GOOD COVERS AND ALGEBRAS ON CONICALLY SMOOTH SPACES

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ABSTRACT. We construct good covers for conically smooth spaces over depth-1 posets. By a result of Karlsson–Scheimbauer–Walde, this implies that, for every such space X, constructible factorisation algebras on X and disk algebras over X coincide. We also give a simplified proof of that result.

CONTENTS

1. INTRODUCTION

Acknowledgments. We thank Tashi Walde for very useful exchanges.

2. GOOD COVERS

Definition 2.1. Let X be a CSS. A *good cover* of X is an open cover $\mathcal{U}$ of X consisting of basics whose finite intersections are also in $\mathcal{U}$.

Definition 2.2. Let Y be a smooth manifold with or without boundary, equipped with a riemannian metric. We call a cover of Y *geodesically convex* if it consists of geodesically convex basics.

Theorem 2.3. *Let X be a CSS over a depth-1 poset. Then X has a good cover.*

Proof. Without loss of generality, suppose X is stratified over $[1]$ and let $M = X_0$ and $N = X_1$. Recall the blow-up of M, the smooth manifold

$$\mathrm{Unzip} \cong L \amalg_{L \times (0,\infty)} N$$

with boundary $L = \partial\mathrm{Unzip}$. We write $\pi \colon L \to M$ for the accompanying proper fibre bundle. Let us equip Unzip with a riemannian metric which splits along the boundary.

Using the paracompactness of M, let $\mathcal{U}$ be a locally finite good cover of M which trivialises π. For each $U \in \mathcal{U}$, let $\epsilon_U > 0$ be the radius of injectivity in the normal direction on the compact closure $\overline{U}$ so that there is a smooth embedding

$$I_U \colon \pi^{-1}U \times [0, \epsilon_U) \hookrightarrow \mathrm{Unzip}$$

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whose restriction $I_U|_{\pi^{-1}U\times\{0\}}$ to $\pi^{-1}U \subset L$ is given by the boundary inclusion, and the path $I_U(q,-)\colon [0,\epsilon_U) \hookrightarrow \mathrm{Unzip}$, for every $q \in \pi^{-1}U$, is the minimising geodesic given by the normal exponential map. More precisely, for ν the inward-pointing unit normal vector field along the boundary, we set $I_U(x,t) = \exp_x(t\nu_x)$.[1] Let

$$C = \bigcup_{U\in\mathcal{U}} \mathrm{Im}(I_U)$$

be the induced 'collar' and consider the cover

$$\mathcal{C} = \{I_U(\pi^{-1}U \times [0,\delta)) : U \in \mathcal{U},\ \delta \leq \epsilon_U\}$$

of C. Let us write $C_{U,\delta} = I_U(\pi^{-1}U \times [0,\delta)) \in \mathcal{C}$. Note that $\mathcal{C}$ is closed under finite intersections since so is $\mathcal{U}$: we have $\pi^{-1}U \cap \pi^{-1}V = \pi^{-1}(U\cap V)$ and

$$C_{U,\delta} \cap C_{V,\delta'} = C_{U\cap V,\min(\delta,\delta')} \in \mathcal{C}$$

since $\min(\delta,\delta') \leq \epsilon_{U\cap V}$ for $\delta \leq \epsilon_U$, $\delta' \leq \epsilon_V$.

Consider now the collection $\mathcal{V}$ of those convex geodesic disks V in N such that $V \cap C_{U,\delta}$ is either empty or a convex geodesic disk for every $C_{U,\delta} \in \mathcal{C}$. Then $\mathcal{V}$ is a good cover of N. To prove this, it suffices to show that every $p \in C$ has a neighbourhood $V_p \in \mathcal{V}$, since convex geodesic disks are closed under finite intersections and so $V \cap V' \cap C_{U,\delta} = V \cap V'' \in \mathcal{V}$. Let now $p \in I_U(\pi^{-1}U \times [0,\delta)) \in \mathcal{C}$, which uniquely determines a point $l_p = \pi(\mathrm{pr}_1(p)) \in \pi^{-1}U \subset L$. Let $W \subset \pi^{-1}U$ be a convex geodesic disk neighbourhood of l_p. Now, since $\mathcal{U}$ is locally finite, so is the cover $\widetilde{\mathcal{C}} = \{I_U(\pi^{-1}U \times [0,\epsilon_U))\}$ of C (which is not necessarily closed under finite intersections), so in particular p is contained within finitely many members $I_{U_i}(\pi^{-1}U_i \times [0,\epsilon_i)) \in \widetilde{\mathcal{C}}$. We necessarily have $p \in I_U(\pi^{-1}U \times [0, \min_i(\epsilon_i)))$ with $U \in \{U_i\}_i$ an open satisfying $\epsilon_U = \min_i(\epsilon_i)$. Then $W \times [0,\min_i(\epsilon_i)) \ni p$ is a convex geodesic half-disk since the riemannian metric on Unzip splits along the boundary, and so we have $p \in V_p = W \times (0, \min_i(\epsilon_i)) \in \mathcal{V}$.

Let us now observe that $\pi^{-1}U \cong L_p \times U$ for $p \in U$ implies that each

$$C(L_p) \times U \cong U \amalg_{\pi^{-1}U} C_{U,\delta} \subseteq X$$

is a basic in X. Thus, writing $\widehat{C_{U,\delta}} = U \amalg_{\pi^{-1}U} C_{U,\delta}$, we obtain that

$$\widehat{\mathcal{C}} = \{\widehat{C_{U,\delta}} : C_{U,\delta} \in \mathcal{C}\}$$

is a cover by basics of the 'tubular neighbourhood'

$$\widehat{C} = \bigcup_{U\in\mathcal{C}} \widehat{C_{U,\epsilon_U}} \subseteq X$$

of M. It is closed intersections since so is $\mathcal{C}$. Finally, since for $V \in \mathcal{V}$ we have $\widehat{C_{U,\delta}} \cap V = C_{U,\delta} \cap V \in \mathcal{V}$, we conclude that

$$\widehat{\mathcal{C}} \cup \mathcal{V}$$

is a good cover of X. □

[1] A global ϵ need not exist unless Unzip is of bounded geometry; see e.g. Schick [Sch01].

3. A proof of...

References

[Sch01] T. Schick. 'Manifolds with Boundary and of Bounded Geometry'. *Mathematische Nachrichten* 223.1 (2001), 103–120.

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