GOOD COVERS AND ALGEBRAS ON CONICALLY SMOOTH SPACES

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Abstract. We construct good covers for conically smooth spaces. By a result of Karlsson–Scheimbauer–Walde, this implies that, for every such space X, constructible factorisation algebras on X and disk algebras over X coincide. We also give a simplified proof of that result.

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1. Introduction

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2. Localities of the full unzip

Definition 2.1. Let X be a CSS. Its *singular locus* $S_X = \bigcup_{k>0} X_k \subset X$ consists of all points in X which have strictly positive depth.

Lemma 2.2. *For every CSS X, the inclusion $S_X \hookrightarrow X$ is a proper constructible embedding of conically smooth spaces.*

Proof. Let $X \to P$ be the stratification of X and let $X \to P \to \mathbb{P} = \mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}^{\mathrm{op}}$ be its depth-dimension stratification [AFT17, Lemma 2.4.10]. The inclusion $\mathbb{Z}^{\mathrm{op}}_{>0} \subset \mathbb{Z}^{\mathrm{op}}$ (where in the subscript $>$ refers to the ordinary ordering and not to its opposite) is a full subcategory inclusion and is thus consecutive [AFT17, Definition 2.3.1], hence so is $\mathbb{P}_+ = \mathbb{Z}^{\mathrm{op}}_{>0} \times \mathbb{Z}^{\mathrm{op}} \hookrightarrow \mathbb{P}$. Since consecutive poset maps are stable under pullbacks, $\mathbb{P}_+ \times_{\mathbb{P}} P \hookrightarrow P$ is likewise consecutive,

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proving that the inclusion $S_X = X|_{\mathbb{P}_+ \times_{\mathbb{P}} P} \hookrightarrow X$ is a constructible map of conically smooth spaces [AFT17, Lemma 3.4.5, Example 3.4.7]. Finally, as $\mathbb{Z}^{\mathrm{op}}_{>0} \subset \mathbb{Z}^{\mathrm{op}}$ is downward-closed, S_X is closed in X, so (since CSSs are Hausdorff) the inclusion map is proper. $\square$

By virtue of Lemma 2.2 we may apply unzip X along S_X according to [AFT17, Proposition 7.3.10].

Proposition 2.3. *The unzip of X is its unzip along its singular locus. That is,* $\mathrm{Unzip}(X) \cong \mathrm{Unzip}_{S_X}(X)$ *and consequently* $\partial\,\mathrm{Unzip}(X) \cong \mathrm{Link}_{S_X}(X)$.

Proof. If $\mathrm{depth}(X) = 0$ the statement holds trivially, so let us suppose that $\mathrm{depth}(X) \leq n \geq 1$ and that the statement holds for all CSSs Y with $\mathrm{depth}(Y) \leq n-1$. The problem is local, so, suppressing euclidean factors for simplicity, let $X = C(Z)$ be a conically smooth basic. By [AFT17, Lemma 7.3.5, (5) and (6)] and by the inductive hypothesis we have $\mathrm{Unzip}(C(Z)) = \mathbb{R}_{\geq 0} \times \mathrm{Unzip}(Z) = \mathbb{R}_{\geq 0} \times \mathrm{Unzip}_{S_Z}(Z)$, as Z has strictly lower depth. On the other hand, since $S_{C(Z)} = C(S_Z)$ by inspection, we have $\mathrm{Unzip}_{S_{C(Z)}} C(Z) = \mathrm{Unzip}_{C(S_Z)} C(Z) = \mathbb{R}_{\geq 0} \times \mathrm{Unzip}_{S_Z}(Z)$ by construction (cf. the proof of [AFT17, Proposition 7.3.10]), proving $\mathrm{Unzip}(C(Z)) = \mathrm{Unzip}_{S_{C(Z)}}(C(Z))$. $\square$

Lemma 2.4. *Let X be a smooth manifold with corners. Then for every boundary collar $I\colon \mathbb{R}_{\geq 0} \times \partial X \hookrightarrow X$ there exists a homeomorphism*

$$\alpha_I\colon X^+ \xrightarrow{\cong} \mathbb{R}_{\geq 0} \times X$$

where $X^+ = \bigcup_{r \in \mathbb{R}_{\geq 0}} [0,r] \times \partial X \cup_{\{r\} \times \partial X} X$.

Proof. Let $I\colon \mathbb{R}_{\geq 0} \times \partial X \hookrightarrow X$ be a collar, which is a homeomorphism onto its image. We have $X = I(\mathbb{R}_{\geq 0} \times \partial X) \cup_{I((0,\infty) \times \partial X)} X^\circ$. Similarly, for each r, we have a diffeomorphism

$$\alpha_r\colon X \to X \smallsetminus I([0,r) \times \partial X) = I([r,\infty) \times \partial X) \cup_{I((r,\infty)\times\partial X)} X \smallsetminus I([0,r] \times \partial X)$$

given by $I(t,q) \mapsto I(t+r,q)$ on $I(\mathbb{R}_{\geq 0} \times \partial X)$ and by the identity on $X \smallsetminus \mathrm{Im}(I)$. We suppress the dependence on I in notation. We obtain a well-defined homeomorphism

$$I \cup \alpha_r\colon [0,r] \times \partial X \cup_{\{r\} \times \partial X} X \to X,$$

well-defined since $\alpha_r(q) = \alpha_r(I(0,q)) = I(r,q)$ for $q \in \partial X$, and thereupon, writing $X^{+r} = [0,r] \times \partial X \cup_{\{r\} \times \partial X} X$, the bijection

$$\begin{aligned} \alpha_I\colon X^+ &\to \mathbb{R}_{\geq 0} \times X, \\ \alpha|_{X^{+r}} &= \{r\} \times I \cup \alpha_r. \end{aligned}$$

We equip X^+ with the induced topology, promoting α to a homeomorphism. $\square$

Lemma 2.5. *Let X be a smooth manifold with corners. Then there is a homeomorphism $\mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times X) \cong \mathbb{R}_{\geq 0} \times X$. Consequently, there is a homeomorphism*

$$\mathrm{Unzip}_{C(L)}(C(Z)) \cong \mathbb{R}_{\geq 0} \times \mathrm{Link}_{C(L)}(C(Z))$$

for $L = S_Z$.

Proof. Using Lemma 2.4 it suffices to provide a homeomorphism

$$\phi\colon \mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times X) \to \bigcup_{r \in \mathbb{R}_{\geq 0}} [0,r] \times \partial X \cup_{\{r\} \times \partial X} X.$$

Noting

$$\partial(\mathbb{R}_{\geq 0} \times X) = \{0\} \times X \cup_{\{0\} \times \partial X} \mathbb{R}_{\geq 0} \times \partial X,$$

we define ϕ to be the following map:

$$\mathbb{R}_{\geq 0} \times \{0\} \times X \ni (t,0,x) \mapsto x \in X \subset [0,t] \times \partial X \cup_{\{t\} \times \partial X} X$$
$$\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \partial X \ni (t,s,q) \mapsto (t,q) \in [0,t+s] \times \partial X.$$

This map and its inverse ϕ^{-1} given by

$$[0,r] \times \partial X \ni (t,q) \mapsto (t, r-t, q) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \partial X$$
$$[0,r] \times \partial X \cup_{\{r\} \times \partial X} X \supset X \ni x \mapsto (r,0,x) \in \mathbb{R}_{\geq 0} \times \{0\} \times X$$

are well-defined and continous. Note that ϕ does not nepend on a collar.

The second statement is the special case where $X = \mathrm{Unzip}_L(Z)$ using Proposition 2.3 and that $\mathrm{Unzip}_{C(L)} C(Z) = \mathbb{R}_{\geq 0} \times \mathrm{Unzip}_L(Z)$ and $\mathrm{Link}_{C(L)} C(Z) = \partial\, \mathrm{Unzip}_{C(L)}(C(Z))$. □

Remark 2.6. In the situation of Lemma 2.4 we will regard X^+ as a smooth manifold with corners with respect to the smooth structure induced by α_I. Up to equivalence, this structure does not depend on the choice of collar. Similarly, in the situation of Lemma 2.5 we will regard $\mathbb{R}_{\geq 0} \times \mathrm{Link}_{C(L)} C(Z)$ as a smooth manifold with corners, tautologically diffeomorphic to $\mathrm{Unzip}_{C(L)} C(Z)$ with respect to the induced smooth structure.

Construction 2.7. Let X be a smooth manifold with corners of dimension n, let $\{(U, \phi_U)\}$ a cover of X by coordinate neighbourhoods where each $\phi_U\colon \mathbb{R}^{n-c_U} \times \mathbb{R}^{c_U}_{\geq 0}$ is a homeomorphism, and let $\{\rho_U\colon X \to [0,1]\}$ be a partition of unity subordinate to this cover. Recall that a collar $I = I(\{U, \phi_U, \rho_U\})\colon \mathbb{R}_{\geq 0} \times \partial X \hookrightarrow X$ is then constructed by defining to be the flow along the nowhere-vanishing inward-pointing vector field $V = \sum \rho_U V_U$ where, in local coordinates, $V_U = \sum_{1 \leq i \leq c_U} \partial_i$ where $\{\partial_i\}$ is the standard basis of $\mathrm{T}_0\mathbb{R}^{c_U}_{\geq 0} \subset \mathrm{T}_0(\mathbb{R}^{n-c_U} \times \mathbb{R}^{c_U}_{\geq 0})$.

Let X and $I = I_{\{U, \phi_U, \rho_U\}}$ be as above. Then there is a canonically induced a collar

$$I^+ = I(\{\mathbb{R}_{\geq 0} \times U, \mathrm{id}_{\mathbb{R}_{\geq 0}} \times \phi_U, \rho_U \circ \mathrm{pr}_X\})\colon \mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times \partial X) \hookrightarrow \mathbb{R}_{\geq 0} \times X$$

on $\mathbb{R}_{\geq 0} \times X$, where $\mathrm{pr}_X \colon \mathbb{R}_{\geq} \times X \to X$ is the coordinate projection. It is the flow along the vector field $V^+ = \sum(\rho_U \circ \mathrm{pr}_X) \cdot V_U^+$ where $V_U^+ = \partial_s + V_U$ where ∂_s is the standard basis of $\mathrm{T}_0\mathbb{R}_{\geq 0} \subset \mathrm{T}_0(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-c_U} \times \mathbb{R}^{c_U}_{\geq 0})$.

Lemma 2.8. *Let X be a smooth manifold with corners and let $I^+ \colon \mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times \partial X) \hookrightarrow \mathbb{R}_{\geq 0} \times X$ be as in Construction 2.7. Then*

$$I^+ = \alpha_I \circ \phi$$

where α_I is as in Lemma 2.4 and ϕ is as in the proof of Lemma 2.5.

Proof. We observe the restrictions

$$\begin{aligned} \mathbb{R}_{\geq 0} \times \{0\} \times X &\xrightarrow{\alpha_I \circ \phi} \mathbb{R}_{\geq 0} \times X \\ (t, 0, x) &\mapsto \begin{cases} (t, I(t_x + t, q)), & x = I(t_x, q) \\ (t, x), & x \in X \smallsetminus \mathrm{Im}(I) \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \partial X &\xrightarrow{\alpha_I \circ \phi} \mathbb{R}_{\geq 0} \times X \\ (t, s, q) &\mapsto (t + s, I(t, q)). \end{aligned}$$

Both maps are the flow (for time t) along the vector field $V^+ = \sum(\rho_U \circ \mathrm{pr}_X) \cdot (\partial_s + V_U)$. □

In the following, $\mathbb{D}^n \subset \mathbb{R}^n$ denotes the unit open n-disk and $C^{<1}(Z) = * \amalg_{\{0\} \times Z} [0,1) \times Z$.

Definition 2.9. We say an embedded basic $\phi \colon \mathbb{R}^k \times C(Z) \hookrightarrow X$ in a conically smooth space X is *extendable* if there exists an embedding $\widehat{\phi} \colon \mathbb{R}^k \times C(Z) \hookrightarrow X$ such that ϕ factors as $\phi \colon \mathbb{R}^k \times C(Z) \to \mathbb{D}^k \times C^{<1}(Z) \hookrightarrow \mathbb{R}^k \times C^{<1}(Z) \xhookrightarrow{\widehat{\phi}} X$ where the first map is the isomorphism given by the cartesian product of the isomorphisms $\mathbb{R}^k \to \mathbb{D}^k$, $x \mapsto \frac{x}{|x|+1}$ and $C(Z) \to C^{<1}(Z)$, $(t, z) \mapsto (\frac{t}{t+1}, z)$.

Definition 2.10. Suppose $\mathcal{U}$ is a cover of a topological space X which is closed under finite intersections. We say $\mathcal{U}$ is *generated* by a cover $\mathcal{V}$ and write $\mathcal{U} = \langle \mathcal{V} \rangle$ if every member of $\mathcal{U}$ is a finite intersection of members of $\mathcal{V}$.

Lemma 2.11. *Every smooth manifold M has a good cover $\mathcal{U}$ which is generated by extendable basics.*

Proof. Equip M with a riemannian metric. We can put $\mathcal{U} = \langle \mathcal{V} \rangle$ for $\mathcal{V} = \{D_p\}_{p \in M}$ where D_p is the convex disk which is the interior of the image of a ball in T_pM under the exponential map, with a radius that is strictly smaller than the radius of injectivity. □

Lemma 2.12. *Let $L = S_Z$ and let $U = C^{<1}(L) \subset C^{<1}(Z) \subset C(Z)$. Let $\pi \colon \mathrm{Link}(C(Z))|_U \to U$ denote the link projection of $\mathrm{Unzip}(C(Z))$ over U. Then there is an isomorphism*

$$\partial\overline{\pi^{-1}U} \cong \mathrm{Link}_L(Z)$$

and an induced conically smooth collar $(0,1] \times \partial\overline{\pi^{-1}U} \hookrightarrow \overline{\pi^{-1}U}$ *which is a refinement onto its image.*

Proof. Using Proposition 2.3 we have $\operatorname{Unzip} C(Z) = \operatorname{Unzip}_S C(Z) = \mathbb{R}_{\geq 0} \times \operatorname{Unzip}_L Z$ and so

$$\begin{aligned}\operatorname{Unzip} C(Z)|_U &= \operatorname{Link}_{C(L)} C(Z)|_{C^{<1}(L)} \\ &= \{0\} \times \operatorname{Unzip}_L Z \cup_{\{0\}\times \operatorname{Link}_L Z} [0,1) \times \operatorname{Link}_L Z\end{aligned}$$

since the projection $\pi\colon \operatorname{Link}_{C(L)} C(Z) = \{0\} \times \operatorname{Unzip}_L Z \cup_{\{0\}\times \operatorname{Link}_L Z} \mathbb{R}_{\geq 0} \times \operatorname{Link}_L Z \to C(L) = *\amalg_{\{0\}\times L} \mathbb{R}_{\geq 0} \times L$ is given by mapping all of $\{0\}\times \operatorname{Unzip}_L Z$ to $*$ and on $\mathbb{R}_{\geq 0} \times \operatorname{Link}_L Z$ by $\mathrm{id}_{\mathbb{R}_{\geq 0}} \times \pi'$ where $\pi'\colon \operatorname{Link}_L Z \to L$ is the link projection. Thus

$$\pi^{-1}\overline{U} = \pi^{-1}C^{\leq 1}(U) = \{0\} \times \operatorname{Unzip}_L Z \cup [0,1] \times \operatorname{Link}_L Z$$

and consequently

$$\pi^{-1}\partial\overline{U} = \pi^{-1}\{1\} \times L = \{1\} \times \operatorname{Link}_L Z, \tag{2.13}$$

proving the first claim. The collar is immediate. □

Example 2.14.

Lemma 2.15. *Let* $\mathbb{R}^n$ *be equipped with a riemannian metric, and let* $c \in \{0,\ldots,n\}$*. Then for all geodesic disks* $D \subset \mathbb{R}^n$ *about the origin the intersections*

(1) $D \cap (\mathbb{R}^{n-c} \times (\mathbb{R}^c \setminus \mathbb{R}^c_{\geq 0}))$
(2) $D \cap (\mathbb{R}^{n-c} \times \mathbb{R}^c_{>0})$

are disks.

Proof. Let us suppose $n = c$ for simplicity. The map $\rho_r\colon B_r \to \mathbb{R}^n$, $v \mapsto \frac{d(0,v)}{r-d(0,v)}v$ is a diffeomorphism from the geodesic disk $B_r = \{x \in \mathbb{R}^n : d(0,x) < r\}$ of radius r about the origin to all of $\mathbb{R}^n$, where d is the metric associated with the given riemannian metric. Now we need only note that ρ_r restricts to diffeomorphisms $B_r \cap \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ and $B_r \cap (\mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}) \to \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}$, and that both targets are disks. □

Lemma 2.16. *Version of Lemma 2.15 for star-shaped (to apply to intersections of convex disks later) using Whitney approximation for functions!*

Proposition 2.17. *Let* $L = S_Z$ *and let* $I = J^+\colon \mathbb{R}_{\geq 0} \times \operatorname{Link}_{C(L)} C(Z) \hookrightarrow \operatorname{Unzip}_{C(L)} C(Z)$ *be the collar induced, according to Construction 2.7, by a collar* $J\colon \mathbb{R}_{\geq 0} \times \operatorname{Link}_L(Z) \hookrightarrow \operatorname{Unzip}_L(Z)$*. Let* U *be as in Lemma 2.12, and let*

$$p \in I(\{1\} \times \partial\overline{\pi^{-1}U}).$$

Then:

(1) *There is a diffeomorphism*

$$\mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times \operatorname{Link}_L Z) \cap I([0,1] \times \overline{\pi^{-1}U}) \cong (1,2] \times J((0,1] \times \operatorname{Link}_L Z)$$

of smooth manifolds with corners.

(2) *There exists a convex disk* $D \subset \operatorname{Unzip}_{C(L)} C(Z)^\circ$ *about* p *such that* $D \cap I([0,1) \times \pi^{-1}U)$ *is a disk.*

Proof. From the proof of Lemma 2.12 we recall that $\overline{\pi^{-1}U} = \{0\} \times \operatorname{Unzip}_L Z \cup_{\{0\}\times \operatorname{Link}_L Z} [0,1] \times \operatorname{Link}_L Z$. By Lemma 2.8 we have $I = \alpha_J \circ \phi$, and observe that

$$\begin{aligned}\phi([0,1] \times \overline{\pi^{-1}U}) &= A' \cup B' \\ &\coloneqq \bigcup_{t\in[0,1]} (X \subset X^{+t}) \cup \{(t,q) \in X^{+(t+s)} : t,s \in [0,1]\}.\end{aligned}$$

Thus $I([0,1] \times \overline{\pi^{-1}U}) = A \cup B \subset \operatorname{Unzip}_{C(L)} C(Z) = \mathbb{R}_{\geq 0} \times \operatorname{Unzip}_L Z$ where $A = \alpha_J(A')$, $B = \alpha_J(B')$. We have

$$A = \{(r,x) \in \mathbb{R}_{\geq 0} \times \operatorname{Unzip}_L Z : r \in [0,1],\ x \in \operatorname{Unzip}_L Z \smallsetminus J([0,r) \times \operatorname{Link}_L Z\}$$

$$B = \{(t+s, J(t,q)) \in \mathbb{R}_{\geq 0} \times \operatorname{Unzip}_L Z : t,s \in [0,1]\}$$

and hence

$$\mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times \operatorname{Link}_L Z) \cap (A \cup B) = \{(t+s, J(t,q)) : t \in (0,1],\ s \in [0,1],\ t+s > 1\}$$

as the intersection with A is empty.

Consider now the diffeomorphism

$$\begin{aligned}\psi = \rho \circ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \rho^{-1} \colon \mathbb{R}_{>1} \times \mathbb{R}_{>0} &\to \mathbb{R}_{>1} \times \mathbb{R}_{>0}, \\ (x,y) &\mapsto ((x-1)y+1, y)\end{aligned}$$

where $\rho\colon \mathbb{R}^2 \to \mathbb{R}_{>1} \times \mathbb{R}_{>0}$, $(x,y) \mapsto (e^x+1, e^y)$. Now ψ fixes $(2,1)$ and satisfies

$$\psi((1,2] \times (0,1]) = \{(t+s,t) : t \in (0,1],\ s \in [0,1],\ t+s > 1\}.$$

Putting

$$\Psi = \mathrm{id}_{\mathbb{R}_{>1}} \times J \circ \psi \times \mathrm{id}_{\operatorname{Link}_L Z} \colon \mathbb{R}_{>1} \times \mathbb{R}_{>0} \times \operatorname{Link}_L Z \to \mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times \operatorname{Link}_L Z)$$

we obtain the map

$$\begin{aligned}(1,2] \times J((0,1] \times \operatorname{Link}_L Z) &\xrightarrow{\cong} (1,2] \times (0,1] \times \operatorname{Link}_L Z \\ &\xrightarrow{\Psi|} \mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times \operatorname{Link}_L Z) \cap I([0,1] \times \overline{\pi^{-1}U}).\end{aligned}$$

The first map, which is a homeomorphism, is tautologically a diffeomorphism of smooth manifolds with corners with respect to the induced smooth structure on its target. With respect to the latter, Ψ is a diffeomorphism as well. Hence the composition is a diffeomorphism. In particular, we obtain the restricted diffeomorphism

$$(1,2) \times J((0,1) \times \operatorname{Link}_L Z) \cong \mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times \operatorname{Link}_L Z) \cap C_U.$$

of smooth manifolds without boundary.

Finally, consider $p = I(1, (1, q)) = (2, J(1, q))$ where $q \in \mathrm{Link}_L Z$ (recall (2.13)). Let $\mathbb{R}^{n-c} \times \mathbb{R}^c_{\geq 0} \cong W \subset \mathrm{Unzip}_L Z$ be a chart neighbourhood of q. Without loss of generality we may assume $c = n$ for simplicity. Up to diffeomorphism we may write the restriction of the collar of $\mathrm{Link}_L Z$ as $J \colon \mathbb{R}_{\geq 0} \times \partial\mathbb{R}^c_{\geq 0}) \hookrightarrow \mathbb{R}^c_{\geq 0}$, given by the flow along the vector field $\sum_{1 \leq i \leq c} \partial_i$ where $\{\partial_i\}$ is the standard basis of $\mathrm{T}_0\mathbb{R}^c_{\geq 0}$ (recall Construction 2.7). We now observe the diffeomorphism

$$J((0,1] \times \partial\mathbb{R}^c_{\geq 0}) = \mathbb{R}^c_{>0} \smallsetminus \left(\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \mathbb{R}^c_{\geq 0} \right) \cong \mathbb{R}^c \smallsetminus \mathbb{R}^c_{>0}$$

of smooth manifolds with corners. Consequently we have $(1, 2] \times J((0, 1] \times \partial\mathbb{R}^c_{\geq 0}) \cong \mathbb{R}^{c+1} \smallsetminus \mathbb{R}^{c+1}_{>0}$. These diffeomorphisms restrict to $J((0, 1) \times \partial\mathbb{R}^c_{\geq 0}) \cong \mathbb{R}^c \smallsetminus \mathbb{R}^c_{\geq 0}$ and so $(1, 2) \times J((0, 1) \times \partial\mathbb{R}^c_{\geq 0}) \cong \mathbb{R}^{c+1} \smallsetminus \mathbb{R}^{c+1}_{\geq 0}$. Hence, the intersection of any convex disk $D \subset \mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times W)$ centred at $(2, J(1, q)) = p$ with $I([0, 1) \times \pi^{-1}U)$ is a disk by the first statement of Lemma 2.15. $\square$

Example 2.18.

Corollary 2.19.

3. Good covers

Theorem 3.1. *Every conically smooth space admits a good cover.*

Proof. Let $n \geq 0$ denote the depth of X. If $n = 0$ then X is smooth and we are done, so suppose $n \geq 1$. Proposition 2.3 gives that $\mathrm{Unzip}(X) \cong \mathrm{Unzip}_S(X)$ and that $\partial = \partial\,\mathrm{Unzip}(X) \cong \mathrm{Link}_S(X)$ where $S = S_X \subset X$ is the union of the depth-k strata of X for all $k \geq 1$. We obtain the pullback-pushout diagram

$$\begin{tikzcd} \partial \arrow[r, hook] \arrow[d, two heads, "\pi"] & \mathrm{Unzip}(X) \arrow[d, two heads] \\ S \arrow[r, hook] & X \end{tikzcd}$$

where every map is a proper constructible bundle. Moreover, $\mathrm{Unzip}(X)$ is a smooth manifold with corners. We equip its interior with a riemannian metric. Note that $S \subset X$ being closed in a compact space is compact, hence so is ∂. As in (the proof of) [AFT17, Proposition 8.2.5] there is a conically smooth collar

$$I \colon \mathbb{R}_{\geq 0} \times \partial \hookrightarrow \mathrm{Unzip}$$

which is a refinement onto its open image. Further, since $\mathrm{depth}(S) < \mathrm{depth}(X)$ we may choose by induction a locally finite good cover $\mathcal{U}$ on S. By Lemma 2.11 we may assume that $\mathcal{U} = \langle \mathbb{U} \rangle$ where $\mathbb{U}$ is a cover of S by extendable basics.

We will write $C_{A,B} = I(A \times \pi^{-1}B)$ where $A \subseteq \mathbb{R}_{\geq 0}$ and $B \subset S$ are subsets, and set $C_U = C_{[0,1),U}$.

Step A. Consider the cover

$$\mathcal{C} = \{C_U : U \in \mathcal{U}\}$$

of $C_{[0,1),S}$, an open neighbourhood of ∂ in Unzip, and the cover $\mathcal{I}$ of Unzip $\smallsetminus C_{[0,1],S}$ consisting of all convex disks therein. Since $C_U \cap C_V = C_{U\cap V}$ and $\mathcal{U}$ is good, $\mathcal{C}$ is closed under finite intersections. We will adjoin to $\mathcal{C} \cup \mathcal{I}$ an open cover $\mathcal{D}$ of $C_{\{1\},S} = \text{Unzip} \smallsetminus \bigcup \mathcal{C} \cup \bigcup \mathcal{I}$ consisting of convex disks whose intersections with members of $\mathcal{C}$ are (not necessarily convex) disks and is itself closed under finite intersections, thus constructing a cover of Unzip which consists of the C_U and disks in the interior Unzip$^\circ$ and is closed under finite intersections.

Let us first observe that such a cover induces a good cover X upon passing from Unzip to X. We claim that

$$\widehat{C_U} = U \amalg_{\pi^{-1}U} C_U \subset X$$

is a basic for each $U \in \mathcal{U}$. To see this, suppose, by ignoring euclidean factors without loss of generality, that $U \cong C(L) \subset C(Z) \subset X$, induced by an inclusion $L \subset Z$ where $L = S_Z$. Then we have that

$$\begin{aligned} \widehat{C_U} \cong C(L) \amalg_{\text{Link}_{C(L)}(C(Z))} \mathbb{R}_{\geq 0} \times \text{Link}_{C(L)}(C(Z)) \\ \hookrightarrow C(L) \amalg_{\text{Link}_{C(L)}(C(Z))} \text{Unzip}_{C(L)}(C(Z)) \end{aligned}$$

is an isomorphism since the collar $\mathbb{R}_{\geq 0} \times \text{Link}_{C(L)}(C(Z)) \hookrightarrow \text{Unzip}_{C(L)}(C(Z))$ is an isomorphism by Lemma 2.5. But

$$C(L) \amalg_{\text{Link}_{C(L)}(C(Z))} \text{Unzip}_{C(L)}(C(Z)) \cong C(Z)$$

by the unzip square of $C(Z)$, showing that

$$\widehat{C_{C(L)}} \cong C(Z)$$

is a basic. We conclude that

$$\widehat{\mathcal{C}} = \{\widehat{C_U} : U \in \mathcal{U}\}$$

is an open cover of S by basics. It is closed under finite intersections since so is $\mathcal{C}$. The cover consisting of $\mathcal{D}$ and $\mathcal{I}$ and the resulting finite intersections with $\widehat{\mathcal{C}}$ descends on X to a cover, contained within the depth-0 stratum X_0 (cf. the last statement of [AFT17, Proposition 7.3.10]), consisting of disks and is closed under finite intersections.

Step B. It remains to construct the cover $\mathcal{D}$ of $C_{\{1\},S}$ with the desired properties. Let $p \in C_{\{1\},S}$ and let

$$k(p) = \#\{U \in \mathcal{U} : p \in \partial\overline{C_{\{1\},U}}\} \geq 0$$

where

$$\partial\overline{C_{\{1\},U}} = \partial C_{\{1\},\overline{U}} = C_{\{1\},\partial\overline{U}}.$$

We will write $\overline{p} \in \partial$ for the image of p under the projection $\partial^i\overline{C_{<1,S}} = C_{\{1\},S} \to \{1\} \times \partial \to \partial$ given by I^{-1}. We will also write $\overline{p} = \pi(\overline{p}) \in S$ by abuse. Let $c\colon \mathrm{Unzip} \to \langle m \rangle$ denote the corner stratification of Unzip so that around $\overline{p}$ there is a chart $\phi\colon \mathbb{R}^{n-|c(p)|} \times \mathbb{R}^{|c(p)|}_{\geq 0} \hookrightarrow \mathrm{Unzip}$ with restriction $\phi|_\partial\colon \partial(\mathbb{R}^{n-|c(p)|} \times \mathbb{R}^{|c(p)|}_{\geq 0}) = \mathbb{R}^{n-|c(p)|} \times \partial\mathbb{R}^{|c(p)|}_{\geq 0} \hookrightarrow \partial$. Using I, we obtain an open neighbourhood

$$I|_{>0} \circ \phi|_\partial\colon (0,\infty) \times \mathbb{R}^{n-|c(p)|} \times \partial\mathbb{R}^{|c(p)|}_{\geq 0} \hookrightarrow \mathrm{Unzip}^\circ$$

such that $p \in I \circ \phi|_\partial(\{1\} \times \partial\mathbb{R}^n_{\geq 0})$. Let us put

$$W = W_\phi = \mathrm{Im}(I|_{>0} \circ \phi|_\partial)$$

and note the diffeomorphism $W \cong \mathbb{R}^n$.

Step C. Suppose now that $k(p) = 0$. By the local finitude of $\mathcal{U}$ we may pick a member $U'_p \in \mathcal{U}$ which contains $\overline{p}$ and intersect only finitely many members of $\mathcal{U}$ non-trivially. This yields

$$U''_p = \bigcap_{\overline{p} \in U} U \cap U'_p \in \mathcal{U},$$

which satisfies $p \in C_{\{1\},U_p}$. Next, consider

$$V_p = \bigcup_{\substack{V \in \mathcal{U},\\ \overline{p} \notin \overline{V},\\ V \cap U''_p \neq \emptyset}} \overline{V}.$$

and put

$$U_p = U''_p \smallsetminus V_p.$$

Since V_p is likewise a finite union it is closed and thus U_p is an open neighbourhood of $\overline{p}$ within S (which need not be in $\mathcal{U}$). We may now pick the chart ϕ above small enough such that $\phi|_\partial$ factors through $\pi^{-1}U_p \subset \partial$ and so $W \cap U_p = \mathrm{Im}(I|_{(0,1)} \circ \phi|_\partial)$. We may now pick a convex disk D about p with $D \subset W$, ensuring $D \cap C_{U_p} = D \cap \mathrm{Im}(I|_{(0,1)} \circ \phi|_\partial)$, and so

$$D \cap C_{U_p} \cong \mathbb{R}^{n-|c(p)|} \times (\mathbb{R}^{|c(p)|} \smallsetminus \mathbb{R}^{|c(p)|}_{\geq 0}) \cong \mathbb{R}^n$$

is a disk by Lemma 2.15. For each $U \in \mathcal{U}$ with $D \cap C_U \neq \emptyset$ we have $D \cap C_U = D \cap C_{U_p}$ by construction, so $D \cap C_U$ is a disk for every $U \in \mathcal{U}$ if it is not empty.

Step D. Suppose now that $k(p) > 0$, and let $U \in \mathcal{U}$ be such that $p \in \partial\overline{C_{\{1\},U}}$. Suppose first that $U \in \mathbb{U}$. Since U is extendable, Proposition 2.17 gives that every geodesic disk D about p contained within a small-enough coordinate neighbourhood M_U satisfies $D \cap C_U$. Consequently, every geodesic disk D about p with

$$D \subset \bigcap_{\substack{U \in \mathbb{U}\\ p \in \partial\overline{C_{\{1\},U}}}} M_U,$$

the intersection being finite by local finitude, satisfies $D \cap C_U = D \cap M_U$ for every $U \in \mathcal{U}$ with $p \in \partial\overline{C_{\{1\},U}}$. $\square$

References

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