

Model Documentation of the Heat Equation

1 Nomenclature

1.1 Nomenclature for Model Equations

t	time
z	space
α	thermal diffusivity
$u(z, t)$	input trajectory
$x(z, t)$	wanted function describing spacial and temporal development of the temperature in matter

2 Model Equations

System Equations:

$$\begin{aligned} \dot{x}(z, t) &= \alpha x''(z, t) & z \in (0, l), t > 0 \\ x(z, 0) &= x_0(z) & z \in [0, l] \\ x(0, t) &= 0 & t > 0 \\ x(l, t) &= u(t) & t > 0 \end{aligned}$$

Parameters: α, l ,

2.1 Assumptions

1. $x_0(z) = 0$

2.2 Exemplary parameter values

Parameter Name	Symbol	Value
number of approximation intervalls	n_{fem}	17
spatial bounds	l	1
temporal bounds	T	1
start of trajectory	y_0	-1
end of trajectory	y_1	4

3 Derivation and Explanation

Approach:

- initial functions $\varphi_1(z), \dots, \varphi_{n+1}(z)$
- test functions $\varphi_1(z), \dots, \varphi_n(z)$
- where the functions $\varphi_1(z), \dots, \varphi_n(z)$ met the homogeneous b.c.
 $\varphi_1(l), \dots, \varphi_n(l) = \varphi_1(0), \dots, \varphi_n(0) = 0$

- only φ_{n+1} can draw the actuation

Approximating the wanted function with

$$x(z, t) = \sum_{i=1}^{n+1} x_i^*(t) \varphi_i(z) \Big|_{x_{n+1}^* = u} = \underbrace{\sum_{i=1}^n x_i^*(t) \varphi_i(z)}_{\hat{x}(z, t)} + \varphi_{n+1}(z) u(t).$$

The weak formulation is given by

$$\begin{aligned} \langle \dot{x}(z, t), \varphi_j(z) \rangle &= a_2 \langle x''(z, t), \varphi_j(z) \rangle \\ &+ a_1 \langle x'(z, t), \varphi_j(z) \rangle + a_0 \langle x(z, t), \varphi_j(z) \rangle \quad j = 1, \dots, n. \end{aligned}$$

Shift of derivation to work with lagrange 1st order initial functions

$$\begin{aligned} \langle \dot{x}(z, t), \varphi_j(z) \rangle &= \overbrace{[a_2 [x'(z, t) \varphi_j(z)]_0^t]_0^t}^{=0} - a_2 \langle x'(z, t), \varphi_j'(z) \rangle \\ &+ a_1 \langle x'(z, t), \varphi_j(z) \rangle + a_0 \langle x(z, t), \varphi_j(z) \rangle \quad j = 1, \dots, n \\ \langle \dot{\hat{x}}(z, t), \varphi_j(z) \rangle + \langle \varphi_{N+1}(z), \varphi_j(z) \rangle \dot{u}(t) &= -a_2 \langle \hat{x}'(z, t), \varphi_j'(z) \rangle - a_2 \langle \varphi_{N+1}'(z), \varphi_j'(z) \rangle u(t) \\ &+ a_1 \langle \hat{x}'(z, t), \varphi_j(z) \rangle + a_1 \langle \varphi_{N+1}'(z), \varphi_j(z) \rangle u(t) + \\ &+ a_0 \langle \hat{x}(z, t), \varphi_j(z) \rangle + a_0 \langle \varphi_{N+1}(z), \varphi_j(z) \rangle u(t) \quad j = 1, \dots, n \end{aligned}$$

leads to state space model for the weights $\mathbf{x}^* = (x_1^*, \dots, x_n^*)^T$

$$\dot{\mathbf{x}}^*(t) = A\mathbf{x}^*(t) + \mathbf{b}_0 u(t) + \mathbf{b}_1 \dot{u}(t).$$

The input derivative can be eliminated through the transformation

$$\bar{\mathbf{x}}^* = \tilde{A}\mathbf{x}^* - \mathbf{b}_1 u$$

with e.g.: $\tilde{A} = I$, and leads to the state space model

$$\begin{aligned} \dot{\bar{\mathbf{x}}}^*(t) &= \tilde{A} \tilde{A}^{-1} \bar{\mathbf{x}}^*(t) + \tilde{A} (A\mathbf{b}_1 + \mathbf{b}_0) u(t) \\ &= \tilde{A} \bar{\mathbf{x}}^*(t) + \tilde{\mathbf{b}} u(t). \end{aligned}$$

References

- [1] Stefan Eckerle, Marcus Riesmeier:
https://pyinduct.readthedocs.io/en/master/examples/rad_dirichlet_fem.html