GOOD COVERS AND ALGEBRAS ON CONICALLY SMOOTH SPACES

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Abstract. We construct good covers for conically smooth spaces. By a result of Karlsson–Scheimbauer–Walde, this implies that, for every such space X, constructible factorisation algebras on X and disk algebras over X coincide. We also give a simplified proof of that result.

Contents

1. Introduction

Acknowledgments. We thank Tashi Walde for very useful exchanges.

2. Localities of the full unzip

Definition 2.1. Let X be a CSS. Its *singular locus* $S_X = \bigcup_{k>0} X_k \subset X$ consists of all points in X which have strictly positive depth.

Lemma 2.2. *For every CSS X, the inclusion $S_X \hookrightarrow X$ is a proper constructible embedding of conically smooth spaces.*

Proof. Let $X \to P$ be the stratification of X and let $X \to P \to \mathbb{P} = \mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}^{\mathrm{op}}$ be its depth-dimension stratification [AFT17, Lemma 2.4.10]. The inclusion $\mathbb{Z}^{\mathrm{op}}_{>0} \subset \mathbb{Z}^{\mathrm{op}}$ (where in the subscript $>$ refers to the ordinary ordering and not to its opposite) is a full subcategory inclusion and is thus consecutive [AFT17, Definition 2.3.1], hence so is $\mathbb{P}_+ = \mathbb{Z}^{\mathrm{op}}_{>0} \times \mathbb{Z}^{\mathrm{op}} \hookrightarrow \mathbb{P}$. Since consecutive poset maps are stable under pullbacks, $\mathbb{P}_+ \times_{\mathbb{P}} P \hookrightarrow P$ is likewise consecutive, proving that the inclusion $S_X = X|_{\mathbb{P}_+ \times_{\mathbb{P}} P} \hookrightarrow X$ is a constructible map of conically smooth spaces [AFT17, Lemma 3.4.5, Example 3.4.7]. Finally,

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as $\mathbb{Z}^{\mathrm{op}}_{>0} \subset \mathbb{Z}^{\mathrm{op}}$ is downward-closed, S_X is closed in X, so (since CSSs are Hausdorff) the inclusion map is proper. $\square$

By virtue of Lemma 2.2 we may apply unzip X along S_X according to [AFT17, Proposition 7.3.10].

Proposition 2.3. *The unzip of X is its unzip along its singular locus. That is,* $\mathrm{Unzip}(X) \cong \mathrm{Unzip}_{S_X}(X)$ *and consequently* $\partial\,\mathrm{Unzip}(X) \cong \mathrm{Link}_{S_X}(X)$.

Proof. If $\mathrm{depth}(X) = 0$ the statement holds trivially, so let us suppose that $\mathrm{depth}(X) \leq n \geq 1$ and that the statement holds for all CSSs Y with $\mathrm{depth}(Y) \leq n-1$. The problem is local, so, suppressing euclidean factors for simplicity, let $X = C(Z)$ be a conically smooth basic. By [AFT17, Lemma 7.3.5, (5) and (6)] and by the inductive hypothesis we have $\mathrm{Unzip}(C(Z)) = \mathbb{R}_{\geq 0} \times \mathrm{Unzip}(Z) = \mathbb{R}_{\geq 0} \times \mathrm{Unzip}_{S_Z}(Z)$, as Z has strictly lower depth. On the other hand, since $S_{C(Z)} = C(S_Z)$ by inspection, we have $\mathrm{Unzip}_{S_{C(Z)}} C(Z) = \mathrm{Unzip}_{C(S_Z)} C(Z) = \mathbb{R}_{\geq 0} \times \mathrm{Unzip}_{S_Z}(Z)$ by construction (cf. the proof of [AFT17, Proposition 7.3.10]), proving $\mathrm{Unzip}(C(Z)) = \mathrm{Unzip}_{S_{C(Z)}}(C(Z))$. $\square$

Lemma 2.4. *Locally finite good cover on S_X. (?)* **unnecessary - second-countability implies more than paracompactness - see [Lee, Smooth, Thm 1.15]**

Lemma 2.5. *Let X be a smooth manifold with corners. Then for every boundary collar $I\colon \mathbb{R}_{\geq 0} \times \partial X \hookrightarrow X$ there exists a homeomorphism*

$$\alpha_I\colon X^+ \xrightarrow{\cong} \mathbb{R}_{\geq 0} \times X$$

where $X^+ = \bigcup_{r \in \mathbb{R}_{\geq 0}} [0,r] \times \partial X \cup_{\{r\}\times\partial X} X$.

Proof. Let $I\colon \mathbb{R}_{\geq 0} \times \partial X \hookrightarrow X$ be a collar, which is a homeomorphism onto its image. We have $X = I(\mathbb{R}_{\geq 0} \times \partial X) \cup_{I((0,\infty)\times\partial X)} X^\circ$. Similarly, for each r, we have a diffeomorphism

$$\alpha_r\colon X \to X \smallsetminus I([0,r)\times\partial X) = I([r,\infty)\times\partial X) \cup_{I((r,\infty)\times\partial X)} X \smallsetminus I([0,r]\times\partial X)$$

given by $I(t,q) \mapsto I(t+r,q)$ on $I(\mathbb{R}_{\geq 0}\times\partial X)$ and by the identity on $X \smallsetminus \mathrm{Im}(I)$. We suppress the dependence on I in notation. We obtain a well-defined homeomorphism

$$I \cup \alpha_r\colon [0,r] \times \partial X \cup_{\{r\}\times\partial X} X \to X,$$

well-defined since $\alpha_r(q) = \alpha_r(I(0,q)) = I(r,q)$ for $q \in \partial X$, and thereupon, writing $X^{+r} = [0,r] \times \partial X \cup_{\{r\}\times\partial X} X$, the bijection

$$\alpha_I\colon X^+ \to \mathbb{R}_{\geq 0} \times X,$$
$$\alpha|_{X^{+r}} = \{r\} \times I \cup \alpha_r.$$

We equip X^+ with the induced topology, promoting α to a homeomorphism. $\square$

Lemma 2.6. *Let X be a smooth manifold with corners. Then there is a homeomorphism $\mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times X) \cong \mathbb{R}_{\geq 0} \times X$. Consequently, there is a homeomorphism*

$$\mathrm{Unzip}_{C(L)}(C(Z)) \cong \mathbb{R}_{\geq 0} \times \mathrm{Link}_{C(L)}(C(Z))$$

for $L = S_Z$.

Proof. Using Lemma 2.5 it suffices to provide a homeomorphism

$$\phi\colon \mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times X) \to \bigcup_{r \in \mathbb{R}_{\geq 0}} [0,r] \times \partial X \cup_{\{r\} \times \partial X} X.$$

Noting

$$\partial(\mathbb{R}_{\geq 0} \times X) = \{0\} \times X \cup_{\{0\} \times \partial X} \mathbb{R}_{\geq 0} \times \partial X,$$

we define ϕ to be the following map:

$$\begin{aligned} \mathbb{R}_{\geq 0} \times \{0\} \times X \ni (t,0,x) &\mapsto x \in X \subset [0,t] \times \partial X \cup_{\{t\} \times \partial X} X \\ \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \partial X \ni (t,s,q) &\mapsto (t,q) \in [0,t+s] \times \partial X. \end{aligned}$$

This map and its inverse ϕ^{-1} given by

$$\begin{aligned} [0,r] \times \partial X \ni (t,q) &\mapsto (t, r-t, q) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \partial X \\ [0,r] \times \partial X \cup_{\{r\} \times \partial X} X \supset X \ni x &\mapsto (r,0,x) \in \mathbb{R}_{\geq 0} \times \{0\} \times X \end{aligned}$$

are well-defined and continous. Note that ϕ does not nepend on a collar.

The second statement is the special case where $X = \mathrm{Unzip}_L(Z)$ using Proposition 2.3 and that $\mathrm{Unzip}_{C(L)} C(Z) = \mathbb{R}_{\geq 0} \times \mathrm{Unzip}_L(Z)$ and $\mathrm{Link}_{C(L)} C(Z) = \partial \, \mathrm{Unzip}_{C(L)}(C(Z))$. □

Remark 2.7. In the situation of Lemma 2.5 we will regard X^+ as a smooth manifold with corners with respect to the smooth structure induced by α_I. Up to equivalence, this structure does not depend on the choice of collar. Similarly, in the situation of Lemma 2.6 we will regard $\mathbb{R}_{\geq 0} \times \mathrm{Link}_{C(L)} C(Z)$ as a smooth manifold with corners, tautologically diffeomorphic to $\mathrm{Unzip}_{C(L)} C(Z)$ with respect to the induced smooth structure.

Construction 2.8. Let X be a smooth manifold with corners of dimension n, let $\{(U, \phi_U)\}$ a cover of X by coordinate neighbourhoods where each $\phi_U\colon \mathbb{R}^{n-c_U} \times \mathbb{R}^{c_U}_{\geq 0}$ is a homeomorphism, and let $\{\rho_U\colon X \to [0,1]\}$ be a partition of unity subordinate to this cover. Recall that a collar $I = I(\{U, \phi_U, \rho_U\})\colon \mathbb{R}_{\geq 0} \times \partial X \hookrightarrow X$ is then constructed by defining to be the flow along the nowhere-vanishing inward-pointing vector field $V = \sum \rho_U V_U$ where, in local coordinates, $V_U = \sum_{1 \leq i \leq c_U} \partial_i$ where $\{\partial_i\}$ is the standard basis of $\mathrm{T}_0 \mathbb{R}^{c_U}_{\geq 0} \subset \mathrm{T}_0(\mathbb{R}^{n-c_U} \times \mathbb{R}^{c_U}_{\geq 0})$.

Let X and $I = I_{\{U, \phi_U, \rho_U\}}$ be as above. Then there is a canonically induced a collar

$$I^+ = I(\{\mathbb{R}_{\geq 0} \times U, \mathrm{id}_{\mathbb{R}_{\geq 0}} \times \phi_U, \rho_U \circ \mathrm{pr}_X\})\colon \mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times \partial X) \hookrightarrow \mathbb{R}_{\geq 0} \times X$$

on $\mathbb{R}_{\geq 0} \times X$, where $\mathrm{pr}_X : \mathbb{R}_{\geq} \times X \to X$ is the coordinate projection. It is the flow along the vector field $V^+ = \sum(\rho_U \circ \mathrm{pr}_X) \cdot V_U^+$ where $V_U^+ = \partial_s + V_U$ where ∂_s is the standard basis of $\mathrm{T}_0\mathbb{R}_{\geq 0} \subset \mathrm{T}_0(\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-c_U} \times \mathbb{R}^{c_U}_{\geq 0})$.

Lemma 2.9. *Let X be a smooth manifold with corners and let $I^+ : \mathbb{R}_{\geq 0} \times \partial(\mathbb{R}_{\geq 0} \times \partial X) \hookrightarrow \mathbb{R}_{\geq 0} \times X$ be as in Construction 2.8. Then*

$$I^+ = \alpha_I \circ \phi$$

where α_I is as in Lemma 2.5 and ϕ is as in the proof of Lemma 2.6.

Proof. We observe the restrictions

$$\begin{aligned} \mathbb{R}_{\geq 0} \times \{0\} \times X &\xrightarrow{\alpha_I \circ \phi} \mathbb{R}_{\geq 0} \times X \\ (t, 0, x) &\mapsto \begin{cases} (t, I(t_x + t, q)), & x = I(t_x, q) \\ (t, x), & x \in X \smallsetminus \mathrm{Im}(I) \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \partial X &\xrightarrow{\alpha_I \circ \phi} \mathbb{R}_{\geq 0} \times X \\ (t, s, q) &\mapsto (t + s, I(t, q)). \end{aligned}$$

Both maps are the flow (for time t) along the vector field $V^+ = \sum(\rho_U \circ \mathrm{pr}_X) \cdot (\partial_s + V_U)$. □

In the following, $\mathbb{D}^n \subset \mathbb{R}^n$ denotes the unit open n-disk and $C^{<1}(Z) = * \amalg_{\{0\} \times Z} [0, 1) \times Z$.

Definition 2.10. We say an embedded basic $\phi : \mathbb{R}^k \times C(Z) \hookrightarrow X$ in a conically smooth space X is *extendable* if there exists an embedding $\widehat{\phi} : \mathbb{R}^n \times C(Z) \hookrightarrow X$ such that ϕ factors as $\phi : \mathbb{R}^k \times C(Z) \to \mathbb{D}^n \times C^{<1}(Z) \hookrightarrow \mathbb{R}^n \times C^{<1}(Z) \xhookrightarrow{\widehat{\phi}} X$ where the first map is the isomorphism given by the cartesian product of the isomorphisms $\mathbb{R}^n \to \mathbb{D}^n$, $x \mapsto \frac{x}{|x|+1}$ and $C(Z) \to C^{<1}(Z)$, $(t, z) \mapsto (\frac{t}{t+1}, z)$.

Definition 2.11. Suppose $\mathcal{U}$ is a cover of a topological space X which is closed under finite intersections. We say $\mathcal{U}$ is *generated* by a cover $\mathcal{V}$ and write $\mathcal{U} = \langle \mathcal{V} \rangle$ if every member of $\mathcal{U}$ is a finite intersection of members of $\mathcal{V}$.

Lemma 2.12. *Every smooth manifold M has a good cover $\mathcal{U}$ which is generated by extendable basics.*

Proof. Equip M with a riemannian metric. We can put $\mathcal{U} = \langle \mathcal{V} \rangle$ for $\mathcal{V} = \{D_p\}_{p \in M}$ where D_p is the convex disk which is the interior of the image of a ball in T_pM under the exponential map, with a radius that is strictly smaller than the radius of injectivity. □

Lemma 2.13. *Let $L = S_Z$ and let $U = C^{<1}(L) \subset C^{<1}(Z) \subset C(Z)$. Let $\pi : \mathrm{Link}(C(Z))|_U \to U$ denote the link projection of* $\mathrm{Unzip}(C(Z))$ *over U. Then there is an isomorphism*

$$\partial \overline{\pi^{-1} U} \cong \mathrm{Link}_L(Z)$$

and an induced conically smooth collar $(0,1] \times \partial\overline{\pi^{-1}U} \hookrightarrow \overline{\pi^{-1}U}$ *which is a refinement onto its image.*

Proof. Using Proposition 2.3 we have $\operatorname{Unzip} C(Z) = \operatorname{Unzip}_S C(Z) = \mathbb{R}_{\geq 0} \times \operatorname{Unzip}_L Z$ and so

$$\begin{aligned}\operatorname{Unzip} C(Z)|_U &= \operatorname{Link}_{C(L)} C(Z)|_{C^{<1}(L)} \\ &= \{0\} \times \operatorname{Unzip}_L Z \cup_{\{0\}\times \operatorname{Link}_L Z} [0,1) \times \operatorname{Link}_L Z\end{aligned}$$

since the projection $\pi\colon \operatorname{Link}_{C(L)} C(Z) = \{0\} \times \operatorname{Unzip}_L Z \cup_{\{0\}\times \operatorname{Link}_L Z} \mathbb{R}_{\geq 0} \times \operatorname{Link}_L Z \to C(L) = * \amalg_{\{0\}\times L} \mathbb{R}_{\geq 0} \times L$ is given by mapping all of $\{0\}\times \operatorname{Unzip}_L Z$ to $*$ and on $\mathbb{R}_{\geq 0} \times \operatorname{Link}_L Z$ by $\mathrm{id}_{\mathbb{R}_{\geq 0}} \times \pi'$ where $\pi'\colon \operatorname{Link}_L Z \to L$ is the link projection. Thus

$$\pi^{-1}\overline{U} = \pi^{-1}C^{\leq 1}(U) = \{0\} \times \operatorname{Unzip}_L Z \cup [0,1] \times \operatorname{Link}_L Z$$

and consequently

$$\pi^{-1}\partial\overline{U} = \pi^{-1}\{1\} \times L = \{1\} \times \operatorname{Link}_L Z, \tag{2.14}$$

proving the first claim. The collar is immediate. □

Example 2.15.

Lemma 2.16. *Let* $\mathbb{R}^n$ *be equipped with a riemannian metric, and let* $c \in \{0, \dots, n\}$. *Then for all geodesic disks* $D \subset \mathbb{R}^n$ *about the origin the intersections*

1. $D \cap (\mathbb{R}^{n-c} \times (\mathbb{R}^c \smallsetminus \mathbb{R}^c_{\geq 0}))$
2. $D \cap (\mathbb{R}^{n-c} \times \mathbb{R}^c_{>0})$

are disks.

Proof. Let us suppose $n = c$ for simplicity. For $r > 0$, recall the diffeomorphism $[0,r) \to [0,\infty)$, $x \mapsto \frac{x}{r-x}$. In the same way, $\rho_r\colon B_r \to \mathbb{R}^n$, $v \mapsto \frac{d(0,v)}{r-d(0,v)}v$ is a diffeomorphism from the disk $B_r = \{x \in \mathbb{R}^n : d(0,x) < r\}$ of radius r about the origin to all of $\mathbb{R}^n$, where d is the metric associated with the given riemannian metric. Now we need only note that ρ_r restricts to diffeomorphisms $B_r \cap \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0}$ and $B_r \cap (\mathbb{R}^n \smallsetminus \mathbb{R}^n_{\geq 0}) \to \mathbb{R}^n \smallsetminus \mathbb{R}^n_{\geq 0}$, and that both targets are disks. □

Proposition 2.17. *Let* $L = S_Z$ *and let* $I = J^+\colon \mathbb{R}_{\geq 0} \times \operatorname{Link}_{C(L)} C(Z) \hookrightarrow \operatorname{Unzip}_{C(L)} C(Z)$ *be the collar induced, according to Construction 2.8, by a collar* $J\colon \mathbb{R}_{\geq 0} \times \operatorname{Link}_L(Z) \hookrightarrow \operatorname{Unzip}_L(Z)$. *Let* U *be as in Lemma 2.13, and let*

$$p \in I(\{1\} \times \partial\overline{\pi^{-1}U}).$$

Then:

(1) *There is a diffeomorphism*

$$\mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times \operatorname{Link}_L Z) \cap I([0,1] \times \overline{\pi^{-1}U}) \cong (1,2] \times J((0,1] \times \operatorname{Link}_L Z)$$

of smooth manifolds with corners.

(2) *There exists a convex disk $D \subset \operatorname{Unzip}_{C(L)} C(Z)^\circ$ about p such that $D \cap I([0,1) \times \pi^{-1}U)$ is a disk.*

Proof. From the proof of Lemma 2.13 we recall that $\overline{\pi^{-1}U} = \{0\} \times \operatorname{Unzip}_L Z \cup_{\{0\}\times \operatorname{Link}_L Z} [0,1] \times \operatorname{Link}_L Z$. By Lemma 2.9 we have $I = \alpha_J \circ \phi$, and observe that

$$\begin{aligned}\phi([0,1] \times \overline{\pi^{-1}U}) &= A' \cup B' \\ &:= \bigcup_{t\in[0,1]} (X \subset X^{+t}) \cup \{(t,q) \in X^{+(t+s)} : t,s \in [0,1]\}.\end{aligned}$$

Thus $I([0,1] \times \overline{\pi^{-1}U}) = A \cup B \subset \operatorname{Unzip}_{C(L)} C(Z) = \mathbb{R}_{\geq 0} \times \operatorname{Unzip}_L Z$ where $A = \alpha_J(A')$, $B = \alpha_J(B')$. We have

$$A = \{(r,x) \in \mathbb{R}_{\geq 0} \times \operatorname{Unzip}_L Z : r \in [0,1],\ x \in \operatorname{Unzip}_L Z \smallsetminus J([0,r) \times \operatorname{Link}_L Z\}$$
$$B = \{(t+s, J(t,q)) \in \mathbb{R}_{\geq 0} \times \operatorname{Unzip}_L Z : t,s \in [0,1]\}$$

and hence

$$\mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times \operatorname{Link}_L Z) \cap (A \cup B) = \{(t+s, J(t,q)) : t \in (0,1],\ s \in [0,1],\ t+s > 1\}$$

as the intersection with A is empty.

Consider now the diffeomorphism

$$\begin{aligned}\psi = \rho \circ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \circ \rho^{-1} \colon \mathbb{R}_{>1} \times \mathbb{R}_{>0} &\to \mathbb{R}_{>1} \times \mathbb{R}_{>0}, \\ (x,y) &\mapsto ((x-1)y + 1, y)\end{aligned}$$

where $\rho\colon \mathbb{R}^2 \to \mathbb{R}_{>1} \times \mathbb{R}_{>0}$, $(x,y) \mapsto (e^x + 1, e^y)$. Now ψ fixes $(2,1)$ and satisfies

$$\psi((1,2] \times (0,1]) = \{(t+s, t) : t \in (0,1],\ s \in [0,1],\ t+s > 1\}.$$

Putting

$$\Psi = \mathrm{id}_{\mathbb{R}_{>1}} \times J \circ \psi \times \mathrm{id}_{\operatorname{Link}_L Z} \colon \mathbb{R}_{>1} \times \mathbb{R}_{>0} \times \operatorname{Link}_L Z \to \mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times \operatorname{Link}_L Z)$$

we obtain the map

$$\begin{aligned}(1,2] \times J((0,1] \times \operatorname{Link}_L Z) &\xrightarrow{\cong} (1,2] \times (0,1] \times \operatorname{Link}_L Z \\ &\xrightarrow{\Psi|} \mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times \operatorname{Link}_L Z) \cap I([0,1] \times \overline{\pi^{-1}U}).\end{aligned}$$

The first map, which is a homeomorphism, is tautologically a diffeomorphism of smooth manifolds with corners with respect to the induced smooth structure on its target. With respect to the latter, Ψ is a diffeomorphism as well. Hence the composition is a diffeomorphism. In particular, we obtain the restricted diffeomorphism

$$(1,2) \times J((0,1) \times \operatorname{Link}_L Z) \cong \mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times \operatorname{Link}_L Z) \cap C_U.$$

of smooth manifolds without boundary.

Finally, consider $p = I(1,(1,q)) = (2, J(1,q))$ where $q \in \operatorname{Link}_L Z$ (recall (2.14)). Let $\mathbb{R}^{n-c} \times \mathbb{R}^c_{\geq 0} \cong W \subset \operatorname{Unzip}_L Z$ be a chart neighbourhood of

q. Without loss of generality we may assume $c = n$ for simplicity. Up to diffeomorphism we may write the restriction of the collar of $\mathrm{Link}_L Z$ as $J\colon \mathbb{R}_{\geq 0} \times \partial\mathbb{R}^c_{\geq 0}) \hookrightarrow \mathbb{R}^c_{\geq 0}$, given by the flow along the vector field $\sum_{1\leq i\leq c} \partial_i$ where $\{\partial_i\}$ is the standard basis of $\mathrm{T}_0\mathbb{R}^c_{\geq 0}$ (recall Construction 2.8). We now observe the diffeomorphism

$$J((0,1] \times \partial\mathbb{R}^c_{\geq 0}) = \mathbb{R}^c_{>0} \smallsetminus \left(\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} + \mathbb{R}^c_{\geq 0}\right) \cong \mathbb{R}^c \smallsetminus \mathbb{R}^c_{>0}$$

of smooth manifolds with corners. Consequently we have $(1,2] \times J((0,1] \times \partial\mathbb{R}^c_{\geq 0}) \cong \mathbb{R}^{c+1} \smallsetminus \mathbb{R}^{c+1}_{>0}$. These diffeomorphisms restrict to $J((0,1) \times \partial\mathbb{R}^c_{\geq 0}) \cong \mathbb{R}^c \smallsetminus \mathbb{R}^c_{\geq 0}$ and so $(1,2) \times J((0,1) \times \partial\mathbb{R}^c_{\geq 0}) \cong \mathbb{R}^{c+1} \smallsetminus \mathbb{R}^{c+1}_{\geq 0}$. Hence, the intersection of any convex disk $D \subset \mathbb{R}_{>1} \times J(\mathbb{R}_{>0} \times W)$ centred at $(2, J(1,q)) = p$ with $I([0,1) \times \pi^{-1}U)$ is a disk by the first statement of Lemma 2.16. □

Example 2.18.

Corollary 2.19.

3. Good covers

Definition 3.1. Let X be a CSS. A *good cover* of X is an open cover $\mathcal{U}$ of X consisting of basics whose finite intersections are also in $\mathcal{U}$.

Definition 3.2. Let Y be a smooth manifold with or without boundary, equipped with a riemannian metric. We call a cover of Y *geodesically convex* if it consists of geodesically convex basics.

Theorem 3.3. *Let X be a CSS over a depth-1 poset. Then X has a good cover.*

Proof. Without loss of generality, suppose X is stratified over $[1]$ and let $M = X_0$ and $N = X_1$. Recall the blow-up of M, the smooth manifold

$$\mathrm{Unzip} \cong L \amalg_{L\times(0,\infty)} N$$

with boundary $L = \partial\mathrm{Unzip}$. We write $\pi\colon L \to M$ for the accompanying proper fibre bundle. Let us equip Unzip with a riemannian metric which splits along the boundary.

Using the paracompactness of M, let $\mathcal{U}$ be a locally finite good cover of M which trivialises π. For each $U \in \mathcal{U}$, let $\epsilon_U > 0$ be the radius of injectivity in the normal direction on the compact closure $\overline{U}$ so that there is a smooth embedding

$$I_U\colon \pi^{-1}U \times [0,\epsilon_U) \hookrightarrow \mathrm{Unzip}$$

whose restriction $I_U|_{\pi^{-1}U\times\{0\}}$ to $\pi^{-1}U \subset L$ is given by the boundary inclusion, and the path $I_U(q,-)\colon [0,\epsilon_U) \hookrightarrow \mathrm{Unzip}$, for every $q \in \pi^{-1}U$, is the minimising geodesic given by the normal exponential map. More precisely, for ν the inward-pointing unit normal vector field along the boundary, we set $I_U(x,t) = \exp_x(t\nu_x)$.[1] Let

$$C = \bigcup_{U\in\mathcal{U}} \mathrm{Im}(I_U)$$

be the induced 'collar' and consider the cover

$$\mathcal{C} = \{I_U(\pi^{-1}U \times [0,\delta)) : U \in \mathcal{U},\ \delta \le \epsilon_U\}$$

of C. Let us write $C_{U,\delta} = I_U(\pi^{-1}U \times [0,\delta)) \in \mathcal{C}$. Note that $\mathcal{C}$ is closed under finite intersections since so is $\mathcal{U}$: we have $\pi^{-1}U \cap \pi^{-1}V = \pi^{-1}(U \cap V)$ and

$$C_{U,\delta} \cap C_{V,\delta'} = C_{U\cap V,\min(\delta,\delta')} \in \mathcal{C}$$

since $\min(\delta,\delta') \le \epsilon_{U\cap V}$ for $\delta \le \epsilon_U$, $\delta' \le \epsilon_V$.

Consider now the collection $\mathcal{V}$ of those convex geodesic disks V in N such that $V \cap C_{U,\delta}$ is either empty or a convex geodesic disk for every $C_{U,\delta} \in \mathcal{C}$. Then $\mathcal{V}$ is a good cover of N. To prove this, it suffices to show that every $p \in C$ has a neighbourhood $V_p \in \mathcal{V}$, since convex geodesic disks are closed under finite intersections and so $V \cap V' \cap C_{U,\delta} = V \cap V'' \in \mathcal{V}$. Let now $p \in I_U(\pi^{-1}U \times [0,\delta)) \in \mathcal{C}$, which uniquely determines a point $l_p = \pi(\mathrm{pr}_1(p)) \in \pi^{-1}U \subset L$. Let $W \subset \pi^{-1}U$ be a convex geodesic disk neighbourhood of l_p. Now, since $\mathcal{U}$ is locally finite, so is the cover $\widetilde{\mathcal{C}} = \{I_U(\pi^{-1}U \times [0,\epsilon_U))\}$ of C (which is not necessarily closed under finite intersections), so in particular p is contained within finitely many members $I_{U_i}(\pi^{-1}U_i \times [0,\epsilon_i)) \in \widetilde{\mathcal{C}}$. We necessarily have $p \in I_U(\pi^{-1}U \times [0,\min_i(\epsilon_i)))$ with $U \in \{U_i\}_i$ an open satisfying $\epsilon_U = \min_i(\epsilon_i)$. Then $W \times [0,\min_i(\epsilon_i)) \ni p$ is a convex geodesic half-disk since the riemannian metric on Unzip splits along the boundary, and so we have $p \in V_p = W \times (0,\min_i(\epsilon_i)) \in \mathcal{V}$.

Let us now observe that $\pi^{-1}U \cong L_p \times U$ for $p \in U$ implies that each

$$C(L_p) \times U \cong U \amalg_{\pi^{-1}U} C_{U,\delta} \subseteq X$$

is a basic in X. Thus, writing $\widehat{C_{U,\delta}} = U \amalg_{\pi^{-1}U} C_{U,\delta}$, we obtain that

$$\widehat{\mathcal{C}} = \{\widehat{C_{U,\delta}} : C_{U,\delta} \in \mathcal{C}\}$$

is a cover by basics of the 'tubular neighbourhood'

$$\widehat{C} = \bigcup_{U\in\mathcal{C}} \widehat{C_{U,\epsilon_U}} \subseteq X$$

of M. It is closed intersections since so is $\mathcal{C}$. Finally, since for $V \in \mathcal{V}$ we have $\widehat{C_{U,\delta}} \cap V = C_{U,\delta} \cap V \in \mathcal{V}$, we conclude that

$$\widehat{\mathcal{C}} \cup \mathcal{V}$$

is a good cover of X. □

[1] A global ϵ need not exist unless Unzip is of bounded geometry; see e.g. Schick [Sch01].

4. A proof of...

References

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