UNIVERSITÄT
LEIPZIG

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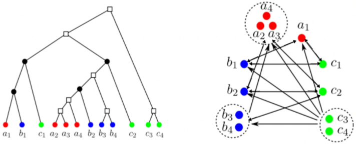
Best match graphs and generalizations

Annachiara Korchmaros
Leipzig Bioinformatics Group

- Gene trees T
  - Rooted trees with leaf coloring $\sigma : L \to S$
  - Leaves set L = mutated copies of a gene
  - Leaves colors S = species



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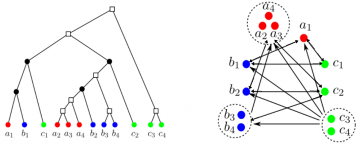
  $\mathrm{lca}(x, y) \preceq \mathrm{lca}(x, z)$ for all $z \in L$ with $\sigma(z) = \sigma(y)$



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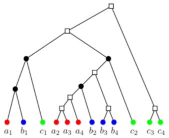
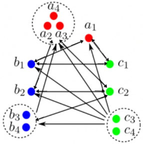
  $V = L$ and $x \to y$ iff y is a best match of x





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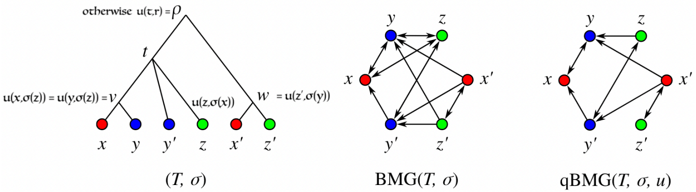
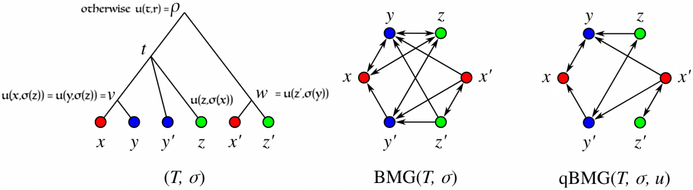


[2] A. Korchmaros *et al.*, *arXiv preprint arXiv:2109.10235* (2021).

Truncation map of T is $u : L \times S \to V$; V (vertex set of T) st

- $u(x, \sigma(x)) = x$
- $u(x, s), s \neq \sigma(x)$ is a node on the path from x to ρ



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$y \in L$ is a **quasi-best match**[2] of $x \in L$ if

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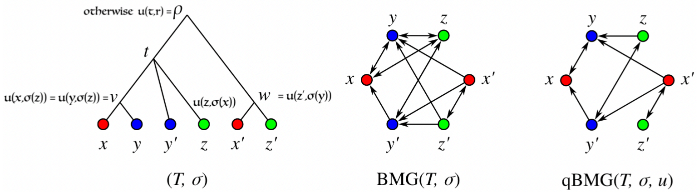

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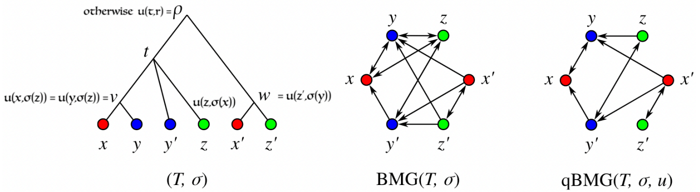

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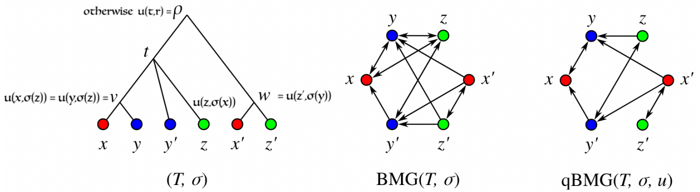
BMGs are qBMGs !! ($u(x, r) = \rho$)



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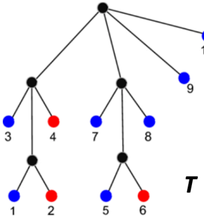
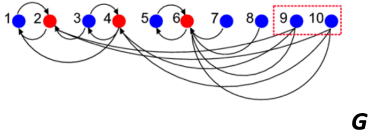
Outline

1. Gene trees
2. 2-BMGs
3. 2-qBMGs

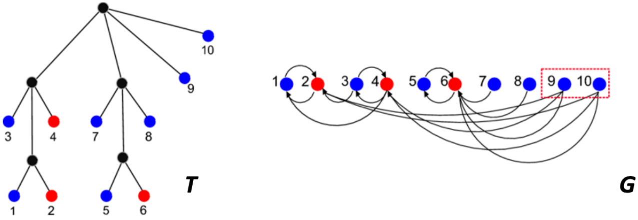
Theorem 1 (Schaller et al. 2021)





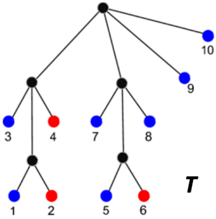
Theorem 1 (Schaller et al. 2021)

1. Let G be a vertex colored digraph. G is a BMG iff its connected components are BMGs on the same color sets.

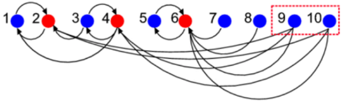


Theorem 1 (Schaller et al. 2021)

1. Let G be a vertex colored digraph. G is a BMG iff its connected components are BMGs on the same color sets.
2. If a connected vertex colored digraph is a BMG, then all its induced subgraphs on 2 colors are BMGs (**2-BMGs**).



T



G

Theorem 2 (Schaller et al. 2021)

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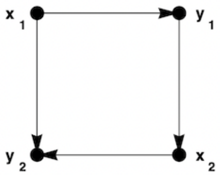
- **N2** G is bitransitive, ie if $x_1y_1, y_1x_2, x_2y_2 \in E \Rightarrow x_1y_2 \in E$.



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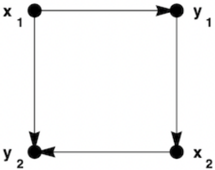
- **N3** If $N(u) \cap N(v) \neq \emptyset \Rightarrow N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.



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$N(x)$ out-neighborhood, $N^-(x)$ in-neighborhood

[3] A. Korchmaros, *Discrete Applied Mathematics* **304**, 397–416 (2021).

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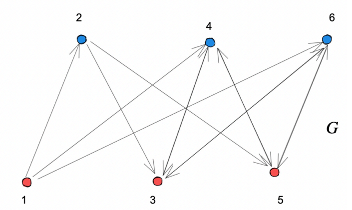
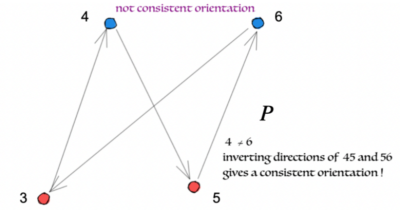
- **Consistent orientation**[3] of G if it preserves $\sim$

Not all orientations are consistent !!





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Theorem 4 (Schaller, Stadler, Hellmuth 2021)

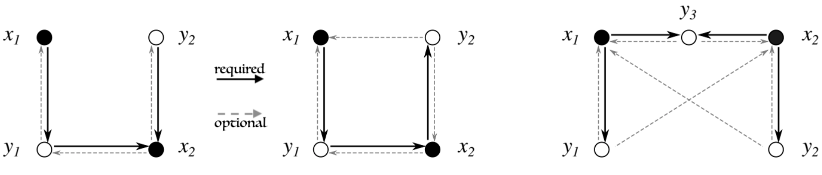
A bipartite (non necessary connected) vertex colored digraph G is 2-BMG iff

[4] D. Schaller *et al.*, *Theoretical Computer Science* **865**, 63–84 (2021).

Theorem 4 (Schaller, Stadler, Hellmuth 2021)

A bipartite (non necessary connected) vertex colored digraph *G* is 2-BMG iff

- *G* is sink-free and
- *G* does not contain an induced F1-, F2-, F3-graphs.



F1-graphs F2-graphs F3-graphs

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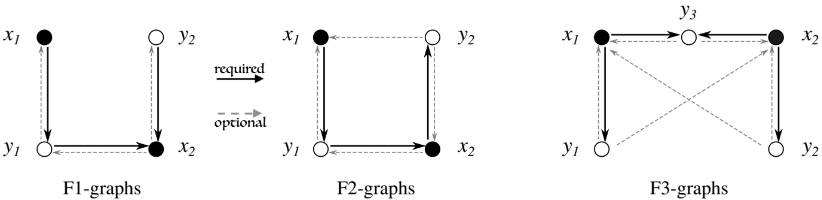

There are 17 non-isomorphic forbidden induced subgraphs[4]

[4] D. Schaller *et al.*, *Theoretical Computer Science* **865**, 63–84 (2021).

- **2-qBMG**[5] is a bipartite qBMG

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Theorem 5 (A.K., Schaller, Stadler, Hellmuth 2021)

A bipartite (connected) vertex-colored digraph *G* is 2-qBMG iff it satisfies N1, N2 and N3.

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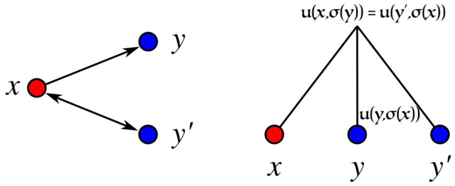

(G, σ) (T, σ, u)

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Proposition 1 (A.K., Schaller, Stadler, Hellmuth 2021)

2-qBMGs form a hereditary digraph class (every induced subgraph of a 2-qBMG is a 2-qBMG).

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2-BMGs do not form an hereditary digraph class
(sink-freeness is not hereditary) !!

Theorem 6 (A.K., Schaller, Stadler, Hellmuth 2021)

Let G be a bipartite (non necessary connected) vertex colored digraph. If G does not contain an induced F1-, F2-, or F3-graph, then G is a 2-qBMG.

Theorem 6 (A.K., Schaller, Stadler, Hellmuth 2021)

Let G be a bipartite (non necessary connected) vertex colored digraph. If G does not contain an induced F1-, F2-, or F3-graph, then G is a 2-qBMG.

Proposition 2 (A.K., Schaller, Stadler, Hellmuth 2021)

Let G be digraph satisfying N1 and N2, then G is bipartite.

Open questions

[6] A. Korchmaros *et al.*, *arXiv preprint arXiv:2109.10235* (2021).

1. Which family of directed graphs G admits an L-coloring σ st (G, σ) is an L-qBMG? ($n = 2$ use N1,N2,N3)

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   $u(x, \sigma(y)) = \rho$ if $xy \in E$; $u(x, \sigma(y)) = x$ if $xy \notin E$.
   (b) What is the qBMG-chromatic number of (G, σ)?
   (minimum number of colors for which G is a qBMG).
   If a L-coloring exists this can be transformed in an $L + 1$-coloring (by arbitrarily splitting a color class)[6].

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Thank you

- Peter Stadler
- David Schaller
- Marc Hellmut

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Is every orientation of G acyclic? No, P has a 4-cycle.