GOOD COVERS AND ALGEBRAS ON CONICALLY SMOOTH SPACES

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Abstract. We construct good covers for conically smooth spaces. By a result of Karlsson–Scheimbauer–Walde, this implies that, for every such space X, constructible factorisation algebras on X and disk algebras over X coincide. We also give a simplified proof of that result.

Contents

1. Introduction

Acknowledgments. We thank Tashi Walde for very useful exchanges.

2. Localities of the full unzip

Definition 2.1. $S_X \subset X$.

Proposition 2.2. $S_X \subset X$ *proper constructible bundle with* $\mathrm{Unzip}_{S_X}(X) \cong \mathrm{Unzip}(X)$ *and* $\mathrm{Link}_{S_X}(X) = \partial\,\mathrm{Unzip}(X)$.

Lemma 2.3. *Locally finite good cover on S_X. (?)* **unnecessary - second-countability implies more than paracompactness - see [Lee, Smooth, Thm 1.15]**

Lemma 2.4. *Let X be a smooth manifold with corners. Then for every boundary collar $I\colon \mathbb{R}_{\geq 0} \times \partial X \hookrightarrow X$ there exists a homeomorphism*

$$\alpha_I\colon X^+ \xrightarrow{\cong} \mathbb{R}_{\geq 0} \times X$$

where $X^+ = \bigcup_{r \in \mathbb{R}_{\geq 0}} [0,r] \times \partial X \cup_{\{r\}\times\partial X} X$.

Proof. Let $I\colon \mathbb{R}_{\geq 0} \times \partial X \hookrightarrow X$ be a collar, which is a homeomorphism onto its image. We have $X = I(\mathbb{R}_{\geq 0} \times \partial X) \cup_{I((0,\infty)\times\partial X)} X^\circ$. Similarly, for each r, we have a diffeomorphism

$$\alpha_r\colon X \to X \smallsetminus I([0,r) \times \partial X) = I([r,\infty)\times\partial X) \cup_{I((r,\infty)\times\partial X)} X \smallsetminus I([0,r]\times\partial X)$$

The authors were supported by the Austrian Science Fund (FWF) through Project no. P 37046.

given by $I(t,q) \mapsto I(t+r,q)$ on $I(\mathbb{R}_{\geq 0} \times \partial X)$ and by the identity on $X \smallsetminus \mathrm{Im}(I)$. We suppress the dependence on I in notation. We obtain a well-defined homeomorphism

$$I \cup \alpha_r \colon [0,r] \times \partial X \cup_{\{r\} \times \partial X} X \to X,$$

well-defined since $\alpha_r(q) = \alpha_r(I(0,q)) = I(r,q)$ for $q \in \partial X$, and thereupon, writing $X^{+r} = [0,r] \times \partial X \cup_{\{r\}\times\partial X} X$, the bijection

$$\alpha_I \colon X^+ \to \mathbb{R}_{\geq 0} \times X,$$
$$\alpha|_{X^{+r}} = \{r\} \times I \cup \alpha_r.$$

We equip X^+ with the induced topology, promoting α to a homeomorphism. □

Lemma 2.5. *Let X be a smooth manifold with corners. Then there is a homeomorphism $\mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times X) \cong \mathbb{R}_{\geq 0} \times X$. Consequently, there is a homeomorphism*

$$\mathrm{Unzip}_{C(L)}(C(Z)) \cong \mathbb{R}_{\geq 0} \times \mathrm{Link}_{C(L)}(C(Z))$$

for $L = S_Z$.

Proof. Using Lemma 2.4 it suffices to provide a homeomorphism

$$\phi\colon \mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times X) \to \bigcup_{r \in \mathbb{R}_{\geq 0}} [0,r] \times \partial X \cup_{\{r\}\times\partial X} X.$$

Noting

$$\partial(\mathbb{R}_{\geq 0} \times X) = \{0\} \times X \cup_{\{0\}\times\partial X} \mathbb{R}_{\geq 0} \times \partial X,$$

we define ϕ to be the following map:

$$\mathbb{R}_{\geq 0} \times \{0\} \times X \ni (t,0,x) \mapsto x \in X \subset [0,t] \times \partial X \cup_{\{t\}\times\partial X} X$$
$$\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \partial X \ni (t,s,q) \mapsto (t,q) \in [0,t+s] \times \partial X.$$

This map and its inverse ϕ^{-1} given by

$$[0,r] \times \partial X \ni (t,q) \mapsto (t, r-t, q) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \partial X$$
$$[0,r] \times \partial X \cup_{\{r\}\times\partial X} X \supset X \ni x \mapsto (r,0,x) \in \mathbb{R}_{\geq 0} \times \{0\} \times X$$

are well-defined and continous. Note that ϕ does not nepend on a collar.

The second statement is the special case where $X = \mathrm{Unzip}_L(Z)$ using Proposition 2.2 and that $\mathrm{Unzip}_{C(L)} C(Z) = \mathbb{R}_{\geq 0} \times \mathrm{Unzip}_L(Z)$ and $\mathrm{Link}_{C(L)} C(Z) = \partial\, \mathrm{Unzip}_{C(L)}(C(Z))$. □

Remark 2.6. In the situation of Lemma 2.4 we will regard X^+ as a smooth manifold with corners with respect to the smooth structure induced by α_I. Up to equivalence, this structure does not depend on the choice of collar. Similarly, in the situation of Lemma 2.5 we will regard $\mathbb{R}_{\geq 0} \times \mathrm{Link}_{C(L)} C(Z)$ as a smooth manifold with corners, tautologically diffeomorphic to $\mathrm{Unzip}_{C(L)} C(Z)$ with respect to the induced smooth structure.

Construction 2.7. Let X be a smooth manifold with corners of dimension n, let $\{(U,\phi_U)\}$ a cover of X by coordinate neighbourhoods where each $\phi_U \colon \mathbb{R}^{n-c_U} \times \mathbb{R}^{c_U}_{\geq 0}$ is a homeomorphism, and let $\{\rho_U \colon X \to [0,1]\}$ be a partition of unity subordinate to this cover. Recall that a collar $I = I(\{U,\phi_U,\rho_U\})\colon \mathbb{R}_{\geq 0} \times \partial X \hookrightarrow X$ is then constructed by defining to be the flow along the nowhere-vanishing inward-pointing vector field $V = \sum \rho_U V_U$ where, in local coordinates, $V_U = \sum_{1 \leq i \leq c_U} \partial_i$ where $\{\partial_i\}$ is the standard basis of $\mathrm{T}_0\mathbb{R}^{c_U}_{\geq 0} \subset \mathrm{T}_0(\mathbb{R}^{n-c_U} \times \mathbb{R}^{c_U}_{\geq 0})$.

Let X and $I = I_{\{U,\phi_U,\rho_U\}}$ be as above. Then there is a canonically induced a collar

$$I^+ = I(\{\mathbb{R}_{\geq 0} \times U, \mathrm{id}_{\mathbb{R}_{\geq 0}} \times \phi_U, \rho_U \circ \mathrm{pr}_X\})\colon \mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times \partial X) \hookrightarrow \mathbb{R}_{\geq 0} \times X$$

on $\mathbb{R}_{\geq 0} \times X$, where $\mathrm{pr}_X\colon \mathbb{R}_{\geq} \times X \to X$ is the coordinate projection. It is the flow along the vector field $V^+ = \sum(\rho_U \circ \mathrm{pr}_X) \cdot V_U^+$ where $V_U^+ = \partial_s + V_U$ where ∂_s is the standard basis of $\mathrm{T}_0\mathbb{R}_{\geq 0} \subset \mathrm{T}_0(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-c_U} \times \mathbb{R}^{c_U}_{\geq 0})$.

Lemma 2.8. *Let X be a smooth manifold with corners and let $I^+\colon \mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times \partial X) \hookrightarrow \mathbb{R}_{\geq 0} \times X$ be as in Construction 2.7. Then*

$$I^+ = \alpha_I \circ \phi$$

where α_I is as in Lemma 2.4 and ϕ is as in the proof of Lemma 2.5.

Proof. We observe the restrictions

$$\mathbb{R}_{\geq 0} \times \{0\} \times X \xrightarrow{\alpha_I \circ \phi} \mathbb{R}_{\geq 0} \times X$$
$$(t, 0, x) \mapsto \begin{cases} (t, I(t_x + t, q)), & x = I(t_x, q) \\ (t, x), & x \in X \smallsetminus \mathrm{Im}(I) \end{cases}$$

and

$$\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \partial X \xrightarrow{\alpha_I \circ \phi} \mathbb{R}_{\geq 0} \times X$$
$$(t, s, q) \mapsto (t + s, I(t, q)).$$

Both maps are the flow (for time t) along the vector field $V^+ = \sum(\rho_U \circ \mathrm{pr}_X) \cdot (\partial_s + V_U)$. □

In the following, $\mathbb{D}^n \subset \mathbb{R}^n$ denotes the unit open n-disk and $C^{<1}(Z) = * \amalg_{\{0\}\times Z} [0,1) \times Z$.

Definition 2.9. We say an embedded basic $\phi\colon \mathbb{R}^k \times C(Z) \hookrightarrow X$ in a conically smooth space X is *extendable* if there exists an embedding $\widehat{\phi}\colon \mathbb{R}^n \times C(Z) \hookrightarrow X$ such that ϕ factors as $\phi\colon \mathbb{R}^k \times C(Z) \to \mathbb{D}^n \times C^{<1}(Z) \hookrightarrow \mathbb{R}^n \times C^{<1}(Z) \xhookrightarrow{\widehat{\phi}} X$ where the first map is the isomorphism given by the cartesian product of the isomorphisms $\mathbb{R}^n \to \mathbb{D}^n$, $x \mapsto \frac{x}{|x|+1}$ and $C(Z) \to C^{<1}(Z)$, $(t, z) \mapsto (\frac{t}{t+1}, z)$.

Definition 2.10. Suppose $\mathcal{U}$ is a cover of a topological space X which is closed under finite intersections. We say $\mathcal{U}$ is *generated* by a cover $\mathcal{V}$ and write $\mathcal{U} = \langle \mathcal{V} \rangle$ if every member of $\mathcal{U}$ is a finite intersection of members of $\mathcal{V}$.

Lemma 2.11. *Every smooth manifold M has a good cover $\mathcal{U}$ which is generated by extendable basics.*

Proof. Equip M with a riemannian metric. We can put $\mathcal{U} = \langle \mathcal{V} \rangle$ for $\mathcal{V} = \{D_p\}_{p \in M}$ where D_p is the convex disk which is the interior of the image of a ball in T_pM under the exponential map, with a radius that is strictly smaller than the radius of injectivity. □

Lemma 2.12. *Suppose $U \subset C(Z)$ be conically smooth basics with $S_{C(Z)} \cong U$, and suppose that $U \cong C(L)$ is extendable. Let $\pi\colon \mathrm{Link}(C(Z))|_U \to U$ denote the link projection of* $\mathrm{Unzip}(C(Z))$ *over U. Then there is an isomorphism*

$$\partial\overline{\pi^{-1}U} \cong \mathrm{Link}_L(Z)$$

and an induced conically smooth collar

$$(0,1] \times \partial\overline{\pi^{-1}U} \hookrightarrow \overline{\pi^{-1}U}$$

which is a refinement onto its image.

Proof. Up to isomorphism we have that $U = C^{<1}(L) \subset C^{<1}(Z) \subset C(Z)$ is induced by the inclusion $L = S_Z \subset Z$. Abbreviating $S = S_{C(Z)}$, the inclusion of U factors as $U \subset S = C(L)$. Now, using Proposition 2.2 we have $\operatorname{Unzip} C(Z) = \operatorname{Unzip}_S C(Z) = \mathbb{R}_{\geq 0} \times \operatorname{Unzip}_L Z$ and so

$$\begin{aligned}\operatorname{Unzip} C(Z)|_U &= \operatorname{Link}_{C(L)} C(Z)|_{C^{<1}(L)} \\ &= \{0\} \times \operatorname{Unzip}_L Z \cup_{\{0\}\times \operatorname{Link}_L Z} [0,1) \times \operatorname{Link}_L Z\end{aligned}$$

since the projection $\pi\colon \operatorname{Link}_{C(L)} C(Z) = \{0\} \times \operatorname{Unzip}_L Z \cup_{\{0\}\times \operatorname{Link}_L Z} \mathbb{R}_{\geq 0} \times \operatorname{Link}_L Z \to C(L) = * \amalg_{\{0\}\times L} \mathbb{R}_{\geq 0} \times L$ is given by mapping all of $\{0\} \times \operatorname{Unzip}_L Z$ to $*$ and on $\mathbb{R}_{\geq 0} \times \operatorname{Link}_L Z$ by $\mathrm{id}_{\mathbb{R}_{\geq 0}} \times \pi'$ where $\pi'\colon \operatorname{Link}_L Z \to L$ is the link projection. Thus

$$\pi^{-1}\overline{U} = \pi^{-1}C^{\leq 1}(U) = \{0\} \times \operatorname{Unzip}_L Z \cup [0,1] \times \operatorname{Link}_L Z$$

and consequently

$$\pi^{-1}\partial\overline{U} = \pi^{-1}\{1\} \times L = \{1\} \times \operatorname{Link}_L Z,$$

proving the first claim. The collar is immediate. □

Example 2.13.

Corollary 2.14.

3. Good covers

Definition 3.1. Let X be a CSS. A *good cover* of X is an open cover $\mathcal{U}$ of X consisting of basics whose finite intersections are also in $\mathcal{U}$.

Definition 3.2. Let Y be a smooth manifold with or without boundary, equipped with a riemannian metric. We call a cover of Y *geodesically convex* if it consists of geodesically convex basics.

Theorem 3.3. *Let X be a CSS over a depth-1 poset. Then X has a good cover.*

Proof. Without loss of generality, suppose X is stratified over $[1]$ and let $M = X_0$ and $N = X_1$. Recall the blow-up of M, the smooth manifold

$$\operatorname{Unzip} \cong L \amalg_{L\times(0,\infty)} N$$

with boundary $L = \partial\operatorname{Unzip}$. We write $\pi\colon L \to M$ for the accompanying proper fibre bundle. Let us equip Unzip with a riemannian metric which splits along the boundary.

Using the paracompactness of M, let $\mathcal{U}$ be a locally finite good cover of M which trivialises π. For each $U \in \mathcal{U}$, let $\epsilon_U > 0$ be the radius of injectivity in the normal direction on the compact closure $\overline{U}$ so that there is a smooth embedding

$$I_U\colon \pi^{-1}U \times [0,\epsilon_U) \hookrightarrow \operatorname{Unzip}$$

whose restriction $I_U|_{\pi^{-1}U\times\{0\}}$ to $\pi^{-1}U \subset L$ is given by the boundary inclusion, and the path $I_U(q,-)\colon [0,\epsilon_U) \hookrightarrow \operatorname{Unzip}$, for every $q \in \pi^{-1}U$, is the minimising geodesic given by the normal exponential map. More precisely, for ν the inward-pointing unit normal vector field along the boundary, we set $I_U(x,t) = \exp_x(t\nu_x)$.[1] Let

$$C = \bigcup_{U\in\mathcal{U}} \mathrm{Im}(I_U)$$

be the induced 'collar' and consider the cover

$$\mathcal{C} = \{I_U(\pi^{-1}U \times [0,\delta)) : U \in \mathcal{U},\ \delta \le \epsilon_U\}$$

of C. Let us write $C_{U,\delta} = I_U(\pi^{-1}U \times [0,\delta)) \in \mathcal{C}$. Note that $\mathcal{C}$ is closed under finite intersections since so is $\mathcal{U}$: we have $\pi^{-1}U \cap \pi^{-1}V = \pi^{-1}(U\cap V)$ and

$$C_{U,\delta} \cap C_{V,\delta'} = C_{U\cap V,\min(\delta,\delta')} \in \mathcal{C}$$

since $\min(\delta,\delta') \le \epsilon_{U\cap V}$ for $\delta \le \epsilon_U$, $\delta' \le \epsilon_V$.

Consider now the collection $\mathcal{V}$ of those convex geodesic disks V in N such that $V \cap C_{U,\delta}$ is either empty or a convex geodesic disk for every $C_{U,\delta} \in \mathcal{C}$. Then $\mathcal{V}$ is a good cover of N. To prove this, it suffices to show that every $p \in C$ has a neighbourhood $V_p \in \mathcal{V}$, since convex geodesic disks are closed under finite intersections and so $V \cap V' \cap C_{U,\delta} = V \cap V'' \in \mathcal{V}$. Let now $p \in I_U(\pi^{-1}U \times [0,\delta)) \in \mathcal{C}$, which uniquely determines a point $l_p = \pi(\mathrm{pr}_1(p)) \in \pi^{-1}U \subset L$. Let $W \subset \pi^{-1}U$ be a convex geodesic disk neighbourhood of l_p. Now, since $\mathcal{U}$ is locally finite, so is the cover $\widetilde{\mathcal{C}} = \{I_U(\pi^{-1}U \times [0,\epsilon_U))\}$ of C (which is not necessarily closed under finite intersections), so in particular p is contained within finitely many members $I_{U_i}(\pi^{-1}U_i \times [0,\epsilon_i)) \in \widetilde{\mathcal{C}}$. We necessarily have $p \in I_U(\pi^{-1}U \times [0, \min_i(\epsilon_i)))$ with $U \in \{U_i\}_i$ an open satisfying $\epsilon_U = \min_i(\epsilon_i)$. Then $W \times [0,\min_i(\epsilon_i)) \ni p$ is a convex geodesic half-disk since the riemannian metric on Unzip splits along the boundary, and so we have $p \in V_p = W \times (0, \min_i(\epsilon_i)) \in \mathcal{V}$.

Let us now observe that $\pi^{-1}U \cong L_p \times U$ for $p \in U$ implies that each

$$C(L_p) \times U \cong U \amalg_{\pi^{-1}U} C_{U,\delta} \subseteq X$$

is a basic in X. Thus, writing $\widehat{C_{U,\delta}} = U \amalg_{\pi^{-1}U} C_{U,\delta}$, we obtain that

$$\widehat{\mathcal{C}} = \{\widehat{C_{U,\delta}} : C_{U,\delta} \in \mathcal{C}\}$$

is a cover by basics of the 'tubular neighbourhood'

$$\widehat{C} = \bigcup_{U\in\mathcal{C}} \widehat{C_{U,\epsilon_U}} \subseteq X$$

of M. It is closed intersections since so is $\mathcal{C}$. Finally, since for $V \in \mathcal{V}$ we have $\widehat{C_{U,\delta}} \cap V = C_{U,\delta} \cap V \in \mathcal{V}$, we conclude that

$$\widehat{\mathcal{C}} \cup \mathcal{V}$$

is a good cover of X. □

[1]A global ϵ need not exist unless Unzip is of bounded geometry; see e.g. Schick [Sch01].

4. A proof of...

References

[Sch01] T. Schick. 'Manifolds with Boundary and of Bounded Geometry'. *Mathematische Nachrichten* 223.1 (2001), 103–120.

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