Factorization Algebras and Factorization Homology on Stratified Spaces

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Abstract

We endeavor to fully describe (locally constant) factorization algebras on stratified spaces, by giving a description of the ∞-operad that underlies them through the use of factorization homology. We also give a result about a collar-gluing property that the ∞-categories of (locally constant) factorization algebras satisfy, which enables novel descriptions of these on more complicated spaces. Finally, we outline a classification scheme for locally constant factorization algebras on some stratified manifolds. Parts of this work were adapted from a Master's thesis dissertation submitted to ETH Zürich for the first author.

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0 Introduction

In this work we will explore the concepts of factorization algebras and factorization homology over stratified spaces. These two concepts, which at first sight will looks very different, will turn out to be closely related. We will strive to keep the discussion at the level of stratified spaces, while making a comparison to results in the smooth case. Some results we will meet have been proven in the smooth case, but with methods which do not (at the present moment) generalize to the stratified case. We will endeavor to point these out and give proofs that hold more generally. The outline of the text is as follows:

§ 1 Stratified Spaces: The first section deals with setting up the theory of stratified spaces that we will use. It is essentially a recap of the necessary results from the work of Ayala, Francis and Tanaka (AFT) [AFT17b], which is one of our main sources. The idea and fundamentals of stratified spaces have existed for a long time in different incarnation beginning with the work of Whitney [Whi92]. The reason why we choose to work with this particular version of stratified spaces is due to them supporting a notion of conical smoothness that is a generalization of smoothness of manifolds. Furthermore, the theory of factorization homology has been developed with these spaces in mind, which eases the process of extending certain results to the stratified case much easier. We will not go into the full details that this incarnation of the theory has to offer, but instead we will only mention results that we will make use of later on. Beyond conical smoothness, this will include tangential structure and its stratified generalization, together with the packaging of both of these data into the concept of an ∞-category of basic singularity types. For the proofs of these statements the reader is encouraged to consult [AFT17b].

§ 2 Factorization Homology: The second section will present the theory of factorization homology. Similar to the first section this will mostly be a recap of useful results developed by Ayala, Francis and Tanaka. Specifically, the stratified space results can be found in [AFT17a], while the results for topological manifolds can be found in [AF15]. For the purposes of the theory we will need the concept of (structured) disk algebras, since this will be the algebraic input. Factorization homology will then combine this with the data of a specific kind of (possibly stratified) manifold into an object that can be viewed either as an invariant of the manifold or an invariant of the algebra. We will explore the key example of factorization homology over the closed, oriented interval $[-1, 1]$, and see how this gives rise to the algebraic concept of the two-sided bar construction. More generally, this underpins the theory because it codifies the property of $\otimes$-excision. Finally, we will also state a classification result [AFT17a, prop.4.8] about a certain type of disk algebras with stratified structure.

§ 3 Factorization Algebras: The third section introduces the concept of factorization algebras, together with its different variations, like prefactorization algebras and locally constant factorization algebras. Of these, the locally constant factorization algebras will be the ones that capture the different possible stratified structures that a manifold can have, and they will be a large focus of this work. The definitions of these will not be novel. They can be found in the works [CG16], [Gin15] and even [AF19]. In fact, in the slightly different context of vertex algebras the idea can be traced back to [BD04]. However, all of the above works take a slightly different perspective in their presentation of the definitions, which we hope to unite in this work. The key result of this section is the following:

Theorem (3.19)**.** *There is an equivalence of ∞-categories between locally constant factorization algebras over a stratified manifold M and $\mathcal{D}\mathsf{isk}_{/M}$-algebras*

$$\int : \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C}), \tag{1}$$

given by factorization homology,

as well as its Corollary 3.21 for the case of ordinary factorization algebras. These show that the ∞-operad governing factorization algebras can be taken to be the slice ∞-operad $\mathcal{D}\mathsf{isk}_{/M}$ (or $\mathsf{Disk}_{/M}$ for the ordinary case). We then introduce some operations that we can do with factorization algebras like the pushforward and restriction, as well as state a result for locally constant factorization algebras on product spaces, which confirms a conjecture of [Gin15]:

Proposition (3.39)**.** *Let M and N be stratified manifolds. There is an equivalence of ∞-categories*

$$\bar{\pi} : \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M\times N}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_N(\mathcal{C})). \tag{2}$$

Finally, we will examine some examples of factorization algebras whose construction is known, specifically, locally constant factorization algebras on the intervals $(-1,1)$, $[-1,1)$ and $[-1,1]$ as well as Euclidean spaces $\mathbb{R}^n$ more generally.

§ 4 Collar-gluings and Disk Algebras: The fourth section presents a novel result about the behavior of certain types of factorization algebras under the operation of collar-gluing in the manifold variable. It essentially shows that we can reproduce factorization algebras on a stratified manifold if we know them on some particular pieces of it together with the data of how to glue them. Formally, we have

Theorem (4.1)**.** *Given a collar-gluing of stratified manifolds $f : M \to [-1,1]$, the ∞-category of locally constant factorization algebras is equivalent to the pullback of ∞-categories*

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C}) \simeq \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M_-}(\mathcal{C}) \underset{\mathcal{F}\mathsf{Alg}^{\mathsf{lc}} M_0\times\mathbb{R}(\mathcal{C})}{\times} cfa_{M_+}(\mathcal{C}). \tag{3}$$

The results of the previous section a very important in the proof of this statement, because we heavily use the properties of factorization homology. A similar version holds for ordinary factorization algebras in the for of Corollary 3.21. This result hugely expands the number of manifolds for which the ∞-category of locally constant factorization algebras can be specified. As a straightforward example, it gives a way to work on, and classify, factorization algebras on all higher spheres S^n. Without it, only the circle has been described to date ([Gin15]) using equivariant factorization algebras and inheritance from covering spaces.

§ 5 Classifying $\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}$s on Defect Manifolds: The fifth section focuses on the stratified structure and gives a way to express what effect this has on the locally constant factorization algebras. Specifically, we look at the data of a smooth manifold M with a distinguished, properly embedded submanifold Σ (also known as a defect submanifold) as the data of a stratified manifold M_Σ. We will show that, along with giving algebras associated to the different pieces, locally constant factorization algebras on these manifolds, in some (precise) sense, also encode a module structure associated with Σ.

Further Questions: The exploration below naturally leads to questions that are beyond the scope of this work. As we will see, factorization algebras only need a topological space to be defined, but locally constant factorization algebras also capture information about the stratified structure. Neither of these, however, capture information about the tangential structure of the manifold, so the question arises:

- Does there exist a version of factorization algebras that usefully capture information both about the stratified structure of a manifold and about its tangential structure?

The recent thesis [Peñ22] could provide a fruitful direction of exploration of the former concept. In a similar vein, [CG16] explain why it's important to have the concept of a G-equivariant factorization algebra, where G is a Lie group. In particular, according to them, this asks for some definition of a smooth action of G on factorization algebras.

- Is there a suitable way to encompass this notion in the formalism presented here?

The beginnings of examining this idea can already be found in the thesis [Mur20], however the development there, while important, is still preliminary.

0.1 Conventions on ∞-categories

The ∞-categories in this work will use the quasi-category model of ∞-category theory introduced by Joyal in [Joy02], which is based on the simplicial sets introduced by [BV]. The work by Lurie in [Lur09a] and [Lur] greatly builds on this theory, and questions about fundamental objects or constructions that are in this text will always be answered in those references. This choice is because a great deal of the theory of factorization homology, as well as factorization algebras deal with ∞-categories of functors, which are most easily defined for quasi-categories.

It is also important to note that we will not be making a notational distinction between topological categories, Kan enriched categories and ∞-categories because of the functors

$$\{\mathsf{Top}\text{—categories}\} \xrightarrow{\mathsf{Sing}} \{\mathsf{Kan}\text{—categories}\} \xrightarrow{\mathsf{N}} \mathfrak{Cat}_\infty, \tag{4}$$

which are both Quillen–equivalences in appropriate ways ([JT07; Ber10]). In particular, we will not distinguish notationally the nerve of an ordinary category from the ordinary category itself. Similarly, we will not distinguish between (topological) multicategories and their ∞-operadic nerve.

1 Stratified Spaces

The spaces that this work will be set in will very often be stratified spaces. Thus, we first need to define what we mean by this concept. There are multiple definitions of stratified spaces in the literature, all of which have their advantages and disadvantages. For our purposes, we will follow the account of [AFT17b]. A compatible source is also [Lur, sec.A.5]. The majority of the work on factorization homology has so far been elaborated using this particular set of definitions, and similarly the stratified spaces used in the literature on factorization algebras are usually subsumed too. Here we will only recap the definitions, results and examples that will be of use to us. Further details are, of course, discussed in the original [AFT17b], where one can find great explanations of the definitions that are needed, and detailed proofs of the theorems that can be proven.

1.1 Fundamentals of Stratified Spaces

Definition 1.1. We will regard posets as topological spaces by declaring that a map $P \to P'$ is continuous if, any only if it is a map of posets. This means that we declare a subset $U \subset P$ to be open if for all $a \in U$, any $b \geq a$ is also in U. This gives a fully faithful functor

$$\mathsf{Poset} \hookrightarrow \mathsf{Top}. \tag{5}$$

Definition 1.2. Let P be a poset. A P-*stratified space* is a topological space X together with a continuous map $X \to P$, which we call the *stratification* of X. The poset P is called the *stratifying poset.* We will sometimes refer to X as the underlying topological space of a stratified space. Furthermore, we denote the preimage of $p \in P$ as X_p and call it the p'th stratum of X[1].

Example 1.3. Any CW complex X can be made into a stratified space $X \to \mathbb{N}$, which sends all points in $X_{\leq k} \setminus X_{\leq k-1}$ to k, where $X_{\leq k}$ is the k-skeleton of X.

Example 1.4. Inspired by the previous example, any topological space X with a filtration by closed subsets $\emptyset \subset X_{\leq 0} \subset \cdots \subset X_{\leq n} = X$, is a stratified space, because the filtration induces a map $X \to \mathbb{N}$, which sends $X_{\leq i} \setminus X_{\leq i-1}$ to i.

Example 1.5. Consider the poset $[1] = \{0 < 1\}$, and the topological space $\mathbb{R}_{\geq 0}$. The standard stratification that we will give to this space is given by $\mathbb{R}_{\geq 0} \to [1]$, that sends $0 \mapsto 0$ and $0 \neq x \mapsto 1$. This space will play an important role for factorization algebras later on.

Example 1.6. Any topological space X can be stratified by its connected components, with P discrete and $|P| = \pi_0(X)$.

Remark 1.7. Given two stratified spaces $X \xrightarrow{s} P$ and $X' \xrightarrow{s'} P'$, we can form the product stratified space $X \times X' \xrightarrow{s \times s'} P \times P'$, where the partial order on the product poset is given by $(p, p') \leq (q, q') \iff (p \leq q) \wedge (p' \leq q')$.

Definition 1.8. Let $X \to P$ and $X' \to P'$ be two stratified spaces. A *continuous stratified map* (or *map of stratified spaces*) is a commutative diagram in Top of type

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ P \arrow[r] & P'. \end{tikzcd} \tag{6}$$

Example 1.9. Every stratified space $(X \to P)$ has a continuous stratified map that forgets the stratification $(X \to P) \to (X \to *)$, giving data equivalent to an unstratified topological space.

Definition 1.10. A continuous stratified map $(f : X \to X', g : P \to P')$ is an *open embedding* if $f : X \to X'$ is an open embedding of topological spaces.

Remark 1.11. The above definitions give us a natural notion of an open cover $\{(U_i \to P_i) \hookrightarrow (X \to P)\}_{i \in I}$ of a stratified space $(X \to P)$, namely, whenever both $\{U_i \hookrightarrow X\}_{i \in I}$ and $\{P_i \hookrightarrow P\}_{i \in I}$ are open covers.

Next, we construct the cone of a stratified space. This plays a very important role in the theory of conically smooth manifolds, because it provides the prototypical departure from smoothness. In other words, the local structure of these will always look like a cone over some space with a possible further thickening by a factor $\mathbb{R}^k$, just as the local structure of smooth manifolds is given by $\mathbb{R}^k$. The cones will be the model spaces for singularities.

Definition 1.12. Let $X \to P$ be a stratified space. Its (*open*) *cone* $\mathsf{C}(X \to P)$ is the stratified space constructed as follows. At the level of topological spaces we define the pushout in Top

$$\mathsf{C}(X) := * \coprod_{X \times \{0\}} X \times \mathbb{R}_{\geq 0}. \tag{7}$$

At the level of posets we define the pushout in Poset

$$\mathsf{C}(P) := * \coprod_{P \times \{0\}} P \times [1] \cong P^{\triangleleft}. \tag{8}$$

The standard stratification of $\mathbb{R}_{\geq 0}$ together with the obvious map $* \to *$ induce a stratification $\mathsf{C}(X) \to \mathsf{C}(P)$ as a map between pushouts.

[1]Sometimes the name stratum is reserved for the connected components of X_p if strata are required to be connected.

Example 1.13. The open cone $\mathsf{C}(S^k)$ on the k-sphere is canonically identified with $\mathbb{R}^{k+1}_*$, which is the stratified space with stratification given by the filtration $\{0\} \subset \mathbb{R}^{k+1}$ as in Example 1.4.

The following will be used in the definition of the ∞-category $\mathcal{M}\mathsf{fld}$ of stratified manifolds.

Definition 1.14. Let $X \to P$ be a stratified space. Its *closed cone* is the stratified space $\overline{\mathsf{C}}(X) \to P^{\triangleleft}$ with $\overline{\mathsf{C}}(X) = * \amalg_{X \times \{0\}} X \times [0,1]$ and the obvious stratification.

The equivalent of a topological manifold in the stratified setting is a C^0 *stratified space*. We will not be fully precise in the definition of these, since this would stray us from our main goal. We will however cite theorems that show that the spaces we consider in this work are C^0 stratified spaces. The following definition is partly cyclic, but we cite it because it provides good intuition for the structure of C^0 stratified spaces.

Definition 1.15. A C^0 *basic*, which we will also often call a *disk*, is a C^0 stratified space of the form $\mathbb{R}^k \times \mathsf{C}(Z)$, where $k \geq 0$, $\mathbb{R}^k$ has the trivial stratification, and Z is a compact C^0 stratified space.

Theorem 1.16 ([AFT17b, Lemma 2.2.2])**.** *Let X be a stratified, second countable, Hausdorff space, and consider the collection of open embeddings*

$$\{U \hookrightarrow X\}, \tag{9}$$

where U ranges over the C^0 basics. Then this collection forms a basis for the topology of X if and only if X is a C^0 stratified space.

Theorem 1.17 ([AFT17b, Corollary 2.3.5])**.** *Let $X \to P$ be a C^0 stratified space. For any $p \in P$, the stratified spaces $X_{\leq p}$, X_p and $X_{\nleq p}$, defined as the relevant preimages, are C^0 stratified spaces. Furthermore, X_p is even a topological manifold.*

For topological spaces we have the concept of (Čech–Lebesgue) covering dimension $\mathsf{dim}_x(X)$ at a point $x \in X$. C^0 stratified spaces have an additional, related concept called depth. It plays a key role in describing these spaces.[2] As the name suggests, it conveys information about how deep the stratification is. For instance, if the stratification is given by a filtration as in Example 1.3, then it coincides with the length of the filtration, or equivalently, with the length of the longest non-trivial chain of arrives in the stratifying poset. Similarly to dim it has a pointwise and a global version.

Definition 1.18. Given a C^0 stratified space $(X \xrightarrow{s} P)$ the *depth at x* is

$$\mathsf{dpt}_x(X) := \mathsf{dim}_x(X) - \mathsf{dim}_x(X_{s(x)}). \tag{10}$$

The depth of X, $\mathsf{dpt}(X)$, in general, is the supremum of the depths over all points. Just as with dimension, the convention is that $\mathsf{dpt}(\emptyset) = -1$.

The 'smooth' version of the above concepts also exists and will be the one that is of main interest to us. As was the case for C^0 stratified spaces, we will not give the precise definition of a *conically smooth stratified space* (or *stratified manifold* for short), since the exact definition is involved. The idea, though, is the same as for smooth manifolds.

A stratified manifold is a C^0 stratified space M equipped with an atlas

$$\{\mathbb{R}^{k_\alpha} \times \mathsf{C}(Z_\alpha) \hookrightarrow M\}_\alpha, \tag{11}$$

[2] For example, we said that Theorem 1.16, as a definition, would be circular. One way to resolve this is by induction on depth since $\mathsf{dpt}(\mathbb{R}^n \times \mathsf{C}(Z)) = \mathsf{dpt}(\mathsf{C}(Z)) = \mathsf{dpt}(Z) + 1$

whose elements we call basics. This atlas has to be a basis for the topology of X, and the transitions maps, open embeddings among basics, have to be conically smooth. General maps of stratified manifolds will then be conically smooth if their representatives in basics are conically smooth. The reason why the definition is involved is, because unlike in the smooth case the presence of the compact spaces Z_α means that we have to use induction on depth to define conical smoothness.

In lieu of a definition we give the following illuminating example:

Example 1.19 ([AFT17a, Example 1.2]). Smooth manifolds fall under the definition of stratified manifolds as those that only have one stratum. A smooth map $f : Z \to Z'$ between compact smooth manifolds gives rise to a conically smooth map $\mathsf{C}(f)$ between the cones of these manifolds. If we also have a smooth map between Euclidean spaces $g : \mathbb{R}^n \to \mathbb{R}^{n'}$ then the map $g \times \mathsf{C}(f) : \mathbb{R}^n \times \mathsf{C}(Z) \to \mathbb{R}^{n'} \times \mathsf{C}(Z')$ is conically smooth. This already allows us to describe stratified manifolds of depth 1.

The next classes of conically smooth maps will be present throughout this work.

Definition 1.20. Let $f : M \to N$ be a conically smooth map.

1. f is an *open embedding* if it is an open map that is conically diffeomorphic onto its image.
2. f is a *refinement* if it's a homeomorphism of underlying topological spaces, and its restriction to each stratum of M is an embedding.
3. Let P_N be the stratifying poset of N. Then f is a *constructible bundle* if, for each $p \in P_N$, the restriction $f|_p : M|_{f^{-1}N_p} \to N_p$ is a fiber bundle of stratified spaces. The map f is a *fiber bundle* if there exists a cover $\{U \hookrightarrow N\}$ of N consisting of open embeddings such that the restriction of f along U is of type $\mathsf{pr}\colon F_U \times U \to U$ for a stratified manifold F_U.

Remark 1.21. We will also on occasion use the concept of *weakly constructible bundle*, which is a continuous stratified map $f : M \to N$ which is a constructible bundle out of a refinement of M, i.e. there is a diagram $M \xleftarrow{r} M' \xrightarrow{s} N$, where there is an equality $f = s \circ r$ of the underlying continuous maps. Constructible bundles are obviously also weakly constructible. Their use owes to the Pushforward Theorem [AFT17a, Theorem 2.25], which states in particular that algebras may be pushed forward along weakly constructible bundles.

We will now introduce two versions of the category of stratified manifolds that will be ubiquitous throughout this work.

Definition 1.22. The ordinary category of stratified manifolds Mfld is the category whose objects are stratified manifolds and whose morphisms are open embeddings between them.

For the next definition we notice that all standard simplices are stratified manifolds, obtained as $\Delta^n \cong \overline{\mathsf{C}}^n(*) \to \overline{\mathsf{C}}^n(\{0\}) \cong [n]$.

Definition 1.23. $\mathcal{M}\mathsf{fld}$ is the ∞-category whose objects are stratified manifolds and whose morphisms are open embeddings among them. It is presented as the Kan-enriched category of stratified manifolds by taking the Kan complex of morphisms between two objects M and N to be defined by

$$\mathsf{Hom}_{\mathcal{M}\mathsf{fld}}(M, N) : [p] \mapsto \mathsf{Emb}_{\Delta^p}(M \times \Delta^p, N \times \Delta^p), \tag{12}$$

where $\mathsf{Emb}_{\Delta^p}(M \times \Delta^p, N \times \Delta^p)$ denotes the embeddings from $M \times \Delta^p$ to $N \times \Delta^p$ that commute on the nose with the projections to Δ^p.

Remark 1.24. The above definition is not immediately parsable, but it can be shown that reducing to the case of smooth manifolds this definition gives equivalent data to the topological enrichment which to morphism spaces assigns the compact-open topology.

Remark 1.25. There is an obvious inclusion $\mathsf{Mfld} \to \mathcal{M}\mathsf{fld}$, which is the identity on objects. In terms of topological categories these categories differ only by the topology of their morphism spaces. The inclusion as a map on morphism spaces is the map from a set with the discrete topology to the same set with a different topology.

Definition 1.26. The ∞-category of basic singularity types $\mathcal{B}\mathsf{sc} \subset \mathcal{M}\mathsf{fld}$ is the full ∞-subcategory of those objects that are basics.

1.2 Stratified Manifolds with Tangential Structure

Tangential structure can be part of the input for factorization homology. Thus, in this section, we will recall the concept of tangential structure, first starting with smooth manifolds and then continuing to the stratified case.

Definition 1.27. Let $\mathcal{M}\mathsf{fld}_n$ be the ∞-category given by the topological category whose objects are smooth n-dimensional manifolds and whose morphisms are embeddings between them, where the morphism spaces have the compact-open topology. It is endowed with the symmetric monoidal structure given by disjoint union.

Given a smooth, n-dimensional manifold M, (the frame bundle of) its tangent bundle is a principal $\mathsf{GL}(n)$-bundle whose bundle isomorphism class corresponds to the homotopy class of its classifying map

$$M \xrightarrow{\tau_M} \mathsf{BO}(n) \xrightarrow[\simeq]{\text{Gram--Schmidt}} \mathsf{BGL}(n), \tag{13}$$

called the *tangent classifier*. Here, $\mathsf{B}G$ is the classifying space of the (topological) group G. Giving the manifold a G-structure, where G is a Lie group with a group homomorphism to $\mathsf{GL}(n)$ (or to $\mathsf{O}(n)$), is to find a lift ϕ of τ_M and a homotopy that commutes

$$\begin{tikzcd} & \mathsf{B}G \arrow[d] \\ M \arrow[ur, dashed, "\phi"] \arrow[r, "\tau_M"'] & \mathsf{BO}(n). \end{tikzcd} \tag{14}$$

A similar picture applies in the stratified case. As a first step, [AF15, Corollary 2.13] shows that $M \mapsto \tau_M$ can be presented as the symmetric monoidal functor

$$\tau : \mathcal{M}\mathsf{fld}_n \xrightarrow{\text{Yoneda}} \mathsf{PShv}(\mathcal{M}\mathsf{fld}_n) \xrightarrow{-|_{\{\mathbb{R}^n\}\subset\mathcal{M}\mathsf{fld}_n}} \mathsf{PShv}(\mathbb{R}^n) \simeq \mathsf{Spaces}_{/\mathsf{BO}(n)}, \tag{15}$$

where the codomain's symmetric monoidal structure is given by coproduct.

Remark 1.28. As noted in [AF15], the Kister–Mazur Theorem ([Kis64]) provides an equivalence $\mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n) \simeq \mathsf{Top}(n)$, where $\mathsf{Top}(n)$ is the group of topological automorphisms of $\mathbb{R}^n$; $\mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n) \simeq \mathcal{M}\mathsf{fld}_n(\mathbb{R}^n, \mathbb{R}^n)$ is the group of self-embeddings of $\mathbb{R}^n$; and both groups are equipped with the compact-open topology. This yields $\mathsf{PShv}(\mathbb{R}^n) \simeq \mathsf{PShv}(\mathsf{BTop}(n))$, which is equivalent to $\mathsf{Spaces}_{/\mathsf{BTop}(n)}$ by straightening–unstraightening (see Remark 1.32).[3] Finally, the inclusion $\mathsf{O}(n) \hookrightarrow \mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n)$ is also an equivalence ([AF19, Proposition 2.2.6]), yielding the displayed equivalence. The first map, the ∞-categorical Yoneda embedding, sends $M \mapsto ((\mathcal{M}\mathsf{fld}_n)_{/M} \to \mathcal{M}\mathsf{fld}_n)$ as a map with target $\mathsf{RFib}(\mathcal{M}\mathsf{fld}_n)$, as in [Lur09a]. The fiber of $(\mathcal{M}\mathsf{fld}_n)_{/M} \to \mathcal{M}\mathsf{fld}_n$ at N is equivalent to the space of embeddings of N into M, and so the restricted presheaf on the full ∞-subcategory at $\mathbb{R}^n$ sends $\mathbb{R}^n$ to the space of embeddings of $\mathbb{R}^n$ into M.

[3]Presheaves are space-valued, that is, $\mathsf{PShv}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Spaces})$.

By fixing a space B with a map $B \to \mathsf{BO}(n)$ (the role of which was previously played by $\mathsf{B}G$), one has

Definition 1.29. The symmetric monoidal ∞-category $\mathcal{M}\mathsf{fld}_n^B$ of n-manifolds with B-structure is the pullback of ∞-categories

$$\begin{tikzcd} \mathcal{M}\mathsf{fld}_n^B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{Spaces}_{/B} \arrow[d] \\ \mathcal{M}\mathsf{fld}_n \arrow[r, "\tau"] & \mathsf{Spaces}_{/\mathsf{BO}(n)}, \end{tikzcd} \tag{16}$$

where the right vertical arrow is induced by the map $B \to \mathsf{BO}(n)$.

Remark 1.30. Since the tangent classifier τ is symmetric monoidal, and since the right vertical map is canonically symmetric monoidal with respect to disjoint union on both $\mathsf{Spaces}_{/B}$ and $\mathsf{Spaces}_{/\mathsf{BO}(n)}$, the newly defined ∞-category does indeed come with a canonical symmetric monoidal structure.

Example 1.31. Classical examples for tangential structure come from principal G-bundles as discussed before. Namely, given a Lie group G together with a smooth homomorphism $G \to \mathsf{GL}(n) \simeq \mathsf{O}(n)$, we have an induced map of spaces $\mathsf{B}G \to \mathsf{BO}(n)$, which gives rise to the following examples:

1. $(G \to \mathsf{O}(n)) = (\mathsf{O}(n) \xrightarrow{\mathrm{id}} \mathsf{O}(n))$ reproduces the case of no tangential structure[4],
2. $(G \to \mathsf{O}(n)) = (\mathsf{SO}(n) \hookrightarrow \mathsf{O}(n))$ is the case of oriented smooth manifolds,
3. $(G \to \mathsf{O}(n)) = (\mathsf{Spin}(n) \to \mathsf{SO}(n) \hookrightarrow \mathsf{O}(n))$ is the case of spin manifolds,
4. $(G \to \mathsf{O}(n)) = (* \to \mathsf{O}(n))$ is the case of framed smooth manifolds.

Moving on to the stratified setting, we note that the stratified version of the full ∞-subcategory $\{\mathbb{R}^n\} \subset \mathcal{M}\mathsf{fld}_n$ is the ∞-category $\mathcal{B}\mathsf{sc}$ of basic singularity types, obtaining

$$\tau : \mathcal{M}\mathsf{fld} \xrightarrow{\text{Yoneda}} \mathsf{PShv}(\mathcal{M}\mathsf{fld}) \xrightarrow{-|_{\mathcal{B}\mathsf{sc}\subset\mathcal{M}\mathsf{fld}}} \mathsf{PShv}(\mathcal{B}\mathsf{sc}). \tag{17}$$

In this case, as in the previous one, given a stratified manifold M, $\tau(M)$ is the functor that to each basic $U \in \mathcal{B}\mathsf{sc}$ assigns the space of conically-smooth open embeddings of U into M.

Remark 1.32. The straightening–unstraightening construction of [Lur09a, §2.2] provides an equivalence between Spaces-valued presheaves on an ∞-category $\mathcal{C}$ and right fibrations[5] thereon:

$$\mathsf{PShv}(\mathcal{C}) \simeq \mathsf{RFib}_{\mathcal{C}}. \tag{19}$$

More details on this equivalence in our context can be found in [AFT17b, §4.2]. Appending this equivalence to the definition of the tangent classifier would say that the value of the tangent classifier on a stratified manifold M is given by the right fibration

$$\tau_M : \mathsf{Entr}(M) := \mathcal{B}\mathsf{sc}_{/M} \to \mathcal{B}\mathsf{sc}, \tag{20}$$

[4] or the contractible choice of a Riemannian metric

[5] A *right fibration* (over $\mathcal{C}$) is a functor $f : \mathcal{E} \to \mathcal{C}$ such that whenever a square of type

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & \mathcal{E} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd} \tag{18}$$

is given, there is a lift as depicted if $0 < i \leq n$.

i.e. the forgetful functor from the over-∞-category over M.[6] The ∞-category $\mathsf{Entr}(M)$ defined here is called the enter-path ∞-category of the stratified manifold M.[7] In fact the statements that we will make next will take on this perspective for the tangent classifier.

Having generalized the tangent classifier to stratified manifolds we can now think about adding structure to them. The key realization is that the data which the map of spaces $B \to \mathsf{BO}(n)$ provided before was the data of a presheaf on $\mathsf{BO}(n)$, standing in as the ∞-category of basics with only one object $\mathbb{R}^n$. The analogous construction in the stratified case is to provide a presheaf on $\mathcal{B}\mathsf{sc}$, which by Remark 1.32 is given by a right fibration over $\mathcal{B}\mathsf{sc}$.

Definition 1.33. An ∞-category of basics is a right fibration $\mathcal{B} \to \mathcal{B}\mathsf{sc}$. It provides both the data of the allowed singularity types and the data of (a generalization of) tangential structure.

Definition 1.34. Given an ∞-category of basics $\mathcal{B} \to \mathcal{B}\mathsf{sc}$, the ∞-category of $\mathcal{B}$-manifolds is the pullback

$$\begin{tikzcd} \mathcal{M}\mathsf{fld}(\mathcal{B}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & (\mathsf{RFib}_{\mathcal{B}\mathsf{sc}})_{/\mathcal{B}} \arrow[d] \\ \mathcal{M}\mathsf{fld} \arrow[r, "\tau"] & \mathsf{RFib}_{\mathcal{B}\mathsf{sc}} \end{tikzcd} \tag{21}$$

where we abbreviated $(\mathcal{B} \to \mathcal{B}\mathsf{sc}) \in \mathsf{RFib}_{\mathcal{B}\mathsf{sc}}$ to $\mathcal{B}$. Similarly, the ordinary category $\mathsf{Mfld}(\mathcal{B})$ of $\mathcal{B}$-manifolds is the further pullback along the inclusion $\mathsf{Mfld} \to \mathcal{M}\mathsf{fld}$.

Remark 1.35. A $\mathcal{B}$-manifold is a stratified manifold M together with a homotopy-lift ϕ

$$\begin{tikzcd} & \mathcal{B} \arrow[d] \\ \mathsf{Entr}(M) \arrow[ur, "\phi"] \arrow[r, "\tau_M"] & \mathcal{B}\mathsf{sc}, \end{tikzcd} \tag{22}$$

i.e., a filler of the 2-horn in the ∞-category $\mathcal{C}\mathsf{at}_\infty$ of ∞-categories given by τ_M and $\mathcal{B} \to \mathcal{B}\mathsf{sc}$.

Remark 1.36. There is an obvious, fully faithful inclusion $\mathcal{B} \hookrightarrow \mathcal{M}\mathsf{fld}(\mathcal{B})$, just as there was in the case without tangential structure.

In the case of smooth manifolds we argued that $\mathcal{M}\mathsf{fld}_n^B$ was defined in a way that allowed it to inherit disjoint union as a symmetric monoidal structure. A similar argument, found in [AFT17a], can be made for the stratified case too:

Proposition 1.37. *Disjoint union endows $\mathcal{M}\mathsf{fld}(\mathcal{B})$ and $\mathsf{Mfld}(\mathcal{B})$ with a symmetric monoidal structure.*

Definition 1.38. A collar-gluing of a stratified manifold M is a weakly constructible bundle $M \xrightarrow{f} [-1,1]$. We denote collar-gluings as

$$M \cong M_- \coprod_{M_0 \times \mathbb{R}} M_+, \tag{23}$$

where $M_- := f^{-1}[-1,1)$, $M_0 := f^{-1}\{0\}$ and $M_+ := f^{-1}(-1,1]$

[6] The notation is slightly misleading since M need not be in $\mathcal{B}\mathsf{sc}$.

[7] A different version of this ∞-cateogry is defined in [Lur, §A.6] called the exit-path ∞-category of the stratified space M. It is shown in [AFR19] that the exit-path ∞-category of [Lur, §A.6] is equivalent to the opposite of the enter-path ∞-category mentioned above.

Remark 1.39. One should think of collar-gluings as analogous to the case of gluing two manifolds with boundary along their boundaries after choosing collars. This is exactly the special thing about them, the overlap region is collared, i.e., looks like a product with $\mathbb{R}$. We also note that the disjoint union is an example of a collar-gluing.

Theorem 1.40 ([AFT17b]). *$\mathcal{M}\mathsf{fld}(\mathcal{B})$ is generated by $\mathcal{B}$ through iteratively forming collar-gluings and taking sequential colimits.*

Remark 1.41. Theorem 1.40 is the analogue of handle decomposition for smooth manifolds. Namely, any compact, smooth manifold can be obtained by handle decomposition, and non-compact smooth manifolds are obtained by sequential colimits of finitary (including compact) smooth manifolds.

1.3 Useful ∞-categories of Basics

Along with the already familiar ∞-categories of basics $\mathsf{D}_n^G := \mathsf{B}G \xrightarrow{\{\mathbb{R}^n\}} \mathcal{B}\mathsf{sc}$, which govern smooth n-manifolds with G-structure, we recall a few more ∞-categories of basics that will be of interest to us.

Example 1.42 ([AFT17b, Example 5.2.5]). We denote by $\mathsf{D}_n \hookrightarrow \mathcal{B}\mathsf{sc}$ or simply D_n the tangential structure given by the full ∞-subcategory inclusion of $\mathcal{B}\mathsf{sc}_{0,n}$, the basics of depth 0 and pure dimension n.[8] Accordingly, D_n-manifolds are topological n-manifolds. As in Remark 1.28, we have an equivalence $\mathsf{D}_n \simeq \mathsf{BGL}(n) \simeq \mathsf{BO}(n)$ of ∞-groupoids.

An ∞-category of basics that will play a major role for the development of factorization homology to come is the particularly simple one describing oriented 1-manifolds with boundary.

Construction 1.43. Consider the ∞-subcategory $\mathsf{D}_1^\partial \subset \mathcal{B}\mathsf{sc}$ whose objects are $\mathbb{R}$ and $\mathbb{R}_{\geq 0} = \mathsf{C}(*)$. There is a unique right fibration $\mathsf{D}_1^{\partial,\mathsf{or}} \to \mathsf{D}_1^\partial$, whose fiber over $\mathbb{R}$ is a point $\{\mathbb{R}\}$, and whose fiber over $\mathbb{R}_{\geq 0}$ is two points $\{\mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}\}$. In other words, $\mathsf{D}_1^{\partial,\mathsf{or}}$ has objects $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\leq 0}$ with their canonical orientations, and the morphisms are orientation preserving embeddings between them. In this way, $\mathsf{D}_1^{\partial,\mathsf{or}}$ is the ∞-category of basics that describes oriented (or, equivalently in this case, framed), smooth 1-manifolds with boundary.

Remark 1.44. Insisting that tangential structures be *right fibrations* over $\mathcal{B}\mathsf{sc}$, as opposed to simply ∞-categories over it as, e.g., in [AFR18], leads to technical issues that lead to our over-simplified notation due to the property of being a right fibration not being invariant under equivalence. For instance, when $G = *$, so that D_n^* controls framed n-manifolds, the map $BG = * \to \mathsf{BO}(n) \simeq \mathsf{D}_n$ is not a right fibration, owing to the observation that right fibrations of Kan complexes are Kan fibrations, and these, when surjective on objects, are epimorphisms. On the other hand, $B* = * \simeq (\mathsf{D}_n)_{/\mathbb{R}^n}$ since over-∞-groupoids are contractible ([Lur24, 018Y]), but $(\mathsf{D}_n)_{/\mathbb{R}^n} \to \mathsf{D}_n$ *is* a right fibration since it is a projection from an over-∞-category ([Lur24, 018F], [Joy02]). Moreover, $\mathsf{D}_n \hookrightarrow \mathcal{B}\mathsf{sc}$ is a right fibration since this full ∞-subcategory is trivially a sieve, so the composition $(\mathsf{D}_n)_{/\mathbb{R}^n} \to \mathcal{B}\mathsf{sc}$ is a right fibration as well. In general, the map $\mathsf{B}G \to \mathsf{BO}(n)$ is not a right fibration unless $G \to \mathsf{O}(n)$ is an epimorphism. However, given the latter, we can consider the presheaf $\mathsf{D}_n^{\mathsf{op}} \to \mathsf{Spaces}$ given by mapping $\mathbb{R}^n$ to the space of G-structures on it, and sending an embedding to the induced pullback map on the respective spaces of G-structures. By unstraightening, this determines a right fibration for which, by abuse, we write $\mathsf{D}_n^G = \mathsf{B}G \to \mathsf{D}_n \hookrightarrow \mathcal{B}\mathsf{sc}$.

[8] A stratified space $X \to P$ has *pure dimension* n ([AFT17b, Definition 2.4.1]) if its local (Čech–Lebesgue) dimension $\mathsf{dim}_x(X)$ at every $x \in X$ is n. Connected topological manifolds are of this type.

Another type of stratified manifold that we will focus on heavily in this work is the case of a smooth manifold with a distinguished, properly embedded, smooth submanifold, which we alternatively call a *defect*. Even though there is a clear picture of what this means we will now formally describe the ∞-category of basics that will give rise to these kinds of manifolds.

Construction 1.45 ([AFT17b, Example 5.2.10])**.** Let $\mathsf{D}_{d\subset n} \subset \mathcal{B}\mathsf{sc}$ be the full ∞-subcategory whose objects are $\mathbb{R}^n$ and $\mathbb{R}^{d\subset n} := \mathbb{R}^d \times \mathsf{C}(S^{n-d-1})$, where $d < n$. We elaborate that the morphism spaces are given as follows:

1. $\mathsf{Hom}_{\mathsf{D}_{d\subset n}}(\mathbb{R}^n, \mathbb{R}^n) = \mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n)$, the space of smooth embeddings of $\mathbb{R}^n$.
2. $\mathsf{Hom}_{\mathsf{D}_{d\subset n}}(\mathbb{R}^{d\subset n}, \mathbb{R}^n) = \emptyset$
3. $\mathsf{Hom}_{\mathsf{D}_{d\subset n}}(\mathbb{R}^n, \mathbb{R}^{d\subset n}) = \mathsf{Emb}(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^d)$, the space of embeddings that miss the defect.
4. $\mathsf{Hom}_{\mathsf{D}_{d\subset n}}(\mathbb{R}^{d\subset n}, \mathbb{R}^{d\subset n}) \simeq \mathsf{O}(d) \times \mathsf{O}(n-d)$. This final morphism space is given differently compared to the other ones because of intricacies related to smoothness around the defect. For more details on the matter there is a discussion in [AFT17b, Example 5.1.7].

That $\mathsf{D}_{d\subset n} \to \mathcal{B}\mathsf{sc}$ is a right fibration is immediate from the fact that the objects of $\mathsf{D}_{d\subset n}$ have at most one defect, which means that they don't receive maps from basics that are not in $\mathsf{D}_{d\subset n}$ already.

Definition 1.46. The ∞-category describing smooth, n-dimensional manifolds M which carry a properly embedded smooth, d-dimensional submanifold Σ is given by $\mathcal{M}\mathsf{fld}(\mathsf{D}_{d\subset n})$.

For later purposes we will also need to define a framed version of these manifolds.

Definition 1.47. The ∞-category of basics describing framed $\mathsf{D}_{d\subset n}$-manifolds is given by the pullback of ∞-categories

$$\begin{tikzcd} \mathsf{D}^*_{d\subset n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{D}^*_{\mathsf{n}} \simeq * \arrow[d] \\ \mathsf{D}_{d\subset n} \arrow[r] & \mathsf{D}_n \simeq \mathsf{BO}(n). \end{tikzcd} \tag{24}$$

Remark 1.48. To explain what the bottom horizontal functor fully is would initially take us into the realm of piecewise linear manifolds, and away from our goal. To not get into those details we note that in fact one can show that $\mathsf{D}^*_{d\subset n} \simeq \Delta^1$ by sending $\mathbb{R}^n \mapsto 0$ and $\mathbb{R}^{d\subset n} \mapsto 1$, which is an extension of the fact that in the case of no defects framings are governed by $\mathsf{D}^*_n \simeq \Delta^0 \simeq *$.

Explaining what exactly we mean by a framed stratified manifold is done in [AFT17b, Example 5.2.12]. According to that example, $\mathsf{D}^*_{d\subset n}$-manifolds are described by the data of a framed smooth n-manifold M, a properly embedded smooth d-submanifold Σ together with a null-homotopy of the Gauss map $\Sigma \to \mathsf{Gr}_d(\mathbb{R}^n)$. In more detail, for a general embedded submanifold $\Sigma \hookrightarrow M$ the Gauss map gives the tangent (or equivalently normal) subspace to Σ at every point. That is encoded as a map $\Sigma \to \mathsf{Gr}_d(\mathsf{T}M)$ to the Grassmann bundle of the tangent bundle of M. The framing of M is a diffeomorphism of type $\mathsf{T}M \cong M \times \mathbb{R}^n$ over, reducing said map to the Gauss map. A null-homotopy of this map is a trivialization of the tangent as well as the normal bundles of Σ in a way that is compatible with the trivialization of the restriction of $\mathsf{T}M$ to Σ.

2 Factorization Homology

Factorization homology (also known as topological chiral homology) is a construction that to the data of a smooth manifold M and an algebra A valued in a symmetric monoidal ∞-category $\mathcal{C}$ assigns an object

$$\int_M A \in \mathcal{C}. \tag{25}$$

The physical interpretation is that given a spacetime M and local observables A, $\int_M A$ returns global observables, though typically not all of them.

This implementation, with M relaxed to a stratified manifold and A to an appropriate type of algebra, is sometimes termed 'alpha' factorization homology, as opposed to the 'beta' version of [AFR18] that evaluates (∞, n)-categories on variframed stratified n-manifolds. In this work, we will be concerned, in this sense, with alpha factorization homology.

Fixing the algebra, the construction is functorial with respect to open embeddings in the manifold variable, which is the starting point to show that it is a chain level homology theory in a generalized sense. The initial chiral version was first introduced in [BD04], and a topological version was introduced by [Lur]. Later, in Theorem 3.19, we will also see how factorization homology serves to construct factorization algebras. The theory of factorization homology in the manner that we will recall here has been developed in [Fra13; AF15; AFT17a]. We will follow the introductory text [AF19].

In the following, and from now on, we let $\mathcal{C}$ be a symmetric monoidal ∞-category that is *$\otimes$-presentable* [AF15], i.e. it satisfies:

1. $\mathcal{C}$ is presentable: it admits colimits, and every object is a filtered colimit of compact objects[9], and
2. the monoidal structure distributes over colimits: for all $c \in \mathcal{C}$, the functor $c \otimes - : \mathcal{C} \to \mathcal{C}$ takes colimit diagrams to colimit diagrams.

Remark 2.1. For developing the theory of factorization homology the assumption that $\mathcal{C}$ has all *sifted*[10] colimits, and the monoidal structure commutes with these, is usually enough. However, if we want the ∞-category of algebras valued in $\mathcal{C}$ to inherit the properties of having sifted colimits that commute with the monoidal structure, then we must require that $\mathcal{C}$ has *all* (small) colimits. This will similarly be the case for factorization algebras. The stronger requirement also doesn't exclude any of the core examples that we are interested in. In fact, in some situations, the full requirement of presentability makes examples tractable.

Example 2.2. The following are examples of $\otimes$-presentable categories:

1. chain complexes $\mathcal{C}\mathsf{h}_{\Bbbk}$ over a commutative ring $\Bbbk$, where equivalences are given by quasi-isomorphism. The symmetric monoidal structure can be both direct sum $\oplus$ or tensor product $\otimes$.
2. Any cocomplete, Cartesian closed ∞-category together with categorical product. (for example Spaces or $\mathcal{C}\mathsf{at}_\infty$).

In particular, $(\mathcal{C}\mathsf{h}_{\Bbbk}, \otimes)$ is the most relevant for physical examples which is why it is the target of choice in [CG16] and [Gin15].

We also fix an ∞-category of basics $\mathcal{B} \to \mathcal{B}\mathsf{sc}$.

2.1 Disk Algebras

Definition 2.3. The symmetric monoidal ∞-category $\mathcal{D}\mathsf{isk}(\mathcal{B}) \subset \mathcal{M}\mathsf{fld}(\mathcal{B})$ is the smallest full symmetric monoidal ∞-subcategory containing $\mathcal{B}$. Namely, the objects of $\mathcal{D}\mathsf{isk}(\mathcal{B})$ are disjoint unions of objects of $\mathcal{B}$.

We will also need the ordinary version of this category in analogy to $\mathcal{M}\mathsf{fld}$ and Mfld.

[9] For the exact meaning of these terms we refer the reader to [Lur09a, §5.3]. For us, it will suffice to give examples of ∞-categories that satisfy these requirements.

[10] Sifted colimits are colimits over a sifted indexing simplicial set K, which means that the diagonal functor of $K \neq \emptyset$ is final

$$\mathsf{colim}(K \to K \times K \to \mathcal{C}) \simeq \mathsf{colim}(K \times K \to \mathcal{C}).$$

Definition 2.4. The symmetric monoidal category $\mathsf{Disk}(\mathcal{B}) \subset \mathsf{Mfld}(\mathcal{B})$ is the smallest full symmetric monoidal ∞-subcategory containing

$$\mathcal{B} \underset{\mathcal{M}\mathsf{fld}(\mathcal{B})}{\times} \mathsf{Mfld}(\mathcal{B}) \subset \mathsf{Mfld}(\mathcal{B}). \tag{26}$$

Remark 2.5. Taking into account all categories of disks and manifolds defined up to now we have a commutative diagram in the ∞-category of symmetric monoidal ∞-categories $\mathcal{C}\mathsf{at}_\infty^\otimes$

$$\begin{tikzcd} \mathsf{Disk}(\mathcal{B}) \arrow[r] \arrow[d] & \mathcal{D}\mathsf{isk}(\mathcal{B}) \arrow[d] \\ \mathsf{Mfld}(\mathcal{B}) \arrow[r] & \mathcal{M}\mathsf{fld}(\mathcal{B}). \end{tikzcd} \tag{27}$$

We will now recall the ∞-category of disk algebras with a chosen tangential structure valued in a symmetric monoidal ∞-category. First, consider, following [Lur], the category Fin_* whose objects are the sets $\langle n \rangle = \{*, 1, \dots, n\}$ and whose morphisms are functions that preserve the distinguished point $*$. For every $i \in \{1, \dots, n\}$ there is a map $\rho^i \colon \langle n \rangle \to \langle 1 \rangle$ that sends i to 1 but is otherwise constant at $* \in \langle 1 \rangle$. A *symmetric monoidal ∞-category* is, after [Lur, Definition 2.0.0.7], a cocartesian fibration[11] $\mathcal{C} \to \mathsf{Fin}_*$ such that the maps $\rho^i \colon \langle n \rangle \to \langle 1 \rangle$, $1 \le i \le n$, induce an equivalence $\mathcal{C}_{\langle n \rangle} \simeq (\mathcal{C}_{\langle 1 \rangle})^{\times n}$ for every $n \ge 0$, where $\mathcal{C}_x$ is the fiber at x. We have abbreviated $\mathsf{N}(\mathsf{Fin}_*)$ to Fin_*, and will mostly abbreviate $\mathcal{C} \to \mathsf{Fin}_*$ to $\mathcal{C}$.

The ∞-category $\mathsf{Fun}^\otimes(\mathcal{O}, \mathcal{C}) \subset \mathsf{Fun}_{/\mathsf{Fin}_*}(\mathcal{O}, \mathcal{C})$ of *symmetric monoidal functors* between symmetric monoidal ∞-categories $\mathcal{O}$, $\mathcal{C}$ is the full sub-∞-category of the ∞-category of functors over Fin_* generated by those functors that preserve cocartesian edges. Following [AFT17a], we write

$$\mathcal{A}\mathsf{lg}_{\mathcal{O}}(\mathcal{C}) := \mathsf{Fun}^\otimes(\mathcal{O}, \mathcal{C})$$

when $\mathcal{O}$, $\mathcal{C}$ are symmetric monoidal ∞-categories.

More generally, an *∞-operad* is a map $\mathcal{C} \to \mathsf{Fin}_*$ that satisfies a generalisation of the properties of a symmetric monoidal ∞-category – see [Lur, Definition 2.1.1.10].

Definition 2.6. The ∞-category of disk algebras valued in $\mathcal{C}$ is the ∞-category of symmetric monoidal functors

$$\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathcal{B})}(\mathcal{C}) := \mathsf{Fun}^\otimes(\mathcal{D}\mathsf{isk}(\mathcal{B}), \mathcal{C}). \tag{28}$$

Remark 2.7. The exact specification of the above ∞-category is not immediate, but the explanations given in [Lur, def.2.0.0.7], [Lur, rem.2.1.2.19] and [Lur, def.2.1.3.7] clarify the matter. A succinct explanation is also given around [AFT17a, def.1.11]. Informally, the objects will be symmetric monoidal functors from $\mathcal{D}\mathsf{isk}(\mathcal{B})$ to $\mathcal{C}$. These are given by functors $F : \mathcal{D}\mathsf{isk}(\mathcal{B}) \to \mathcal{C}$ together with maps

$$F(U \sqcup V) \xrightarrow{\simeq} F(U) \otimes F(V), \tag{29}$$

and such that the swap in $\mathcal{D}\mathsf{isk}(\mathcal{B})$, $U \sqcup V \simeq V \sqcup U$, is sent to the swap in $\mathcal{C}$, $F(U) \otimes F(V) \simeq F(V) \otimes F(U)$, in a way that is compatible with the maps in (29). A good informal explanation of this and other topics related to disk algebras can be found in the lecture notes [Tan].

These algebras will be the algebraic input of factorization homology. To convince ourselves that this is not a huge limitation on the types of algebras that can be evaluated we have:

[11] A *cocartesian fibration* ([Lur24, Definition 01T5]) $q \colon \mathcal{A} \to \mathcal{B}$ between simplicial sets is one such that every edge $f \in \mathcal{A}_1$ is *(q-)cocartesian*, that is, any horn of type $\Lambda^n_0 \to \mathcal{A}$, $n \ge 2$, whose restriction to $\{0 < 1\}$ is f, admits a filler if its projection to $\mathcal{B}$ admits a filler. By (un)straightening, such a q corresponds to a $\mathcal{C}\mathsf{at}_\infty$-valued copresheaf $\mathcal{B} \to \mathcal{C}\mathsf{at}_\infty$.

Proposition 2.8 ([AFT17a, prop.2.12])**.** *Let* $(\mathcal{B} \to \mathcal{B}\mathsf{sc}) = (* \xrightarrow{\{\mathbb{R}^n\}} \mathcal{B}\mathsf{sc})$*, namely the* ∞*-category of basics describing framed smooth* n*-manifolds. There is an equivalence of* ∞*-categories*

$$\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathcal{B})}(\mathcal{C}) \simeq \mathcal{A}\mathsf{lg}_{\mathbb{E}_n}(\mathcal{C}). \tag{30}$$

Furthermore, if the ∞*-category of basics is rather* $(\mathcal{B} \to \mathcal{B}\mathsf{sc}) = (\mathsf{BO}(n) \xrightarrow{\{\mathbb{R}^n\}} \mathcal{B}\mathsf{sc})$*, the one describing unstructured smooth manifolds, then there is an equivalence of* ∞*-categories*

$$\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathcal{B})}(\mathcal{C}) \simeq \mathcal{A}\mathsf{lg}_{\mathbb{E}_n}(\mathcal{C})^{\mathsf{O}(n)}, \tag{31}$$

with the (homotopy) $\mathsf{O}(n)$*-invariants, where the action of* $\mathsf{O}(n)$ *is given by change of framing.*

Remark 2.9. For more details on $\mathbb{E}_n$-algebras and why these algebras encompass large portions of the examples one usually cares about see [Lur, sec.5.1]. Roughly speaking, $\mathbb{E}_1$-algebras are homotopy associative algebras and with increasing n we get increasing levels of homotopy commutativity. Having said that, the first part of the proposition is just the classical statement that the little n-disks operad and the little n-cubes operad are equivalent. Through the work of [Lur09b] and [Sch14], $\mathbb{E}_n$-algebras also serve as input for constructing fully extended TQFTs by using factorization homology.

Definition 2.10. For $M \in \mathcal{M}\mathsf{fld}(\mathcal{B})$ a $\mathcal{B}$-manifold the slice ∞-category of disks over M is defined by

$$\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M} := \mathcal{D}\mathsf{isk}(\mathcal{B}) \underset{\mathcal{M}\mathsf{fld}(\mathcal{B})}{\times} \mathcal{M}\mathsf{fld}(\mathcal{B})_{/M}. \tag{32}$$

Remark 2.11. $\mathcal{D}\mathsf{isk}(\mathcal{B})$ is a symmetric monoidal ∞-category, however there is no way to inherit this structure to $\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}$. Intuitively, this is because disjoint union cannot serve as a monoidal structure now that the disks are equipped with an embedding into a given manifold M; they could be such that they intersect. However, as explained in [AFT17a, not.1.21], we *can* equip $\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}$ with the structure of an ∞-operad because the symmetric monoidal unit $\emptyset$ of $\mathcal{D}\mathsf{isk}(\mathcal{B})$ and $\mathcal{M}\mathsf{fld}(\mathcal{B})$ is initial. This is also the case for $\mathcal{M}\mathsf{fld}(\mathcal{B})_{/M}$ itself.

In more details, following [AFT17a, ex.2.5], objects of $\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}$ are finite sets of open embeddings of disks $(U \hookrightarrow M)$. A morphism between two open embeddings $(U \hookrightarrow M) \to (V \hookrightarrow M)$ is specified by an open embedding $U \hookrightarrow V$ together with an isotopy between $U \hookrightarrow M$ and $U \hookrightarrow V \hookrightarrow M$. A morphism $((U_1 \hookrightarrow M), (U_2 \hookrightarrow M)) \to (V \hookrightarrow M)$ is given by an open embedding $U_1 \sqcup U_2 \hookrightarrow V$ together with two isotopies from $U_1 \hookrightarrow M$ to $U_1 \hookrightarrow U_1 \sqcup U_2 \hookrightarrow M$, and from $U_2 \hookrightarrow M$ to $U_2 \hookrightarrow U_1 \sqcup U_2 \hookrightarrow M$.

Remark 2.12. The constructions of the slice category also holds, word for word, in the case of $\mathsf{Disk}(\mathcal{B})_{/M}$ and $\mathsf{Mfld}(\mathcal{B})_{/M}$. The fact that they are ∞-operads is also true. The key difference is what the objects and morphisms are, say, in $\mathsf{Disk}(\mathcal{B})_{/M}$ as compared to $\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}$. The objects are the same as before, i.e. finite sets of open embeddings into M. However, the morphisms now contain less information. A morphism $(U \hookrightarrow M) \to (V \hookrightarrow M)$ is specified only by open embedding $U \hookrightarrow V$, such that $(U \hookrightarrow M) = (U \hookrightarrow V \hookrightarrow M)$, and no further information. This also holds for higher arity morphisms too.

That is, a morphism in $\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}$ is a diagram in $\mathcal{D}\mathsf{isk}(\mathcal{B})$

$$\begin{tikzcd} \coprod_i U_i \arrow[rr] \arrow[dr] & & U \arrow[dl] \\ & M, & \end{tikzcd} \tag{33}$$

which is homotopy commutative, while in $\mathsf{Disk}(\mathcal{B})_{/M}$ a morphism is a diagram as above that commutes on the nose[12].

Definition 2.13. We will refer to the ∞-category of algebras over the ∞-operads $\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}$, $\mathsf{Disk}(\mathcal{B})_{/M}$, $\mathcal{M}\mathsf{fld}(\mathcal{B})_{/M}$ and $\mathsf{Mfld}(\mathcal{B})_{/M}$ in the usual way as

$$\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}}(\mathcal{C}) \qquad \mathcal{A}\mathsf{lg}_{\mathsf{Disk}(\mathcal{B})_{/M}}(\mathcal{C}) \qquad \mathcal{A}\mathsf{lg}_{\mathcal{M}\mathsf{fld}(\mathcal{B})_{/M}}(\mathcal{C}) \qquad \mathcal{A}\mathsf{lg}_{\mathsf{Mfld}(\mathcal{B})_{/M}}(\mathcal{C}). \tag{34}$$

Remark 2.14. These algebras are defined as described in [Lur] for a general ∞-operad. For a superficial understanding what will be most important for us is that algebras $\mathcal{A}\mathsf{lg}_{\mathcal{O}}(\mathcal{C})$, where $\mathcal{O}$ and $\mathcal{C}$ are ∞-operads, are described as functors $\mathcal{O} \to \mathcal{C}$ that satisfy some additional requirements[13]. The following propositions from [Lur] gives another perspective on how to look at these algebras.

Proposition 2.15 ([Lur, prop.2.2.4.9])**.** *Let $\mathcal{O}$ be an ∞-operad and let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then there is an equivalence*

$$\mathsf{Fun}^{\otimes}(\mathsf{Env}(\mathcal{O}), \mathcal{C}) \xrightarrow{\simeq} \mathcal{A}\mathsf{lg}_{\mathcal{O}}(\mathcal{C}), \tag{36}$$

between the ∞-categories of $\mathcal{O}$-algebras in $\mathcal{C}$ and symmetric monoidal functors from the symmetric monoidal envelope[14] *of $\mathcal{O}$ to $\mathcal{C}$.*

Remark 2.16. The ∞-operad $\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}$, so defined, does not actually depend on the $\mathcal{B}$-structure; there is an equivalence

$$\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M} \simeq \mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc})_{/M}, \tag{37}$$

where M on the right-hand side is the underlying stratified manifold of the $\mathcal{B}$-manifold also called M. This is essentially because the ∞-cateogry of basics $\mathcal{B} \to \mathcal{B}\mathsf{sc}$ is defined as a right fibration. Intuitively, those disks that have an open embedding into M admit a $\mathcal{B}$-structure by restriction since M admits one. From here, since over ∞-groupoids are contractible ([Lur24, 018Y]), there is an equivalence between the singleton space consisting of this inherited $\mathcal{B}$-structure and the space of $\mathcal{B}$-structures the disk originally had. The argument, of course, holds for $\mathsf{Disk}(\mathcal{B})_{/M}$ too, as well as $\mathcal{M}\mathsf{fld}(\mathcal{B})_{/M}$ and $\mathsf{Mfld}(\mathcal{B})_{/M}$. This is why, when talking about these slice ∞-operads we will simplify the notation down to $\mathcal{D}\mathsf{isk}_{/M}$, $\mathsf{Disk}_{/M}$, $\mathcal{M}\mathsf{fld}_{/M}$ and $\mathsf{Mfld}_{/M}$.

Essentially the same argument as above, namely that over ∞-groupoids are contractible, also has another important consequence that we will need to make use of later:

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[12] Regarding notation, [Lur] introduces both the ∞-operad $\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}$ and $\mathsf{Disk}(\mathcal{B})_{/M}$, however there they are denoted by $\mathbb{E}_M^{\otimes}$ and $\mathrm{N}(\mathrm{Disk}(M))^{\otimes}$, respectively.

[13] Specifically, a functor F between the underlying ∞-categories is an algebra if it lies over Fin_*

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "F"] \arrow[dr] & & \mathcal{C} \arrow[dl] \\ & \mathsf{Fin}_* & \end{tikzcd}, \tag{35}$$

and if it carries inert morphisms in $\mathcal{O}$ to inert morphisms in $\mathcal{C}$.

[14] A full definition of the symmetric monoidal envelope of an ∞-operad can be found in [Lur, sec.2.2.4]. For our purposes the two important facts about Env are that it is a left adjoint to the forgetful functor $U : \mathcal{C}\mathsf{at}_{\infty}^{\otimes} \to \mathcal{O}\mathsf{p}_{\infty}$, and that the underlying ∞-category of $\mathsf{Env}(\mathcal{O})$ is the ∞-subcategory of $\mathcal{O}$ spanned by the active morphisms. Informally, the symmetric monoidal structure of $\mathsf{Env}(\mathcal{O})$ is given by concatenating these active morphisms [Lur, rem.2.2.4.6].

Lemma 2.17. *Every open embedding $e : N \hookrightarrow M$ induces an equivalence of ∞-categories*

$$(\mathcal{M}\mathsf{fld}_{/M})_{/e} \simeq \mathcal{M}\mathsf{fld}_{/N}. \tag{38}$$

The same holds for Mfld*, as well as,* $\mathcal{D}\mathsf{isk}$ *and* Disk*.*

The next result is of key importance to the technical side of the theory, and will be used extensively throughout. It alone, is essentially the reason for the appearance of weakly constructible bundles in a lot of later developments.

Lemma 2.18 ([AFT17a, lem.2.24])**.** *Let $f : M \to N$ be a weakly constructible bundle. There is a functor of ∞-operads*

$$f^{-1} : \mathcal{D}\mathsf{isk}_{/N} \to \mathcal{M}\mathsf{fld}_{/M}, \tag{39}$$

which acts on objects by sending $V \hookrightarrow N$ to $f^{-1}V \hookrightarrow M$.

Remark 2.19. Because of Remark 2.16, the above construction works for manifolds with any general $\mathcal{B}$-structure, with no further requirements on the weakly constructible bundle f.

2.2 Definition of Factorization Homology

In the previous subsection we defined and explored the algebraic input of factorization homology. We now know that disk algebras are defined at their core as a functor that evaluates disks. Thus, we now gain the appreciation that factorization homology is a way to extend the input of this functor to all stratified manifolds instead of just stratified disks. It comes as no surprise then that factorization homology is defined as a Kan extension:

Definition 2.20. *(Absolute) factorization homology* is a functor that is a left adjoint to the restriction along $\mathcal{D}\mathsf{isk}(\mathcal{B}) \hookrightarrow \mathcal{M}\mathsf{fld}(\mathcal{B})$

$$\int : \mathsf{Fun}^{\otimes}(\mathcal{D}\mathsf{isk}(\mathcal{B}), \mathcal{C}) \rightleftarrows \mathsf{Fun}^{\otimes}(\mathcal{M}\mathsf{fld}(\mathcal{B}), \mathcal{C}). \tag{40}$$

Similarly, given a stratified manifold M, *(relative) factorization homology* is a functor that is a left adjoint to the restriction along $\mathcal{D}\mathsf{isk}_{/M} \hookrightarrow \mathcal{M}\mathsf{fld}_{/M}$

$$\int : \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}(\mathcal{C}) \rightleftarrows \mathcal{A}\mathsf{lg}_{\mathcal{M}\mathsf{fld}_{/M}}(\mathcal{C}). \tag{41}$$

The bulk of the task is then to show that such an adjoint exists and to try to find more concrete forms of factorization homology that would be easier to compute with. Being a left Kan extension one guess on the side of computation would be that we can find its values via colimits, as is usual for pointwise left Kan extensions. Determining under which assumptions factorization homology, in both its forms, exists and is given by a colimit is one of the major results of [AF15] (for the smooth case) and [AFT17a] (for the stratified case). Specifically lemmas [AFT17a, lem.2.16] and [AFT17a, lem.2.17] are great encapsulations of all the conditions that are needed.

The upshot is that, with our setup and assumptions on $\mathcal{C}$, the guess that the value of factorization homology can be calculated by a colimit is correct and given by the standard expressions for left Kan extensions

$$\int_M A \simeq \mathsf{colim}\left(\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M} \to \mathcal{D}\mathsf{isk}(\mathcal{B}) \xrightarrow{A} \mathcal{C}\right) \tag{42}$$

in the absolute case, and

$$\int_{N \hookrightarrow M} A \simeq \mathsf{colim}\left((\mathcal{D}\mathsf{isk}_{/M})_{/(N \hookrightarrow M)} \to \mathcal{D}\mathsf{isk}_{/M} \xrightarrow{A} \mathcal{C} \right) \tag{43}$$

in the relative case. Namely, the left Kan extension of underlying ∞-categories lifts to a symmetric monoidal, in the absolute case, and operadic, in the relative case, left Kan extension. In the relative case, we will almost always abuse notation and write $\int_N A$ instead of $\int_{N \hookrightarrow M} A$, but because of Lemma 2.17 this is not much of an abuse. Furthermore, factorization homology is also shown to be fully faithful due to this being true for the inclusion $\mathcal{D}\mathsf{isk}(\mathcal{B}) \to \mathcal{M}\mathsf{fld}(\mathcal{B})$, as well as, the relative version with slices.

Remark 2.21. In fact, lemmas [AFT17a, lem.2.16] and [AFT17a, lem.2.17] are general enough that they already encompass the alternative definitions with Disk and $\mathsf{Disk}_{/M}$, so factorization homology is just as easily defined in that context too.

2.3 Disk Algebras over Oriented Intervals

We now focus on describing disk algebras over oriented intervals (or equivalently, framed intervals) with and without boundary. Not only is this an important example for what disk algebras look like, but it's also important in defining what it means to say that factorization homology is excisive. The following constructions can be found in the originals [AF15; AFT17a].

We want to show that disk algebras with $\mathsf{D}_1^{\partial,*}$-structure, as defined in Construction 1.43, can be described algebraically. For this we set up two ∞-operads $\mathsf{Assoc}^{\mathsf{RL}}$ and O^{RL}.

Construction 2.22. Let $\mathsf{Assoc}^{\mathsf{RL}}$ denote the multicategory[15] with three objects R, A and L. To describe the multi-morphisms let I be an arbitrary finite list of the objects of $\mathsf{Assoc}^{\mathsf{RL}}$. We have:

1. $\mathsf{Assoc}^{\mathsf{RL}}(I, A)$ is empty if either R or L appear on the list I; otherwise it is given by the number of total orders of I.
2. $\mathsf{Assoc}^{\mathsf{RL}}(I, R)$ is empty if either L appears on the list or R appears on the list more than once; otherwise it is given by the number of total orders of I such that the possible element R is a minimum.
3. $\mathsf{Assoc}^{\mathsf{RL}}(I, L)$ is empty if either R appears on the list or L appears on the list more than once; otherwise it is given by the number of total orders of I such that the possible element L is a minimum.

Composition of multi-morphisms is given by concatenation.

Remark 2.23. More informally, what the Construction 2.22 says is that the only multi-morphism domains possible are $\emptyset$, $(A, A, \dots, A)$, $(R, A, \dots, A)$ or $(A, \dots, A, L)$ and that there is one multi-morphism for each permutation of the A entries. This makes it clear that algebras over this ∞-operad will be unital, (homotopy) associative algebras, together with a unital left and a unital right module.

Construction 2.24. Let O^{RL} be the (ordinary) category whose objects are totally ordered finite sets $(I, \leq)$ with two distinguished subsets $R \subset I \supset L$, such that each element of R is a minimum and each element of L is a maximum (consequently, $|R| \leq 1$ and $|L| \leq 1$). The morphisms $f : (I, \leq, R, L) \to (I', \leq', R', L')$ are order preserving maps $f : (I, \leq) \to (I', \leq')$ which also preserve the distinguished subsets $f(R) = R'$, $f(L) = L'$. Concatenation of total orders gives O^{RL} a multicategory structure.

[15] A *multicategory* is like an (ordinary) category except that morphisms (now called multi-morphisms) are allowed to have multiple objects in their domain while still having only a single object in their codomain.

Remark 2.25. The ∞-operads $\mathsf{Assoc}^{\mathsf{RL}}$ and O^{RL} are clearly related, though there are differences. The major difference is that O^{RL} has objects that have both a distinguished maximal and distinguished minimal element, while in $\mathsf{Assoc}^{\mathsf{RL}}$ only one of them exists at a time. In fact, we will see that their symmetric monoidal envelopes are equivalent. Therefore, considering them separately also gives some intuition for the symmetric monoidal envelope, so we choose to do this for expositional reasons. The following propositions make it geometrically clear why this is the case.

Proposition 2.26. *There are equivalences of symmetric monoidal ∞-categories between $\mathsf{D}_1^{\partial,*}$-structured disks and the symmetric monoidal envelope of $\mathsf{Assoc}^{\mathsf{RL}}$*

$$\mathsf{Disk}(\mathsf{D}_1^{\partial,*}) \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}(\mathsf{D}_1^{\partial,*}) \xrightarrow[\simeq]{[-]} \mathsf{Env}(\mathsf{Assoc}^{\mathsf{RL}}). \tag{44}$$

The functor $[-]$ assigns $[\mathbb{R}_{\geq 0}] = R$, $[\mathbb{R}] = A$ and $[\mathbb{R}_{\leq 0}] = L$.

Proof. The first equivalence comes by inspection of the morphism spaces which are discrete in this particular case. Since $\mathsf{Assoc}^{\mathsf{RL}}$ is also born of a discrete nature we only need to find a bijection of morphism spaces. By the construction of $\mathsf{Assoc}^{\mathsf{RL}}$ such a bijection is immediate. □

Proposition 2.27. *There are equivalences of ∞-operads*

$$\mathsf{Disk}(\mathsf{D}_1^{\partial,*})_{/[-1,1]} \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}(\mathsf{D}_1^{\partial,*})_{/[-1,1]} \xrightarrow{\simeq} \mathsf{O}^{\mathsf{RL}} \tag{45}$$

that descend to the equivalences of Proposition 2.26.

Proof. As in the proof of Proposition 2.26 the first equivalence is clear. The idea to keep in mind is that since the boundary of $[-1,1]$ consists of a left point and a right point, there can only be oriented embeddings of disks into $[-1,1]$ with at most one of each of $[-1,1)$ and $(-1,1]$. These are then clearly the objects that map to the respective distinguished subsets. Observing the definition of O^{RL} clearly offers a bijection on morphism spaces since they are by definition order preserving maps, and the embeddings are oriented. It is trivial to check that these maps lie over Fin_* and that they send active morphisms to active morphisms completing the proof. □

In fact in the simple case of intervals, that we're considering here, even more is true:

Proposition 2.28. *There are equivalences of symmetric monoidal ∞-categories*

$$\mathsf{Env}(\mathsf{O}^{\mathsf{RL}}) \xrightarrow{\simeq} \mathsf{Env}(\mathcal{D}\mathsf{isk}(\mathsf{D}_1^{\partial,*})_{/[-1,1]}) \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}(\mathsf{D}_1^{\partial,*}) \xrightarrow{\simeq} \mathsf{Env}(\mathsf{Assoc}^{\mathsf{RL}}) \tag{46}$$

Proof. The only thing to show is the middle equivalence. This follows from Lemma 3.24, which we will introduce later. □

Remark 2.29. A few times in this work we will encounter the fact that a final functor $F : \mathcal{A} \to \mathcal{B}$ between ∞-categories induces an equivalence

$$\mathsf{colim}(G) \xrightarrow{\simeq} \mathsf{colim}(G \circ F), \tag{47}$$

for each functor $G : \mathcal{B} \to \mathcal{C}$, whenever both colimits exist. We remark here that these equivalences are actually even natural in G. To explain this, we first notice that the equivalences in Equation (47) actually arise from equivalences between ∞-categories

$$\mathcal{C}_{G/} \xrightarrow{\simeq} \mathcal{C}_{G \circ F/}, \tag{48}$$

as discussed in [Lur09a, prop.4.1.1.8], where the functor is the obvious one. Furthermore, [Lur24, 016H] tells us that the slice construction is functorial, which, after restricting, defines the functors

$$S : \mathsf{Fun}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathcal{C}\mathsf{at}_\infty \qquad \text{and} \qquad S' : \mathsf{Fun}(\mathcal{A}, \mathcal{C}) \longrightarrow \mathcal{C}\mathsf{at}_\infty, \tag{49}$$

whose value on objects is given by $G \mapsto \mathcal{C}_{G/}$. We can now state that by naturality in G we mean that the functors of Equation (48) assemble into an equivalence of functors

$$S \xrightarrow{\simeq} S' \circ F^* \tag{50}$$

in $\mathsf{Fun}(\mathsf{Fun}(\mathcal{B}, \mathcal{C}), \mathcal{C}\mathsf{at}_\infty)$. But this is obviously the case, given the description of the maps in Equation (48), and amounts to nothing more than bookkeeping.

Proposition 2.30. *There is an equivalence of functors between the two-sided bar construction*

$$\mathsf{B} : \mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{C}) \to \mathcal{C} \tag{51}$$

and factorization homology over the oriented closed interval

$$\int_{[-1,1]} : \mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{C}) \simeq \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/[-1,1]}}(\mathcal{C}) \xrightarrow{\int} \mathcal{A}\mathsf{lg}_{\mathcal{M}\mathsf{fld}_{/[-1,1]}}(\mathcal{C}) \xrightarrow{\mathsf{ev}_{[-1,1]}} \mathcal{C}. \tag{52}$$

In particular, given a unital, (homotopy) associative algebra A, a unital right A-module R and a unital left A-module L, there is an equivalence of objects

$$\int_{[-1,1]} (R, A, L) \xrightarrow{\simeq} R \bigotimes_A L. \tag{53}$$

Remark 2.31. The two-sided bar construction here gives the left derived tensor product. We, however, do not make a notational distinction between this left derived tensor product and the standard one since it should be clear from context. The left derived version will always appear due to some two-sided bar construction.

Proof. The equivalence at the level of objects is exactly [AFT17a, prop.2.34], whose proof we recount here for completeness.

For the purposes of the proof introduce the shorthand notation $\mathcal{D} := \mathcal{D}\mathsf{isk}(\mathsf{D}_1^{\partial,*})$. There is a functor $\Delta^{\mathsf{op}} \hookrightarrow \mathsf{O}^{\mathsf{RL}}$ from the opposite of the simplex category, whose essential image consists of those objects for which the distinguished subsets with a maximal and minimal element are non-empty. This functor is final because adjoining a minimum and maximum gives a left adjoint to it. We recognize the simplicial object

$$\mathsf{B}_\bullet(R, A, L) : \Delta^{\mathsf{op}} \hookrightarrow \mathsf{O}^{\mathsf{RL}} \simeq \mathcal{D}_{/[-1,1]} \to \mathcal{D} \xrightarrow{(R,A,L)} \mathcal{C} \tag{54}$$

as the two-sided bar construction. Since $\mathcal{C}$ has colimits and $\Delta^{\mathsf{op}} \hookrightarrow \mathsf{O}^{\mathsf{RL}}$ is final, the geometric realization of the above is

$$\mathsf{B}(R, A, L) \equiv R \bigotimes_A L = |\mathsf{B}_\bullet(R, A, L)| \simeq \mathsf{colim}(\mathcal{D}_{/[-1,1]} \to \mathcal{D} \xrightarrow{(R,A,L)} \mathcal{C}) = \int_{[-1,1]} (R, A, L). \tag{55}$$

Since the equivalences above come from an argument about finality, Remark 2.29 tells us that they are actually even natural in the algebra variable (R, A, L), so that the equivalence lifts to the level of functors. □

2.4 Pushforward

The construction of Lemma 2.18 gives a functor which allows us to change the stratified manifold over which we work. This brings about the important operation of pushforward:

Theorem 2.32 ([AFT17a, thm.2.25]). *Let $f : M \to N$ be a weakly constructible bundle. There is a pushforward functor*

$$f_* : \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}(\mathcal{C}) \longrightarrow \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/N}}(\mathcal{C}), \tag{56}$$

which takes a $\mathcal{D}\mathsf{isk}_{/M}$-algebra A to

$$f_*A : \mathcal{D}\mathsf{isk}_{/N} \xrightarrow{f^{-1}} \mathcal{M}\mathsf{fld}_{/M} \xrightarrow{\int_{-} A} \mathcal{C}. \tag{57}$$

This functor is such that there is a canonical equivalence in $\mathcal{C}$

$$\int_M A \simeq \int_N f_*A \tag{58}$$

Remark 2.33. In particular, the above applies to disk algebras with any $\mathcal{B}$-structure, since a $\mathcal{D}\mathsf{isk}(\mathcal{B})$-algebra A can be viewed as a $\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}$-algebra for any $\mathcal{B}$-manifold M by

$$\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M} \to \mathcal{D}\mathsf{isk}(\mathcal{B}) \xrightarrow{A} \mathcal{C}. \tag{59}$$

Remark 2.34. As was the case with the bar construction the equivalence of Theorem 2.32, comes about through an argument about a final functor, this means that, through Remark 2.29 there is even an equivalence between functors

$$\int_M -, \int_N f_*- : \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}(\mathcal{C}) \longrightarrow \mathcal{C}. \tag{60}$$

2.5 ⊗-excision and Homology Theories

Now that we have a pushforward operation, we can finally explicitly state what excision is in this context. This is because a collar-gluing of a stratified manifold M is defined exactly as a weakly constructible bundle $f : M \to [-1, 1]$.

Definition 2.35. Let $F : \mathcal{M}\mathsf{fld}(\mathcal{B}) \to \mathcal{C}$ be a symmetric monoidal functor. Also let M be a $\mathcal{B}$-manifold and $f : M \to [-1, 1]$ be a collar-gluing of M. This data allows for the construction of a canonical morphism in $\mathcal{C}$

$$F(M_-) \bigotimes_{F(M_0 \times \mathbb{R})} F(M_+) \longrightarrow F(M), \tag{61}$$

where M_-, M_0 and M_+ are as usual. If this morphism is an equivalence for each collar-gluing then we say that F satisfies ⊗-excision.

Proof. The definition comes with a claim that the canonical morphism in $\mathcal{C}$ exists. This we need to show. The data of a collar-gluing $f : M \to [-1, 1]$ allows us to pushforward F to

$$f_*F : \mathcal{D}\mathsf{isk}(\mathsf{D}_1^{\partial,*})_{/[-1,1]} \xrightarrow{f^{-1}} \mathcal{M}\mathsf{fld}(\mathcal{B})_{/M} \to \mathcal{M}\mathsf{fld}(\mathcal{B}) \xrightarrow{F} \mathcal{C}. \tag{62}$$

By Proposition 2.26 we know that this data defines an $\mathsf{Assoc}^{\mathsf{RL}}$-algebra in $\mathcal{C}$ whose right module, algebra and left module are exactly $F(M_-)$, $F(M_0 \times \mathbb{R})$ and $F(M_+)$. Further using Proposition 2.30 we have

$$F(M_-) \bigotimes_{f(M_0 \times \mathbb{R})} F(M_+) \simeq \int_{[-1,1]} f_*F \longrightarrow F(f^{-1}[-1, 1]) = F(M), \tag{63}$$

where the morphism is given by the fact that factorization homology is a colimit. □

Remark 2.36. A functor $F : \mathcal{M}\mathsf{fld}(\mathcal{B}) \to \mathcal{C}$ that satisfies $\otimes$-excision and respects sequential colimits is called a homology theory in the terminology of [AF15]. We will not comment on this direction of development further except as a justification for the name of factorization homology. If in the above proof F was given by factorization homology, then in the final step the pushforward would provide the further necessary equivalence so that:

Proposition 2.37 ([AFT17a]). *Factorization homology is a homology theory. In particular, for a collar-gluing of a $\mathcal{B}$-manifold M, and a $\mathcal{D}\mathsf{isk}(\mathcal{B})$-algebra A*

$$\int_{M_-} A \bigotimes_{\int_{M_0\times\mathbb{R}} A} \int_{M_+} A \xrightarrow{\simeq} \int_M A. \tag{64}$$

Remark 2.38. The fact that factorization homology is $\otimes$-excisive is extremely computationally useful, and is the way we evaluate factorization homology in practice.

Example 2.39. Checking the definitions we can see that evaluating basics $U \in \mathcal{D}\mathsf{isk}(\mathcal{B}) \hookrightarrow \mathcal{M}\mathsf{fld}(\mathcal{B})$ with factorization homology is the same as simply evaluating the disk with the disk algebra. This is simply because factorization homology is a left Kan extension.

The first nontrivial example is then the oriented circle. We know that the algebras that we will be evaluating with are $\mathcal{D}\mathsf{isk}(\mathsf{D}_1^{\partial,*})$-algebras (or even simpler the restriction to $\mathcal{D}\mathsf{isk}(\mathsf{D}_1^*)$ the ones without boundary). The oriented circle can be presented as a collar-gluing

$$S^1 \cong \mathbb{R} \coprod_{S^0\times\mathbb{R}} \mathbb{R} \cong \mathbb{R} \coprod_{\bar{\mathbb{R}}\sqcup\mathbb{R}} \mathbb{R}, \tag{65}$$

where $\bar{\mathbb{R}}$ denotes $\mathbb{R}$ with the opposite orientation to the standard one. Factorization homology then gives

$$\int_{S^1} A \simeq \int_{\mathbb{R}} A \bigotimes_{\int_{\bar{\mathbb{R}}} A \otimes \int_{\mathbb{R}} A} \int_{\mathbb{R}} A \simeq A \bigotimes_{A\otimes A^{\mathrm{op}}} A, \tag{66}$$

where the result is known in the literature as the Hochschild homology (or more properly chains) of the associative algebra A.

2.6 Classification of $\mathcal{D}\mathsf{isk}(\mathsf{D}^*_{d\subset n})$-algebras

All of the above statements about factorization homology and disk algebras are quite general, and they do not say anything specific about a stratified structure, if there is one. Namely, had we stuck to smooth manifolds (possibly with boundary) we could have made similar statements. In this section we describe a result that clearly uses the stratified structure, and quantifies some additional information it can encode. It is a classification statement about disk algebras whose structure $\mathsf{D}^*_{d\subset n}$ is provided by Definition 1.47.

Theorem 2.40 ([AFT17a, prop.4.8]). *There is a pullback diagram*

$$\begin{tikzcd}
\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathsf{D}^*_{d+1})}\left(\int_{S^{n-d-1}\times\mathbb{R}^{d+1}} A, \mathsf{Z}(B)\right) \arrow[rr] \arrow[d] & & \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathsf{D}^*_{d\subset n})} \arrow[d] \\
* \arrow[r, "\{(A,B)\}"] & \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathsf{D}^*_n)} \times \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathsf{D}^*_d)}, &
\end{tikzcd} \tag{67}$$

where we have omitted the target ∞-category $\mathcal{C}$ for clarity. In other words, the data of a $\mathcal{D}\mathsf{isk}(\mathsf{D}^*_{d\subset n})$-algebra is equivalent to giving a triple (A, B, α), of a $\mathcal{D}\mathsf{isk}(\mathsf{D}^*_n)$-algebra A, a $\mathcal{D}\mathsf{isk}(\mathsf{D}^*_d)$-algebra B and a map of $\mathcal{D}\mathsf{isk}(\mathsf{D}^*_{d+1})$-algebras

$$\alpha : \int_{S^{n-d-1}\times\mathbb{R}^{d+1}} A \longrightarrow \mathsf{Z}(B). \tag{68}$$

Remark 2.41. The above theorem is a slight generalization of, and heavily relies on, the proof of the higher Deligne conjecture on Hochschild cohomology as in [Lur, sec.5.3]. Thus, from Proposition 2.8, we know that it is very important to the theorem to work with framed disks since these are the ones that reproduce the $\mathbb{E}_n$-algebras of the higher Deligne conjecture. The proof of the proposition is not easily generalizable to other tangential structures. The associative algebra $\mathsf{Z}(B)$ is the center of B as defined in [Lur, def.5.3.1.6], so that even the statement, which implies that $\mathsf{Z}(B)$ is more than associative, relies on the special structure provided by framing.

Remark 2.42. By definition, the center $\mathsf{Z}(B)$ is the universal object that acts on B, i.e. all objects that act on B factor through the action of the center $\mathsf{Z}(B)$ on B. Keeping this in mind, we see that, informally, Theorem 2.40 is saying that the structure of coupling the defect algebra B to the bulk algebra A is by giving B a suitable module structure over A. The complication then lies in proving what this suitable module structure is.

3 Factorization Algebras

3.1 Definition of Factorization Algebras

Factorization algebras are a rigorous way to capture the idea of assigning an object to each piece of a space, together with a local-to-global principle that allows one to find out about larger pieces from smaller ones. The first ideas around factorization algebras were introduced by [BD04] in the context of vertex algebras in CFTs. These were geometric in nature. The topological version that we present was then introduced by Lurie and subsequently further developed in [CG16], especially for the purposes of mathematical physics. Here we will present these ideas following [CG16] and also [AF19]. A similar exposition can be found in [Gin15].

Remark 3.1. We note that we have fixed a symmetric monoidal ∞-category $\mathcal{C}$ that is $\otimes$-presentable to serve as the ∞-category that our algebras will be valued in.

Definition 3.2. Let X be a topological space. We regard $\mathcal{O}\mathsf{pns}(X)$ as a multicategory (and consequently as an ∞-operad) whose objects are the open sets of X, and through the assignment of a unique multi-morphism from $\{U_i\}_{i\in I}$ to U if the U_i are pairwise disjoint and if $\bigcup_{i\in I} U_i \subset U$. A *prefactorization algebra F on X with values in* $\mathcal{C}$ is an algebra in $\mathcal{C}$ over this ∞-operad

$$F \in \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(X)}(\mathcal{C}). \tag{69}$$

Remark 3.3. It is obvious that in the case of (stratified) manifolds we have already seen alternative notation for the ∞-operad $\mathcal{O}\mathsf{pns}(X)$ before, namely $\mathsf{Mfld}_{/M}$. This makes a connection to the standard notation used around factorization homology, and is something that we'll make use of later.

Remark 3.4. Let's unwind the above definition to understand the underlying data needed to define a prefactorization algebra. On the level of objects, for every open set U we need to assign an object $F(U) \in \mathcal{C}$. On the level of morphisms, given a pairwise disjoint set of opens $\{U_i\}_{i\in I}$ and an open U, such that the U_i all lie in U we need to assign a morphism in $\mathcal{C}$

$$\bigotimes_{i\in I} F(U_i) \xrightarrow{F(\{U_i\},U)} F(U). \tag{70}$$

As the notation suggests and the operadic symmetry condition guarantees, the maps $F(\{U_i\}, U)$ only depend on the set of U_i and not on any particular order of the U_i, which is also allowed on the left-hand side by the symmetric monoidal structure of $\mathcal{C}$. Furthermore, the operadic associativity condition imposes that given pairwise disjoint $\{U_i\}_{i\in I}$ that all lie in U, and for each $i \in I$, pairwise disjoint $\{V_{i,j}\}_{j\in J_i}$ that all lie in U_i, there is a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} \bigotimes_{(i,j)} F(V_{i,j}) \arrow[rr, "F(\{V_{i,j}\},U)"] \arrow[dr, "\bigotimes_i F(\{V_{i,j}\},U_i)"'] & & F(U) \\ & \bigotimes_i F(U_i) \arrow[ur, "F(\{U_i\},U)"'] & \end{tikzcd} \quad , \tag{71}$$

where the (i,j) tensor product runs over all possible pairs $i \in I$ and $j \in J_i$. Finally, the operadic unitarity condition actually tells us that the morphisms $F(\{U\}, U)$ are equivalent to the identity morphism.

We also get out that an algebra morphism $\phi : F \to G$ is simply a family of maps $\phi(U) : F(U) \to G(U)$ for each open set U, such that it respects the operations of the algebra. Namely, for each multi-morphism $\{U_i\}_{i\in I} \to U$ there is a commuting square

$$\begin{tikzcd} \bigotimes_{i\in I} F(U_i) \arrow[r, "F(\{U_i\},U)"] \arrow[d, "\bigotimes_{i\in I} \phi(U_i)"'] & F(U) \arrow[d, "\phi(U)"] \\ \bigotimes_{i\in I} G(U_i) \arrow[r, "G(\{U_i\},U)"'] & G(U). \end{tikzcd} \tag{72}$$

Remark 3.5. There is an obvious inclusion functor from the poset $\mathsf{Opns}(X)$ of open subsets of X, ordered by inclusion, to the multicategory $\mathcal{O}\mathsf{pns}(X)$, namely, the one which hits only those multi-morphisms $\{U_i\}_{i\in I} \to U$, where the cardinality of the finite set I is 1.

A factorization algebra will be a prefactorization algebra that further satisfies a certain gluing condition that lets us construct its value on 'larger' sets if we know it on 'smaller' sets. More formally, this will be a kind of cosheaf condition, that we describe now, following [Wei99; AF19].

We endow the poset $\mathsf{Opns}(X)$ with a Grothendieck topology called the *Weiss* Grothendieck topology [Wei99]. In this topology a sieve $\mathcal{U} \subset \mathsf{Opns}(X)_{/U}$ on U is a covering sieve if for each finite subset $S \subset U$, there is an object $(e : V \to U) \in \mathcal{U}$ for which $S \subset e(V)$. In other words, a family $\{V_i \to U\}_{i\in I}$ is a Weiss cover of U if every set of finitely many points in U is contained in some V_i. Contrast this with the standard Grothendieck topology on the poset of opens $\mathsf{Opns}(X)$, in which instead of a finite set we have a one-element set. Thus, every Weiss cover is a cover in the standard sense, but not necessarily the other way around.

Definition 3.6. The ∞-category of $\mathcal{C}$*-valued Weiss (homotopy) cosheaves on* X is the full ∞-subcategory

$$\mathsf{c}\mathcal{S}\mathsf{hv}_X^{\mathsf{W}}(\mathcal{C}) \subset \mathsf{Fun}(\mathsf{Opns}(X), \mathcal{C}), \tag{73}$$

of the ∞-category of copresheaves, consisting of those functors $F : \mathsf{Opns}(X) \to \mathcal{C}$ for which, for each Weiss covering sieve $\mathcal{U} \subset \mathsf{Opns}(X)_{/U}$, the canonical functor

$$\mathcal{U}^{\triangleright} \to \mathsf{Opns}(X)_{/U} \to \mathsf{Opns}(X) \xrightarrow{F} \mathcal{C}, \tag{74}$$

where $\mathcal{U}^{\triangleright} \to \mathsf{Opns}(X)_{/U}$ is the functor from the colimit cone that assigns U to the colimit object, is a colimit diagram.

Remark 3.7. In other words, F is a Weiss cosheaf if it is a functor that sends colimits (which, in this case, are unions) of Weiss covers $\{U_i \hookrightarrow U\}_{i\in I}$ to colimits in $\mathcal{C}$

$$F(U) \simeq F\left(\bigcup_{i\in I} U_i\right) \simeq \mathrm{colim}_{i\in I} F(U_i). \tag{75}$$

Remark 3.8. The category $\mathsf{Opns}(X)$ can be replaced with any other category that supports an analogue of the Weiss Grothendieck topology. In the case of (stratified) manifolds, $\mathsf{Opns}(X)$ is already the same as $\mathsf{Mfld}_{/X}$, be we could just as well define Weiss cosheaves on Mfld, i.e. $\mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W}}(\mathsf{Mfld}, \mathcal{C})$. If we're working with the ∞-categories $\mathcal{M}\mathsf{fld}$ or $\mathcal{M}\mathsf{fld}_{/X}$, then we define cosheaves on them through the pullback

$$\begin{tikzcd}
\mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W}}(\mathcal{M}\mathsf{fld}, \mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathsf{Fun}(\mathcal{M}\mathsf{fld}, \mathcal{C}) \arrow[d] \\
\mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W}}(\mathsf{Mfld}, \mathcal{C}) \arrow[r] & \mathsf{Fun}(\mathsf{Mfld}, \mathcal{C}),
\end{tikzcd} \tag{76}$$

and similarly for the relative case $\mathcal{M}\mathsf{fld}_{/X}$.

Definition 3.9. The ∞-category of *($\mathcal{C}$-valued) factorization algebras on X*[16] is the full ∞-subcategory of $\mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(X)}$ as in the pullback

$$\begin{tikzcd}
\mathcal{F}\mathsf{Alg}_X(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(X)}(\mathcal{C}) \arrow[d] \\
\mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W}}_X(\mathcal{C}) \arrow[r] & \mathsf{Fun}(\mathsf{Opns}(X), \mathcal{C}).
\end{tikzcd} \tag{77}$$

Proof. The previous definition implies that the top horizontal functor is fully faithful, which is something we need to prove. The bottom horizontal functor is fully faithful, by the definition of cosheaf, and the right vertical functor forgets the operations with arity higher than 1. The key observation is that a Weiss cover is pairwise disjoint only when it consists of a single subset. Thus, the cosheaf condition does not affect the algebra morphisms and an equivalence of morphism spaces is trivial to find, namely one is given by the identity map. □

Remark 3.10. A factorization algebra is a prefactorization algebra whose restriction to $\mathsf{Opns}(X)$ is a Weiss cosheaf. This means that the definition can be rewritten in an equivalent way as with all cosheaves. Following [nLa23] (which presents the dual case of sheaves) we can express the condition through the Čech nerve. Let $\mathcal{U}_U = \{U_i \to U\}_{i\in I}$ be a (Weiss) cover of an open set U and F be a copresheaf, and define the simplicial object

$$\check{\mathsf{C}}_\bullet(\mathcal{U}_U, F) := \left(\coprod_{i\in I} F(U_i) \leftleftarrows \coprod_{i,j\in I} F(U_i\cap U_j) \Lleftarrow \coprod_{i,j,k\in I} F(U_i\cap U_j\cap U_k) \cdots \right), \tag{78}$$

where the maps are induced by the value of F on the inclusions of opens. There are canonical maps $F(U_i) \to F(U)$, so that at the level of simplicial objects there is a canonical map

$$\check{\mathsf{C}}_\bullet(\mathcal{U}_U, F) \longrightarrow F(U)_\bullet, \tag{79}$$

[16]These are called homotopy factorization algebras in [CG16], however we do not consider the lax version of factorization algebras that they also present.

to the constant simplicial complex at $F(U)$. Since $\mathcal{C}$ has (sifted) colimits we can take the geometric realization to obtain a morphism

$$\check{\mathsf{C}}(\mathcal{U}_U, F) \longrightarrow F(U). \tag{80}$$

F is a (homotopy) cosheaf if, and only if, this morphism is an equivalence for every open set U and every Weiss cover of it $\mathcal{U}_U$.

All of our definitions above are valid for all topological spaces, so in particular, also the stratified spaces we introduced at the beginning. For the next definition of locally constant factorization algebras though, it will be important to specify more information about the space. In other words, they will capture information about the stratification.

There is a general definition of locally constant algebra objects in the context of any ∞-operad given in [Lur, def.2.3.3.20], which, of course, applies here. In our case one can also take a perspective more akin to [Lur, def.A.1.12] by defining locally constant cosheaves and then pulling back this property to factorization algebras. This is what we will do, along with focusing our attention to stratified (and therefore also to smooth) manifolds. The definitions we will make can in a lot of cases also be expanded for any C^0 stratified space or topological manifold.

Definition 3.11. The ∞-category of locally constant Weiss cosheaves on a stratified manifold M (or, equivalently, on $\mathsf{Mfld}_{/M}$) is the full ∞-subcategory

$$\mathsf{c}\mathcal{S}\mathsf{hv}_M^{\mathsf{W,lc}}(\mathcal{C}) \subset \mathsf{c}\mathcal{S}\mathsf{hv}_M^{\mathsf{W}}(\mathcal{C}), \tag{81}$$

of those Weiss cosheaves whose underlying functor $F : \mathcal{B}\mathsf{sc}_{/M} \to \mathcal{C}$ satisfies

$$f(U) \simeq f(V) \Rightarrow F(U) \simeq F(V), \tag{82}$$

for all $U, V \in \mathcal{B}\mathsf{sc}_{/M}$, where $f : \mathcal{B}\mathsf{sc}_{/M} \to \mathcal{B}\mathsf{sc}$ denotes the forgetful functor.

Remark 3.12. Thus, a functor is locally constant if it takes open embeddings of basics $V \hookrightarrow U$, with equivalent stratifications to equivalences in $\mathcal{C}$. We emphasize this last point about stratification; the disks have to have equivalent stratifications inherited from the stratified space. In the case of $\mathbb{R}_{\geq 0}$, for example, open embeddings like $(a, b) \to [0, c)$, for $0 \leq a < b \leq c$, would *not* be taken to equivalences since the disks have different stratifications.

Remark 3.13. Since being locally constant according to the previous definition is only a property of the underlying functor the definition can easily be used to define locally constant prefactorization algebras too.

Definition 3.14. The ∞-category $\mathcal{F}\mathsf{Alg}_M^{\mathsf{lc}}(\mathcal{C})$ of *locally constant factorization algebras on a stratified manifold M valued in* $\mathcal{C}$ is the full ∞-subcategory of $\mathcal{F}\mathsf{Alg}_M(\mathcal{C})$ as in the pullback

$$\begin{tikzcd}
\mathcal{F}\mathsf{Alg}_M^{\mathsf{lc}}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{F}\mathsf{Alg}_M(\mathcal{C}) \arrow[d] \\
\mathsf{c}\mathcal{S}\mathsf{hv}_M^{\mathsf{W,lc}}(\mathcal{C}) \arrow[r] & \mathsf{c}\mathcal{S}\mathsf{hv}_M^{\mathsf{W}}(\mathcal{C}).
\end{tikzcd} \tag{83}$$

Remark 3.15. The ∞-categories of factorization algebras of all varieties acquire a symmetric monoidal structure from the symmetric monoidal structure of $\mathcal{C}$ pointwise

$$(F \otimes G)(U) = F(U) \otimes G(U). \tag{84}$$

The only subtlety is the cosheaf condition, where the symmetric monoidal structure is induced only if we also take into account that in $\mathcal{C}$ colimits commute with its symmetric monoidal structure.

The existence of colimits is also inherited from $\mathcal{C}$ pointwise through the collection of evaluation functors $\{\mathsf{ev}_U : \mathcal{F}\mathsf{Alg}_X(\mathcal{C}) \to \mathcal{C}\}$, one for each open $U \subset X$.

3.2 The ∞-operad of Factorization Algebras

Now that we have introduced all the varieties of factorization algebras, it's a natural and useful question to ask what the ∞-operad governing them is. For prefactorization algebras, the answer comes by definition, but this is not the case for factorization algebras. In the case of stratified manifolds, we will show that the ∞-operad that governs locally constant factorization algebras can be taken to be $\mathcal{D}\mathsf{isk}_{/M}$, and the one that governs ordinary factorization algebras can be taken to be $\mathsf{Disk}_{/M}$. This result is immensely helpful in relating factorization algebras to results in the area of disk algebras. We will see one such example in § 4.

Proposition 3.16. *Given a stratified manifold M factorization homology provides a functor between the ∞-categories of $\mathcal{D}\mathsf{isk}_{/M}$-algebras and locally constant factorization algebras $\mathcal{F}\mathsf{Alg}_M^{\mathsf{lc}}$*

$$\int : \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_M}(\mathcal{C}) \longrightarrow \mathcal{F}\mathsf{Alg}_M^{\mathsf{lc}}(\mathcal{C}). \tag{85}$$

Proof. We adopt a proof strategy similar to [AF19, prop.3.14]. Unlike there, we work in the relative case of $\mathcal{D}\mathsf{isk}_{/M}$ instead of $\mathcal{D}\mathsf{isk}(\mathcal{B})$. Since the ∞-category of factorization algebras is defined as a pullback, we will look for a commutative diagram involving $\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}$, so that, by the universal property of the pullback, we get the stated functor.

We first observe that, by definition, the ∞-operad $\mathcal{O}\mathsf{pns}(M) := \mathsf{Mfld}_{/M}$. There are clear functors of ∞-operads

$$\mathcal{D}\mathsf{isk}_{/M} \hookrightarrow \mathcal{M}\mathsf{fld}_{/M} \longleftarrow \mathsf{Mfld}_{/M}. \tag{86}$$

At the level of algebras over them, they give rise to the commutative diagram

$$\begin{tikzcd}
\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}(\mathcal{C}) \arrow[d] & \mathcal{A}\mathsf{lg}_{\mathcal{M}\mathsf{fld}_{/M}}(\mathcal{C}) \arrow[l] \arrow[r] \arrow[d] & \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(M)}(\mathcal{C}) \arrow[d] \\
\mathsf{Fun}(\mathcal{D}\mathsf{isk}_{/M}, \mathcal{C}) & \mathsf{Fun}(\mathcal{M}\mathsf{fld}_{/M}, \mathcal{C}) \arrow[l] \arrow[r] & \mathsf{Fun}(\mathcal{O}\mathsf{pns}(M), \mathcal{C}),
\end{tikzcd} \tag{87}$$

where the vertical arrows are forgetful functors. Factorization homology as a left Kan extension gives one the adjoints indicated below

$$\begin{tikzcd}
\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}(\mathcal{C}) \arrow[r, bend left, "\int"] \arrow[d] & \mathcal{A}\mathsf{lg}_{\mathcal{M}\mathsf{fld}_{/M}}(\mathcal{C}) \arrow[l, bend left] \arrow[r] \arrow[d] & \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(M)}(\mathcal{C}) \arrow[d] \\
\mathsf{Fun}(\mathcal{D}\mathsf{isk}_{/M}, \mathcal{C}) \arrow[r, bend left, "\int"] & \mathsf{Fun}(\mathcal{M}\mathsf{fld}_{/M}, \mathcal{C}) \arrow[l, bend left] \arrow[r] & \mathsf{Fun}(\mathcal{O}\mathsf{pns}(M), \mathcal{C}).
\end{tikzcd} \tag{88}$$

A result found in [AFT17b, thm.1.2.5] (where they use slightly different notation and call locally constant sheaves constructible) gives us the equivalence

$$\mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W}}(\mathcal{M}\mathsf{fld}_{/M}, \mathcal{C}) \simeq \mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W},\mathsf{lc}}(\mathsf{Mfld}_{/M}, \mathcal{C}) =: \mathsf{c}\mathcal{S}\mathsf{hv}_M^{\mathsf{W},\mathsf{lc}}(\mathcal{C}), \tag{89}$$

which at the level of underlying functors, is exactly the restriction along the functor $\mathcal{O}\mathsf{pns}(M) \to$

$\mathcal{M}\mathsf{fld}_{/M}$, so that we can append this to our commutative diagram

$$\begin{tikzcd}
\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}(\mathcal{C}) \arrow[r, bend left, "\int"] \arrow[d] & \mathcal{A}\mathsf{lg}_{\mathcal{M}\mathsf{fld}_{/M}}(\mathcal{C}) \arrow[l, bend left] \arrow[r] \arrow[d] & \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(M)}(\mathcal{C}) \arrow[d] \\
\mathsf{Fun}(\mathcal{D}\mathsf{isk}_{/M}, \mathcal{C}) \arrow[r, bend left, "\int"] \arrow[dr, dashed] & \mathsf{Fun}(\mathcal{M}\mathsf{fld}_{/M}, \mathcal{C}) \arrow[l, bend left] \arrow[r] & \mathsf{Fun}(\mathcal{O}\mathsf{pns}(M), \mathcal{C}) \\
 & \mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W}}(\mathcal{M}\mathsf{fld}_{/M}, \mathcal{C}) \arrow[u] \arrow[r, "\simeq"] & \mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W,lc}}_{M}(\mathcal{C}). \arrow[u]
\end{tikzcd} \tag{90}$$

If we can find the dashed functor which makes the diagram commute, as indicated above, we would be done with the construction. This amounts to showing that the left Kan extension $\int F$ of a functor $F \in \mathsf{Fun}(\mathcal{D}\mathsf{isk}_{/M}, \mathcal{C})$ along $\mathcal{D}\mathsf{isk}_{/M} \hookrightarrow \mathcal{M}\mathsf{fld}_{/M}$ is automatically a Weiss cosheaf. By the definition of $\mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W}}(\mathcal{M}\mathsf{fld}_{/M}, \mathcal{C})$, this means that the restriction of $\int F$ to $\mathsf{Fun}(\mathsf{Mfld}_{/M}, \mathcal{C})$ satisfies the cosheaf property for every Weiss sieve $\mathcal{U} \subset (\mathsf{Mfld}_{/M})_{/e}$, where $e : N \hookrightarrow M$ is an open embedding. Here we notice that for the Weiss property to even make sense, we are implicitly using Lemma 2.17

Since $\mathcal{C}$ has all relevant colimits the left Kan extension $\int F$ is given pointwise by

$$\int_e F \simeq \mathsf{colim}((\mathcal{D}\mathsf{isk}_{/M})_{/e} \to \mathcal{D}\mathsf{isk}_{/M} \xrightarrow{F} \mathcal{C}), \tag{91}$$

so what we need to show is that this colimit is equivalent to the colimit

$$\mathsf{colim}(\mathcal{U} \to (\mathsf{Mfld}_{/M})_{/e} \to \mathsf{Mfld}_{/M} \to \mathcal{M}\mathsf{fld}_{/M} \xrightarrow{\int F} \mathcal{C}), \tag{92}$$

for each covering sieve $\mathcal{U}$. Since we aren't actually trying to get the value of these colimits we can rewrite Equation (91) into the idiosyncratic form

$$\mathsf{colim}((\mathcal{D}\mathsf{isk}_{/M})_{/e} \to (\mathcal{M}\mathsf{fld}_{/M})_{/e} \to \mathcal{M}\mathsf{fld}_{/M} \xrightarrow{\int F} \mathcal{C}), \tag{93}$$

which is easier to compare to Equation (92). Using the fact that $\mathsf{Disk}_{/M} \to \mathcal{D}\mathsf{isk}_{/M}$ is a final functor, which is a consequence of [AFT17a, prop.2.22], and the equivalences of Lemma 2.17 we can rewrite Equation (93) further into

$$\mathsf{colim}((\mathsf{Disk}_{/M})_{/e} \to (\mathcal{D}\mathsf{isk}_{/M})_{/e} \to (\mathcal{M}\mathsf{fld}_{/M})_{/e} \to \mathcal{M}\mathsf{fld}_{/M} \xrightarrow{\int F} \mathcal{C}) \tag{94}$$

$$\simeq \mathsf{colim}((\mathsf{Disk}_{/M})_{/e} \to (\mathsf{Mfld}_{/M})_{/e} \to \mathsf{Mfld}_{/M} \to \mathcal{M}\mathsf{fld}_{/M} \xrightarrow{\int F} \mathcal{C}). \tag{95}$$

Denoting the functor $\mathsf{Mfld}_{/N} \simeq (\mathsf{Mfld}_{/M})_{/e} \to \mathsf{Mfld}_{/M} \to \mathcal{M}\mathsf{fld}_{/M} \xrightarrow{\int F} \mathcal{C}$ by K_F and comparing with Equation (92) we need to show

$$\mathsf{colim}(\mathcal{U} \to \mathsf{Mfld}_{/N} \xrightarrow{K_F} C) \simeq \mathsf{colim}(\mathsf{Disk}_{/N} \to \mathsf{Mfld}_{/N} \xrightarrow{K_F} C), \tag{96}$$

but this already follows from part 2 of the proof of [AF19, prop.2.22], and is equivalent to the statement that the functor

$$\mathsf{colim}(\mathcal{U} \to \mathsf{Mfld}_{/N} \to \mathsf{Mfld} \xrightarrow{\mathcal{D}\mathsf{isk}_{/-}} \mathcal{C}\mathsf{at}_\infty) \to \mathcal{D}\mathsf{isk}_{/N}, \tag{97}$$

is final. □

Remark 3.17. Intuitively, the key idea for the statement and proof is that $\mathsf{Disk}_{/N}$ is a basis for the Weiss Grothendieck topology on manifolds. This is essentially because for each finite set of points $S \subset N$, and each Weiss cover of it U_S, we can take small enough disks $\{D_s\}_{s\in S}$ around each point, by virtue of N being a manifold, and their disjoint union will cover S, as well as satisfy $\sqcup_{s\in S} D_s \subset U_S$. The proof of [AF19, prop.2.22] makes this precise, and the above proof transports that idea to the relative case.

From the proof of Proposition 3.16 it is easy to see that we can make a similar statement for ordinary factorization algebras too:

Corollary 3.18. *Given a stratified manifold M factorization homology provides a functor*

$$\int : \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}(\mathcal{C}) \longrightarrow \mathcal{F}\mathsf{Alg}_M(\mathcal{C}). \tag{98}$$

Theorem 3.19. *For each stratified manifold M, the previously constructed factorization homology functor provides an equivalence of ∞-categories between $\mathcal{D}\mathsf{isk}_{/M}$-algebras and locally constant factorization algebras on M. Equivalently, the ∞-operad governing locally constant factorization algebras can be taken to be $\mathcal{D}\mathsf{isk}_{/M}$*

$$\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C}). \tag{99}$$

Proof. To show essential surjectivity of $\int$, for each $F \in \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C})$ we will find an equivalence $\int F| \simeq F$ in $\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C})$, with $F|$ to be defined later. Since $\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C}) \to \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(M)}$ is fully faithful, this amounts to finding equivalences of prefactorization algebras, namely a family of equivalences

$$\int_U F| \xrightarrow[\simeq]{\phi(U)} F(U), \tag{100}$$

one for each $(U \hookrightarrow M) \in \mathsf{Mfld}_{/M}$, which preserve the multiplicative structure (see Equation (72)). Since F is a locally constant factorization algebra the functor

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C}) \to \mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W,lc}}(\mathsf{Mfld}_{/M}, \mathcal{C}) \simeq \mathsf{c}\mathcal{S}\mathsf{hv}^{\mathsf{W}}(\mathcal{M}\mathsf{fld}_{/M}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{M}\mathsf{fld}_{/M}, \mathcal{C}), \tag{101}$$

which we omit in the notation, allows us to consider F as a functor from $\mathcal{M}\mathsf{fld}_{/M}$. Furthermore, since $\mathsf{Disk}_{/U}$ is a Weiss sieve we can write

$$F(U) \simeq \mathsf{colim}(\mathsf{Disk}_{/U} \to \mathsf{Disk}_{/M} \to \mathsf{Mfld}_{/M} \to \mathcal{M}\mathsf{fld}_{/M} \xrightarrow{F} \mathcal{C}). \tag{102}$$

By consulting the commutative diagram

$$\begin{tikzcd}
\mathsf{Disk}_{/U} \arrow[r] \arrow[d] & \mathcal{D}\mathsf{isk}_{/U} \arrow[d] & \\
\mathsf{Disk}_{/M} \arrow[r] \arrow[d] & \mathcal{D}\mathsf{isk}_{/M} \arrow[r, "F|"] \arrow[d] & \mathcal{C}, \\
\mathsf{Mfld}_{/M} \arrow[r] & \mathcal{M}\mathsf{fld}_{/M} \arrow[ur, "F"'] &
\end{tikzcd} \tag{103}$$

and using the fact that $\mathsf{Disk}_{/U} \to \mathcal{D}\mathsf{isk}_{/U}$ is final ([AFT17a, prop.2.22]) immediately gives us that

$$F(U) \simeq \mathsf{colim}(\mathcal{D}\mathsf{isk}_{/U} \to \mathcal{D}\mathsf{isk}_{/M} \xrightarrow{F|} \mathcal{C}) =: \int_U F|. \tag{104}$$

To preserve the multiplicative structure, these equivalences have to be natural, and there also has to be a commutative diagram

$$\begin{tikzcd} \otimes_i \int_{U_i} F| \arrow[r, "\simeq"] \arrow[d, "\otimes_i \phi(U_i)"', "\simeq"] & \int_{\sqcup_i U_i} F| \arrow[d, "\simeq"', "\phi(\sqcup_i U_i)"] \\ \otimes_i F(U_i) \arrow[r, "\simeq"] & F(\sqcup_i U_i), \end{tikzcd} \tag{105}$$

for each collection of disjoint open sets $\{U_i\}$. Naturality is immediate from the construction because, as already used, each inclusion of open subsets $V \hookrightarrow U$ gives a full subcategory inclusion $\mathsf{Disk}_{/V} \hookrightarrow \mathsf{Disk}_{/U}$. On the other hand, the existence of the commutative diagram is guaranteed by the equivalence

$$\mathsf{Disk}_{/\sqcup_i U_i} \xrightarrow{\simeq} \bigtimes_i \mathsf{Disk}_{/U_i}. \tag{106}$$

To show full faithfulness we need to find equivalences of morphism spaces

$$\mathsf{Hom}_{\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}(\mathcal{C})}(A, B) \simeq \mathsf{Hom}_{\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C})}(\int A, \int B), \tag{107}$$

for each $A, B \in \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}$. However, since the functors

$$\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}_{/M}}(\mathcal{C}) \xrightarrow{\text{f.f.}} \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}(\mathcal{C}) \qquad \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C}) \xrightarrow{\text{f.f.}} \mathcal{A}\mathsf{lg}_{\mathsf{Mfld}_{/M}}(\mathcal{C}) \tag{108}$$

are both fully faithful we are reduced to finding equivalences

$$\mathsf{Hom}_{\mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}(\mathcal{C})}(A, B) \simeq \mathsf{Hom}_{\mathcal{A}\mathsf{lg}_{\mathsf{Mfld}_{/M}}(\mathcal{C})}(\int A, \int B). \tag{109}$$

The existence of these is exactly the requirement that factorization homology in the usual sense is fully faithful, which is part of the statement of [AFT17a, lem.2.17] □

Remark 3.20. One perspective on this result, provided by [AFT17a, thm.2.43], is that the theory of locally constant factorization algebras captures as much information as (generalized) homology theory in the sense of [AFT17a, def.2.37].

As before, the case of ordinary factorization algebras gets a similar statement whose proof goes along the same lines:

Corollary 3.21. *The ∞-operad governing factorization algebras on a stratified manifold M can be taken to be* $\mathsf{Disk}_{/M}$*,*

$$\mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{F}\mathsf{Alg}_M(\mathcal{C}). \tag{110}$$

Remark 3.22. A version of Theorem 3.19 was proven as [GTZ14, thm.6] in the case of smooth manifolds. The proof there, however, made crucial use of choosing a Riemannian metric on the manifold and constructing geodesically convex neighborhoods. This technique is, at least presently, not extendable to the stratified case, but our proof above subverts the need for it.

Remark 3.23. As noted at the beginning of §2, the physical interpretation of this version of factorization homology is a way to take a spacetime M and its local observables A, and construct *some*, and typically not all, of the global observables. The previous Theorem 3.19 and Corollary 3.21 transfer this observation to factorization algebras, which aligns with the observation of [CG16] that the current version of factorization algebras can only deal with perturbative quantum field theory.

The ∞-operad $\mathcal{D}\mathsf{isk}_{/M}$ is, in general, harder to deal with than its symmetric monoidal counterpart $\mathcal{D}\mathsf{isk}(\mathcal{B})$. The next lemma outlines a certain situation where this is not the case.

Lemma 3.24. *Let M be a stratified manifold. If we can find an ∞-category of basics $\mathcal{B}_M$ such that M admits a $\mathcal{B}_M$-structure and such that*

$$\mathsf{Hom}_{\mathcal{M}\mathsf{fld}(\mathcal{B}_M)}(V, M) \simeq * \tag{111}$$

for all $V \in \mathcal{B}_M$, then there is an equivalence of symmetric monoidal ∞-categories

$$\mathsf{Env}(\mathcal{D}\mathsf{isk}_{/M}) \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}(\mathcal{B}_M). \tag{112}$$

Remark 3.25. If we limit M to be a basic then the requirements of the lemma are exactly that M is final in $\mathcal{B}_M$.

Proof. We follow the proof of [AFT17a, cor.2.33]. Using Remark 2.16, the forgetful functor from the slice ∞-operad, which is a map of ∞-operads, is what will provide the equivalence

$$\mathcal{D}\mathsf{isk}_{/M} \simeq \mathcal{D}\mathsf{isk}(\mathcal{B}_M)_{/M} \longrightarrow \mathcal{D}\mathsf{isk}(\mathcal{B}_M). \tag{113}$$

This functor is an equivalence on maximal ∞-subgroupoids because for any $V \in \mathcal{D}\mathsf{isk}(\mathcal{B}_M)$ the space of morphisms to M is contractible, so, up to equivalence it is unique. Full faithfulness is provided by the fact that the functor on the active ∞-subcategories $\mathcal{D}\mathsf{isk}_{/M} \to \mathcal{D}\mathsf{isk}(\mathcal{B}_M)$ is a right fibration. □

The lemma says that for stratified manifolds M that satisfy the conditions, the classification of locally constant factorization algebras on M, i.e. algebras over $\mathcal{D}\mathsf{isk}_{/M}$ simplifies to the classification of $\mathcal{D}\mathsf{isk}(\mathcal{B}_M)$-algebras in the usual sense

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C}) \simeq \mathsf{Fun}^{\otimes}(\mathsf{Env}(\mathcal{D}\mathsf{isk}_{/M}), \mathcal{C}) \simeq \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathcal{B}_M)}(\mathcal{C}). \tag{114}$$

This is an improvement because $\mathcal{D}\mathsf{isk}(\mathcal{B}_M)$-algebras are easier to work with, one reason being that $\mathcal{D}\mathsf{isk}(\mathcal{B}_M)$ has the benefit of being not just an ∞-operad but even a symmetric monoidal ∞-category.

The results of Theorem 3.19 and Corollary 3.21 can also be seen as lowering the amount of data that is necessary to say that we have defined a factorization algebra on a stratified manifold. There is a similar, but slightly weaker result by [CG16], which nonetheless works for all Hausdorff topological spaces. We recount this result here for comparison.

Definition 3.26. A *factorizing basis* of a topological space X is a basis $\{U_i \hookrightarrow X\}_{i\in I}$ for the topology of X which is closed under finite intersections, and which satisfies that for each finite set $S \subset X$, there exists a finite sub-collection of pairwise disjoint open subsets $\{U_j \hookrightarrow X\}_{j\in J}$, for which $S \subset \bigsqcup_{j\in J} U_j$.

Definition 3.27. We will call factorization algebras defined only on the open subsets of an open cover $\mathcal{U}$ (in particular, for example, a factorizing basis), a $\mathcal{U}$-factorization algebra and the ∞-category of such $\mathcal{F}\mathsf{Alg}_{\mathcal{U}}(\mathcal{C})$.

Theorem 3.28 ([CG16])**.** *Given a Hausdorff topological space X, there is an equivalence of ∞-categories between the ∞-categories of $\mathcal{U}$-factorization algebras defined on a factorizing basis $\mathcal{U}$ and factorization algebras defined on the full topological space X*

$$\begin{tikzcd} \mathcal{F}\mathsf{Alg}_X(\mathcal{C}) \arrow[r, bend left, "-|_{\mathcal{U}}"] & \mathcal{F}\mathsf{Alg}_{\mathcal{U}}(\mathcal{C}), \arrow[l, bend left, "\mathrm{ext}"] \end{tikzcd} \tag{115}$$

where the top functor is given by restriction.

Remark 3.29. Looking into the proof of this statement we see that the value of the extension on open sets $U \subset X$ is defined as

$$\mathrm{ext}(F)(U) := \check{\mathsf{C}}(\mathcal{U}_U, F), \tag{116}$$

where $\mathcal{U}_U$ is a Weiss cover of U generated by $\mathcal{U}$. That such a cover exists is already guaranteed by the Hausdorffness of X and the basis property of $\mathcal{U}$. Namely, given any finite set $S \subset U$, Hausdorffness gives us the existence of pairwise disjoint open sets $\{V_s \ni s\}_{s\in S}$, while the basis property of $\mathcal{U}$ gives us open subsets $\{U_s \in \mathcal{U}\}_{s\in S}$ such that $s \in U_s \subset V_s$. $\sqcup_{s\in S} U_s$ then gives an open set that covers S.

Remark 3.30. The proof of Theorem 3.28 can be found in [CG16, ch.7.2], and we will not reproduce it here. The target category there is the category of chain complexes, but it holds more generally in $\mathcal{C}$ by considering simplicial objects and their geometric realizations (which exist by the colimit assumption on $\mathcal{C}$) instead.

As mentioned, in the stratified manifold case, this is an easy corollary of the proof of Proposition 3.16, where we show that taking all disks as the Weiss cover of choice reproduces the result that we would have gotten with any other Weiss cover. It is important to point out though that disks are not necessarily closed under intersections, which means that we don't have the opposite implication, and the methods to prove Proposition 3.16 are necessary.

3.3 Operations on Factorization Algebras

There are a few important operations that we can do with factorization algebras (and the other mentioned variants) that will be important for our considerations. They allow us to compare factorization algebras from different spaces and will play a big role in any kind of classification statement one might make.

Proposition 3.31. *Given a continuous map $f : X \to Y$ between topological spaces, there are pushforward functors*

$$f_* : \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(X)} \longrightarrow \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(Y)} \qquad\qquad f_* : \mathcal{F}\mathsf{Alg}_X \longrightarrow \mathcal{F}\mathsf{Alg}_Y, \tag{117}$$

which, on objects, are given by the prescription

$$f_*F(U) := F(f^{-1}U). \tag{118}$$

Proof. By the definition of continuity, the case of prefactorization algebras is trivial. Thus, the only thing we are left to show is that if F satisfies the Weiss cosheaf property then so does f_*F. Given a Weiss cover $\{U_i \hookrightarrow U\}_{i\in I}$ of U, we observe that $\{f^{-1}U_i \hookrightarrow f^{-1}U\}_{i\in I}$ is a Weiss cover of $f^{-1}U$. Namely, if $S \subset f^{-1}U$ is a finite subset of U, then $f(S)$ is contained in some U_i by the Weiss cover property of $\{U_i \hookrightarrow U\}_{i\in I}$. But in that case S has to be contained in $f^{-1}U_i$, giving it the Weiss cover property too. Thus, by the definition of the pushforward on objects the Weiss cosheaf property is preserved. □

The case of locally constant factorization algebras is different when it comes to the pushforward. This is because it's not immediate that local constancy is preserved when pushed forward. However, if we limit the types of maps we pushforward with we can still construct a functor. The following two results give the flavor of what is required.

Proposition 3.32 ([Gin15, prop.15])**.** *If $f : X \to Y$ is a locally trivial fibration between smooth manifolds then the pushforward functor exists even between the ∞-categories of locally constant factorization algebras*

$$f_* : \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_X \longrightarrow \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_Y. \tag{119}$$

Corollary 3.33. *Let $f : M \to N$ be a constructible bundle of stratified manifolds. Then the pushforward functor for factorization homology of Theorem 2.32 serves as a pushforward*

$$f_* : \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M \longrightarrow \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_N, \tag{120}$$

by using Theorem 3.19.

Remark 3.34. Having defined the pushforward functor of factorization algebras we can consider the map $\mathsf{p} : M \to *$. Given any $A \in \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}}$ the factorization algebra F_A generated by A satisfies

$$\int_M A = F_A(M) \simeq (\mathsf{p}_* F_A)(*). \tag{121}$$

The one point manifold is a very simple space, which makes the concepts of prefactorization algebras, factorization algebras and locally constant factorization algebras coincide. All of them are in fact given by a pointed object of the target category. Evaluation at $*$ simply returns the underlying object. Thus, in a sense evaluating factorization homology on a space M is the same procedure as pushing-forward by the map p. Versions of factorization homology that can evaluate lower dimensional manifolds (as compared to the structure of the disk algebra), for example as in [AFT17a, cor.2.29], can then be seen as relaxing this pushforward from $*$ to a manifold with more structure.

Lemma 3.35. *Let $u : X \to \hat{X}$ be the morphism of Example 1.9 that forgets the stratification. The functor $u_* : \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_X(\mathcal{C}) \to \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\hat{X}}(\mathcal{C})$ has a fully faithful left adjoint.*

Proof. Consider the functor $\hat{-} : \mathcal{F}\mathsf{Alg}_{\hat{X}} \to \mathcal{F}\mathsf{Alg}_X$, which acts as $\hat{F}(U) := F(u(U))$, namely it returns algebras that evaluate open subsets by forgetting their stratification first. We claim this functor is the left adjoint of u_*. Indeed, given a morphism $(\phi : \hat{F} \to G) \in \mathcal{F}\mathsf{Alg}_X$, i.e. a family of morphisms $\phi(U) : \hat{F}(U) \to G(U)$ for all opens U in X, it is easy to verify that we can construct a morphism of algebras over $\hat{X}$, $\hat{\phi}(u(U)) : F(u(U)) \to u_*G(u(U))$, since $\hat{F}(U) = F(u(U))$ and $u_*G(u(U)) = G(u^{-1}(u(U))) = G(U)$. The last equality holds because u is an injective map. In fact, this last observation also proves that the unit of this adjunction is an isomorphism, granting the full faithfulness. □

Remark 3.36. In fact, the above proof is true more generally for any map of stratified spaces that is a homeomorphism of the underlying topological spaces (so, in particular, any refinement). This means that locally constant factorization algebras record the data of coarser stratified structures as ∞-subcategories. Forgetting local constancy, the proof, of course, applies to factorization algebras of all flavors, however for the types that don't detect the stratified structure the above functor is even an equivalence, and not just fully-faithful, and is of no interest.

In the opposite direction, we do not, in general, have a pullback functor. [CG16] provide a construction of a pullback in the case of open immersions, but we will only focus on the case of restrictions. Namely, given an open subset $U \subset X$ of a topological space X, and given a factorization algebra $F \in \mathcal{F}\mathsf{Alg}_X$ we can clearly define a factorization algebra $F|_U \in \mathcal{F}\mathsf{Alg}_U$ by restricting to those open subsets that are fully contained in U. Furthermore, this clearly holds not only for factorization algebras, but prefactorization algebras and locally constant factorization algebras too.

In the special case of product spaces $X \times Y$, we now know that the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$, give us pushforwards. For π_1, for example, this essentially uses the fact that if we can evaluate all open sets of $X \times Y$ then we can definitely evaluate the open sets that look like $U \times Y$, with $U \subset X$ an open set. But in this case we can do even more, because we can even evaluate the more granular open subsets $U \times V$ with $U \subset X$ and $V \subset Y$ opens. More formally, we claim:

Proposition 3.37. *Let X and Y be topological spaces. There is a functor from the ∞-category of prefactorization algebras on the product $X \times Y$ to the ∞-category of prefactorization algebras on X valued in the category of prefactorization algebras on Y*

$$\bar{\pi} : \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(X\times Y)}(\mathcal{C}) \longrightarrow \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(X)}(\mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(Y)}(\mathcal{C})). \tag{122}$$

This functor descends to factorization algebras too

$$\bar{\pi} : \mathcal{F}\mathsf{Alg}_{X\times Y}(\mathcal{C}) \longrightarrow \mathcal{F}\mathsf{Alg}_X(\mathcal{F}\mathsf{Alg}_Y(\mathcal{C})). \tag{123}$$

Remark 3.38. The conditions we imposed on $\mathcal{C}$ allowed us, in Remark 3.15, to inherit a symmetric monoidal structure on the ∞-categories of factorization algebras of all varieties. It also allowed us to inherit the existence of colimits. This is the reason why the above can even be stated.

Proof. Indeed, given opens $U \subset X$ and $V \subset Y$, the values we assign are $(\bar{\pi}F)(U)(V) = F(U \times V)$. Fixing an open subset $U \subset X$, we obviously get a prefactorization algebra $(\pi_2|_U)_*(F|_U) \in \mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(Y)}$ by pushing forward with $\pi_2|_U : U \times Y \to Y$, which shows that $\bar{\pi}F$ is valued in $\mathcal{A}\mathsf{lg}_{\mathcal{O}\mathsf{pns}(Y)}$. The only thing left to show is that $\bar{\pi}F$ has the structure of a prefactorization algebra on X. To do this, for each multi-morphism $(\{U_i\}_{i\in I} \to U) \in \mathcal{O}\mathsf{pns}(X)$ we assign a map $\bar{\pi}F(\{U_i\}_{i\in I}, U)$ of factorization algebras on Y. Such an algebra map is specified by giving its value on all opens of Y separately. We can thus assign

$$\bar{\pi}F(\{U_i\}_{i\in I}, U)(V) = F(\{U_i \times V\}_{i\in I}, U \times V), \tag{124}$$

which uses the projections $\pi_1|_V : X \times V \to X$ to assign $\pi_1|_V^{-1}(U) = U \times V$.

It is not hard to check that the above also holds for the example of factorization algebras on topological spaces, by using Theorem 3.28. However, we'll focus on manifolds, in which case, the construction from above immediately works since we now have to do it for $\mathsf{Disk}_{/X}$ instead of $\mathsf{Mfld}_{/X}$. The only things to mention is that products of disks are again disks [AFT17b, cor.3.4.9]. $\square$

Proposition 3.39. *Let M and N be stratified manifolds. The functor $\bar{\pi}$ descends to locally constant factorization algebras, and moreover, it is an equivalence of ∞-categories*

$$\bar{\pi} : \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M\times N}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_N(\mathcal{C})). \tag{125}$$

Remark 3.40. The above proposition also appears as [Gin15, prop.18]. The equivalence proven there is for the case of smooth manifolds and relies on the local proof of [Lur] for $\mathbb{R}^n$. Using Theorem 3.19, the statement is implied by the fact that there is an equivalence of ∞-operads

$$\mathcal{D}\mathsf{isk}_{/M} \otimes \mathcal{D}\mathsf{isk}_{/N} \xrightarrow{\simeq} \mathcal{D}\mathsf{isk}_{/M\times N}, \tag{126}$$

with the tensor product of ∞-opeards. Again for smooth manifolds this is found in [Lur, ex.5.4.5.5].

Proof. The proof of existence of the functor goes along the same lines as the proof of Proposition 3.37 because of Theorem 3.19. The thing to take into account is that the projection maps are both, trivially, constructible bundles so by Corollary 3.33, pushing forward by them yields locally constant factorization algebras again. This combined with the fact that for any ∞-operads $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{C}$ we have an equivalence

$$\mathcal{A}\mathsf{lg}_{\mathcal{O}}(\mathcal{A}\mathsf{lg}_{\mathcal{O}'}(\mathcal{C})) \simeq \mathcal{A}\mathsf{lg}_{\mathcal{O}'}(\mathcal{A}\mathsf{lg}_{\mathcal{O}}(\mathcal{C})) \tag{127}$$

allows us to conclude that the functor lands in locally constant factorization algebras

$$\bar{\pi} : \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{X\times Y}(\mathcal{C}) \longrightarrow \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{X}(\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{Y}(\mathcal{C})). \tag{128}$$

One way to show the equivalence is to use factorization homology, as in Theorem 3.19, to write a locally constant factorization algebra as $\int_{\_} A \in \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{X\times Y}(\mathcal{C})$. Then using the Fubini theorem [AFT17a, cor.2.29][17] for factorization homology we get essential surjectivity because there are equivalences

$$\int_{U\times V} A \xrightarrow{\simeq} \int_U \int_V \bar{\pi} A \tag{129}$$

between objects of $\mathcal{C}$. As argued in the proof of Theorem 3.19, these equivalences are natural in the open set variables, which means that they rise to equivalences at the level of locally constant factorization algebras. In fact, looking at the proof of [AFT17a, thm.2.25], which is what underlies the construction of the above equivalences, we see that they arise from an argument about a final functor. By Remark 2.29 this means that the equivalences are actually natural in the algebra variable, so it is immediate that the equivalences of individual locally constant factorization algebras lift to a full equivalence of ∞-categories. $\square$

3.4 Factorization Algebras on Intervals

As was the case with factorization homology, so too with factorization algebras we now turn to a few very important constructions of locally constant factorization algebras on intervals. Namely, we will consider how we can construct a locally constant factorization algebra on the spaces $\mathbb{R}$ (a.k.a. the open interval) and $\mathbb{R}_{\geq 0}$ (a.k.a. the half-open interval). These will be a very important backbone for intuition about locally constant factorization algebras, as well as a key part of proofs later on. We finish with factorization algebras for the closed interval $[-1, 1]$.

Construction 3.41. Consider a unital, (homotopy) associative algebra A in the category $\mathcal{C}$. For a first pass we use Theorem 3.28 to construct a factorization algebra out of this data. We first check that the disks $\mathsf{Disk}_{/\mathbb{R}}$ of $\mathbb{R}$ form a factorizing basis. This is true since the disks are always a basis for any manifold, they are factorizing since we have all disjoint unions, and they are closed under finite intersections because the disks of $\mathbb{R}$ are intervals. Therefore, by Theorem 3.28, we need only assign values to these. Given any disk U we assign the same value, $F_A(U) := A$. Similarly, given a multi-morphism $\sqcup_{i\in I} U_i \hookrightarrow U$, with I non-empty, we assign the map given by the multiplication of A

$$F_A(\sqcup_{i\in I} U_i \hookrightarrow U) := \otimes_{i\in I} A \to A. \tag{130}$$

The implicit convention, is that the ordering of the intervals, provided by the standard orientation (or framing) of $\mathbb{R}$, determines the order in which we multiply. Associativity forbids any other ordering except the completely opposite one, which would yield the opposite multiplication, and thus the opposite algebra. Finally, in the case of $\emptyset \hookrightarrow U$, we assign the map that chooses the unit $\mathbb{1} \to A$. At this point we have constructed a prefactorization algebra on $\mathbb{R}$. To use Theorem 3.28 we need to first check the Weiss cosheaf condition. But because intervals are closed under finite intersections the Čech nerve definition of Remark 3.10 makes this easy; the simplicial object $\check{\mathsf{C}}_\bullet(\mathsf{Disk}_{/\mathbb{R}}, F_A)$ in all degrees will be a coproduct of copies of A. Thus, F_A is indeed a $\mathsf{Disk}_{/\mathbb{R}}$-factorization algebra and can be extended to a factorization algebra on $\mathbb{R}$. By definition, it is also obviously locally constant.

[17] The notation used in the citation is technically abusive because factorization homology is evaluating lower dimensional manifolds, but it is clear that this is to be understood exactly as the evaluation of the pushforward factorization algebra on the lower dimensional manifold (see Remark 3.34).

In this simple case the disks are closed under finite intersections, so the construction works out. More generally, one would need to find another factorizing basis. In the case of smooth manifolds one possible way is to choose a Riemannian metric and take the geodesically convex neighborhoods. An alternative is to get rid of the property of being closed under finite intersections altogether. It turns out, because of Theorem 3.19, that this is indeed possible, and that we can always stick to using disks even in the most general situations.

In whatever way we go about it, it is immediate that we have constructed a functor $\mathcal{A}\mathsf{lg}_{\mathbb{E}_1}(\mathcal{C}) \to \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}}(\mathcal{C})$ from the ∞-category of unital, (homotopy) associative algebras, or $\mathbb{E}_1$-algebras to the ∞-category of locally constant factorization algebras on $\mathbb{R}$. In fact, we claim that this functor is actually an equivalence, i.e., that every locally constant factorization algebra on $\mathbb{R}$ is equivalent to one induced like in the construction.

Proposition 3.42. *There is an equivalence of ∞-categories*

$$\mathcal{A}\mathsf{lg}_{\mathbb{E}_1}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}}(\mathcal{C}). \tag{131}$$

Proof. What is left to show is the essential surjectivity of the functor. This holds because every disk in $\mathbb{R}$ is diffeomorphic to $\mathbb{R}$ itself. So, given $F \in \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}}$, we can get an underlying object for our $\mathbb{E}_1$-algebra as $A = F(\mathbb{R})$. The previous observation and the local constancy then force the value on each disk to be equivalent to this. The space of embeddings of two disks $\mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}$ is, up to homotopy, equivalent to two points, which provide exactly the multiplication map $A \otimes A \to A$ of the algebra and its opposite multiplication. Associativity, up to homotopy, comes about exactly because all embeddings of any number of disks can, up to homotopy, be factorized through the embedding of two disks at a time. As before, the unit comes from the map $F(\emptyset \hookrightarrow \mathbb{R}) = (\mathbb{1} \to A)$. Thus, we have constructed an $\mathbb{E}_1$-algebra from the given data, and the proposition holds. □

Construction 3.43. Consider a unital, (homotopy) associative algebra A in the category $\mathcal{C}$, as above, together with a unital right module M of A in the category $\mathcal{C}$. From this data we will construct a locally constant factorization algebra $F_{(M,A)}$ on the stratified space $\mathbb{R}_{\geq 0}$. Considering Construction 3.41 and the open embedding $\mathbb{R} \cong \mathbb{R}_{>0} \hookrightarrow \mathbb{R}_{\geq 0}$, we will choose the restriction to be $F_{(M,A)}|_{\mathbb{R}} = F_A$, i.e. any disk that doesn't include the point $0 \in \mathbb{R}_{\geq 0}$, we regard as a disk of $\mathbb{R}$ through the given map, and we assign the value A to it like in the Construction 3.41. This leaves us only with the stratified disks U_* that contain the point $0 \in \mathbb{R}_{\geq 0}$. To these we assign the module $F_{(M,A)}(U_*) = M$. Since there can be at most one stratified disk that includes the point $0 \in \mathbb{R}_{\geq 0}$, the maps that we have to assign values to look like $U_* \sqcup (\sqcup_{i\in I} U_i) \hookrightarrow U'_*$, but this is exactly what is provided by the right module structure

$$F_{(M,A)}(U_* \sqcup (\sqcup_{i\in I} U_i) \hookrightarrow U'_*) = M \otimes (\otimes_{i\in I} A) \to M. \tag{132}$$

Of course the pointing of the module comes from $F_{(M,A)}(\emptyset \hookrightarrow U_*) = \mathbb{1} \to M$. As before, even more is true:

Proposition 3.44. *There is an equivalence of ∞-categories between the ∞-categories of $\mathbb{E}_1$-algebras together with a right module and locally constant factorization algebras on the half-open interval*

$$\mathsf{RMod}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}_{\geq 0}}(\mathcal{C}). \tag{133}$$

Proof. The proof goes along the same lines as the proof of Proposition 3.42. We can find the module by evaluating on $\mathbb{R}_{\geq 0}$, and we can find the algebra by evaluating on $\mathbb{R}_{\geq 0} \setminus \{0\} \cong \mathbb{R}$, as a specified stratum of $\mathbb{R}_{\geq 0}$. □

Remark 3.45. The fact that we chose left modules to provide the data in Proposition 3.44, does not play much of a role because a left module is exactly a right module of the opposite algebra, giving $\mathsf{LMod}(\mathcal{C}) \simeq \mathsf{RMod}(\mathcal{C})$. Where the distinction comes up, is if we want to fix the $\mathbb{E}_1$-algebra away from $0 \in \mathbb{R}_{\geq 0}$ beforehand. In that case we have to use one of the two possible conventions on the order of multiplication. If we choose the increasing convention then the subset containing $0 \in \mathbb{R}_{\geq 0}$ is on the left making it a right module, while with the opposite convention it would be a left module of the opposite algebra.

It's not hard to extend Construction 3.41 and Construction 3.43 to the closed interval $[-1, 1]$ by hand, but through Theorem 3.19 we can see that Proposition 2.27 and Proposition 2.28 already do all the necessary work.

Corollary 3.46. *There is an equivalence between the ∞-categories of locally constant factorization algebras on the closed interval and algebras over the ∞-operad* $\mathsf{Assoc}^{\mathsf{RL}}$

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{[-1,1]}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{C}). \tag{134}$$

Remark 3.47. Another perspective on why all of these constructions look the way they do is provided by Lemma 3.24. It applies in all of the above cases, namely:

1. For $\mathbb{R}$, the convenient ∞-category of basics is $\mathcal{B} = \mathsf{D}_1^*$,
2. For $[-1, 1]$ the convenient ∞-category of basics is $\mathcal{B} = \mathsf{D}_1^{\partial,*}$.
3. For $\mathbb{R}_{\geq 0}$, we could define a convenient ∞-category of basics which describes framed 1-dimensional manifolds which only have a right boundary (a concept that we can state because of the framing).

We can see that even though factorization algebras, as we have defined them here, don't detect the tangential structure of the manifold, it is usually favorable, if possible, to give the manifold some structure anyway. In particular, the rigid structure of framing is very useful because it massively reduces the spaces of allowed open embeddings.

3.5 Factorization Algebras on Euclidean Spaces

In this section we want to describe locally constant factorization algebras on Euclidean spaces $\mathbb{R}^n$. By Lemma 3.24 we can hope for an ∞-category of basics such that we simplify the situation down to $\mathcal{D}\mathsf{isk}(\mathcal{B})$-algebras. In fact, since $\mathbb{R}^n$ is frameable an ∞-category of basics that we can choose is $\mathsf{D}_n^* \simeq * \xrightarrow{\{\mathbb{R}^n\}} \mathcal{B}\mathsf{sc}$. Specifically, this makes us focus on framed n-dimensional disk algebras

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^n}(\mathcal{C}) \simeq \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathsf{D}_n^*)}(\mathcal{C}). \tag{135}$$

Theorem 3.48 ([Lur, thm.5.5.4.10])**.** *There is an equivalence of ∞-categories between the ∞-categories of locally constant factorization algebras on $\mathbb{R}^n$ and $\mathbb{E}_n$-algebras*

$$\mathcal{A}\mathsf{lg}_{\mathbb{E}_n}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^n}(\mathcal{C}). \tag{136}$$

Proof. With our setup, thanks to Theorem 3.19 and Lemma 3.24, it is sufficient to show that there is an equivalence of ∞-operads

$$\mathcal{D}\mathsf{isk}(\mathsf{D}_n^*) \xrightarrow{\simeq} \mathbb{E}_n. \tag{137}$$

But this is just the classical fact, that the little n-disks operad and the little n-cubes operad are equivalent, in disguise. $\square$

Remark 3.49. We already encountered the $n = 1$ version of the proposition above when we discussed locally constant factorization algebras on the open interval in Proposition 3.42, and the construction there gives us an intuition about how to actually construct the locally constant factorization algebra from the data of the $\mathbb{E}_n$-algebra. The theorem formalizes that this assignment is, in fact, an equivalence.

Remark 3.50 ([Fra13, prop.3.16], [Lur, ex.5.5.4.16]). In the case of Euclidean spaces like $\mathbb{R}^n$, essentially because of translation invariance, a special thing happens when we restrict away from the origin. That is, we know that given a locally constant factorization algebra F on $\mathbb{R}^n$ we can restrict it to any open subset like, for example, $\mathbb{R}^n \setminus \{0\} \cong S^{n-1} \times \mathbb{R}$ to get another locally constant factorization algebra $F|_{S^{n-1}\times\mathbb{R}}$. By Proposition 3.39 we know that $F|_{S^{n-1}\times\mathbb{R}} \in \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}}(\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{S^{n-1}}(\mathcal{C}))$, so that in particular it is an $\mathbb{E}_1$-algebra. If we pushforward to $\mathbb{R}$, by the discussion at Remark 3.34 we could denote the resulting $\mathbb{E}_1$-algebra by

$$\int_{S^{n-1}} F. \tag{138}$$

What [Fra13, prop.3.16] shows is then:

Proposition 3.51. *Given a locally constant factorization algebra F on $\mathbb{R}^n$, its universal enveloping algebra is given by $\int_{S^{n-1}} F$, i.e. there is an equivalence of ∞-categories*

$$\mathsf{Mod}^{\mathbb{E}_n}_F(\mathcal{C}) \xrightarrow{\simeq} \mathsf{LMod}_{\int_{S^{n-1}} F}(\mathcal{C}). \tag{139}$$

This result goes towards an explanation of the appearance of factorization homology over higher spheres in the classification statement of Theorem 2.40.

We can view a Euclidean space $\mathbb{R}^{m+n}$ as a product space $\mathbb{R}^m \times \mathbb{R}^n$, in which case Proposition 3.39 specializes to give the famous result of Dunn additivity [Dun88] which can be found in the following form in [Lur]:

Proposition 3.52. *There is an equivalence of ∞-categories*

$$\mathcal{A}\mathsf{lg}_{\mathbb{E}_{n+m}}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{A}\mathsf{lg}_{\mathbb{E}_n}(\mathcal{A}\mathsf{lg}_{\mathbb{E}_m}(\mathcal{C})). \tag{140}$$

Remark 3.53. The proof of Proposition 3.39, when specialized to ordinary manifolds, as given in [Gin15] actually depends on Dunn additivity to work, however the proof presented above and the one in [Lur] are independent.

4 Collar-gluings and Factorization Algebras

Given a stratified manifold M, and a $\otimes$-presentable ∞-category $\mathcal{C}$, we defined the ∞-category $\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C})$ of locally constant factorization algebras on M valued in $\mathcal{C}$. We also noticed, in Theorem 3.19, that factorization homology is involved in constructing these algebras from local data. This prompts us to ask if we can say something about the behavior of this ∞-category under collar-gluings in the manifold variable. In this section we will prove that such a result exists in both the cases of locally constant and ordinary factorization algebras.

4.1 The Gluing Theorem

Theorem 4.1. *Given a collar-gluing of stratified manifolds $f : M \to [-1, 1]$, the ∞-category of $\mathcal{D}\mathsf{isk}_{/M}$-algebras is equivalent to the pullback of ∞-categories*

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_M(\mathcal{C}) \simeq \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M_-}(\mathcal{C}) \bigtimes_{\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M_0\times\mathbb{R}}(\mathcal{C})} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M_+}(\mathcal{C}). \tag{141}$$

As in previous cases the proof of the above theorem will have a corollary that is easier to prove and whose proof goes along the same lines

Corollary 4.2. *A collar-gluing of stratified manifolds $f : M \to [-1,1]$ induces an equivalence of ∞-categories*

$$\mathcal{F}\mathsf{Alg}_M(\mathcal{C}) \simeq \mathcal{F}\mathsf{Alg}_{M_-}(\mathcal{C}) \underset{\mathcal{F}\mathsf{Alg}_{M_0 \times \mathbb{R}}(\mathcal{C})}{\bigtimes} \mathcal{F}\mathsf{Alg}_{M_+}(\mathcal{C}). \tag{142}$$

Before me move on to the proof of the above two results we pause to record a few useful lemmas that will play a role in it.

Lemma 4.3. *A collar-gluing $f : M \to [-1,1]$ of a stratified manifold M induces a functor*

$$\bar{f} : \mathsf{Mfld}_{/M} \longrightarrow \mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathsf{Mfld}_{/M}), \tag{143}$$

which on objects acts as $(N \hookrightarrow M) \mapsto (N_- \hookrightarrow M, N_0 \times \mathbb{R} \hookrightarrow M, N_+ \hookrightarrow M)$. Moreover, this functor is even a map of ∞-operads.

Proof. We recall that the underlying ∞-category of $\mathsf{Mfld}_{/M}$ is actually (the nerve of) an ordinary category. In particular, morphisms are given by the data of an embedding $N \hookrightarrow N'$ such that $N \hookrightarrow N' \hookrightarrow M = N \hookrightarrow M$ on the nose.

A collar-gluing of M restricts to a collar-gluing of any N that has an embedding into M like the objects of our category. On the right-hand side, at the level of objects, the associative algebra structure is provided by embeddings that differ in the $\mathbb{R}$ factor and the module maps are provided by embeddings of the middle $N_0 \times \mathbb{R}$ into N_- and N_+. At the level of morphisms we send the morphism $e : N \hookrightarrow N'$ to

$$(e_- : N_- \hookrightarrow N'_-,\ e_0 : N_0 \times \mathbb{R} \hookrightarrow N'_0 \times \mathbb{R},\ e_+ : N_+ \hookrightarrow N'_+). \tag{144}$$

The existence of these maps can be stated as the existence of a unique factorization, as in e.g.

$$\begin{tikzcd} N_- \arrow[r, hook] \arrow[dr, hook, dashed, "e_-"'] & N \arrow[r, hook, "e"] & N', \\ & N'_- \arrow[ur, hook] & \end{tikzcd} \tag{145}$$

where the diagram commutes on the nose. Such an assignment clearly plays well with the algebra structure of the right-hand side, through the existence of (on the nose) commutative diagrams like e.g.

$$\begin{tikzcd} N_0 \times \mathbb{R} \arrow[r, hook, "e_0"] \arrow[d, hook] & N'_0 \times \mathbb{R} \arrow[d, hook] \\ N_- \arrow[r, hook, "e_-"] & N'_-, \end{tikzcd} \tag{146}$$

so that we clearly get a functor

$$\bar{f} : \mathsf{Mfld}_{/M} \longrightarrow \mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathsf{Mfld}_{/M}). \tag{147}$$

That the functor is a map of ∞-operads is clear from the fact that the key role on both sides is played by disjoint union. □

Remark 4.4. In the above lemma it is important that we stuck to the comparatively simple $\mathsf{Mfld}_{/M}$ instead of $\mathcal{M}\mathsf{fld}_{/M}$. It is not immediately obvious how to deal with the higher morphisms in the latter, as can be seen from the form of Equation (145). The issues are related to the ones that appear in the construction of the pushforward functor Theorem 2.32. A simple case to ponder is an isotopy that maps one embedding of a disk in $M_- \setminus (M_0 \times \mathbb{R})$ to another one in $M_+ \setminus (M_0 \times \mathbb{R})$.

Lemma 4.5. *The bar construction* $\mathsf{B} : \mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{C}) \longrightarrow \mathcal{C}$ *is a symmetric monoidal functor.*

Proof. The symmetric monoidal structure of $\mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{C})$ is inherited from $\mathcal{C}$ pointwise. We denote the objects of $\mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{C})$ as (R, A, L), and use $\mathsf{B}_\bullet(R, A, L) : \Delta^{\mathsf{op}} \to \mathcal{C}$ for the simplicial object of the bar construction. Using the language of [Lur, def.2.1.2.7], to prove that B is symmetric monoidal amounts to showing that for each basepoint-preserving map of based finite sets $f : I_* \to J_*$ there is a commutative diagram of ∞-categories

$$\begin{tikzcd} \mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{C})^I \arrow[r, "\mathsf{B}^I"] \arrow[d, "f_*"] & \mathcal{C}^I \arrow[d, "f_*"] \\ \mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{C})^J \arrow[r, "\mathsf{B}^J"] & \mathcal{C}^J. \end{tikzcd} \tag{148}$$

Following [Lur, rem.2.1.2.2], each f admits a unique factorization as a composition of an active and an inert map. We can also further uniquely decompose the active maps into surjective active maps and injective active maps. So we will examine the above for each type of map separately. The case of inert maps is trivial as always, because of the simplicity of f_*. The case of injective active maps is essentially guaranteeing that B maps the monoidal unit to the monoidal unit. In this case the existence of the commutative diagram is provided by

$$\mathsf{B}(\mathbb{1}_{\mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{C})}) = \mathsf{colim}(\mathsf{B}_\bullet(\mathbb{1}, \mathbb{1}, \mathbb{1})) \simeq \mathsf{colim}(\Delta^{\mathsf{op}} \xrightarrow{\{\mathbb{1}\}} \mathcal{C}) \simeq \mathbb{1}. \tag{149}$$

Finally, the case of surjective active morphisms describes what happens when we use the monoidal product to multiply. In this case we will look at the model calculation for the active map $f : \langle 2\rangle \to \langle 1\rangle$. All other possibilities go along similar lines. Given two objects $H = (R, A, L)$ and $H' = (R', A', L')$ we have

$$\begin{aligned} \mathsf{B}(H \otimes H') &= \mathsf{colim}(\mathsf{B}_\bullet((R, A, L) \otimes (R', A', L'))) \\ &= \mathsf{colim}(\mathsf{B}_\bullet(R \otimes R', A \otimes A', L \otimes L')) \\ &\simeq \mathsf{colim}(\mathsf{B}_\bullet(R, A, L) \otimes \mathsf{B}_\bullet(R', A', L')) \\ &= \mathsf{colim}(\Delta^{\mathsf{op}} \to \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \xrightarrow{\mathsf{B}_\bullet(R,A,L)\otimes \mathsf{B}_*(R',A',L')} \mathcal{C}) \\ &\simeq \mathsf{colim}(\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \xrightarrow{\mathsf{B}_\bullet(R,A,L)\otimes \mathsf{B}_*(R',A',L')} \mathcal{C}) \\ &\simeq \mathsf{colim}(\mathsf{B}_\bullet(R, A, L)) \otimes \mathsf{colim}(\mathsf{B}_*(R', A', L')) \\ &= \mathsf{B}(H) \otimes \mathsf{B}(H'), \end{aligned} \tag{150}$$

where $\Delta^{\mathsf{op}} \to \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}$ is the diagonal functor. The first equivalence comes from the symmetric monoidal structure of $\mathcal{C}$ dues to reshuffling in the simplicial object. The second equivalence uses the finality of the diagonal functor which comes from the fact that Δ^{op} is sifted. Finally, the third equivalence exists because the symmetric monoidal structure of $\mathcal{C}$ commutes with (sifted) colimits. $\square$

Proof of Corollary 4.2. We will first focus on the case of factorization algebras and prove the statement of Corollary 4.2. Moreover, because of Corollary 3.21, this means that we will focus on $\mathsf{Disk}_{/M}$-algebras.

The data of a collar-gluing provides open embeddings $M_0 \times \mathbb{R} \hookrightarrow M_-$ and $M_0 \times \mathbb{R} \hookrightarrow M_+$, where M_-, M_0 and M_+ have the usual meanings as in Definition 1.38. This can be used to construct the cospan of restriction functors $\mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_-}}(\mathcal{C}) \to \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_0\times\mathbb{R}}}(\mathcal{C}) \leftarrow \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_+}}(\mathcal{C})$. Using this data we can define the pullback ∞-category

$$\mathcal{P} := \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_-}}(\mathcal{C}) \underset{\mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_0\times\mathbb{R}}}(\mathcal{C})}{\times} \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_+}}(\mathcal{C}). \tag{151}$$

The further open embeddings of M_-, $M_0 \times \mathbb{R}$ and M_+ into M again give rise to restriction functors that form a(n on the nose) commutative square

$$\begin{tikzcd} \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_+}}(\mathcal{C}) \arrow[d] \\ \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_-}}(\mathcal{C}) \arrow[r] & \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_0 \times \mathbb{R}}}(\mathcal{C}), \end{tikzcd} \tag{152}$$

which, by the universal property of the pullback, gives us a canonical functor $\rho : \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}(\mathcal{C}) \to \mathcal{P}$. Our job is to find an inverse to this functor.

Consider the following construction. Given the data of an object $(A_-, A_+) \in \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_-}}(\mathcal{C}) \times \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M_+}}(\mathcal{C})$ such that $A_-|_{M_0 \times \mathbb{R}} = A_+|_{M_0 \times \mathbb{R}} =: A_0$ we want to construct a functor $A : \mathsf{Disk}_{/M} \to \mathcal{C}$. We do this by relying on factorization homology and the bar construction. Given $(U \hookrightarrow M) \in \mathsf{Disk}_{/M}$ we define

$$A(U) := \int_{U_-} A_- \bigotimes_{\int_{U_0 \times \mathbb{R}} A_0} \int_{U_+} A_+, \tag{153}$$

where U_-, U_0 and U_+ are defined through the inherited collar-gluing $U \hookrightarrow M \to [-1, 1]$. Well-definition of the bar construction is provided if we have the appropriate left and right module structures on $\int_{U_-} A_-$ and $\int_{U_+} A_+$ over the associative algebra $\int_{U_0 \times \mathbb{R}} A_0$. Pushforward along the restrictions $f|_{U_-} : U_- \to [-1, 1)$ and $f|_{U_+} : U_+ \to (-1, 1]$, together with the properties of factorization homology provides these left and right module structures. Furthermore, the associative algebra object is exactly the appropriate one since consecutive restrictions commute on the nose. The above argument is essentially the statement that there is a functor

$$\begin{aligned} \gamma : \mathcal{P} \xrightarrow{\int} & \mathcal{A}\mathsf{lg}_{\mathsf{Mfld}_{/M_-}}(\mathcal{C}) \underset{\mathcal{A}\mathsf{lg}_{\mathsf{Mfld}_{/M_0 \times \mathbb{R}}}(\mathcal{C})}{\times} \mathcal{A}\mathsf{lg}_{\mathsf{Mfld}_{/M_+}}(\mathcal{C}) \\ & \xrightarrow{-\circ \bar{f}} \mathsf{Fun}(\mathsf{Mfld}_{/M}, \mathcal{A}\mathsf{lg}_{\mathsf{Assoc}^{\mathsf{RL}}}(\mathcal{C})) \xrightarrow{\mathsf{B}\circ -} \mathsf{Fun}(\mathsf{Mfld}_{/M}, \mathcal{C}) \to \mathsf{Fun}(\mathsf{Disk}_{/M}, \mathcal{C}), \end{aligned} \tag{154}$$

where $\bar{f}$ is the functor from Lemma 4.3.

Our first order of business is to show that γ actually even lands in $\mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}(\mathcal{C})$, and not just $\mathsf{Fun}(\mathsf{Disk}_{/M}, \mathcal{C})$. Since we have expressed the construction $A = \gamma(A_-, A_+)$ functorially, we can see that A will inherit the structure of an algebra from the fact that factorization homology and the bar construction are symmetric monoidal functors (see Lemma 4.5 for the latter), together with Lemma 4.3 that lends this property to $\bar{f}$. With that we have constructed a functor

$$\gamma : \mathcal{P} \longrightarrow \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}(\mathcal{C}). \tag{155}$$

Or strategy will be to show that the pair of functors ρ and γ give rise to an equivalence by first checking that this is the case on maximal ∞-subgroupoids and subsequently that it is also so on morphism spaces. Towards the first goal we need to show that:

1. $\rho(\gamma(A_-, A_+))$ reproduces (A_-, A_+) up to equivalence, and
2. $\gamma(\rho(A))$ reproduces A up to equivalence.

We now check these in order. Checking (1) amounts to evaluating disks U which are contained in only one of the pieces M_-, $M_0 \times \mathbb{R}$ and M_+. For the case of $U_0 \times \mathbb{R} = \emptyset = U_+$, we have

$$A(U) = \int_U A_- \bigotimes_{\mathbb{1}} \mathbb{1} \simeq A_-(U), \tag{156}$$

and similarly if $U_- = \emptyset = U_0 \times \mathbb{R}$. Alternatively, for the case of $U_- = U_0 \times \mathbb{R} = U_+$ we instead have

$$A(U) = \int_U A_0 \bigotimes_{\int_U A_0} \int_U A_0 \simeq A_0(U). \tag{157}$$

If U is such that $U_+ = \emptyset$, but $U_0 \times \mathbb{R} \neq \emptyset$ then

$$A(U) = \int_U A_- \bigotimes_{\int_{U_0\times\mathbb{R}} A_-} \mathbb{1} \simeq \int_U A_- \bigotimes_{\int_{U_0\times\mathbb{R}} A_-} \int_{\emptyset} A_- \simeq \int_U A_- = A_-(U), \tag{158}$$

where the last equivalence is provided by the collar-gluing property of factorization homology. This observation is also what confirms (2). Namely,

$$\int_U A \simeq \int_{U_-} A \bigotimes_{\int_{U_0\times\mathbb{R}} A} \int_{U_+} A \simeq \int_{U_-} A_- \bigotimes_{\int_{U_0\times\mathbb{R}} A_0} \int_{U_+} A_+, \tag{159}$$

where the first equivalence comes from collar-gluing, while the second is a rewriting using the fact that restricting the algebra doesn't change the value of factorization homology it just changes what we can evaluate with it.

Finally, we want to show that ρ is fully faithful, i.e. we want to show that

$$\mathsf{Hom}_{\mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}(\mathcal{C})}(A, B) \underset{\gamma}{\overset{\rho}{\rightleftarrows}} \mathsf{Hom}_{\mathcal{P}}((A_-, A_+), (B_-, B_+)) \tag{160}$$

represents an equivalence of spaces for each $A, B \in \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}(\mathcal{C})$. That is, we have to do similar computations as (1) and (2) above, but now on the level of morphisms. Nothing new happens in the calculations involving restrictions that don't involve multiple pieces, since their logic essentially only depends on the properties of the bar construction when restricting. In the rest of the calculations we crucially used the collar-gluing property of factorization homology. That this property continues to hold at the morphism level is a consequence of the fact that we made sure to elevate the statement of Proposition 2.30 to the level of functors, together with the pushforward property which also holds functorially à la Remark 2.34. Namely, these provide equivalences of functors

$$\int_U - \simeq \int_{[-1,1]} (f|_U)_* - \simeq \mathsf{B} \circ (f|_U)_* = \int_{U_-} (-)_- \bigotimes_{\int_{U_0\times\mathbb{R}} (-)_0} \int_{U_+} (-)_+, \tag{161}$$

which is exactly the collar-gluing property extended to the level of morphisms. This shows the equivalence at the level of factorization algebras. □

Proof of Theorem 4.1. In the case that all ∞-categoies involved are locally constant we will extend the previous proof to provide the equivalence. Since we know that restriction does not destroy local constancy, extending the result to the case of locally constant factorization algebras simply requires us to show that the gluing γ also preserves this property. Let $U \hookrightarrow V$ be two disks of M. What we have to check is that there is an equivalence

$$\int_U \gamma(A_-, A_+) \xrightarrow{\simeq} \int_V \gamma(A_-, A_+) \tag{162}$$

in $\mathcal{C}$. We will achieve this if we find compatible equivalences,

$$\int_{U_-} A_- \simeq \int_{V_-} A_- \qquad \int_{U_0\times\mathbb{R}} A_0 \simeq \int_{V_0\times\mathbb{R}} A_0 \qquad \int_{U_+} A_+ \simeq \int_{V_+} A_+, \tag{163}$$

on each of the pieces $(U_- \hookrightarrow V_-), (U_0 \times \mathbb{R} \hookrightarrow V_0 \times \mathbb{R}), (U_+ \hookrightarrow V_+)$, and combine them using the bar construction. If these pieces were disks then the above equivalences would be given by the local constancy of the constituent locally constant factorization algebras.

Generically though, these pieces will not be disks again, so we can't simply inherit the local constancy property immediately. To alleviate this issue we make use of Theorem 1.40 which assures us that every manifold can be obtained by collar-gluings and sequential colimits if we start with disks. In fact, since our largest manifold of interest is a disk, which is finitary, we only need collar-gluings, and not sequential colimits. Thus, we can work inductively and iterate the cutting and gluing process until the pieces become disks. The only thing that we need to check is that through one inductive step the equivalences remain compatible.

Specifically, by compatibility here we mean that there is a commutative diagram

$$\begin{tikzcd} \int_{U_-} A_- \arrow[d, "\simeq"] & \int_{U_0\times\mathbb{R}} A_0 \arrow[l] \arrow[r] \arrow[d, "\simeq"] & \int_{U_+} A_+ \arrow[d, "\simeq"] \\ \int_{V_-} A_- & \int_{V_0\times\mathbb{R}} A_0 \arrow[l] \arrow[r] & \int_{V_+} A_+, \end{tikzcd} \tag{164}$$

where the horizontal arrows are induced by the relevant embeddings. A sufficient condition to guarantee compatibility is that all collar-gluings are defined as collar-gluings of M and then inherited to every open submanifold through its embedding in M. Focusing on the final two consecutive collar-gluings we get a commutative diagram, which we schematically denote

$$\begin{tikzcd} -+ & 0+ \arrow[l] \arrow[r] & ++ \\ -0 \arrow[u] \arrow[d] & 00 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & +0 \arrow[u] \arrow[d] \\ -- & 0- \arrow[l] \arrow[r] & +-, \end{tikzcd} \tag{165}$$

one for each of U and V. Stacking these two with the equivalence as connecting arrows is again a commutative diagram. This is enough information to guarantee that after acting with the bar construction in one direction we again get compatible equivalences. The key point being that we consider the collar-gluings on all pieces in both orders to get all the necessary arrows.

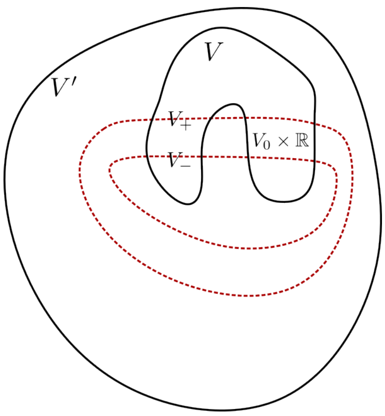

Figure 1: Extending a collar-gluing.

The final point to check is that we have not limited ourselves by insisting to only consider collar-gluings which are inherited from ones on M. This is best shown in Figure 1. Namely, given the disk $V \hookrightarrow M$, we can take a larger disk V', which is guaranteed to exist from the fact that M is a manifold. The figure then shows that given any collar-gluing of V we can find one particular collar-gluing of M that extends the one of V. Since V' is a disk, the figure does not obscure any detail.

$\square$

Remark 4.6. The simple case when the collar-gluing is a disjoint union, and the manifolds are smooth can also be found as [Lur, ex.5.4.5.4].

4.2 An Outside View of Factorization Algebras (Work In Progress)

After the results of the previous section we remark on a general view that we can take for the ∞-categories of (locally constant) factorization algebras. Namely, we notice factorization algebras can be cast as a functor

$$\mathcal{F}\mathsf{Alg}\_(\mathcal{C}), \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}\_(\mathcal{C}) : \mathsf{Mfld} \longrightarrow \mathcal{C}\mathsf{at}_\infty^{\mathsf{op}}, \tag{166}$$

where the action on morphisms is given by restriction, which is what gives rise to the $^{\mathsf{op}}$. In the case of disjoint union, Theorem 4.1 and Corollary 4.2 then provide the equivalences that are needed to make these into symmetric monoidal functors, where the symmetric monoidal product on $\mathcal{C}\mathsf{at}_\infty^{\mathsf{op}}$ is given by *coproduct*.

We know from Theorem 1.40 that any manifold can be constructed from basics by iterated collar-gluings and sequential colimits. Having described what happens to a locally constant factorization algebras under collar-gluings of manifolds, we now turn to sequential colimits. The gluing result has an ordinary (nonhomotopy) limit. Is this also the output of the (co)bar construction?

Proposition 4.7. *The functors $\mathcal{F}\mathsf{Alg}(\mathcal{C})$ and $\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}(\mathcal{C})$ respect sequential colimits.*

Proof. The proof will go along the same lines for both, so we will focus on factorization algebras, and shorten notation to $\mathcal{F} := \mathcal{F}\mathsf{Alg}(\mathcal{C}) : \mathsf{Mfld} \to \mathcal{C}\mathsf{at}_\infty^{\mathsf{op}}$. We will also use the convention that we work in $\mathcal{C}\mathsf{at}_\infty$ instead of its opposite. In these terms, the statement we need to prove is that the canonical functor

$$\mathcal{F}(M) \xrightarrow[\alpha]{\simeq} \lim(\mathcal{F}(M_0) \leftarrow \mathcal{F}(M_1) \leftarrow \dots) \tag{167}$$

is an equivalence of ∞-categories, where $M := \cup_{i\geq 0} M_i$. All functors used to make the diagram are restriction functors.

An object of the limit is a collection $(A_0, A_1, \dots)$ of factorization algebras on $M_0, M_1, \dots$, respectively, together with equivalences $\eta_i : A_i \simeq A_{i+1}|_{M_i}$ for $i \geq 0$. Towards showing essential surjectivity of α we will construct the factorization algebra A on M given by the following prescription. Using Corollary 3.21, we will take the disk algebra incarnation of the relevant ∞-categories, so that we want to construct $A \in \mathcal{A}\mathsf{lg}_{\mathsf{Disk}_{/M}}$. Given any disk $U \subset M$, there will be an integer i, such that $U \subset M_i$; we take the smallest such integer, call it i_U, and define

$$A(U) := A_{i_U}(U). \tag{168}$$

Given a morphism $U \hookrightarrow V$ we know that $i_U \leq i_V$, and the morphism we assign is

$$A(U \hookrightarrow V) := \left(A_{i_U}(U) \xrightarrow{\eta_{i_U}(U)} \cdots \xrightarrow{\eta_{i_V - 1}(U)} A_{i_V}(U) \xrightarrow{A_{i_V}(U \hookrightarrow V)} A_{i_V}(V) \right). \tag{169}$$

The same construction works for multimorphisms too, e.g. for $U \sqcup U' \hookrightarrow V$ we would have

$$\begin{aligned} A(U \sqcup U' \hookrightarrow V) := \Bigg(& A_{i_U}(U) \otimes A_{i_{U'}}(U') \xrightarrow{\eta_{i_U}(U) \otimes \mathbb{1}} \cdots \xrightarrow{\eta_{i_V - 1}(U) \otimes \mathbb{1}} A_{i_V}(U) \otimes A_{i_{U'}}(U') \\ & \xrightarrow{\mathbb{1} \otimes \eta_{i_{U'}}(U')} \cdots \xrightarrow{\mathbb{1} \otimes \eta_{i_V - 1}(U')} A_{i_V}(U) \otimes A_{i_V}(U') \to A_{i_V}(U \sqcup U') \xrightarrow{A_{i_V}(U \sqcup U' \hookrightarrow V)} A_{i_V}(V) \Bigg). \end{aligned} \tag{170}$$

This describes the data of a $\mathsf{Disk}_{/M}$-algebra. When applying the above construction to locally constant factorization algebras it's also clear that the property of local constancy is inherited since then all the arrows in eq. (169) are equivalences.

To show essential surjectivity of α we need to find equivalences $\epsilon_i : A|_{M_i} \xrightarrow{\simeq} A_i$ for all $i \geq 0$, which fit into a commutative diagram

$$\begin{tikzcd} & A|_{M_i} = (A|_{M_{i+1}})|_{M_i} \arrow[dl, "\epsilon_i"'] \arrow[dr, "\epsilon_i|_{M_i}"] & \\ A_i \arrow[rr, "\eta_i"] & & A_{i+1}|_{M_i}, \end{tikzcd} \tag{171}$$

where we have remembered that consecutive restrictions commute on the nose. The equivalence ϵ_i is given by its components $\epsilon_i(U) : A|_{M_i} \to A_i(U)$, which we define as

$$\epsilon_i(U) := \left(A|_{M_i}(U) = A_{i_U}(U) \xrightarrow{\eta_{i_U}(U)} \cdots \xrightarrow{\eta_{i-1}(U)} A_i(U) \right). \tag{172}$$

That these assemble into a natural transformation is provided by commutative diagrams like

$$\begin{tikzcd} A_{i_U}(U) \arrow[r, "\eta_{i-1}(U)\circ\cdots\circ\eta_{i_U}(U)"] \arrow[d, "\eta_{i_V-1}(U)\circ\cdots\circ\eta_{i_U}(U)"'] & A_i(U) \arrow[d, "="] \\ A_{i_V}(U) \arrow[r, "\eta_{i-1}(U)\circ\cdots\circ\eta_{i_V}(U)"] \arrow[d] & A_i(U) \arrow[d] \\ A_{i_V}(V) \arrow[r, "\eta_{i-1}(V)\circ\cdots\circ\eta_{i_V}(V)"] & A_i(V). \end{tikzcd} \tag{173}$$

That these equivalences ϵ_i fit into diagrams like in eq. (171) is obvious from their definition.

Showing full faithfulness of α is the same for ordinary and locally constant factorization algebras since the later are a full ∞-subcategory of the former. What we have to show is that

$$\mathsf{Hom}_{\mathcal{F}(M)}(A, B) \xrightarrow{\alpha} \mathsf{Hom}_{\mathsf{lim}}(\alpha(A), \alpha(B)) \simeq \mathsf{lim}_{i\geq 0}(\mathsf{Hom}_{\mathcal{F}(M_i)}(A_i, B_i)) \tag{174}$$

is an equivalence of spaces. In the last equivalence we made use of the essential surjectivity from above. The same idea as for the construction of A work here too, the main point being that any disk U will have a finite i_U.

... □

Considering Theorem 1.40 together with the statements of Corollary 3.21 (or Theorem 3.19 in the locally constant case), together with the above Proposition 4.7, we have an algorithm of how to compute the ∞-category of factorization algebras on any manifold if we know it on the basics. In other words, the theory of (locally constant) factorization algebras is fully determined by the assignment

$$U \mapsto \mathcal{F}\mathsf{Alg}_U(\mathcal{C}), \tag{175}$$

for each disk $U \in \mathcal{B}\mathsf{sc}$.

After a simplification down to disks, in the locally constant case, lemma Lemma 3.24 points towards a result like the following:

Conjecture 4.8. *The theory of factorization algebras is fully determined, using factorization homology, by the data of a functor*

$$\mathsf{Bsc} \longrightarrow \mathcal{C}\mathsf{at}_\infty^{\mathsf{op}}. \tag{176}$$

In the locally constant case, we can even say that to each basic U this functor attaches the ∞-category $\mathsf{Fun}^{\otimes}(\mathcal{D}\mathsf{isk}(\mathsf{B}_U), \mathcal{C})$ of disk algebras with B_U structure, where B_U is the sieve poset whose maximal element is U.

Remark 4.9. The ∞-category of basics B_U for our purposes provides a kind of generalization of framing, which makes calculation easier.

4.3 Some Examples of Gluing

Example 4.10. With the ingredients we have introduced up to now, specifically in § 3.4, we can do a nontrivial check of the gluing theorem. This is because we can write a collar-gluing of the closed interval $[-1,1] \cong [-1,1) \coprod_{(-1,1)} (-1,1] \cong \mathbb{R}_{\geq 0} \coprod_{\mathbb{R}} \mathbb{R}_{\leq 0}$. We note here that we have written $\mathbb{R}_{\leq 0}$ and $\mathbb{R}_{\geq 0}$ to emphasize the maps forming the decomposition. At the level of unstructured objects these two are isomorphic (see also Remark 3.45). For this example, the theorem says that locally constant factorization algebras on the closed interval should be given by

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{[-1,1]}(\mathcal{C}) \simeq \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}_{\geq 0}}(\mathcal{C}) \underset{\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}}(\mathcal{C})}{\times} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}_{\leq 0}}(\mathcal{C}) \simeq \mathsf{RMod}(\mathcal{C}) \underset{\mathcal{A}\mathsf{lg}_{\mathbb{E}_1}(\mathcal{C})}{\times} \mathsf{LMod}(\mathcal{C}), \tag{177}$$

where the functors are the ones that forget the module. At the level of objects, this is the data of an algebra with a right module $(A,R) \in \mathsf{RMod}(\mathcal{C})$ and an algebra with a left module $(B,L) \in \mathsf{LMod}(\mathcal{C})$, such that the algebras are equal $A = B$. But this is exactly the same as the description we gave in terms of an algebra with a left and a right module, i.e. an algebra over $\mathsf{Assoc}^{\mathsf{RL}}$, in Corollary 3.46.

Example 4.11. A highly nontrivial check of the gluing theorem is given by the circle S^1. Using results from [CG16, prop.4.0.1], namely inheritance from a covering space, [Gin15, sec.5.5] characterizes locally constant factorization algebras on the circle S^1 as

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{S^1}(\mathcal{C}) \simeq \mathsf{Aut}(\mathcal{A}\mathsf{lg}_{\mathbb{E}_1}(\mathcal{C})), \tag{178}$$

i.e. $\mathbb{E}_1$-algebras together with a self-equivalence. This result is not generalizable to higher spheres because it uses the technology of covering spaces. Theorem 4.1 allows us to recover this nontrivial result. To wit, the circle can be exhibited as a collar-gluing $S^1 \cong \mathbb{R} \coprod_{S^0 \times \mathbb{R}} \mathbb{R} \cong \mathbb{R} \coprod_{\mathbb{R} \sqcup \mathbb{R}} \mathbb{R}$. For locally constant factorization algebras this means that

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{S^1}(\mathcal{C}) \simeq \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}}(\mathcal{C}) \underset{\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}\sqcup\mathbb{R}}(\mathcal{C})}{\times} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}}(\mathcal{C}) \simeq \mathcal{A}\mathsf{lg}_{\mathbb{E}_1}(\mathcal{C}) \underset{\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}\sqcup\mathbb{R}}(\mathcal{C})}{\times} \mathcal{A}\mathsf{lg}_{\mathbb{E}_1}(\mathcal{C}). \tag{179}$$

Since a disjoint union is a collar-gluing we also have that $\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}\sqcup\mathbb{R}} \simeq \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}}(\mathcal{C}) \times \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}}(\mathcal{C}) \simeq \mathcal{A}\mathsf{lg}_{\mathbb{E}_1}(\mathcal{C}) \times \mathcal{A}\mathsf{lg}_{\mathbb{E}_1}(\mathcal{C})$. Let's denote the restriction functors that restrict to the first and the second component of the disjoint union $\mathbb{R} \sqcup \mathbb{R}$ by $|_1$ and $|_2$ respectively. Going back to Equation (179), the data of a locally constant factorization algebra on S^1 is equivalent to the data of two $\mathbb{E}_1$-algebras A and B such that their restrictions $A|_1 = B|_1$ and $A|_2 = B|_2$ agree. Since all of these are $\mathbb{E}_1$-algebras there are equivalences with the restrictions forming

$$\begin{tikzcd}
 & A|_1 = B|_1 & \\
A \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & & B \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\
 & A|_2 = B|_2 &
\end{tikzcd} \tag{180}$$

That is we have the data of two algebras A and B, and two equivalences from A to B. It is standard that this data is equivalent (in one direction by composing) to the data of an $\mathbb{E}_1$-algebra with a self-equivalence.

Example 4.12. Unlike the covering space result for the circle [Gin15, sec.5.5] which doesn't generalize to higher spheres, the gluing theorem has no such restriction. Just as for the circle, higher spheres can also be described as a collar-gluing $S^n \cong \mathbb{R}^n \coprod_{S^{n-1}\times\mathbb{R}} \mathbb{R}^n$. Thus, we can iterate the procedure from Example 4.11 to get the locally constant factorization algebras on all higher spheres. For example, for the two-sphere S^2 we have

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^2}(\mathcal{C}) \underset{\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{S^1\times\mathbb{R}}(\mathcal{C})}{\times} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^2}(\mathcal{C}) \simeq \mathcal{A}\mathsf{lg}_{\mathbb{E}_2}(\mathcal{C}) \underset{\mathcal{A}\mathsf{lg}_{\mathbb{E}_1}(\mathsf{Aut}(\mathcal{A}\mathsf{lg}_{\mathbb{E}_1}(\mathcal{C})))}{\times} \mathcal{A}\mathsf{lg}_{\mathbb{E}_2}(\mathcal{C}), \tag{181}$$

namely, a locally constant factorization algebra on S^2 is equivalent to two $\mathbb{E}_2$-algebras, such that they restrict on the fattened equator to the same $\mathbb{E}_2$-algebra equipped with an $\mathbb{E}_1$-automorphism. It is nontrivial to describe this data in a simpler way as was possible for S^1.

From this example we can easily see that beyond the dimensionality n of the space, which is locally kept track of as an $\mathbb{E}_n$-algebra, locally constant factorization algebras for the higher spheres also keep track of higher coherent (self-)equivalences, which imitate triangulations of the spheres.

Example 4.13. In § 5 we will see that fiber bundles are important for the classification of some stratified manifolds. Here we will look at line bundles over the circle as a toy example, namely, the cylinder C and Möbius band M. The cylinder, as a trivial fiber bundle, is isomorphic to the product space $S^1 \times \mathbb{R}$. By Proposition 3.39 together with the results of Example 4.11, locally constant factorization algebras on it are given by $\mathbb{E}_2$-algebras together with an $\mathbb{E}_1$-self-equivalence

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{C}(\mathcal{C}) \simeq \mathsf{Aut}_{\mathbb{E}_1}(\mathcal{A}\mathsf{lg}_{\mathbb{E}_2}(\mathcal{C})). \tag{182}$$

Switching our attention to the Möbius band M, we can use a similar logic as in Example 4.11 to find a collar gluing of M. The pieces are the same as they would be for the cylinder, namely $M_- \cong M_+ \cong \mathbb{R}^2$, and $M_0 \times \mathbb{R} \cong \mathbb{R}^2 \sqcup \mathbb{R}^2$, but they are glued differently because of the twist. Using the notation from Example 4.11 the diagram becomes

$$\begin{tikzcd}[column sep=small] & A|_1 = B|_1^{\mathsf{op_f}} \simeq B^{\mathsf{op_f}} \\ A \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] & \\ & A|_2 = B|_2 \simeq B, \end{tikzcd} \tag{183}$$

where $^{\mathsf{op_f}}$ is the opposite functor in the fiber direction. Thus, instead of an $\mathbb{E}_1$-*self*-equivalence we have an equivalence with the opposite algebra.

We should remark that the fact that we can take the opposite algebra in one direction only is a consequence of the Dunn additivity of Proposition 3.52. Without it, we wouldn't be able to combine the opposite algebra in one direction with the original algebra in the other direction into a well-defined algebra in two dimensions.

5 Towards Classifying $\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}$ on Defect Manifolds

Our goal in this section is to lay out a possible roadmap for classifying locally constant factorization algebras on stratified manifolds. Specifically we will focus on the case of stratified manifolds M_Σ, which are described by a smooth, n-dimensional manifold M together with a distinguished smooth, properly embedded, d-dimensional submanifold $\Sigma \hookrightarrow M$, such that $d < n$. In other words we are considering objects $M_\Sigma \in \mathcal{M}\mathsf{fld}(\mathsf{D}_{d\subset n})$ as discussed in Definition 1.46. More general statements can be achieved by iterating the procedure and explicating the information of more defects, stratum by stratum, in the stratified manifold.

Remark 5.1. Having already considered the classification of disk algebras with $\mathsf{D}^*_{d\subset n}$-structure in Theorem 2.40, one might wonder whether the classification we are after is already done by simply using the factorization homology functor, and porting over the classification of disk algebras to the setting of locally constant factorization algebras. There are two issues with this direction of inquiry. One problem is that not all $\mathsf{D}_{d\subset n}$-manifolds are frameable since this is not even the case for smooth manifolds; framed disk algebras like in Theorem 2.40 can only evaluate framed stratified manifolds.

The second problem is that having classified all $\mathcal{D}\mathsf{isk}(\mathcal{B})$-algebras does not immediately give a classification of $\mathcal{D}\mathsf{isk}_{/M}$-algebras for a given manifold M. There is a functor

$$\mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathcal{B})}(\mathcal{C}) \longrightarrow \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathcal{B})_{/M}}(\mathcal{C}), \tag{184}$$

which is induced by the forgetful functor $\mathcal{D}\mathsf{isk}_{/M} \to \mathcal{D}\mathsf{isk}(\mathcal{B})$, but this functor will, in general, not be an equivalence. We can see this, for example, by using Remark 2.16 which tells us that the left-hand side depends on $\mathcal{B}$, while the right-hand side doesn't.

5.1 Euclidean Spaces with Defects

In §3.5 we discussed locally constant factorization algebras on Euclidean spaces of different dimension n, and we saw that they presented the data of $\mathbb{E}_n$-algebras. In §3.4, on the other hand we focused on 1 dimensional manifolds, but with different possible stratification structures, and we saw that the data encoded in defects was a module structure. Here, in the spirit of this section, we work to extend both of these developments to the case of Euclidean spaces with the specific kind of defect that we have limited ourselves to, namely, $\mathbb{R}^{d\subset n}$.

Despite Remark 5.1, we know from Lemma 3.24 that sometimes the classification results for disk algebras can be used for factorization algebras. In particular, this is the case for $\mathbb{R}^{d\subset n}$, since it is final in the ∞-category of basics $\mathcal{B} = \mathsf{D}^*_{d\subset n}$, as long as $d < n-1$. This gives us

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^{d\subset n}}(\mathcal{C}) \simeq \mathcal{A}\mathsf{lg}_{\mathcal{D}\mathsf{isk}(\mathsf{D}^*_{d\subset n})}(\mathcal{C}). \tag{185}$$

In fact, technically, the only cases we need to consider are when $d = 0$ — the case of pointed Euclidean spaces, which are alternatively denoted $\mathbb{R}^n_*$. This is because of the isomorphism

$$\mathbb{R}^{d\subset n} \cong \mathbb{R}^{n-d}_* \times \mathbb{R}^d, \tag{186}$$

together with Proposition 3.39, which describes locally constant factorization algebras on product spaces. Since $\mathbb{R}^n_*$ is a basic the collar-gluing property Theorem 4.1 can't help us to analyze the situation. We are left with two cases: $n = 1$ and $n > 1$. To get a better grasp on the data, we first explore the more complicated case of $n > 1$ by examining the ∞-operad $\mathcal{D}\mathsf{isk}(\mathsf{D}^*_{0\subset n})$, when $n > 1$.

Observation 5.2. For the purposes of the observation we will shorten notation to $\mathcal{D}_* := \mathcal{D}\mathsf{isk}(\mathsf{D}^*_{0\subset n})$. Since the defect is a point there can be at most one disk that contains it per morphism. Thus, the morphism spaces to consider are

$$\mathsf{Hom}_{\mathcal{D}_*}((\mathbb{R}^n)^{\sqcup i}, \mathbb{R}^n), \qquad \mathsf{Hom}_{\mathcal{D}_*}((\mathbb{R}^n)^{\sqcup i}, \mathbb{R}^n_*) \qquad \text{and} \qquad \mathsf{Hom}_{\mathcal{D}_*}((\mathbb{R}^n)^{\sqcup i} \sqcup \mathbb{R}^n_*, \mathbb{R}^n_*). \tag{187}$$

The first of these is equivalent to the space $\mathsf{Emb}^*((\mathbb{R}^n)^{\sqcup i}, \mathbb{R}^n)$ of framed open embeddings, which makes it is obvious that restricting to non-defect disks would give the data of an $\mathbb{E}_n$-algebra. This is related to Lemma 3.35, which says that forgetting the stratification results in obtaining a full ∞-subcategory of locally constant factorization algebras. When $i = 0$, the third space above is equivalent to a point, which makes it clear that restricting exclusively to pointed disks

$\mathbb{R}^n_*$ would give rise to plain objects[18] (or $\mathbb{E}_0$-algebras). The structure that connects these two algebras is encoded in the rest of the morphism spaces. In fact the second and third space above are equivalent,

$$\mathsf{Hom}_{\mathcal{D}_*}((\mathbb{R}^n)^{\sqcup i}, \mathbb{R}^n_*) \simeq \mathsf{Hom}_{\mathcal{D}_*}((\mathbb{R}^n)^{\sqcup i} \sqcup \mathbb{R}^n_*, \mathbb{R}^n_*) \simeq \mathsf{Emb}^*((\mathbb{R}^n)^{\sqcup i}, \mathbb{R}^n \setminus \{0\}), \tag{188}$$

for all $i \in \mathbb{N}$, where $\mathsf{Emb}^*((\mathbb{R}^n)^{\sqcup i}, \mathbb{R}^n \setminus \{0\})$ is the space of framed embeddings of i n-dimensional disks into the punctured $\mathbb{R}^n$.

The isomorphism $\mathbb{R}^n \setminus \{0\} \cong S^{n-1} \times \mathbb{R}$ together with an examination of [Fra13, prop.3.16], which we recalled in Proposition 3.51, and its proof leads us to suggest that the $\mathbb{E}_n$-algebra A and the $\mathbb{E}_0$-algebra B are further such that B is a module over the universal enveloping algebra $U_A \simeq \int_{S^{n-1}} A$ as an $\mathbb{E}_1$-algebra. Remembering that we limited the whole discussion to the case when $n > 1$, we propose:

Theorem 5.3. *Let n be bigger than 1. There is an equivalence between the ∞-category of locally constant factorization algebras on $\mathbb{R}^n_*$ and the ∞-category of $\mathbb{E}_n$-modules that fits into the commutative diagram*

$$\begin{tikzcd} \mathsf{Mod}^{\mathbb{E}_n}(\mathcal{C}) \arrow[r, "\simeq"] \arrow[d] & \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^n_*}(\mathcal{C}) \arrow[d, "(3.35)"] \\ \mathcal{A}\mathsf{lg}_{\mathbb{E}_n}(\mathcal{C}) \arrow[r, "\simeq"] & \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^n}(\mathcal{C}). \end{tikzcd} \tag{189}$$

Remark 5.4. In the above $\mathsf{Mod}^{\mathbb{E}_n}$ is the ∞-category of $\mathbb{E}_n$-modules, which is fibered over $\mathcal{A}\mathsf{lg}_{\mathbb{E}_n}$ the ∞-category of $\mathbb{E}_n$-algebras. Informally speaking, the objects of $\mathsf{Mod}^{\mathbb{E}_n}$ are pairs (A, M) consisting of an $\mathbb{E}_n$-algebra A and an A-module M. However, to make things precise we, as always, use the definitions in [Lur, ch.3].

Proof. From eq. (185) we know that when $n > 1$ we have

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^n_*}(\mathcal{C}) \simeq \mathcal{A}\mathsf{lg}_{\mathcal{D}_*}(\mathcal{C}), \tag{190}$$

where we used the notation from Observation 5.2. Specializing Theorem 2.40 to the case at hand, and rewriting gives the left pullback square of the diagram

$$\begin{tikzcd} \mathsf{Mod}^{\mathbb{E}_n}_A(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{A}\mathsf{lg}_{\mathcal{D}_*}(\mathcal{C}) \arrow[r, "\simeq"] \arrow[d] & \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^n_*}(\mathcal{C}) \arrow[d, "(3.35)"] \\ * \arrow[r, "\{A\}"] & \mathcal{A}\mathsf{lg}_{\mathbb{E}_n}(\mathcal{C}) \arrow[r, "\simeq"] & \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^n}(\mathcal{C}). \end{tikzcd} \tag{191}$$

The right square obviously commutes by construction. Since, by definition, $\mathsf{Mod}^{\mathbb{E}_n}(\mathcal{C})$ is fibered over $\mathcal{A}\mathsf{lg}_{\mathbb{E}_n}(\mathcal{C})$, with fibers $\mathsf{Mod}^{\mathbb{E}_n}_A(\mathcal{C})$ for A an $\mathbb{E}_n$-algebra, the above diagram gives exactly the desired statement. $\square$

Remark 5.5. The study of locally constant factorization algebras on $\mathbb{R}^n_*$ was also undertaken in [Gin15, sec.6.3]. Theorem 5.3, as it appears above, is slightly different in its statement compared to the corresponding [Gin15, cor.8]. We believe that this is due to the fact that using the homotopy pullback is not what the proof provided in the reference actually does. The remaining discrepancy is then explained by our Theorem 4.1, which glues locally constant factorization algebras.

[18] We've omitted mentioning the fact that, because of morphisms like $\emptyset \to U$, all of the above algebraic objects will be pointed. We have done this to avoid confusion with the use of pointed in the sense of a point defect of the space.

Remark 5.6. The case $n = 1$ is special because the higher stratum $\mathbb{R} \setminus \{0\}$ is now disconnected. Looking back at the construction leading up to Proposition 3.44 and at Remark 3.45 we can guess that a statement like the following can hold true:

Proposition 5.7. *There is an equivalence between the ∞-category of locally constant factorization algebras on $\mathbb{R}_*$ and the ∞-category consisting of two $\mathbb{E}_1$-algebras A and B together with a pointed, left $A \otimes B^{\mathsf{op}}$-module M*

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}_*} \simeq \mathsf{LMod}(\mathcal{C}) \underset{\mathcal{C}_{\mathbb{1}/}}{\times} \mathsf{RMod}(\mathcal{C}), \tag{192}$$

where the functors $\mathsf{LMod}(\mathcal{C}) \to \mathcal{C}_{\mathbb{1}/}$ and $\mathsf{RMod}(\mathcal{C}) \to \mathcal{C}_{\mathbb{1}/}$ are the ones giving the underlying (pointed) object of a module.

5.2 The General Case of M_Σ

Locally, near the defect, the locally constant factorization algebra is, of course, going to look like a locally constant factorization algebra over $\mathbb{R}^{d \subset n}$. However, it is possible that the global topology introduces some twisting of this local understanding. Due to Theorem 4.1 and the properties of stratified manifolds, in any particular case, it's possible to find some list of collar-gluings that reduce the problem down to the local case, but this doesn't tell us anything about the general structure. Despite this, there is a particular collar-gluing that is the obvious first step if we're looking at a general manifold M_Σ. We will now describe this.

The data of an embedding $\Sigma \hookrightarrow M_\Sigma$ allows us to construct a regular neighborhood of our defect, echoing the tubular neighborhood theorem, now in the case of stratified spaces.

Theorem 5.8 ([AFT17b, prop.8.2.3, prop.8.2.5])**.** *Let $\Sigma \hookrightarrow M_\Sigma$ be a proper, constructible embedding of stratified manifolds. There exists a stratified map*

$$\mathsf{C}(\pi) \to M_\Sigma \tag{193}$$

under $\Sigma \hookrightarrow M_\Sigma$ such that:

1. *the image is open, and*
2. *$\mathsf{C}(\pi)$ is the fiberwise open cone of the constructible bundle $\mathsf{L}_\Sigma \xrightarrow{\pi} \Sigma$ giving the link.*

If Σ is a stratum of M_Σ then the map $\mathsf{C}(\pi) \to M_\Sigma$ is even an open embedding.

Remark 5.9. In more detail, specialized to our situation, the regular neighborhood $\mathsf{C}(\pi)$ is constructed as the pushout of stratified manifolds

$$\begin{tikzcd} \mathsf{L}_\Sigma \times \{0\} \arrow[r, "\pi"] \arrow[d, hook, "\mathrm{id}\times 0"'] & \Sigma \arrow[d] \\ \mathsf{L}_\Sigma \times \mathbb{R}_{\geq 0} \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & \mathsf{C}(\pi), \end{tikzcd} \tag{194}$$

where L_Σ is the link of Σ in M_Σ (see [AFT17b]). In our case of interest where the stratified manifold is M_Σ, the link L_Σ takes a simple form; it can always be constructed as the sphere bundle of the normal bundle of Σ when embedded into M as smooth manifolds.

Example 5.10. For the case of $(\Sigma \hookrightarrow M) = S^1 \hookrightarrow \mathbb{R}^3$, i.e. a circle defect in Euclidean 3-space, the link L_Σ would be a torus.

Construction 5.11. The existence of $\mathsf{C}(\pi)$ allows us to define a collar-gluing for the stratified manifold M_Σ

$$M_\Sigma \cong (M \setminus \Sigma) \coprod_{\mathsf{L}_\Sigma \times \mathbb{R}} \mathsf{C}(\pi), \tag{195}$$

where the map $\mathsf{L}_\Sigma \times \mathbb{R} \to M \setminus \Sigma$ is provided by $\mathsf{L}_\Sigma \times \mathbb{R} \cong \mathsf{L}_\Sigma \times \mathbb{R}_{\geq 0} \cong \mathsf{C}(\pi) \setminus \Sigma \hookrightarrow M \setminus \Sigma$. We also note that the properness of the embedding $\Sigma \hookrightarrow M$ makes $M \setminus \Sigma$ an open submanifold of M.

Theorem 4.1 allows us to use the collar-gluing from above to write the equivalence of ∞-categories

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M_\Sigma}(\mathcal{C}) \simeq \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M\setminus\Sigma}(\mathcal{C}) \bigtimes_{\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathsf{L}_\Sigma\times\mathbb{R}}(\mathcal{C})} \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathsf{C}(\pi)}(\mathcal{C}), \tag{196}$$

as a first step in the classification of locally constant factorization algebras on M_Σ. Since we have focused on stratified manifolds with the particular form M_Σ, the locally constant factorization algebras $\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M\setminus\Sigma}$ in this decomposition already live on a smooth manifold, rather than on a stratified one. Similarly, this is the case for the locally constant factorization algebras $\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathsf{L}_\Sigma\times\mathbb{R}}$. The more prescient question is about the classification of locally constant factorization algebras on the one remaining stratified space $\mathsf{C}(\pi)$.

By [AFT17b, ex.3.6.6] $\mathsf{C}(\pi)$ is always a bundle over Σ, fibered over basics. For our particular setup this means that $\mathsf{C}(\pi)$ a fiber bundle over Σ with typical fiber $\mathbb{R}^{n-d}_*$. Thus, the classification comes down to the classification of locally constant factorization algebras on stratified fiber bundles.

It is outside the scope of this work to finish the classification as outlined above. However, there is a special case that we can already tackle, which is the case of framed stratified manifolds. This is because, as discussed in Definition 1.47, $\mathsf{D}^*_{d\subset n}$-manifolds come with a trivialization of the normal bundle of the distinguished submanifold Σ. This, in turn implies trivializations for $\mathsf{C}(\pi)$ and $\mathsf{L}_\Sigma \times \mathbb{R}$ as bundles over Σ with typical fibers $\mathbb{R}^{n-d}_*$ and $S^{n-d-1} \times \mathbb{R}$, respectively. In other words, there are isomorphisms

$$\mathsf{C}(\pi) \cong \Sigma \times \mathbb{R}^{n-d}_* \qquad \text{and} \qquad \mathsf{L}_\Sigma \times \mathbb{R} \cong \Sigma \times S^{n-d-1} \times \mathbb{R}. \tag{197}$$

However, Proposition 3.39 already gives us a way to classify locally constant factorization algebras on product spaces, if we know them on the product components. Having classified locally constant factorization algebras on the pointed Euclidean space in Theorem 5.3, we already have the result that

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathsf{C}(\pi)}(\mathcal{C}) \simeq \mathsf{Mod}^{\mathbb{E}_{n-d}}(\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\Sigma}(\mathcal{C})). \tag{198}$$

Similarly, we also know that for the link we have

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathsf{L}_\Sigma\times\mathbb{R}}(\mathcal{C}) \simeq \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^{n-d}\setminus\{0\}}(\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\Sigma}(\mathcal{C})), \tag{199}$$

exactly in such a way that the functor $\mathsf{Mod}^{\mathbb{E}_{n-d}}(\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\Sigma}(\mathcal{C})) \to \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^{n-d}\setminus\{0\}}(\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\Sigma}(\mathcal{C}))$ is the one given by $\int_{S^{n-d-1}} -$, which to an algebra assigns its universal enveloping algebra.

Putting it all together, we have the equivalence

$$\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M_\Sigma}(\mathcal{C}) \simeq \mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{M\setminus\Sigma}(\mathcal{C}) \bigtimes_{\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\mathbb{R}^{n-d}\setminus\{0\}}(\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\Sigma}(\mathcal{C}))} \mathsf{Mod}^{\mathbb{E}_{n-d}}(\mathcal{F}\mathsf{Alg}^{\mathsf{lc}}_{\Sigma}(\mathcal{C})). \tag{200}$$

This description confirms the observation that what a locally constant factorization algebra on a defect manifold encodes is an algebra away from the defect, an algebra on the defect and a module structure relating them. Additionally, it exactly specifies what that module structure should be given by. In the case where the stratified manifold isn't framed we expect that the above structure will be 'twisted' as seen in Example 4.13 for the case of the Möbius band.

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