

# Model Documentation of the Heat Equation

## 1 Nomenclature

### 1.1 Nomenclature for Model Equations

$t$	time
$z$	space
$\alpha$	thermal diffusivity
$u(z, t)$	input trajectory
$x(z, t)$	wanted function describing spacial and temporal development of the temperature

## 2 Model Equations

System Equations:

$$\begin{aligned}\dot{x}(z, t) &= \alpha x''(z, t) & z \in (0, l), t > 0 \\ x(z, 0) &= x_0(z) & z \in [0, l] \\ x(0, t) &= 0 & t > 0 \\ x(l, t) &= u(t) & t > 0\end{aligned}$$

Parameters:  $\alpha$

### 2.1 Assumptions

1.  $x_0(z) = 0$

### 2.2 Exemplary parameter values

Parameter Name	Symbol	Value
thermal diffusivity	$\alpha$	1

## 3 Derivation and Explanation

Approach [1]:

- initial functions  $\varphi_1(z), \dots, \varphi_{n+1}(z)$
- test functions  $\varphi_1(z), \dots, \varphi_n(z)$
- where the functions  $\varphi_1(z), \dots, \varphi_n(z)$  met the homogeneous boundary conditions  
 $\varphi_1(l), \dots, \varphi_n(l) = \varphi_1(0), \dots, \varphi_n(0) = 0$
- only  $\varphi_{n+1}$  can draw the actuation

Approximating the wanted function with

$$x(z, t) = \sum_{i=1}^{n+1} x_i^*(t) \varphi_i(z) \Big|_{x_{n+1}^*=u} = \underbrace{\sum_{i=1}^n x_i^*(t) \varphi_i(z)}_{\hat{x}(z, t)} + \varphi_{n+1}(z) u(t).$$

The weak formulation is given by

$$\begin{aligned} \langle \dot{x}(z, t), \varphi_j(z) \rangle &= a_2 \langle x''(z, t), \varphi_j(z) \rangle \\ &+ a_1 \langle x'(z, t), \varphi_j(z) \rangle + a_0 \langle x(z, t), \varphi_j(z) \rangle \quad j = 1, \dots, n. \end{aligned}$$

Shift of derivation to work with lagrange 1st order initial functions

$$\begin{aligned} \langle \dot{x}(z, t), \varphi_j(z) \rangle &= \overbrace{[a_2 [x'(z, t) \varphi_j(z)]_0]_0}^{=0} - a_2 \langle x'(z, t), \varphi_j'(z) \rangle \\ &+ a_1 \langle x'(z, t), \varphi_j(z) \rangle + a_0 \langle x(z, t), \varphi_j(z) \rangle \quad j = 1, \dots, n \\ \langle \dot{\hat{x}}(z, t), \varphi_j(z) \rangle + \langle \varphi_{N+1}(z), \varphi_j(z) \rangle \dot{u}(t) &= -a_2 \langle \hat{x}'(z, t), \varphi_j'(z) \rangle - a_2 \langle \varphi_{N+1}'(z), \varphi_j'(z) \rangle u(t) \\ &+ a_1 \langle \hat{x}'(z, t), \varphi_j(z) \rangle + a_1 \langle \varphi_{N+1}'(z), \varphi_j(z) \rangle u(t) + \\ &+ a_0 \langle \hat{x}(z, t), \varphi_j(z) \rangle + a_0 \langle \varphi_{N+1}(z), \varphi_j(z) \rangle u(t) \quad j = 1, \dots, n \end{aligned}$$

leads to state space model for the weights  $\mathbf{x}^* = (x_1^*, \dots, x_n^*)^T$

$$\dot{\mathbf{x}}^*(t) = A\mathbf{x}^*(t) + \mathbf{b}_0 u(t) + \mathbf{b}_1 \dot{u}(t).$$

The input derivative can be eliminated through the transformation

$$\bar{\mathbf{x}}^* = \tilde{A}\mathbf{x}^* - \mathbf{b}_1 u$$

with e.g.:  $\tilde{A} = I$ , and leads to the state space model

$$\begin{aligned} \dot{\bar{\mathbf{x}}}^*(t) &= \tilde{A}A\tilde{A}^{-1}\bar{\mathbf{x}}^*(t) + \tilde{A}(A\mathbf{b}_1 + \mathbf{b}_0)u(t) \\ &= \tilde{A}\bar{\mathbf{x}}^*(t) + \bar{\mathbf{b}}u(t). \end{aligned}$$

## References

- [1] Stefan Ecklebe, Marcus Riesmeier:  
[https://pyinduct.readthedocs.io/en/master/examples/rad\\_dirichlet\\_fem.html](https://pyinduct.readthedocs.io/en/master/examples/rad_dirichlet_fem.html)