

- if $(\boldsymbol{A})_{i\,j} = (\boldsymbol{A})_{j\,i}$ there is a solid (dotted) edge between i and j plotted as $i \overset{\omega}{\text{—}} j$ ($i \overset{\omega}{\cdots} j$) if $\omega\,(\{i, j\}) < 0$ ($\omega\,(\{i, j\}) > 0$) respectively,
- if $\left|(A)_{i\,j}\right| < \left|(A)_{j\,i}\right|$ there is a solid (dotted) arrow between i and j plotted as $i \xrightarrow{\omega} j$ ($i \overset{\omega}{\dashrightarrow} j$) if $\omega\,(i, j) < 0$ ($\omega\,(i, j) > 0$) respectively.

We may omit this label by defaulting, if $|\omega\,(\{i, j\})| = 1$ ($|\omega\,(i, j)| = 2$) and there is an edge (arrow) between i and j respectively. We say that an integer matrix $\boldsymbol{A} \in \boldsymbol{M_n}(\mathbf{Z})$ is *admissible* if is locally symmetrizable, $\left|(\boldsymbol{A})_{i\,j}\right| < 3$ for all $i, j \leq n$ and has no parallel edges.

A	1	2	3	4	5	6
1	2	−1	1	0	0	0
2	−1	2	−1	0	0	0
3	1	−1	2	−1	−1	1
4	0	0	−1	2	1	−1
5	0	0	−1	1	2	−1
6	0	0	2	−2	−2	2



bigr (A)

Figure 1. A quasi-Cartan matrix A and its associated bigraph

Remark 1.1. There is no bijection between quasi-Cartan matrices and bigraphs. As a counterexample, consider the directed cycle $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 1$ which represents the following matrix $\boldsymbol{A}$ with no symmetrizer (i.e. $\boldsymbol{D}\,\boldsymbol{A}$ is symmetric if and only if $\boldsymbol{D}$ is the zero matrix):

$$A = \begin{bmatrix} 2 & -1 & -2 \\ -2 & 2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$$

but A is admissible.

1.2. Some properties of the symmetrizer of a quasi-Cartan matrix

Two quasi-Cartan matrices $\boldsymbol{A}, \boldsymbol{A}' \in \mathbf{qC_n}$ are defined to be $\mathbf{Z}$-*equivalent (we denote it by* $\boldsymbol{A} \sim \boldsymbol{A}'$*) if there exists a* $\mathbf{Z}$*−invertible integer matrix* $\boldsymbol{M} \in \boldsymbol{M_n}(\mathbf{Z})$ *and symmetrizers* D, $D' \in \boldsymbol{M_n}(\mathbf{Z})$ *such that* $\boldsymbol{D}'\,\boldsymbol{A}' = \boldsymbol{M}^{\mathrm{T}}\,(\boldsymbol{D}\,\boldsymbol{A})\,\boldsymbol{M}$ *and* $\boldsymbol{D}' = \boldsymbol{P}^{\mathrm{T}}\,\boldsymbol{D}\,\boldsymbol{P}$ *where* $\boldsymbol{P}$ *is a permutation matrix.*

As in [3], we consider the elementary matrix $\boldsymbol{T}^{\sigma}_{s\,r} = \boldsymbol{I} + \sigma\,\boldsymbol{e}_s\,\boldsymbol{e}_r{}^{\mathrm{T}} \in \boldsymbol{M_n}(\mathbf{Z})$ which is called an *inflation* if $s \neq r$ and $\sigma < 0$, *deflation* if $s \neq r$ and $\sigma > 0$, and *sign inversion* if $s = r$ and $\sigma = -1$. We also consider the *transposition* matrix $\boldsymbol{P}_{s\,r} := \boldsymbol{I}_n - (\boldsymbol{e}_s - \boldsymbol{e}_r)\,(\boldsymbol{e}_s - \boldsymbol{e}_r)^{\mathrm{T}} \in \boldsymbol{M_n}(\mathbf{Z})$. We say that $\boldsymbol{A}, \boldsymbol{A}' \in \mathbf{qC_n}$ are $\mathbb{G}$-*equivalent* if they are $\mathbf{Z}$-*equivalent* and $\boldsymbol{M}$ is a product of inflations, deflations, transpositions and sign inversions. A matrix $\boldsymbol{A} \in \mathbf{qC_n}$ is called *connected* if and only if the underlying graph of the bigraph $\mathbf{bigr}\,(\boldsymbol{A})$ associated to $A \in \boldsymbol{M_n}(\mathbf{Z})$ is connected. The following lemma is a "quasi-Cartan analogue" of the Lemma 2.56 in [2].

Lemma 1.2. Let $\boldsymbol{A} \in \mathbf{qC_n}$ be a connected quasi-Cartan matrix. Then the associated diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{DA}$ is symmetric is unique up to a multiplicative scalar.

Proof:
Suppose that $\boldsymbol{D}$ and $\boldsymbol{D}'$ are two diagonal matrices with positive diagonal entries such that $\boldsymbol{P} = \boldsymbol{DA}$ and $\boldsymbol{P}' = \boldsymbol{D}'\boldsymbol{A}$ are symmetric. Then $\boldsymbol{P}$ and $\boldsymbol{P}' = \boldsymbol{D}'\boldsymbol{D}^{-1}\boldsymbol{P}$ are both symmetric. Write $\boldsymbol{D}'\boldsymbol{D}^{-1} = \mathrm{diag}\,(b_1, \ldots, b_n)$. For each i and j $(i \neq j)$, we have

$$b_i\,(\boldsymbol{P})_{i\,j} = (\boldsymbol{P}')_{i\,j} = (\boldsymbol{P})'_{j\,i} = b_j\,(\boldsymbol{P})_{j\,i} = b_j\,(\boldsymbol{P})_{i\,j}\,.$$

Thus either $(\boldsymbol{P})_{i\,j} = 0$ or $b_i = b_j$, i.e., $(\boldsymbol{A})_{i\,j} = 0$ or $b_i = b_j$. Since $\boldsymbol{A}$ is connected, there exists a path between i and j, i.e., there is indices $i_1, \dots i_k$ such that $\boldsymbol{A}_{\boldsymbol{i}\,\boldsymbol{i_1}} \neq 0$, $\boldsymbol{A}_{\boldsymbol{i_l}\,\boldsymbol{i_{l+1}}} \neq 0$ for all $1 \leq l < k$ and $\boldsymbol{A}_{\boldsymbol{i_k}\,\boldsymbol{j}} \neq 0$. Therefore we obtain $b_i = b_{i_1} = \cdots = b_k = b_j$ for all i, j $(i \neq j)$ in the set $\{1, \dots, n\}$. Thus the diagonal entries of $\boldsymbol{D}'$ are proportional to the diagonal entries of $\boldsymbol{D}$. □

Definition 1.3. If $\boldsymbol{A} \in \mathbf{qC_n}$ is a quasi-Cartan matrix, then the unique diagonal matrix $\boldsymbol{D} \in \boldsymbol{M_n}(\mathbf{Z})$ with positive integer diagonal entries and minimal trace, such that $\boldsymbol{DA}$ is symmetric it is defined to be the symmetrizer of $\boldsymbol{A}$.

Lemma 1.4. Let $\boldsymbol{A} \in \mathbf{qC_n}$ a quasi-Cartan matrix and $\boldsymbol{D}$ a symmetrizer of A. Then:

$$\boldsymbol{D}^{-1}\left(\boldsymbol{T}_{r\,s}^{-(\boldsymbol{A})_{s\,r}}\right)\boldsymbol{D} = \boldsymbol{T}_{r\,s}^{-(\boldsymbol{A})_{r\,s}}$$

Proof:
Let $\boldsymbol{B} = \boldsymbol{D}^{-1}(\boldsymbol{T}_{r\,s}^{-(\boldsymbol{A})_{s\,r}})\,\boldsymbol{D}$ then $(\boldsymbol{B})_{r\,s} = -\,(\boldsymbol{D})_{r\,r}^{-1}\,(\boldsymbol{D})_{s\,s}\,(\boldsymbol{A})_{s\,r}$. But $-\,(\boldsymbol{A})_{s\,r}\,(\boldsymbol{D})_{s\,s} = -\,(\boldsymbol{A})_{r\,s}\,(\boldsymbol{D})_{r\,r}$, from which we obtain $-\,(\boldsymbol{A})_{s\,r}\,(\boldsymbol{D})_{s\,s}\,(\boldsymbol{D})_{r\,r}^{-1} = -\,(\boldsymbol{A})_{r\,s}$. □

As a consequence of Lemma 1.4, we get the following corollary.

Corollary 1.5. For any $\boldsymbol{A} \in \mathbf{qC_n}$ the matrix $\boldsymbol{A}' = \boldsymbol{T}_{r\,s}^{-(\boldsymbol{A})_{r\,s}}\,\boldsymbol{A}\,\boldsymbol{T}_{s\,r}^{-(\boldsymbol{A})_{s\,r}}$ is $\mathbf{Z}$-equivalent to $\boldsymbol{A}$. Moreover,

$$(\boldsymbol{A})'_{i\,j} = \begin{cases} (\boldsymbol{A})_{i\,j} & \text{if } i \neq r \text{ and } j \neq r; \\ (\boldsymbol{A})_{r\,j} - (\boldsymbol{A})_{r\,s}\,(\boldsymbol{A})_{s\,j} & \text{if } i = r \text{ and } j \neq r; \\ (\boldsymbol{A})_{i\,r} - (\boldsymbol{A})_{s\,r}\,(\boldsymbol{A})_{i\,s} & \text{if } i \neq r \text{ and } j = r; \\ (\boldsymbol{A})_{r\,r} & \text{if } i = r \text{ and } j = r. \end{cases} \tag{1}$$

Lemma 1.6. Let $\boldsymbol{A} \in \mathbf{qC_n}$ be a quasi-Cartan matrix, with symmetrizer $\boldsymbol{D}$ and let $\boldsymbol{T} \in \left\{\boldsymbol{T}_{r\,r}^{-1}, \boldsymbol{P}_{s\,r}, \boldsymbol{T}_{s\,r}^{-(\boldsymbol{A})_{s\,r}}\right\}$. If $\boldsymbol{A}' = \boldsymbol{D}^{-1}\boldsymbol{T}^{T}\,\boldsymbol{D}\,\boldsymbol{A}\,\boldsymbol{T}$ then $\boldsymbol{A}' \in \mathbf{qC_n}$ and the symmetrizer of $\boldsymbol{A}'$ is

$$\boldsymbol{D}' = \begin{cases} \boldsymbol{T}^{T}\,\boldsymbol{D}\,\boldsymbol{T} & \text{if } \boldsymbol{T} = \boldsymbol{P}_{s\,r} \\ \boldsymbol{D} & \text{if } \boldsymbol{T} \in \left\{\boldsymbol{T}_{r\,r}^{-1}, \boldsymbol{T}_{s\,r}^{-(\boldsymbol{A})_{s\,r}}\right\}. \end{cases}$$

Proof:
Let $\boldsymbol{T} = \boldsymbol{T}_{s\,r}^{-(\boldsymbol{A})_{s\,r}}$. Then $\boldsymbol{D}\,\boldsymbol{A}' = \left(\boldsymbol{T}_{s\,r}^{-(\boldsymbol{A})_{s\,r}}\right)^{\mathrm{T}}\boldsymbol{D}\,\boldsymbol{A}\,\boldsymbol{T}_{s\,r}^{-(\boldsymbol{A})_{s\,r}}$is symmetric and $(\boldsymbol{A}')_{i\,i} = 2$. If $\boldsymbol{D}'$ is the symmetrizer of $\boldsymbol{A}$, then $\mathrm{tr}\,(\boldsymbol{D}') \leq \mathrm{tr}\,(\boldsymbol{D})$, were $\mathrm{tr}\,(\boldsymbol{A})$ is the trace of $\boldsymbol{A}$. Now Lemma 1.4 yields

$$\boldsymbol{A}' = \boldsymbol{T}_{r\,s}^{-(\boldsymbol{A})_{r\,s}} A \boldsymbol{T}_{s\,r}^{-(\boldsymbol{A})_{s\,r}}.$$

Hence we obtain:

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{T}_{r\,s}^{-(\boldsymbol{A}')_{r\,s}} A' \boldsymbol{T}_{s\,r}^{-(\boldsymbol{A}')_{s\,r}} \\ &= (\boldsymbol{D}')^{-1}\,(\boldsymbol{T}_{s\,r}^{-(\boldsymbol{A}')_{s\,r}})^{T}\,\boldsymbol{D}' A' \boldsymbol{T}_{s\,r}^{-(\boldsymbol{A}')_{s\,r}} \end{aligned}$$

and $tr(\boldsymbol{D}) \leq tr(\boldsymbol{D}')$.

If $\boldsymbol{T} = \boldsymbol{P}_{s\,r}$ then $\boldsymbol{D}'\boldsymbol{A}' = \boldsymbol{P}_{s\,r}{}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{A}\boldsymbol{P}_{s\,r}$, consequently $\boldsymbol{D}' = \boldsymbol{P}_{s\,r}{}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{P}_{s\,r}$. Finally, if $\boldsymbol{T} = \boldsymbol{T}_{r\,r}^{-1}$ then $\boldsymbol{D} = \boldsymbol{D}'$. □

2. Dynkin type and roots of positive integral quadratic forms

Let $\boldsymbol{A}$ be a quasi-Cartan matrix in $\mathbf{qC_n}$ and D the symmetrizer of A, we associate to DA an integral quadratic form $q_{\boldsymbol{D}\boldsymbol{A}} : \mathbf{Z}^n \to \mathbf{Z}$ defined by

$$q_{\boldsymbol{D}\boldsymbol{A}}(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\,\boldsymbol{D}\,\boldsymbol{A}\,\boldsymbol{x} = \sum_{i=1}^{n} q_i\,x_i^2 + \sum_{1\le i<j\le n} q_{i\,j}\,x_i\,x_j$$

where $q_i = (\boldsymbol{D})_{i\,i}$, $q_{i\,j} = (\boldsymbol{A})_{i\,j}\,(\boldsymbol{D})_{i\,i}$, and $\frac{q_{i\,j}}{q_i} \in \mathbf{Z}$ for all i [7]. We shall say that $q_{\boldsymbol{D}\boldsymbol{A}}$ is *positive definite* (equivalently, A is positive definite,) if $q_{\boldsymbol{D}\boldsymbol{A}}(x) > 0$ for all non-zero vectors in $\mathbf{Z}^n$. We say that $q_{\boldsymbol{D}\boldsymbol{A}}$ is a *unit form* if $q_i = 1$ for all i. We say that $\boldsymbol{x} \in \mathbf{Z}^n$ is a root of $q_{\boldsymbol{D}\boldsymbol{A}}$ if $q_{\boldsymbol{D}\boldsymbol{A}}(\boldsymbol{x}) = q_i$ for some $1 \le i \le n$. Given $\boldsymbol{A} \in \mathbf{qC_n}$ and $\boldsymbol{D}$ the symmetrizer of $\boldsymbol{A}$. By a *positive root* we mean the root $\boldsymbol{x} \neq 0$ of $q_{\boldsymbol{D}\boldsymbol{A}}$, with $x_i \ge 0$. We denote by $\mathcal{R}(q_{\boldsymbol{D}\boldsymbol{A}}) \subseteq \mathbf{Z}^n$ the set of all roots of $q_{\boldsymbol{D}\boldsymbol{A}}$, and by $\mathcal{R}^+(q_{\boldsymbol{D}\boldsymbol{A}}) \subseteq \mathbf{N}^n$ the subset of $\mathcal{R}(q_{\boldsymbol{D}\boldsymbol{A}})$ consisting of the positive roots. Note that $\mathcal{R}(q_{\boldsymbol{D}\boldsymbol{A}}) = \{\boldsymbol{x} \in \mathbf{Z}^n \mid q_{\boldsymbol{D}\boldsymbol{A}}\boldsymbol{x} = 1\}$ if $q_{\boldsymbol{D}\boldsymbol{A}}$ is a unit form. The integral quadratic form $q_{\boldsymbol{D}\boldsymbol{A}}$ is called *connected* if and only if the matrix A is connected. The Theorem 2.1 was proved in [5], it also follows from the results of Makuracki [6]. We have included a proof inspired in Theorem 8.26, [8] for convenience to the reader (see also [9] for a proof in the case of unit forms).

Theorem 2.1. Every positive definite integral quadratic form $q_{\boldsymbol{D}\boldsymbol{A}}$ is $\mathbb{G}$-equivalent up to isomorphism to a positive integral quadratic form $q_{\boldsymbol{D}\boldsymbol{A}_\Delta}$, where the bigraph Δ is a disjoint union of Dynkin diagrams.

Proof:
Let $q_{\boldsymbol{D}\boldsymbol{A}}$ be a positive integral quadratic form where $\boldsymbol{A}$ is a quasi-Cartan matrix. By Proposition 8.19, [8] the set $\mathcal{R}(q_{\boldsymbol{D}\boldsymbol{A}})$ is finite. Assume that there is a coefficient $q_{ij} > 0$ with $i \neq j$ and let $q_{\boldsymbol{D}\boldsymbol{A}'}(x) = q_{\boldsymbol{D}\boldsymbol{A}}(T_{i\,j}^{(-\boldsymbol{A})_{i\,j}} \cdot x)$, it follows that $q_{\boldsymbol{D}\boldsymbol{A}'}(\boldsymbol{e_a}) = q_{\boldsymbol{D}\boldsymbol{A}}(\boldsymbol{e_a})$ for all $a \neq j$, where the set of column vectors $\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}$ is the canonical basis of $\mathbb{Z}^n$. For $a = j$ we have that $q_{\boldsymbol{D}\boldsymbol{A}'}\left(\boldsymbol{e_j}\right) = q_{\boldsymbol{D}\boldsymbol{A}}\left((-\boldsymbol{A})_{i\,j}\,\boldsymbol{e_i} + \boldsymbol{e_j}\right) = (\boldsymbol{A})_{i\,j}^2\,q_i + q_j - q_i\,(\boldsymbol{A})_{i\,j}^2 = q_j$. As in Theorem 8.26, [8], for each $\boldsymbol{x} \in \mathbf{Z}^n$ denote $\boldsymbol{x'} = (T_{i\,j}^{(-\boldsymbol{A})_{i\,j}})^{-1}\boldsymbol{x}$ (equivalently $\boldsymbol{x'} = T_{i\,j}^{(\boldsymbol{A})_{i\,j}}\boldsymbol{x}$). Then $q_{\boldsymbol{D}\boldsymbol{A}'}(\boldsymbol{x'}) = q_{\boldsymbol{D}\boldsymbol{A}}(\boldsymbol{x})$ and we obtain an injective map $\mathcal{R}^+(q_{\boldsymbol{D}\boldsymbol{A}}) \to \mathcal{R}^+(q_{\boldsymbol{D}\boldsymbol{A}'})$, $\boldsymbol{x} \mapsto \boldsymbol{x'}$, since $\boldsymbol{x'_a} = \boldsymbol{e_a}^{\mathrm{T}} T_{i\,j}^{(\boldsymbol{A})_{i\,j}}\boldsymbol{x} = \boldsymbol{e_a}^T\boldsymbol{x} = \boldsymbol{x_a}$ for all $a \neq i$ and $\boldsymbol{x'_i} = \boldsymbol{e_i}^T T_{i\,j}^{(-\boldsymbol{A})_{i\,j}}\boldsymbol{x} = \left((\boldsymbol{A})_{i\,j}\,\boldsymbol{e_i} + \boldsymbol{e_j}\right)^T \boldsymbol{x} = (\boldsymbol{A})_{i\,j}\,\boldsymbol{x_i} + \boldsymbol{x_j} \ge 0$. However this map is not surjective since, $\boldsymbol{e_j} \in \mathcal{R}^+(q_{\boldsymbol{D}\boldsymbol{A}'})$ but $T_{i\,j}^{(-\boldsymbol{A})_{i\,j}}\boldsymbol{e_j} = (-\boldsymbol{A})_{i\,j}\,\boldsymbol{e_i} + \boldsymbol{e_j} \notin \mathcal{R}^+(q_{\boldsymbol{D}\boldsymbol{A}})$. Therefore $|\mathcal{R}^+(q_{\boldsymbol{D}\boldsymbol{A}})| < |\mathcal{R}^+(q_{\boldsymbol{D}\boldsymbol{A}'})| \le |\mathcal{R}(q_{\boldsymbol{D}\boldsymbol{A}'})| = |\mathcal{R}(q_{\boldsymbol{D}\boldsymbol{A}})|$. After many finitely steps we must end with an integral form $q_{\boldsymbol{D}\boldsymbol{A}^{(k)}}$ such that $q_{ij}^{(k)} \le 0$ for all $i \neq j$. □

If $\boldsymbol{A}$ is a positive definite, connected quasi-Cartan matrix in $\mathbf{qC_n}$, then by the Theorem 2.1 there exists a Dynkin diagram Δ (see Figure 2) such that $\boldsymbol{A} \sim \boldsymbol{A_\Delta}$. The Dynkin diagram Δ is called the *Dynkin type* of A. It follows that two integral quadratic forms and therefore, two quasi-Cartan matrices $\boldsymbol{A}$ and $\boldsymbol{A'}$ are $\mathbb{G}$-*equivalent* if and only if they are $\mathbf{Z}$-*equivalent*.
The following result gives us a technique to prove that the inflations algorithm has polynomial running time.

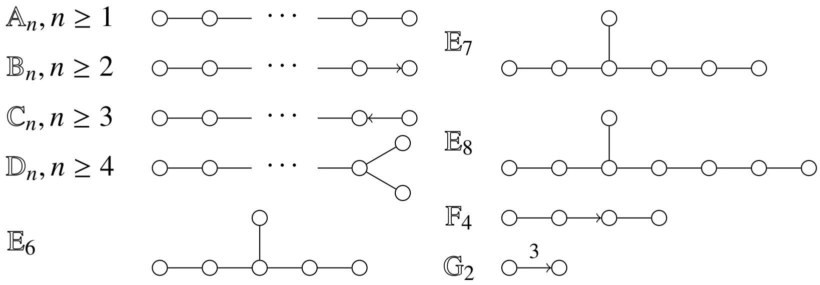


Figure 2. Dynkin diagrams

Corollary 2.2. If $q^{(0)} := q_{\boldsymbol{DA}}$ is positive definite connected integral quadratic form such that there is a coefficient $q_{ij} > 0$ with $i \neq j$ then there exists a sequence of inflations $T_1, \dots, T_k$ such that $q^{(i)}(x) = q^{(i-1)}(T_i \cdot x)$, $|\mathcal{R}^+\left(q^{(i-1)}\right)| < |\mathcal{R}^+\left(q^{(i)}\right)|$ for all $i \in \{1, \dots k\}$ and the bigraph associated to the matrix $\boldsymbol{A_k}$ is a Dynkin diagram, where $\boldsymbol{A_k}$ is the Cartan matrix such that $q^{(k)}(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A_k}\boldsymbol{x}$.

Now we recall the following definition.

Definition 2.3. A *root system* (in the sense of Bourbaki [10]) in a finite-dimensional real inner space V with inner product $(\cdot,\cdot)$ and norm squared $|\cdot|^2$ (see [2]), is a finite set Π of non-zero elements of V such that:

$R1)$ Π spans V,

$R2)$ the orthogonal transformations $s_\alpha(\phi) = \phi - \frac{2(\phi,\alpha)}{|\alpha|^2}\alpha$, for $\alpha \in \Pi$ and $\phi \in V$, carry Π to itself,

$R3)$ $\frac{2(\phi,\alpha)}{|\alpha|^2}$ is an integer whenever α and ϕ are in Π.

A root system Π is called *irreducible* if it admits no non-trivial disjoint decomposition $\Pi = \Pi_1 \cup \Pi_2$ with every member of Π_1 orthogonal to every member of Π_2 [2].

Remark 2.4. Assume that Δ is any of the Dynkin graphs $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4$, and $\mathbb{G}_2$. Let $D \in \boldsymbol{M_n}(\mathbf{Z})$ be the symmetrizer of the Cartan matrix A_Δ, and $q_{DA_\Delta} : \mathbf{Z}^n \to \mathbf{Z}$ the associated quadratic form. It follows from [11], [9], [12], [13] that, for $\Delta \neq \mathbb{C}_n$, the set $\mathcal{R}(q_{DA_\Delta})$ of roots of q_{DA_Δ} is an irreducible and reduced root system in the sense of Bourbaki [10] and the Dynkin graph associated with the system coincides with Δ. Moreover, $\mathcal{R}(q_{DA_\Delta})$ is also Φ_A-mesh root system in the sense of [12], where Φ_A is the Coxeter transformation associated to the Coxeter matrix of the bigraph B_A. Furthermore, in case when $\Delta = \mathbb{C}_n$ for $n \geq 5$, the set $\mathcal{R}(q_{DA_\Delta})$ is not a root system of Dynkin type $\mathbb{C}_n$. In Lemma 2.6 we prove that the set $\mathcal{R}^+\left(q_{DA_{\mathbb{C}_n}}\right)$ has cardinality $\frac{n}{4}\left(n^3 - 6n^2 + 15n - 6\right)$, whereas the set of positive roots of each irreducible root system of the Dynkin type $\mathbb{C}_n$ has cardinality n^2, see [12].

One of the purpose of the paper is to show that the Inflations Algorithm runs in polynomial time for positive definite integral quadratic forms. We will show this in the next section.

Lemma 2.5. Let $q_{\boldsymbol{D}\mathbb{B}_n} : \mathbf{Z}^n \to \mathbf{Z}$ be the integral quadratic form associated to $\mathbb{B}_n$. Then $q_{\boldsymbol{D}\mathbb{B}_n}$ has n^2 positive roots.

Although this Lemma follows from the results of Kasjan et al [13], [14], see also Knapp [2] and Makuracki et al [6], we include a proof for convenience of the readers.

Proof:
The quadratic form $q_{\boldsymbol{D}\mathbb{B}_n}$ may be written as:

$$q_{\boldsymbol{D}\mathbb{B}_n}(x_1, \dots, x_n) = x_1^2 + \sum_{1 \le i < n} (x_i - x_{i+1})^2$$

Since all the addends have positive coefficients, $q_{\boldsymbol{D}\mathbb{B}_n}(\boldsymbol{x}) = 2$ can occur if and only if exactly two addends are equal to 1 and, in the same manner $q_{\boldsymbol{D}\mathbb{B}_n}(\boldsymbol{x}) = 1$ if and only if exactly one addend has value 1. The following table shows how this two cases may be achieved:

Root form	Count
$(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$	$\frac{(n-2)(n-1)}{2}$
$(0, \dots, 0, 1, \dots, 1, 2, \dots, 2)$	$\frac{(n-2)(n-1)}{2}$
$(1, \dots, 1, 2, \dots, 2)$	$(n-1)$
$(1, \dots, 1, 0, \dots, 0)$	$(n-1)$
$(0, \dots, 0, 1, \dots, 1)$	$(n-1)$
$(1, \dots, 1)$	1
Total	n^2

□

Lemma 2.6. Let $q_{\boldsymbol{D}\mathbb{C}_n} : \mathbf{Z}^n \to \mathbf{Z}$ $(n \ge 5)$ be the integral quadratic form associated to $\mathbb{C}_n$. Then $q_{\boldsymbol{D}\mathbb{C}_n}$ has $\frac{n}{4}\left(n^3 - 6n^2 + 15n - 6\right)$ positive roots.

Proof:
The set of positive roots of $\mathbb{C}_n$ are the $\boldsymbol{x} \in \mathbf{N}^n$ such that:

$$\sum_{1 \le i \le n-1} x_i^2 - \sum_{1 \le i < n-1} x_i\, x_{i+1} + 2x_n^2 - 2x_{n-1}\, x_n \in \{1, 2\}$$

or equivalently

$$x_1^2 + \sum_{1 \le i < n-1} (x_i - x_{i+1})^2 + (x_{n-1} - 2x_n)^2 \in \{2, 4\} \tag{2}$$

Let us first compute the positive $\boldsymbol{x} \in \mathbf{N}^n$ such that the polynomial (2) takes the value 2. Using a similar argument as in the previous lemma the n-tuples are the following:

Root form	Count
$(0,\dots,0,1,\dots,1,0,\dots,0)$	$\frac{(n-2)(n-1)}{2}$
$(1,\dots,1,2,\dots,2,1)$	$(n-2)$
$(1,\dots,1,0,\dots,0)$	$(n-1)$
$(0,\dots,0,1,\dots,1,1)$	$(n-2)$
$(1,\dots,1)$	1

If the polynomial (2) evaluated in x is equal to 4, the roots are of the form:

Root form	Count
$(0,\dots,0,1,\dots,1,0,\dots,0,1,\dots,1,2,\dots 2,1)$	
$(0,\dots,0,1,\dots,1,2,\dots,2,1,\dots,1,2,\dots 2,1)$	$\frac{1}{2}\sum_{0\leq k\leq n-6}(k+1)(n-5-k)(n-4-k)$
$(0,\dots,0,1,\dots,1,2,\dots,2,3,\dots,3,2,\dots 2,1)$	
$(0,\dots,0,1,\dots,1,2,\dots,2,3,\dots,3,4,\dots 4,2)$	
$(0,\dots,0,1,\dots,1,0,\dots,0,1,\dots 1,0,\dots,0)$	
$(0,\dots,0,1,\dots,1,2,\dots,2,1,\dots,1,0,\dots 0)$	$\frac{1}{2}\sum_{0\leq k\leq n-5}(k+1)(n-4-k)(n-3-k)$
$(0,\dots,0,1,\dots,1,0,\dots,0,1,\dots 1,1)$	
$(0,\dots,0,1,\dots,1,2,\dots,2,1,\dots 1,1)$	
$(0,\dots,0,1,\dots,1,2,\dots,2,3,\dots 3,1)$	$\frac{1}{2}\sum_{0\leq k\leq n-5}(n-4-k)(n-3-k)$
$(0,\dots,0,1,\dots,1,2,\dots,2,3,\dots 3,2)$	
$(1,\dots,1,2,\dots,2,3,\dots,3,4,\dots 4,2)$	
$(1,\dots,1,0,\dots,0,1,\dots,1,0,\dots 0)$	
$(1,\dots,1,2,\dots,2,1,\dots,1,0,\dots 0)$	$\frac{1}{2}\sum_{0\leq k\leq n-4}(n-3-k)(n-2-k)$
$(1,\dots,1,2,\dots,2,3,\dots,3,2,\dots 2)$	
$(1,\dots,1,0,\dots,0,1,\dots,1,1)$	
$(1,\dots,1,2,\dots,2,3,\dots,3,1)$	$\frac{(n-3)(n-2)}{2}$
$(0,\dots,01)$	1
Total	$\frac{n}{4}\left(n^3-6n^2+15n-6\right)$

□

The following theorem is a direct consequence of [8, Prop. 8.25], [2], [12], and Lemmas 2.5 and 2.6.

Theorem 2.7. Let $q = q_{\boldsymbol{DA}}$ be an integral form such that **bigr** $(\boldsymbol{A})$ is a Dynkin diagram. Then $|\mathcal{R}^+(q)|$ is as indicated in the following list:

bigr $(\boldsymbol{A})$	$\mathbb{A}_n$	$\mathbb{B}_n$	$\mathbb{C}_n$ $(n \geq 5)$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$\lvert\mathcal{R}^+(q)\rvert$	$\frac{n}{2}(n+1)$	n^2	$\frac{n}{4}\left(n^3-6n^2+15n-6\right)$	$n(n-1)$	36	63	120	24	6

3. The naive inflations algorithm runs in polynomial time

The proof of Theorem 2.1 (see[5]) contains the classical inflations algorithm (Algorithm 3.1), which transforms any positive-definite quasi-Cartan matrix into a **Z**-equivalent Cartan matrix.

Algorithm 3.1: Inflations algorithm

Input: A positive-definite quasi-Cartan matrix $\boldsymbol{A}$

while *there exists a non-diagonal entry* $(\boldsymbol{A})_{s\,r} > 0$ **do**

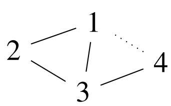
  $\boldsymbol{A} \leftarrow \boldsymbol{T}_{r\,s}^{-(\boldsymbol{A})_{r\,s}}\, \boldsymbol{A}\, \boldsymbol{T}_{s\,r}^{-(\boldsymbol{A})_{s\,r}}$

The Corollary 2.2 implies that the maximum number of iterations cannot exceed $\ell(\boldsymbol{A}) := \left|\mathcal{R}^{+}(q_{\Delta})\right|$ where Δ is the Dynkin type of $\boldsymbol{A}$. In [15] and [13] an exponential upper bound of $n \cdot 6^{n+1}$ was reported; nonetheless the same paper suggests that the inflations algorithm is much faster in practice. Indeed, Theorem 2.7 suggests that the worst case for the inflations algorithm happens when $\boldsymbol{A}$ has Dynkin type $\mathbb{C}_n$, yielding a maximum of $\ell(\boldsymbol{A}) = \mathrm{O}\left(n^4\right)$ iterations. Also, when restricted to symmetric matrices the worst case is $\mathbb{D}_n$ with a maximum of $\mathrm{O}\left(n^2\right)$ iterations.

A naive implementation of the algorithm would look for the non-diagonal entry $(\boldsymbol{A})_{s\,r} > 0$ by scanning the matrix $\boldsymbol{A}$ in each iteration. Thus, the running time of the naive inflations algorithm is $\mathrm{O}\left(n^6\right)$, or just $\mathrm{O}\left(n^4\right)$ when restricted to the symmetric case.

Notice that the input matrix of the inflations algorithm must be positive-definite. Whenever a non positive definite matrix is given as input, the algorithm may loop endlessly, as shown in the following example.

Example 3.1. Let $\boldsymbol{A}$ be the quasi-Cartan matrix associated to the following bigraph:

$\mathbf{bigr}(\boldsymbol{A}) =$ 

Let m mod n denote the reminder after division of m by n. An endless loop of Algorithm 3.1 on input $\boldsymbol{A}$ ensues by choosing, at each iteration k, the indices $s = (k \bmod 3) + 1$ and $r = 4$.

4. Improvements for the inflations algorithm

We may improve on the inflations algorithm to handle integer matrices and report the finite Dynkin type if it has one; all in just $\mathrm{O}\left(n^3\right)$ operations. In this paper we will assume that $\mathbf{bigr}(\boldsymbol{A})$ is connected. To handle disconnected bigraphs we may apply the suggested method for each connected component as in [3]. A vertex is a *source* (*sink*) if all of its incidences are arrows coming out of (pointing to) it. We say that a vertex x of a bigraph G is *distinguished* if it is either a source or a sink, and all the arrows of G are incident on x. Notice that by removing the distinguished vertex from both $\mathbb{B}_n$ and $\mathbb{C}_n$ we get $\mathbb{A}_{n-1}$; also notice that $\mathbb{F}_4$ is the only positive definite mixed bigraph without a distinguished vertex.

Lemma 4.1. Let $\boldsymbol{A}$ be a non symmetric admissible matrix of size $n \times n$ such that $\mathbf{bigr}\,(\boldsymbol{A})$ has no distinguished vertex; then either $\boldsymbol{A}$ has Dynkin type $\mathbb{F}_4$ or it is not positive definite.

Proof:
Immediate by Lemma 1.6 by noticing that the symmetrizer is preserved by the elementary matrix operations (inflation, deflation and sign inversion). □

Following [3], for any $\boldsymbol{A} \in \mathbf{qC_n}$ and $G = \mathbf{bigr}\,(\boldsymbol{A})$ let us define a graph morphism *flation* given by $T_{s\,r}\,(G) = \mathbf{bigr}\left(\left(\boldsymbol{T}_{s\,r}^{-(\boldsymbol{A})_{r\,s}}\right)^{\mathrm{T}} \boldsymbol{A}\, \boldsymbol{T}_{s\,r}^{-(\boldsymbol{A})_{s\,r}}\right)$. We say that G is *solid-arched* if all of its arrows have a solid line style.

Lemma 4.2. If G has a distinguished vertex, is solid-arched and contains a dotted edge $u \cdots v$ then either $T_{u\,v}\,(G)$ or $T_{v\,u}\,(G)$ is solid-arched.

Proof:
For simplicity assume that the distinguished vertex x is a sink; the case where x is a source is analogous. We now have three cases depending on the vertices u and v:

1. If neither u nor v are tails of an arrow directed towards x then both $T_{u\,v}\,(G)$ and $T_{v\,u}\,(G)$ are solid-arched.

2. If both u and v are tails of an arrow (i.e. if there exists solid arrows $u \longrightarrow x$ and $v \longrightarrow x$) then $T_{u\,v}\,(G)$ erases the edge $v \longrightarrow x$ whereas $T_{v\,u}\,(G)$ erases $u \longrightarrow v$. Either case yields a solid-arched bigraph.

3. If only one of u and v is the tail of an arrow, assume without any loss of generality that such vertex is u. Then there cannot exists an edge v—x; thus, $T_{v\,u}\,(G)$ leaves the arrow $u \longrightarrow v$ unchanged. □

Lemma 4.3. Let $\boldsymbol{A}$ be an admissible matrix. If $\mathbf{bigr}\,(\boldsymbol{A})$ has a distinguished vertex, then $\boldsymbol{A} \in \mathbf{qC_n}$.

Proof:
If $r \in \{1, \ldots, n\}$ is the distinguished vertex of $\mathbf{bigr}\,(\boldsymbol{A})$ and $D = diag(d_1, \ldots, d_n)$, then if r is a source the matrix D is defined as follows: $d_i = \begin{cases} 2 & \text{if } i = r\,, \\ 1 & \text{if } i \neq r\,. \end{cases}$ If r is a sink then the matrix D is given by $d_i = \begin{cases} 1 & \text{if } i = r\,, \\ 2 & \text{if } i \neq r\,. \end{cases}$ □

4.1. Summary of the suggested algorithm

The following algorithm works in three stages on an input matrix $\boldsymbol{A} \in M_n(\mathbf{Z})$:

Input An integer matrix $\boldsymbol{A} \in M_n(\mathbf{Z})$ with diagonal entries $(\boldsymbol{A})_{i\,i} = 2$.

Stage 1 Preprocess $\boldsymbol{A}$ to check whether it is an admissible matrix; if $\boldsymbol{A}$ is a non-symmetric, check if it has a distinguished vertex, remove any dotted arrow by using flations. Also, during this stage build a data structure which helps locating dotted edges.

Stage 2 Perform the inflations algorithm up to a maximum number of iterations; updating $\boldsymbol{A}$ at each iteration (see Lemma 1.6, Lemma 4.3 and equation (1)) and consider the following, if $\boldsymbol{A}$ is non-symmetric choose $(\boldsymbol{A})_{s\,r} > 0$ such that $T_{sr}(\mathbf{bigr}\,(A))$ is solid-arched (see Lemma 4.2). Finally, give maintenance to the data structure.

Stage 3 Perform at most two graph transversal in order to decide if any Dynkin diagram match the bigraph.

We detail the algorithm below.

4.2. Preprocessing A

This part of the algorithm is composed by two principal steps; the filtrate and the efficient search of dotted edges. After the first step we shall know about the admissibility and symmetry of $\boldsymbol{A} \in M_n(\mathbf{Z})$ and also its distinguished vertex if it has one. In the second step we build a data structure to localize any dotted edge in linear time. After that we transform every dotted arrow of $\mathbf{bigr}\,(\boldsymbol{A})$ into solid. The following algorithm takes a matrix $\boldsymbol{A}$ with diagonal entries $(\boldsymbol{A})_{i\,i} = 2$ and vertices i, j such that $|\,(\boldsymbol{A})_{i\,j}\,| = 2$ as input and returns *distinguished* $\in \{i, j\}$ if one of those is a sink or a source vertex and a variable P where:

- **distinguished:** is initialized with None and will store the possible distinguished vertex if it exists,
- **P:** is a boolean variable which holds the value False if the matrix $\boldsymbol{A}$ has not finite Dynkin type and True otherwise.

4.2.1. Filtering A

We start recovering each vertex i checking step by step if $\left|(\boldsymbol{A})_{i\,j}\right| < 3$ for all j and the locally symmetric conditions. If any of this items is not accomplished then $\boldsymbol{A}$ is not admissible and therefore $\boldsymbol{A}$ has not finite Dynkin type. During this process while the admissible conditions for $\boldsymbol{A}$ holds, check the symmetry and if there exists any non symmetric entry, then check that $\boldsymbol{A}$ has a distinguished vertex for $n \neq 4$. If it is not the case then by Lemma 1.6 $\boldsymbol{A}$ has not finite Dynkin type. If $n = 4$ this condition is not necessary. First of all we have to create the following variables to store this information:

- **Sym:** is a boolean variable which holds the value True if the matrix is symmetric and False if it is not.
- **Admissible:** is a boolean variable that return True if the matrix is admissible and False if it is not.

Algorithm 4.2 contains a detailed procedure for this filtering step.

Algorithm 4.1: Algorithm to find the possible distinguished vertex.

```
Algorithm findistinguished(A, i, j, N, P, distinguished)
    src=issource(A, i, j, N)
    snk=issink(A, i, j, N)
    if src = −1 and snk = −1 and N ≠ 4 then
        P:=False
    if src = −1 and snk ≠ −1 then
        distinguished := snk
    if src ≠ −1 and snk = −1 then
        distinguished := src
    return distinguished, P
Function issink(A, i, j, N)
    z := 1
    sink := i
    while z ≤ N and sink = i do
        if (A)_{iz} ≠ 0 and (|(A)_{iz}| ≠ 2 or |(A)_{zi}| ≠ 1) and z ≠ i then
            sink := −1
        z++
    return sink
Function issource(A, i, j, N)
    z := 1
    source := j
    while z ≤ N and source = j do
        if (A)_{zj} ≠ 0 and (|(A)_{jz}| ≠ 1 or |(A)_{zj}| ≠ 2) and z ≠ j then
            source := −1
        z++
    return source
```

Remark 4.4. Note that if $P :=$ False the admissibility and the symmetry are not necessarily checked in the entire matrix.

4.2.2. Efficient search of dotted and multiple edges

We refer as multiple or parallel edges between the vertices v_i and v_j if $\left|(\boldsymbol{A})_{ji}\right| = \left|(\boldsymbol{A})_{ij}\right| > 1$. Recall that the inflations algorithm searches for a dotted edge $s \cdots r$ at each iteration and then performs an inflation $T_{sr}\,(\mathbf{bigr}\,(\boldsymbol{A}))$. Recall that a flation acts only in column and row r of $\boldsymbol{A}$. Also, notice that no bigraph G with multiple edges can be positive definite (see [3, sec. 1.2] for a short proof). Ideally, the inflations algorithm should search for both the dotted edge or a pair of vertices with a bundle of parallel edges at each iteration. If parallel edges exists then the algorithm can stops immediately: G has no Dynkin type.

Algorithm 4.2: Filtering $\boldsymbol{A}$

Data: An integer matrix $\boldsymbol{A}$ with diagonal entries $(\boldsymbol{A})_{i\,i} = 2$.
Result: Admissible, $\boldsymbol{A}$, Sym, distinguished, P

Admissible:=True, Sym:=True, distinguished:=None, P:=True, $i := 1$
while $i \leq N$ **and** P **do**
    $j := 1$
    while $j \leq N$ **and** P **do**
        if $i \neq j$ **then**
            if $\left(\left|(\boldsymbol{A})_{i\,j}\right| = 2 \text{ and } \left|(\boldsymbol{A})_{j\,i}\right| \neq 1\right)$ **or** $\operatorname{sgn}\left((\boldsymbol{A})_{i\,j}\right) \neq \operatorname{sgn}\left((\boldsymbol{A})_{j\,i}\right)$ **or** $\left|(\boldsymbol{A})_{i\,j}\right| \geq 3$ **then**
                Admissible:=False; P:=False
            if $(\boldsymbol{A})_{i\,j} \neq (\boldsymbol{A})_{j\,i}$ **then**
                Sym:=False
            if $(\boldsymbol{A})_{i\,j} = 2$ **and** *distinguished:*=None **then**
                distinguished, P=`findistinguished`($\boldsymbol{A}, i, j, N, P$, *distinguished*)
            if $\left((\boldsymbol{A})_{i\,j} = 2 \text{ and } \textit{distinguished} \neq \text{None}\right)$ **and** $i, j \neq$ *distinguished* **then**
                P:=False
        $j++$
    $i++$

With a data structure which gives us information about the whereabouts of the dotted edges we may speed the search of the dotted edge. One such data structure is a binary array $\boldsymbol{g} = (g_1, g_2, \ldots, g_n)$ defined by

$$g_i = \begin{cases} 1 & \text{if there exists at least one dotted edge ending in the vertex } i; \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

After performing an inflation $T_{s\,r}(\mathbf{bigr}(\boldsymbol{A}))$ we have to give maintenance to $\boldsymbol{g}$. This is possible by checking rows and columns $i,\ i \in \{s, r\}$ of $\boldsymbol{A}$, if there is another dotted edge ending in vertex i then $g_i = 1$, else $g_i = 0$. Let x be the distinguished vertex of G. If G is a bigraph which is not solid-arched then by applying $T_{u\,u}$ for all dotted arrows $u \dashrightarrow x$ (or $x \dashrightarrow u$ if x is a source) we get an equivalent solid-arched bigraph.

4.3. Inflations for the symmetric case

First assume that $\boldsymbol{A}$ is symmetric. From Theorem 2.7 follows that if $\boldsymbol{A}$ is a positive definite quasi-Cartan matrix then any chain of inflations has length at most:

$$\ell(n) = \begin{cases} \frac{n}{2}\,(n+1) & \text{if } n < 4 \\ n\,(15\,n - 168) + 504 & \text{if } 6 \leq n \leq 8 \\ n\,(n-1) & \text{otherwise} \end{cases} \tag{4}$$

This means that if $\boldsymbol{A}$ is a symmetric admissible matrix but the Inflations Algorithm has not come to an end after $\ell(n)$ iterations then either $\boldsymbol{A}$ is not positive definite or is not a quasi-Cartan matrix.

4.4. Inflations for the non symmetric case

For the non symmetric case another approach can be used. After the Stage 1 if $\boldsymbol{A}$ is non symmetric then $\boldsymbol{A}$ have a distinguished vertex or $n \leq 4$. If $n \leq 4$ we apply the Inflations Algorithm to $\boldsymbol{A}$ if the algorithm has not come to an end after 24 iterations then $\boldsymbol{A}$ has not a finite Dynkin type. Otherwise, let G be solid-arched. By sequentially applying Lemma 4.2 we may use the inflations algorithm without ever introducing a dotted arrow. Thus, the non-symmetric case can be solved by applying inflations on the symmetric bigraph $G - x$ using at most $\ell(n-1)$ iterations.

4.5. Recovering the Dynkin type

Assuming that the main loop of the inflations method ended successfully in less than $\ell(n)$ steps, there will be no dotted edges nor dotted arrows in G; but we still have to check whether G is a Dynkin diagram and output the Dynkin type. In order to do so consider the following classification method:

1. Perform a graph transversal on the underlying graph of G and in doing so compute a list of arrows and a list of the vertices with exactly one or three neighbours; if the transversal reveals any cycle, any parallel edges, any vertex with more than 3 neighbours, or more than one arrow, then G is not a Dynkin diagram. The graph transversal can be performed by using a depth-first search algorithm (see [16] for a detailed explanation).

2. If there exists only a vertex u with exactly 3 neighbours then:

   (a) if G contains any arrow then G is not a Dynkin diagram;

   (b) perform a breadth first search starting at u and for each vertex v compute the distance $d(u, v)$ measured in the number of edges of the shortest path from u to v;

   - if there exists at most one vertex at distance 2 from u then G is $\mathbb{D}_n$;
   - if $6 \leq n \leq 8$, there exists at most one vertex at distance 3 and exactly two vertices at distance 2 from u, then G is $\mathbb{E}_n$.

   (c) G is not a Dynkin diagram.

3. Let x and y be the two vertices with exactly one neighbour (we have established that G is a path since it is connected, contains no cycles, and no vertices with more than two neighbours).

   - If G has exactly one arrow $u \longrightarrow v$:

     (a) If $v \in \{x, y\}$ then G is $\mathbb{B}_n$

     (b) If $u \in \{x, y\}$ then G is $\mathbb{C}_n$

     (c) If $n = 4$ and $u, v \notin \{x, y\}$ then G is $\mathbb{F}_4$

     (d) G is not a Dynkin diagram

   - G is $\mathbb{A}_n$

4.6. Analysis of the suggested algorithm

In this subsection we briefly analyse the algorithm. **Stage 1** was proposed in subsections 4.2 and 4.2.2. By scanning the matrix $\boldsymbol{A} \in \mathbf{qC_n}$ prior to the main loop of the inflations algorithm we may compute $\boldsymbol{g}$ or find a bundle of parallel edges in $\mathrm{O}\left(n^2\right)$ time. The search of a dotted edge can be done in $\mathrm{O}(n)$ time by looking for an i such that $g_i = 1$ and then scanning the i-th row of $\boldsymbol{A}$. Finally, after each inflation T_{sr} we may search for a dotted edge (or parallel edges) ending in r by scanning the r-th row in $\mathrm{O}(n)$ time. From which Stage 1 can be implemented in $\mathrm{O}\left(n^2\right)$ steps. The maximum number of iterations in **Stage 2** was shown in subsections 4.4 and 4.3 $\mathrm{O}\left(n^2\right)$ at cost $\mathrm{O}(n)$ each. Finally, from subsection 4.5 we have that **Stage 3** can be implemented by doing breadth-first search twice, for a total of $\mathrm{O}\left(n^2\right)$ steps. Clearly, Stage 2 is the slowest, with a running time of $\mathrm{O}\left(n^3\right)$; thus, the time complexity of the suggested algorithm is $\mathrm{O}\left(n^3\right)$.

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