Bachelor of Science

Semester :- 5

Group theory

Group Theory

○ Groupoid or Binary Algebra Definition

- A non-empty set G equipped with one binary operation is called groupoid

  - i.e. G is a Groupoid is closed for * .

It is denated by (G,*).

For Example :- (N,+), (Z, -), (Q, x) etc.

Note :- Groupoid is also called Quasi Group.

○ Semi Group :- Definition

- An Algebraic Structure (G, *) is called a semi Group if the binary operation * satisfy associative Property

i.e. :- [G] $(a * b) * c = a * (b * c), \forall a \in G$

Ex.1

The algebraic structures (N,+), (Z,+), (Z, x), (Q, x) are semi groups but the structure (Z, -) is not so because subtraction, (-) is associative.

Ex. 2

The structures (p(s), ∪) and (P(s), ∩) where P(s) is the power set of a set S are Semi Groups as both the Operations Union (U) an intersection (∩) are associative.

Monoid : Definition

- A Semi Group is called monoid if there exist an identity element ‘e’ in G such that

  [G2] $e * a = a * e = a, \forall\ a \in G$

Ex.1

The Semi Group (N, x) is Monoid because 1 is the identity for the multiplication. But the Semi Group (N, +) is not because 0 is the identity for addition is not in N.

Ex. 2

The Semi Group (P(s), ∪) and (P(s), ∩) are monoid because Φ and S are the identity respectively for union (∪) and (∩) in P(s).

Group : Definition

- An algebraic structure of set G and a binary operation * defined in G i.e. (G, *) is called a group if * satisfies the following postulate :

[G1] Closure :

$$a \in G, b \in G \Rightarrow a * b \in G,\ \forall a, b \in G$$

[G2] Associativity :

- The composition * is associative in G
  i.e. $(a * b) * c = a * (b * c),\ \forall a, b, c \in G$

[G3] Existence of Identity :

- There exist an identity element e in G such that

$$e * a = a * e = a,\ \forall a \in G$$

[G4] Existence of Inverse :

- Each element of G is invertible, i.e., for every $a \in G$, there exist a^{-1}in G such that
  $$a * a^{-1} = a^{-1} * a = \mathrm{e} \text{ (Identity)}$$
- Thus group (G, *) is a monoid in which each of its element is ...

Abelian group or Commutative group : Definition

- A group $(G,*)$ is said to be abelian or commutative if * is commutative also. A group $(G,*)$ is an abelian group, if

[G4] Commutativity :

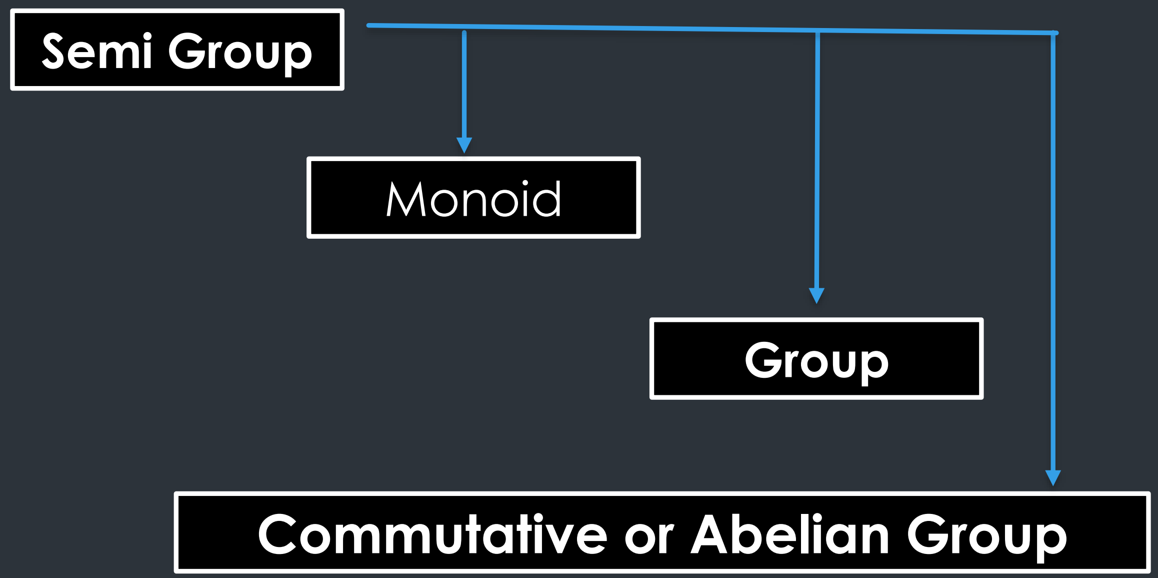
$$a * b = b * a, \ \forall a, b \in G$$

G1. Associativity

G2. Existence of Identity

G3. Existence of Inverse

G4. Commutativity



Finite and Infinite groups :

- A group (G, *) is said to be finite if its underlying set G is finite set and a group which is not finite is called an infinite group.

Order of a Group : Definition

- The number of elements in a finite group is called the order of the group.
- It is denoted by O(G).
- If (G,*) is in infinite group, then it is said to be of infinite order.

Example – 1

Show that the set of integers forms an abelian group under addition.

Solution :

Let $G = \{0 + 1 + 2 + 3 \ldots\}$

$\boldsymbol{G_1}$ **:Closure :**

let $a, b \in G \Rightarrow a + b \in G$

Closure law is satisfied.

$\boldsymbol{G_2}$ **:Associative** :

let $a, b, c \in G$,

$$a + (b + c) = (a + b) + c$$

Associative law is satisfied.

$\boldsymbol{G_3}$ **:Identity** : let $a \in G$ and 0 be the identity

$$a + 0 = 0 + a = a$$

G4: Inverse :

$-a$ is the inverse of a

$$a + (-a) = -a + a = 0$$

$\boldsymbol{G_5}$:Commutative :

let $a, b \in G$

$$a + b = b + a$$

$(G, +)$ is an abelian group under addition.

Example – 2

G is the set of rationals except -1. Binary operation $*$ is defined by $\boldsymbol{a * b = a + b + ab}$. Show that it is a group.

Solution :

G_1 :Let $a, b \in G \Rightarrow a * b = a + b + ab \in G$

Closure is satisfied.

G_2 :Let $a, b, c \in G$

$$a * (b * c) = (a * b) * c$$

L.H.S.: $b * c = b + c + bc = x$

$$a * x = a + x + ax$$

$$= a + b + c + bc + a(b + c + bc)$$

$$= a + b + c + ab + bc + ca + abc$$