Model Documentation of the Heat Equation

1 Nomenclature

1.1 Nomenclature for Model Equations

t	time
z	space
α	thermal diffusivity
$u(z,t)$	input trajectory
$x(z,t)$	wanted function describing spacial and temporal development of the temperature in matter

2 Model Equations

System Equations:

$$\begin{aligned}
\dot{x}(z,t) &= \alpha x''(z,t) && z \in (0,l), t > 0 \\
x(z,0) &= x_0(z) && z \in [0,l] \\
x(0,t) &= 0 && t > 0 \\
x(l,t) &= u(t) && t > 0
\end{aligned}$$

Parameters: $\alpha, l,$

2.1 Assumptions

1. $x_0(z) = 0$

2.2 Exemplary parameter values

Parameter Name	Symbol	Value
number of approximation intervalls	n_{fem}	17
spatial bounds	l	1
temporal bounds	T	1
start of trajectory	y_0	-1
end of trajectory	y_1	4

3 Derivation and Explanation

Approach:

- inital functions $\varphi_1(z), ..., \varphi_{n+1}(z)$
- test functions $\varphi_1(z), ..., \varphi_n(z)$
- where the functions $\varphi_1(z), .., \varphi_n(z)$ met the homogeneous b.c.
  $\varphi_1(l), .., \varphi_n(l) = \varphi_1(0), .., \varphi_n(0) = 0$

- only φ_{n+1} can draw the actuation

Approximating the wanted function with

$$x(z,t) = \sum_{i=1}^{n+1} x_i^*(t)\varphi_i(z)\Big|_{x_{n+1}^*=u} = \underbrace{\sum_{i=1}^{n} x_i^*(t)\varphi_i(z)}_{\hat{x}(z,t)} + \varphi_{n+1}(z)u(t).$$

The weak formulation is given by

$$\begin{aligned}\langle \dot{x}(z,t), \varphi_j(z)\rangle &= a_2\langle x''(z,t), \varphi_j(z)\rangle \\ &\quad + a_1\langle x'(z,t), \varphi_j(z)\rangle + a_0\langle x(z,t), \varphi_j(z)\rangle \qquad j = 1, ..., n.\end{aligned}$$

Shift of derivation to work with lagrange 1st order initial functions

$$\begin{aligned}\langle \dot{x}(z,t), \varphi_j(z)\rangle &= \overbrace{[a_2[x'(z,t)\varphi_j(z)]_0^l}^{=0} - a_2\langle x'(z,t), \varphi_j'(z)\rangle \\ &\quad + a_1\langle x'(z,t), \varphi_j(z)\rangle + a_0\langle x(z,t), \varphi_j(z)\rangle \qquad j = 1, ..., n \\ \langle \dot{\hat{x}}(z,t), \varphi_j(z)\rangle + \langle \varphi_{N+1}(z), \varphi_j(z)\rangle \dot{u}(t) &= -a_2\langle \hat{x}'(z,t), \varphi_j'(z)\rangle - a_2\langle \varphi_{N+1}'(z), \varphi_j'(z)\rangle u(t) \\ &\quad + a_1\langle \hat{x}'(z,t), \varphi_j(z)\rangle + a_1\langle \varphi_{N+1}'(z), \varphi_j(z)\rangle u(t) + \\ &\quad + a_0\langle \hat{x}(z,t), \varphi_j(z)\rangle + a_0\langle \varphi_{N+1}(z), \varphi_j(z)\rangle u(t) \qquad j = 1, ..., n\end{aligned}$$

leads to state space model for the weights $\boldsymbol{x}^* = (x_1^*, ..., x_n^*)^T$

$$\dot{\boldsymbol{x}}^*(t) = A\boldsymbol{x}^*(t) + \boldsymbol{b}_0 u(t) + \boldsymbol{b}_1 \dot{u}(t).$$

The input derivative can be eliminated through the transformation

$$\bar{\boldsymbol{x}}^* = \tilde{A}\boldsymbol{x}^* - \boldsymbol{b}_1 u$$

with e.g.: $\tilde{A} = I$, and leads to the state space model

$$\begin{aligned}\dot{\bar{\boldsymbol{x}}}^*(t) &= \tilde{A}A\tilde{A}^{-1}\bar{\boldsymbol{x}}^*(t) + \tilde{A}(A\boldsymbol{b}_1 + \boldsymbol{b}_0)u(t) \\ &= \bar{A}\bar{\boldsymbol{x}}^*(t) + \bar{\boldsymbol{b}}u(t).\end{aligned}$$

References

[1] Stefan Ecklebe, Marcus Riesmeier:
https://pyinduct.readthedocs.io/en/master/examples/rad_dirichlet_fem.html