Numerical Approximations to the Airy functions

Aleksandar Ivanov

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1 Problem statement

By using a combination of the Taylor series approximation and the asymptotic approximation for the Airy Ai and Bi functions find the most efficient way to calculate their values on the whole real axis with an **absolute** error of less than 10^{-10}. Do the same taking into account **relative** error and see if an error of less than 10^{-10} is achievable.

2 Mathematical preparation

To get the Taylor series around $x = 0$ for both of the Airy functions we need to introduce the two series

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!} \,, \tag{1}$$

$$g(x) = \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} \,. \tag{2}$$

Using these series and the fact that at $x = 0$ the Airy functions have the values

$$\alpha = \mathrm{Ai}(0) = \mathrm{Bi}(0)/\sqrt{3} \approx 0.355028053887817239$$
$$\beta = -\mathrm{Ai}'(0) = \mathrm{Bi}'(0)/\sqrt{3} \approx 0.258819403792806798$$

we get the representations

$$\mathrm{Ai}(x) = \alpha f(x) - \beta g(x) \,, \tag{3}$$

$$\mathrm{Bi}(x) = \sqrt{3}\,\big(\alpha f(x) + \beta g(x)\big) \,. \tag{4}$$

Theoretically, the series converge for all real x (and even complex x!), but practically they are usually useful in a neighborhood around 0.

For absolutely large arguments we can use an asymptotic approximation [2] instead. For that we define the auxiliary asymptotic series

$$L(z) \sim \sum_{s=0}^{\infty} \frac{u_s}{z^s} \,, \tag{5}$$

$$P(z) \sim \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s}}{z^{2s}} \,, \tag{6}$$

$$Q(z) \sim \sum_{s=0}^{\infty} (-1)^s \frac{u_{2s+1}}{z^{2s+1}} \,, \tag{7}$$

where the symbol u_s is defined as

$$u_s = \frac{\Gamma(3s + \frac{1}{2})}{54^s s!\, \Gamma(s + \frac{1}{2})}. \tag{8}$$

If we further define the variable $\xi = \frac{2}{3}|x|^{\frac{3}{2}}$, we get the series

$$\mathrm{Ai}(x) \sim \frac{\mathrm{e}^{-\xi}}{2\sqrt{\pi} x^{1/4}}\, L(-\xi) \,, \tag{9}$$

$$\mathrm{Bi}(x) \sim \frac{\mathrm{e}^{\xi}}{\sqrt{\pi} x^{1/4}}\, L(\xi) \tag{10}$$

for large positive argument, and

$$\mathrm{Ai}(x) \sim \frac{\left[\sin(\xi - \pi/4)\, Q(\xi) + \cos(\xi - \pi/4)\, P(\xi)\right]}{\sqrt{\pi}(-x)^{1/4}} \,, \tag{11}$$

$$\mathrm{Bi}(x) \sim \frac{\left[-\sin(\xi - \pi/4)\, P(\xi) + \cos(\xi - \pi/4)\, Q(\xi)\right]}{\sqrt{\pi}(-x)^{1/4}} \tag{12}$$

for large negative argument.

3 Methods

We will be calculating the functions using the `mpmath` package [1] in Python keeping 20 significant digits. We will then compare to the built-in Airy functions from the package `scipy.special`.

For the Taylor series, we will be summing terms until they become smaller, in absolute value, than the error we're aiming for, namely, 10^{-10}.

For the asymptotic series, we will continue the sum until one of two things happens; either the terms become smaller than the error, as before, or the terms start increasing. We do this since asymptotic series generically diverge, so it may happen that we never reach our desired error.

4 Airy Ai function

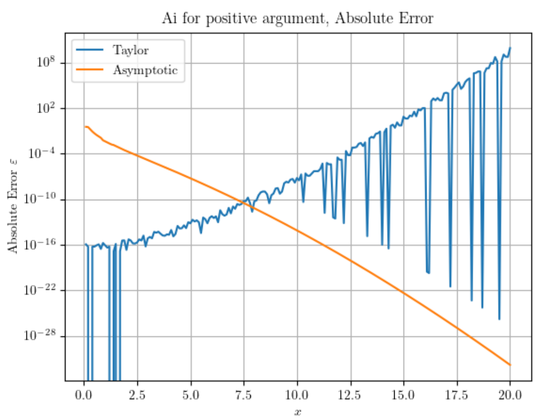

Figure 1: Absolute error for positive x for the Airy Ai function.

The Airy Ai function is bounded everywhere on the real axis, which makes it somewhat easier to handle.

For positive x, figure 1 shows us the absolute error compared to the built-in function for the Taylor series and the asymptotic series. As expected, the asymptotic series is better for larger x while the Taylor series is better for smaller x. We see that the changeover happens at around $x \approx 7.5$.

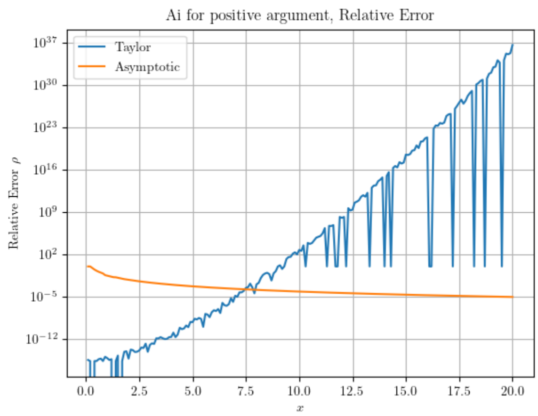

Figure 2: Relative error for positive x for the Airy Ai function.

Although, if we wanted to loosen our 10^{-10} requirement, figure 3, which plots the time of execution, shows us that we could use the asymptotic series for even smaller x and gain on time efficiency.

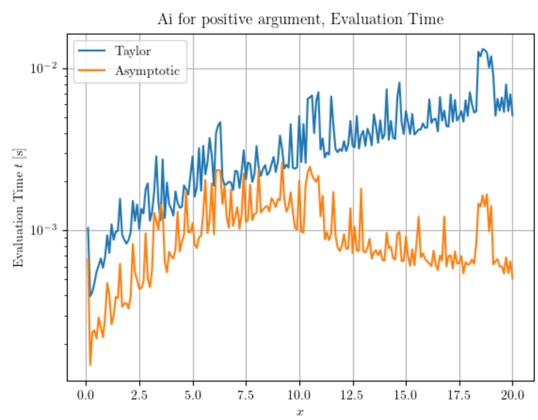

Figure 3: Evaluation time for positive x for the Airy Ai function.

For the relative error, we have the same trend but we can't reach an error of less than 10^{-4} at around the changeover value. (fig. 2)

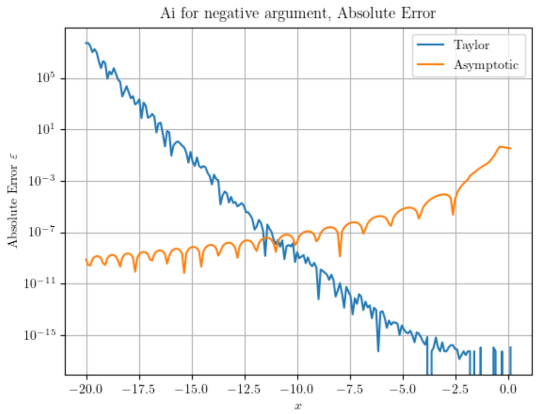

Figure 4: Absolute error for negative x for the Airy Ai function.

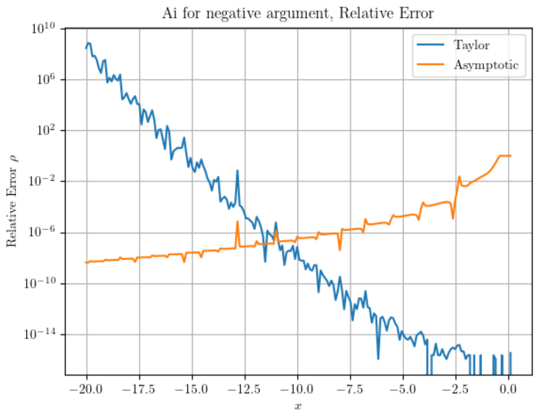

Figure 5: Relative error for negative x for the Airy Ai function.

For negative x the situation is similar, but now the changeover happens at around $x \approx -11.0$, where the absolute error is around 10^{-7} as is the relative one. Again, the asymptotic series is more time efficient over the whole domain we're looking at, but it's not at all accurate around $x = 0$. The corresponding graphs for negative x are figures 4, 5 and 6.

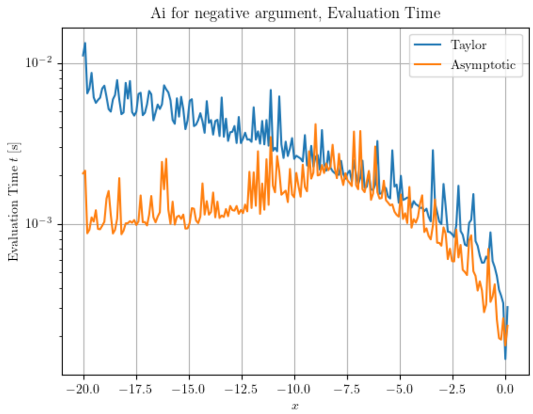

Figure 6: Evaluation time for negative x for the Airy Ai function.

5 Airy Bi function

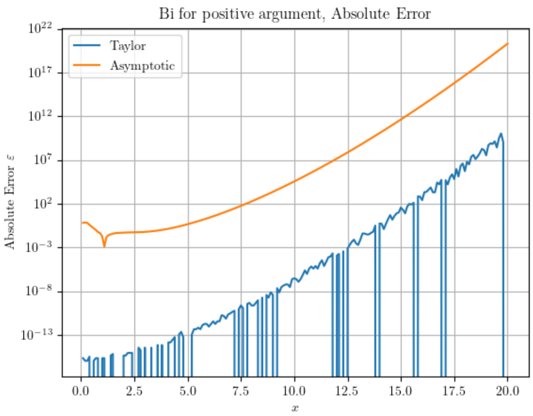

Figure 7: Absolute error for positive x for the Airy Bi function.

The Airy Bi function is more difficult to handle since it's exponentially increasing as x becomes large and positive.

For negative x Bi behaves similarly to Ai so figures 9 and 10, which show us the absolute and relative errors, being very similar to the ones from Ai is not surprising; we have the same changeover at $x \approx -11.0$ and same absolute error of 10^{-4}.

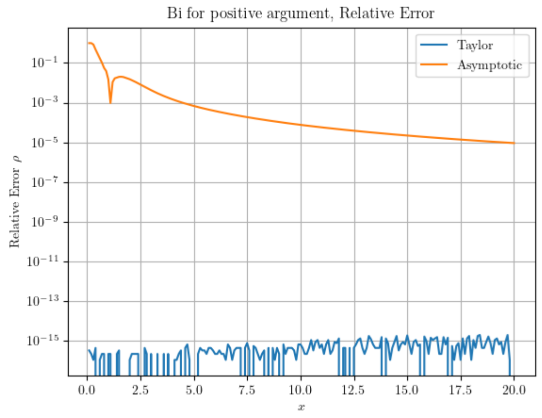

Figure 8: Relative error for positive x for the Airy Bi function.

As with the Ai function the evaluation time of the asymptotic series is almost always less than the Taylor series and getting smaller as we go off to infinity, since we need less terms there.

The positive x absolute and relative errors for Bi are shown in figures 7 and 8, respectively. Since Bi is exponentially growing, the better measure for us is the relative error. The asymptotic series, for values larger than 5.0, gives a respectable error of 10^{-3} and and shrinking as we go off to infinity.

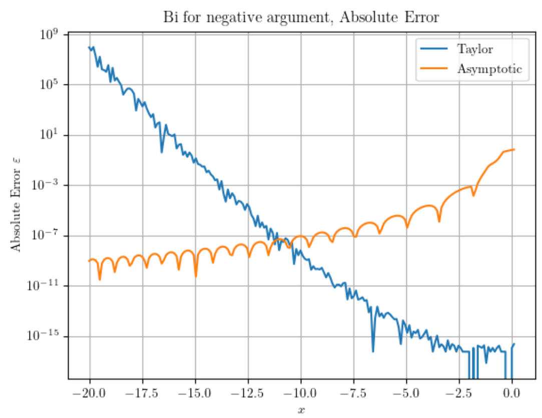

Figure 9: Absolute error for negative x for the Airy Bi function.

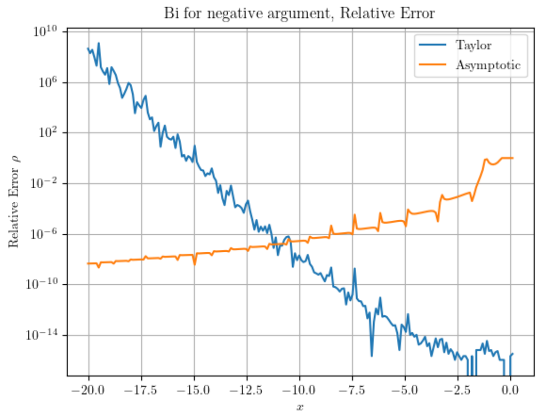

Figure 10: Relative error for negative x for the Airy Bi function.

Surprisingly the Taylor series, unlike in the Ai case, does much better, keeping the relative error at around 10^{-15} for quite a range of x. For this, though, we pay a time price, as the Taylor series needs more and more terms to reach the error bound.

6 Additional question

Find the first 100 zeros $\{a_i\}_{i=1}^{100}$ of the Airy Ai function and the first 100 zeros $\{b_i\}_{i=1}^{100}$ of the Airy Bi function with $x < 0$. Compare your values with the formulas

$$a_s = -\phi\left(\frac{3\pi(4s-1)}{8}\right), \tag{13}$$

$$b_s = -\phi\left(\frac{3\pi(4s-3)}{8}\right), \tag{14}$$

where

$$\phi(z) \sim z^{2/3}\Big(1 + \frac{5}{48}\,z^{-2} - \frac{5}{36}\,z^{-4} + \frac{77125}{82944}\,z^{-6} - \frac{108056875}{6967296}\,z^{-8} + \dots\Big). \tag{15}$$

7 Finding zeros

To find the actual zeros we use the `findroot` method of the package `mpmath` and use the approximate zeros from the formula as an approximation for `findroot`. Plotting the absolute and relative errors as a function of the zeros then gives figures 11 and 12. We can see that the asymptotic formula is very good from around 5.0 onward.

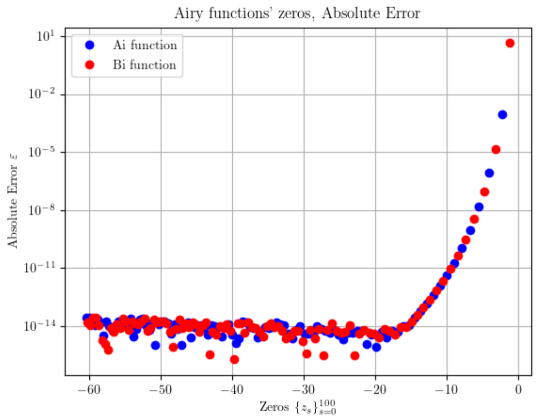

Figure 11: Absolute error for zeros of the Airy functions.

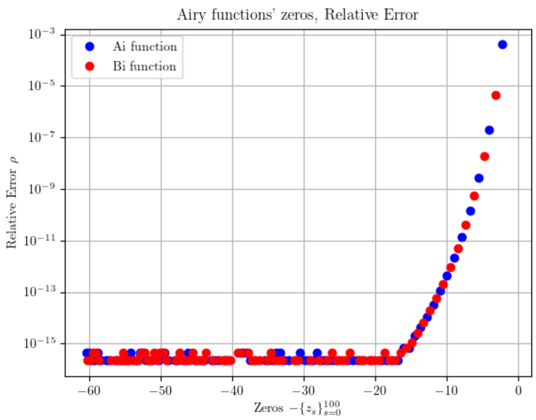

Figure 12: Relative error for zeros of the Airy functions.

The first 40 calculated zeros are given in table 1.

s	a_s	b_s	s	a_s	b_s
1	-2.33811	-1.17371	21	-21.2248	-20.8825
2	-4.08795	-3.27109	22	-21.9014	-21.5644
3	-5.52056	-4.83074	23	-22.5676	-22.2357
4	-6.78671	-6.16985	24	-23.2242	-22.8971
5	-7.94413	-7.37676	25	-23.8716	-23.549
6	-9.02265	-8.49195	26	-24.5103	-24.192
7	-10.0402	-9.53819	27	-25.1408	-24.8266
8	-11.0085	-10.5299	28	-25.7635	-25.4531
9	-11.936	-11.477	29	-26.3788	-26.0721
10	-12.8288	-12.3864	30	-26.987	-26.6838
11	-13.6915	-13.2636	31	-27.5884	-27.2885
12	-14.5278	-14.1128	32	-28.1833	-27.8866
13	-15.3408	-14.9371	33	-28.772	-28.4784
14	-16.1327	-15.7392	34	-29.3548	-29.0641
15	-16.9056	-16.5214	35	-29.9318	-29.644
16	-17.6613	-17.2855	36	-30.5033	-30.2182
17	-18.4011	-18.0331	37	-31.0695	-30.787
18	-19.1264	-18.7655	38	-31.6306	-31.3506
19	-19.8381	-19.4839	39	-32.1867	-31.9092
20	-20.5373	-20.1892	40	-32.7381	-32.463

Table 1: First 40 zeros of the Airy functions.

References

[1] Fredrik Johansson. *mpmath Documentation*. [Last accessed: 13.10.2020]. 2018. URL: `http://mpmath.org/doc/1.1.0/`.

[2] NIST - National Insitute for Standard and Technology. *Asymptotic Expansions for Airy Functions*. [Last accessed: 13.10.2020]. 2020. URL: `https://dlmf.nist.gov/9.7`.