INSTITUT DE MATHÉMATIQUES

Integral Cohomology of Finite Postnikov Towers

Thèse de doctorat

présentée à la

Faculté des Sciences de
l'Université de Lausanne

par

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LAUSANNE
2002

La dernière démarche de la raison est de reconnaître qu'il y a une infinité de choses qui la surpassent.

Pensées, 267 – *Blaise Pascal*
(1623-1662)

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Abstract

By the work of H. Cartan, it is well known that one can find elements of arbitrarilly high torsion in the integral (co)homology groups of an Eilenberg-MacLane space $K(G,n)$, where G is a non-trivial abelian group and $n \geqslant 2$.

The main goal of this work is to extend this result to H-spaces having more than one non-trivial homotopy group.

In order to have an accurate hold on H. Cartan's result, we start by studying the duality between homology and cohomology of 2-local Eilenberg-MacLane spaces of finite type. This leads us to some improvements of H. Cartan's methods in this particular case.

Our main result can be stated as follows. Let X be an H-space with two non-trivial finite 2-torsion homotopy groups. Then X does not admit any exponent for its reduced integral graded (co)homology group.

We construct a wide class of examples for which this result is a simple consequence of a topological feature, namely the existence of a weak retract $X \to K(G,n)$ for some abelian group G and $n \geqslant 2$.

We also generalize our main result to more complicated stable 2-stage Postnikov systems, using the Eilenberg-Moore spectral sequence and analytic methods involving Betti numbers and their asymptotic behaviour.

Finally, we investigate some guesses on the non-existence of homology exponents for finite Postnikov towers. We conjecture that Postnikov pieces do not admit any (co)homology exponent.

This work also includes the presentation of the "Eilenberg-MacLane machine", a C++ program designed to compute explicitely all integral homology groups of Eilenberg-MacLane spaces.

Résumé

Depuis le séminaire H. Cartan de 1954-55, il est bien connu que l'on peut trouver des éléments de torsion arbitrairement grande dans l'homologie entière des espaces d'Eilenberg-MacLane $K(G,n)$ où G est un groupe abélien non trivial et $n \geqslant 2$.

L'objectif majeur de ce travail est d'étendre ce résultat à des H-espaces possédant plus d'un groupe d'homotopie non trivial.

Dans le but de contrôler précisément le résultat de H. Cartan, on commence par étudier la dualité entre l'homologie et la cohomologie des espaces d'Eilenberg-MacLane 2-locaux de type fini. On parvient ainsi à raffiner quelques résultats qui découlent des calculs de H. Cartan.

Le résultat principal de ce travail peut être formulé comme suit. Soit X un H-espace ne possédant que deux groupes d'homotopie non triviaux, tous deux finis et de 2-torsion. Alors X n'admet pas d'exposant pour son groupe gradué d'homologie entière réduite.

On construit une large classe d'espaces pour laquelle ce résultat n'est qu'une conséquence d'une caractéristique topologique, à savoir l'existence d'un rétract faible $X \to K(G,n)$ pour un certain groupe abélien G et $n \geqslant 2$.

On généralise également notre résultat principal à des espaces plus compliqués en utilisant la suite spectrale d'Eilenberg-Moore ainsi que des méthodes analytiques faisant apparaître les nombres de Betti et leur comportement asymptotique.

Finalement, on conjecture que les espaces qui ne possèdent qu'un nombre fini de groupes d'homotopie non triviaux n'admettent pas d'exposant homologique.

Ce travail contient par ailleurs la présentation de la"machine d'Eilenberg-MacLane", un programme C++ conçu pour calculer explicitement les groupes d'homologie entière des espaces d'Eilenberg-MacLane.

Remerciements

Je remercie cordialement M. le Prof. Dominique Arlettaz d'avoir accepté de diriger ce travail de thèse. Son aide constante et inconditionnelle, ses connaissances, ses qualités humaines et son amitié m'ont soutenu durant toute les phases de ma recherche. Je le remercie de m'avoir donné la possibilité de voyager, de rencontrer de nombreux mathématiciens et d'avoir ainsi réuni les meilleures conditions pour l'élaboration de ce travail. Mes remerciements vont également à Mme Chantal Arlettaz, ainsi qu'à Mathieu et Vincent, pour leur hospitalité et leur gentillesse.

Je remercie M. le Prof. Carles Casacuberta ainsi que M. le Prof. Jérôme Scherer d'avoir si spontanément accepté d'expertiser ce travail. Il n'aurait jamais abouti sans leurs nombreux conseils, corrections et commentaires qui furent tant d'encouragements. Je tiens finalement à remercier M. le Prof. Frederick R. Cohen ainsi que M. le Prof. Wojciech Chachólski pour toutes les discussions que nous avons eues et pour leurs excellents conseils.

Je remercie chaleureusement M. le Prof. Bernard Testa d'avoir présidé le Jury.

Ce travail a été partiellement financé par le Fonds National pour la Recherche Scientifique (FNRS).

CHAPTER 1

Introduction

Let X be a connected space. One can both consider its graded homotopy group, $\pi_*(X)$, and its graded reduced integral homology group, $\widetilde{H}_*(X;\mathbb{Z})$. If there exists an integer $h \geqslant 1$ such that $h \cdot \pi_*(X) = 0$, we then say that X has a *homotopy exponent*. Analogously, if there exists an integer $e \geqslant 1$ such that $e \cdot \widetilde{H}_*(X;\mathbb{Z}) = 0$, we then say that X has a *homology exponent*.

The general problem posed by D. Arlettaz is to know whether there is a relationship between homotopy exponents and homology exponents. For instance, is it true that a space with a homotopy exponent has a homology exponent too? In this case, how are these two exponents related, if they are? Or conversely, is it possible for a space without a homotopy exponent to admit a homology exponent?

In the present work, we focus on spaces with homotopy exponents and ask if they have a homology exponent.

The first example of such a space which come to mind is an Eilenberg-MacLane space $K(G,n)$, with G a finite group, abelian if $n \geqslant 2$.

It is very well known that if G is a finite group, then $K(G,1)$ has a homology exponent. Actually we can prove, by using a transfer argument, that $\mathrm{Card}(G) \cdot \widetilde{H}^*(K(G,1);\mathbb{Z}) = 0$.

By H. Cartan's work [**15**], it is well known that $K(G,n)$ has no homology exponent if $n \geqslant 2$. This result gives a drastic answer to D. Arlettaz's question: the existence of a homotopy exponent does not imply the existence of a homology exponent in general. But is it always the case for 1-connected spaces?

For instance, and as an obvious consequence of the Künneth formula, products of Eilenberg-MacLane spaces, which are called *generalized Eilenberg-MacLane spaces* (GEM), neither have a homology exponent.

The purpose of this work is then to investigate the integral (co)homology of spaces which have a finite Postnikov tower, i.e. spaces with finitely many non-trivial homotopy groups (these spaces are also called *Postnikov pieces*), and which are not GEM's.

Our main results (see Section 1.3) deal with spaces with two non-trivial homotopy groups and with stable 2-stage Postnikov systems (see p. 4 for a definition). These spaces turn out not to have a homology exponent, although they have a homotopy exponent.

In order to work in an affordable and manageable technical framework, we will only compute at the prime 2. For simplicity, our results then focus on 2-local Postnikov pieces of finite type, and most of the time with an H-space structure.

1.1. Basic definitions and notations

Let $X = \{X_*\}$ be a graded object. The m-th **shift** is the graded object $\Sigma^m X$ given by $(\Sigma^m X)_n = X_{n+m}$. We also set

$$X^{\text{ even}} \text{ or } X^+ = \{\text{even degree elements}\} \text{ and}$$
$$X^{\text{ odd}} \text{ or } X^- = \{\text{odd degree elements}\}.$$

Every object can be seen as a graded one, concentrated in degree zero.

Unless otherwise specified, a **space** will mean a pointed, connected and simple topological space with the homotopy type of a CW-complex of finite type.

We will denote by $K(G,n)$ the **Eilenberg-MacLane space** with single non-trivial homotopy group isomorphic to G in dimension n (G abelian if $n \geqslant 2$). If $U = \bigoplus_{i\geqslant 1} U_i$ is a graded group, then we will denote by KU the generalized Eilenberg-MacLane space (GEM) given by the (weak) product $\prod_{i\geqslant 1} K(U_i, i)$. For instance, we have $K\Sigma^m\mathbb{F}_2 = K(\mathbb{Z}/2, m)$, since $\mathbb{F}_2$ can be seen as a graded object concentrated in degree zero.

Since we will only consider simple spaces, we will only deal with abelian fundamental groups and it will always be possible to consider the **Postnikov tower** and the **k-invariants** of a space. Let us recall that the Postnikov tower of a space X looks like:

$$\begin{tikzcd}
 & \vdots \arrow[d, dotted, no head] & \\
 & X[n+1] \arrow[d, "\gamma_n"] & K(\pi_{n+1}X, n+1) \arrow[l] \\
X \arrow[ur, "\alpha_{n+1}"] \arrow[r, "\alpha_n"] \arrow[dr, "\alpha_{n-1}"] \arrow[ddr, "\alpha_1"'] & X[n] \arrow[d, "\gamma_{n-1}"] & K(\pi_n X, n) \arrow[l] \\
 & X[n-1] \arrow[d, dotted, no head] & K(\pi_{n-1}X, n-1) \arrow[l] \\
 & X[1] & K(\pi_1 X, 1), \arrow[l, "\simeq"']
\end{tikzcd}$$

where $\alpha_n : X \to X[n]$ is the n-th **Postnikov section** and $X[n]$ is given by the homotopy pullback along the $(n+1)$-th k-invariant $k^{n+1}(X) \in H^{n+1}(X[n-1]; \pi_n X) \cong [X[n-1], K(\pi_n X, n+1)]$ and the path-loop fibration over $K(\pi_n X, n+1)$:

$$\begin{tikzcd}
X[n] \arrow[r] \arrow[d, "\gamma_{n-1}"'] & PK(\pi_n X, n+1) \simeq * \arrow[d] \\
X[n-1] \arrow[r, "k^{n+1}(X)"'] & K(\pi_n X, n+1).
\end{tikzcd}$$

We will say that a space X is a **Postnikov piece** if its Postnikov tower is finite, i.e. if $X \simeq X[m]$ for some positive integer m or, equivalently, if it has finitely many non-trivial homotopy groups.

A 2-**stage Postnikov system** is a Postnikov piece given by the following homotopy pullback along the path-loop fibration:

$$\begin{tikzcd} KV \arrow[r, equal] \arrow[d] & KV \arrow[d] \\ X \arrow[r] \arrow[d] & PK\Sigma V \simeq * \arrow[d] \\ KU \arrow[r, "k"'] & K\Sigma V, \end{tikzcd}$$

where U, V are two graded abelian groups and KU, KV their associated GEM. The system is called **stable** if the map $k : KU \to K\Sigma V$ is an H-map. In this case, the space X is an H-space.

A space X will be said to have a **homology exponent** if there exists an integer $e \geqslant 1$ such that $e \cdot \widetilde{H}^*(X;\mathbb{Z}) = 0$.

Let G be a non-trivial finitely generated 2-torsion abelian group, i.e. a finite 2-torsion abelian group. We will say that G is of **rank** $\mathrm{rank}(G) = l$ and of **type** $(s_1, \dots, s_l)$ with $s_1 \geqslant \cdots \geqslant s_l$ if its decomposition (unique up to permutation) into a direct sum of 2-primary cyclic groups is given by

$$G \cong \bigoplus_{j=1}^{l} \mathbb{Z}/2^{s_j} \text{ with } s_j \geqslant 1 \text{ for all } 1 \leqslant j \leqslant l.$$

Let X be a space. Its **mod-2 cohomology Bockstein spectral sequence** $\{B_r^*, d_r\}$ is the spectral sequence given by the exact couple

$$\begin{tikzcd} H^*(X;\mathbb{Z}) \arrow[rr, "(\cdot 2)_*"] & & H^*(X;\mathbb{Z}) \arrow[dl, "(\mathrm{red}_2)_*"] \\ & H^*(X;\mathbb{F}_2) \arrow[ul, "\partial"] & \end{tikzcd}$$

which is the long exact sequence in cohomology induced by the short exact sequence of coefficients

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\mathrm{red}_2} \mathbb{Z}/2 \longrightarrow 0.$$

The first page is $B_1^* \cong H^*(X;\mathbb{F}_2)$ and the first differential is given by $d_1 = (\mathrm{red}_2)_*\partial = \beta$, the **Bockstein homomorphism**. The spectral sequence converges to $(H^*(X;\mathbb{Z})/\mathrm{torsion}) \otimes \mathbb{F}_2$. Moreover, if $x \in B_r^n$ is such that $d_r x \neq 0 \in B_r^{n+1}$, then $d_r x$ detects an element of order 2^r in $H^{n+1}(X;\mathbb{Z})$, i.e. there exists an element y of order 2^r in $H^{n+1}(X;\mathbb{Z})$ such that $\mathrm{red}_2(y) = d_r x$. Analogously, there is a mod-2 homological Bockstein spectral sequence.

Let H be an **Hopf algebra** on a field k, with multiplication $\mu : H \otimes H \to H$, comultiplication $\Delta : H \to H \otimes H$, augmentation $\epsilon : H \to k$ and unit $\eta : k \to H$ (see [**41**] for the definitions). The **augmentation ideal** of H is denoted by

$$\bar{H} = \ker \epsilon : H \to k.$$

We will denote by

$$\begin{aligned} QH &= \bar{H}/\mu(\bar{H} \otimes \bar{H}) \\ &= \operatorname{coker} \mu : \bar{H} \to \bar{H} \otimes \bar{H} \end{aligned}$$

the graded modules of **indecomposables** elements of H. We will denote

$$\begin{aligned} PH &= \{x \in \bar{H} \mid \Delta(x) = x \otimes 1 + 1 \otimes x\} \\ &= \ker \Delta : \bar{H} \to \bar{H} \otimes \bar{H} \end{aligned}$$

the graded modules of **primitives** elements of H. The modules of indecomposable and primitive elements of the Hopf algebra H are related to those of the dual Hopf algebra H^{dual} as follows:

$$\begin{aligned} (QH)^{\text{dual}} &\cong P(H^{\text{dual}}), \\ (PH)^{\text{dual}} &\cong Q(H^{\text{dual}}). \end{aligned}$$

Let H and H' be two associative Hopf algebras. If $H \subset H'$ and $\mu(\bar{H} \otimes H') = \mu(H' \otimes \bar{H})$, we say that H is **normal** in H'. This condition ensures that $k \otimes_H H'$ $(= H'/\mu(\bar{H} \otimes H'))$ is a Hopf algebra, which we denote $H'//H$. Actually, the obvious map $H' \to H'//H$ is the **cokernel** of the inclusion $H \subset H'$.

If $\varphi : H \to H'$ is an epimorphism, then, under mild hypothesis (for instance if H is bicommutative), φ admits a **kernel** denoted by $\ker \varphi : H\backslash\backslash\varphi \to H$ (see [**41**, 3.5. Definitions, pp. 223-224]), [**41**, 3.6. Proposition, p. 224] or [**57**, Proposition 2.1 (2), pp. 64-65]). We also denote it by $H\backslash\backslash H'$ when φ is clear from the context.

The **Milnor-Moore theorem** states that there is an exact sequence of graded modules $0 \longrightarrow P(\xi H) \longrightarrow PH \longrightarrow QH$, where ξH is the image of the **Frobenius map** $\xi : x \mapsto x^2$. The Hopf algebra H is said to be **primitively generated** if $PH \to QH$ in the above exact sequence is an epimorphism. The prefered reference is [**41**].

A finite non-empty sequence of non-negative integers $I = (a_0, \ldots, a_k)$ is **admissible** if $a_i \geqslant 2a_{i+1}$ for all $0 \leqslant i \leqslant k-1$. Its **stable degree** is defined by $\deg_{\text{st}}(I) = \sum_{i=0}^{k} a_i$ and its **excess** by $e(I) = 2a_0 - \deg_{\text{st}}(I)$.

1.2. Historical survey

In order to make precise what is already known and what is our contribution, we provide here a very short historical survey on the homotopy and homology theories of finite CW-complexes. Then we give the most important known results about the (co)homology of Postnikov pieces.

1.2.1. Homotopy theory of finite CW-complexes. The homotopy theory of finite CW-complexes has been an extensive subject of study since the 1930's. Computing the homotopy groups of such topological spaces is a very difficult problem in general. It suffices to think about spheres to be convinced.

In 1953, J.-P. Serre provided one of the most celebrated results in homotopy theory. He proved in [**53**] that every non-contractible simply connected finite CW-complex has infinitely many homotopy groups. Actually, he was able to establish that there are infinitely many homotopy groups containing a subgroup isomorphic to $\mathbb{Z}/2$ or $\mathbb{Z}$. His method involved the knowledge of the mod-2 cohomology of the Eilenberg-MacLane spaces and the asymptotic behavior of the related Betti numbers.

In 1984, C. A. McGibbon and J. A. Neisendorfer proved in [**37**], without requiring the space to be of finite type, that one can find infinitely many copies of $\mathbb{Z}/p$, p any prime, in the homotopy groups of the space. The proof relies on H. Miller's solution of the D. Sullivan conjecture [**40**].

All these results have then been generalized in several ways by a number of authors, all utilizing the theory of unstable modules over the Steenrod algebra as developed by J. Lannes and L. Schwartz [**32**] [**33**]. See for instance the works of N. Oda and Z.-Y. Yosimura [**47**], Y. Félix, S. Halperin, J.-M. Lemaire and J.-C. Thomas [**24**], W. G. Dwyer and C. W. Wilkerson [**22**], S. Wenhuai [**61**], and J. Grodal [**25**].

Recently, C. Casacuberta considered finite CW-complexes with finitely many non-trivial homotopy groups, without any *a priori* restriction on the fundamental group. Natural examples are wedges of circles, rationalizations of spheres and finite products of any of these. He proved that the homotopy groups of the universal cover of such spaces are $\mathbb{Q}$-vector spaces. An open question is to know if there exists such a space which is not a $K(G, 1)$ for some group G. Along these lines, C. Casacuberta proved that the fundamental group is necessarily torsion-free and cannot contain any abelian subgroup of infinite rank.

C. Broto, J. A. Crespo [**11**], and L. Saumell [**12**] considered non-simply connected H-spaces with more general finiteness conditions rather than "simply" requiring the space to be finite dimensional. For instance, they proved that H-spaces with noetherian mod-p cohomology are extensions of finite mod-p H-spaces or Eilenberg-MacLane spaces.

1.2.2. Homology theory of finite CW-complexes. The homology theory of finite CW-complexes has also been extensively studied since the 1930's. Actually, the focus at the very beginning was on the homology theory of Lie groups.

As mathematicians went along, it soon became apparent that some of the obtained results did not really depend on the entire Lie group structure but rather only on the much more general concept of finite H-spaces.

The first motivating result in the cohomology theory of finite H-spaces was discovered by H. Hopf [**27**]. Given a Lie group or an H-space X, the H-space structure induces a Hopf algebra structure on $H^*(X;\mathbb{Q})$. The very strong restriction on X to be finite dimensional forces this Hopf algebra to be a rational exterior algebra on finitely many generators of odd degrees.

Throughout the 1950's and early 1960's, a number of interesting general properties were obtained. See for instance the works by A. Borel [**3**] [**4**] [**5**], R. Bott [**7**] [**8**] [**9**] and R. Bott and H. Samelson [**10**]. Notably, we have Borel's mod-p version of the preceeding Hopf result. It may be interesting to mention that throughout this period, the only known examples of connected finite H-spaces were products of Lie groups, the sphere S^7 and the real projective space $\mathbb{R}P^7$.

From a chronological point of view, finite H-space theory has occured in two waves. The first one consists in the work in the 1960's of J. F. Adams, W. Browder, A. Clark, J. R. Hubbuck, J. Milnor, J. C. Moore, J. D. Stasheff and E. Thomas among others. The second wave consists of the work in the 1970's and 1980's of J. F. Adams, J. R. Harper, R. Kane, J. P. Lin, C. W. Wilkerson and A. Zabrodsky among others. Let us look at some results which are interesting for our own purposes.

In J. F. Adams' celebrated Hopf Invariant One paper [**1**] he proved that S^1, S^3 and S^7 are the only spheres carrying an H-space structure. The problem of determining the connectivity of a finite H-space has a long history. W. Browder [**13**] developped the notion of ∞-*implications* in the Bockstein spectral sequence of an H-space. He then proved, under the assumption that X is a finite H-space, that $Sq^1 P^{\text{even}} H^*(X;\mathbb{F}_2) = 0$, where $PH^*(X;\mathbb{F}_2)$ denotes the module of primitive elements. The action of the Steenrod algebra on the cohomology of a finite H-space is therefore severely restricted. A consequence is that any simply connected finite H-space is actually 2-connected. E. Thomas [**60**] proved that finite H-spaces with primitively generated mod 2 cohomology have the first non-trivial homotopy group in degree 1, 3, 7 or 15. J. P. Lin [**35**] generalized all the above work proving that the first non-trivial homotopy group of a finite H-space occurs in degree 1, 3, 7 or 15. A. Clark [**16**] proved that finite loop spaces have the first non-trivial homotopy group in degree 1 or 3.

1.2.3. Known results on Postnikov pieces.

Calculation of $H^*(K(\mathbb{Z}/2^r, n); \mathbb{F}_2)$ – work of J.-P. Serre. J.-P. Serre [**53**] computed the free graded commutative algebra structure of the mod-2 cohomology of Eilenberg-MacLane spaces. He showed that this is a graded polynomial algebra on generators given by iterated Steenrod squares. For instance, he proved the following result:

Theorem. *Let $n \geqslant 1$. The graded $\mathbb{F}_2$-algebra $H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2)$ is isomorphic to the graded polynomial $\mathbb{F}_2$-algebra on generators $Sq^I u_n$, where I covers all the admissible sequences of excess $e(I) < n$ and where $u_n \in H^n(K(\mathbb{Z}/2, n); \mathbb{F}_2)$ is the fundamental class.*

His proof was done by means of the Serre spectral sequence and a theorem of A. Borel [**3**] on transgressive elements.

He also studied the behavior of the Poincaré series $P(H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2), t)$ around $t = 1$ and then proved his celebrated result on the homotopy groups of a finite CW-complex.

Calculation of $H^*(K(G, n); \mathbb{Z})$ – séminaire H. Cartan 1954-1955. The goal of the seminar [**15**] was to compute the homology and cohomology of Eilenberg-MacLane spaces explicitely. The method consists in constructing a tensor product of *elementary complexes*. An elementary complex is itself a tensor product of a polynomial and an exterior algebra generated by a pair of *admissible words*. At the end, the (co)homology of this construction is isomorphic to the (co)homology of the associated Eilenberg-MacLane space.

Let us remark that, in the mod-2 case, the cohomology is given by a tensor product of a polynomial and an exterior algebra, and the exterior part is not trivial. Therefore, as an algebra, this description is not isomorphic to the one given by J.-P. Serre. Nevertheless, we will prove that the two descriptions agree when we restrict our attention only to the graded vector space structures.

Using H. Cartan's method, it is easy to establish that the reduced integral homology of an Eilenberg-MacLane space $K(G, n)$, with G any finitely generated non-trivial abelian group and $n \geqslant 2$, is a graded group which does not admit an exponent. In other words we have:

Theorem. *Let G be a finitely generated non-trivial abelian group and $n \geqslant 2$. The Eilenberg-MacLane space $K(G, n)$ has no homology exponent.*

It is interesting that even if G admits an exponent, $\widetilde{H}_*(K(G, n); \mathbb{Z})$ does not. See for instance the tables of computations for $H_*(K(Z/2^r, n); \mathbb{Z})$ in the Appendix C.

The Künneth formula immediately implies the following consequence:

Corollary. *A simply connected GEM has no homology exponent.*

Mod-2 cohomology of 2-stage Postnikov systems. L. Kristensen [**29**] [**30**] computed the mod-2 cohomology, as a graded vector space, of certain 2-local loop spaces with two non-trivial homotopy groups. His computations were carried out by means of J.-P. Serre's spectral sequence arguments.

His method can be described as follows. The spectral sequence argument of J.-P. Serre giving the cohomology of $K(\mathbb{Z}/2, 2)$ relies heavily on the fact that the transgression commutes with the Steenrod squares. L. Kristensen needed to have some information about the image of the differentials on the $Sq^i x$ when x lies in the fibre, even if x is not transgressive (provided the differentials on x were known). He succeeded in his computations by equipping the whole spectral sequence with Steenrod operations. This idea to have $\mathcal{A}_2$ acting on a spectral sequence was exploited by various authors like D. L. Rector [**49**], W. M. Singer [**54**] [**55**] and W. G. Dwyer [**21**].

L. Smith [**56**] studied the Eilenberg-Moore spectral sequence of stable 2-stage Postnikov systems. He then recovered and generalized all the results of L. Kristensen in a very elegant way.

The next step in the study of the mod-2 cohomology of such spaces with two non-trivial homotopy groups was to fetch more structure than simply the graded vector space one. A lot of authors have studied this problem.

R. J. Milgram [**39**] gave the unstable $\mathcal{A}_2$-module structures of the fibres of the Steenrod squares $Sq^n : K(\mathbb{Z}/2, n + k) \to K(\mathbb{Z}/2, 2n + k)$, where $n \geqslant 1$ and $k \geqslant 0$.

J. R. Harper [**26**] gave the graded Hopf algebra structure of these spaces and recovered some of the mod-2 Steenrod algebra action.

L. Kristensen and E. K. Pedersen [**31**] gave a very efficient method for determining all these additional structures. Their idea was to express the $\mathcal{A}_2$-module structure in terms of Massey products in $\mathcal{A}_2$. Thus they were able to describe completely the unstable $\mathcal{A}_2$-module structure and they also proved that the results of R. J. Milgram were incorrect.

1.2.4. Some Other Related and Motivating Results.

On associative and commutative H-spaces. J. C. Moore [**44**] proved the following general result:

Theorem. *Let X be an associative and commutative connected H-space. Then all the k-invariants of X are trivial and thus X is a GEM.*

As a corollary, the Künneth formula yields the following result:

Corollary. *If X is an associative and commutative simply connected H-space of finite type, then X has no homology exponent.*

R. Levi's PhD thesis. In his PhD thesis which was published in [**34**], R. Levi studied the homotopy type of p-completed classifying spaces of the form $BG_p^\wedge$ for G a finite p-perfect group, p a prime. He constructed an algebraic analogue of Quillen's "plus" construction for differential graded coalgebras. He then proved that the loop spaces $\Omega BG_p^\wedge$ admit integral homology exponents. More precisely, he proved the following result:

Theorem. *Let G be a finite p-perfect group of order $p^r \cdot m$, m prime to p. Then*

$$p^r \cdot \widetilde{H}_*(\Omega BG_p^\wedge; \mathbb{Z}_{(p)}) = 0.$$

He also showed that his bound is best possible for groups G containing a Sylow 2-subgroup isomorphic to a dihedral or a semidihedral group. He then proved that in general $BG_p^\wedge$ admits infinitely many non-trivial k-invariants, and thus in particular $\pi_* BG_p^\wedge$ is non-trivial in arbitrarily high dimensions. His method for proving this last result is based on a version of H. Miller's theorem improved by J. Lannes and L. Schwartz [**32**].

It may be interesting to ask if this last result remains true for a more general class of spaces, namely those for which a homology exponent exists. In other words, when is it possible for a Postnikov piece to admit a homology exponent? Our work shows that it is "rarely" the case, in a sense that will be made precise later.

"A short walk in the Alps" with F. R. Cohen and F. P. Peterson. Let $\Omega f : \Omega X \to K(\mathbb{Z}/2, n)$ be a loop map. If Ωf has a section, the space ΩX splits as a product $K(\mathbb{Z}/2, n) \times \Omega F$ where F denotes the homotopy theoretic fibre of f. The order of the torsion in the homology of ΩF is then bounded by the order of the torsion in the homology of ΩX. In the more general case where sections fail to exist, it sometimes happens that the behavior of the torsion is more complicated.

In their article [**19**], F. R. Cohen and F. P. Peterson gave examples of loop maps $\Omega f : \Omega X \to K(\mathbb{Z}/2, n)$, $n \geqslant 2$, with the property that $(\Omega f)^* : H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2) \to H^*(\Omega X; \mathbb{F}_2)$ is a monomorphism and such that Ωf does not admit a section. In particular, using the classical result on the structure of Hopf algebras that a connected Hopf algebra is free over a sub-Hopf algebra, they proved that $H^*(\Omega X; \mathbb{F}_2)$ is a free module over $H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2)$.

The examples are mainly provided by $\Omega\Sigma(\mathbb{R}P^\infty)^n \to K(\mathbb{Z}/2,n)$, the canonical multiplicative extension of Serre's map $e : (\mathbb{R}P^\infty)^n \to K(\mathbb{Z}/2,n)$, and by $\Omega\Sigma BSO(3) \to K(\mathbb{Z}/2,2)$, the canonical multiplicative extension of the second Stiefel-Whitney class in the mod-2 cohomology of $BSO(3)$ in the case $n=2$.

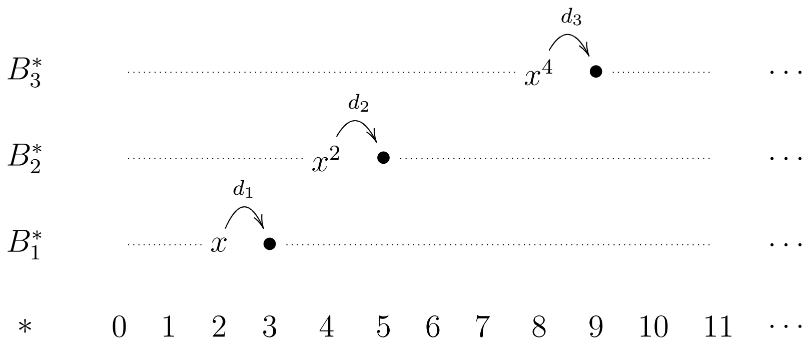
Let us remark that $\Sigma(\mathbb{R}P^\infty)^n$ and $\Sigma BSO(3)$ are not Postnikov pieces. Then it may be interesting to ask if there exist a Postnikov piece X and a map $f : X \to K(\mathbb{Z}/2,n)$, $n \geqslant 2$, with the properties that X is not a GEM and f^* is a monomorphism. We will give a positive answer to that question.

1.3. Statement of the main results

In the first part of this work, we shall study Cartan's work [**15**] on (co)homology of Eilenberg-MacLane spaces. We mainly focus on elements in the mod-2 (co)homology of Eilenberg-MacLane spaces which behave in such a way that they are of key interest for our purpose, namely the study of homology exponents. We call such elements *transverse*, and the accurate definition runs as follows:

1.3.1. Definition. Let X be a space and $\{B_r^*, d_r\}$ be its mod-2 cohomology Bockstein spectral sequence. Let n and r be two positive integers. An element $x \in B_r^n$ is said to be **ℓ-transverse** if $d_{r+l}x^{2^l} \neq 0 \in B_{r+l}^{2^l n}$ for all $0 \leqslant l \leqslant \ell$. An element $x \in B_r^n$ is said to be **∞-transverse**, or simply **transverse**, if it is ℓ-transverse for all $\ell \geqslant 0$. We will also speak of **transverse implications** of an element $x \in B_r^n$.

Suppose that $x \in B_1^2$ is ∞-transverse. Let us picture how the transverse implications of x look like in the Bockstein spectral sequence:



Thus every transverse element gives rise to 2-torsion of arbitrarily high order in the integral cohomology of X. Our strategy for disproving the existence of a homology exponent for a space will then consist in exhibiting a transverse element in its mod-2 cohomology Bockstein spectral sequence. In the special case of Eilenberg-MacLane spaces, we have the following key result (see Section 2.4):

1.3.2. Theorem. *Let G be a non-trivial finitely generated 2-torsion abelian group of type $(s_1, \dots, s_l)$ and let $n \geqslant 2$. Consider the Eilenberg-MacLane space $K(G, n)$ and its mod-2 cohomology Bockstein spectral sequence $\{B_r^*, d_r\}$. Suppose that one of the following assumptions holds:*

- *n is even and $x \in B_{s_j}^n$ is 0-transverse for any $1 \leqslant j \leqslant l$,*
- *$x \in P^{\text{even}} B_1^*$ is 0-transverse ($Sq^1 x \neq 0$).*

Then x is ∞-transverse.

Let us remark that a 0-transverse implication does not imply ∞-transverse implications for more general H-spaces. More precisely, the fact that $x \in P^{\text{even}} H^*(X; \mathbb{F}_2)$ is such that $Sq^1 x \neq 0$ does not always force x to be ∞-transverse. A counterexample is given by $X = BSO$ and $x = w_2$, the second Stiefel-Withney class in $H^2(BSO; \mathbb{F}_2)$. See Section 2.5.

As a corollary of the previous theorem, it is then possible to give a new proof of H. Cartan's original result:

1.3.3. Corollary. *Let G be a non-trivial finitely generated 2-torsion abelian group and $n \geqslant 2$. The Eilenberg-MacLane space $K(G, n)$ has no homology exponent.*

We shall provide a proof of this corollary in Section 2.4 for the special case $G = \mathbb{Z}/2$; this is particularly interesting since the argument shows where transverse implications arise.

H. Cartan's method allows us actually to compute the (co)homology groups of the $K(G, n)$ spaces accurately. A glance at the tables in the Appendix C gives some "heuristic" formulae about the exponents of these groups.

1.3.4. Proposition. *The exponent of the homology group $H_{2^r m}(K(\mathbb{Z}/2, 2); \mathbb{Z})$, with m odd, is exactly 2^r. Moreover, one has the following formula:*

$$\exp \widetilde{H}_n(K(\mathbb{Z}/2,2);\mathbb{Z}) = \begin{cases} n_{(2)} & \text{if } n \geqslant 2 \text{ is even,} \\ 2 & \text{if } n \geqslant 5 \text{ is odd,} \\ 1 & \text{if } n \in \{0,1,3\}, \end{cases}$$

where $n_{(2)}$ denotes the 2-primary part of the integer n.

1.3.5. Proposition. *The exponent of the cohomology group $H^{2^r m+1}(K(\mathbb{Z}/2, 2); \mathbb{Z})$, with m odd, is exactly 2^r. Moreover, one has the following formula:*

$$\exp \widetilde{H}^n(K(\mathbb{Z}/2,2);\mathbb{Z}) = \begin{cases} (n-1)_{(2)} & \text{if } n \geqslant 3 \text{ is odd,} \\ 2 & \text{if } n \geqslant 6 \text{ is even,} \\ 1 & \text{if } n \in \{0,1,2,4\}, \end{cases}$$

where $(n-1)_{(2)}$ denotes the 2-primary part of the integer $(n-1)$.

These two results, as well as the cohomology Bockstein spectral sequence of $K(\mathbb{Z}/2, 2)$, will be discussed in Section 3.2.

In the second part of this work, we shall consider Postnikov pieces with more than only one non-trivial homotopy group, in other words, Postnikov pieces which are not Eilenberg-MacLane spaces. One can classify this family of spaces into two classes. Actually, some Postnikov pieces retract onto an Eilenberg-MacLane space and this class is then very easy to study from our homology viewpoint. To illustrate our purpose, let us consider the following example (see Section 4.1):

1.3.6. Example. Let X be the space given by the fibration

$$X \xrightarrow{i} K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,2) \xrightarrow{k} K(\mathbb{Z}/2,4),$$

where its single non-trivial k-invariant is

$$\begin{aligned} k \in & [K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,2), K(\mathbb{Z}/2,4)] \\ \cong & H^4(K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,2); \mathbb{F}_2) \\ \cong & H^4(K(\mathbb{Z}/2,2); \mathbb{F}_2) \otimes \mathbb{F}_2 \\ & \oplus H^2(K(\mathbb{Z}/2,2); \mathbb{F}_2) \otimes H^2(K(\mathbb{Z}/2,2); \mathbb{F}_2) \\ & \oplus \mathbb{F}_2 \otimes H^4(K(\mathbb{Z}/2,2); \mathbb{F}_2) \\ \cong & \mathbb{F}_2\{u_2^2 \otimes 1, u_2 \otimes v_2, 1 \otimes v_2^2\} \end{aligned}$$

given by $k = u_2 \otimes v_2$ where u_2 and v_2 are the fundamental classes of both copies of $K(\mathbb{Z}/2,2)$. The space X has only two non-trivial homotopy groups $\pi_2(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\pi_3(X) \cong \mathbb{Z}/2$.

1.3.7. Theorem. *The space X of Example 1.3.6 has the following properties:*

1. *X is not a GEM,*
2. *X is not an H-space,*
3. *X retracts (weakly) onto the Eilenberg-MacLane space $K(\mathbb{Z}/2,2)$, i.e. there exist maps $f : X \to K(\mathbb{Z}/2,2)$ and $g : K(\mathbb{Z}/2,2) \to X$ such that $fg \simeq \mathrm{id}_{K(\mathbb{Z}/2,2)}$,*
4. *$f^* : H^*(K(\mathbb{Z}/2,2); \mathbb{F}_2) \to H^*(X; \mathbb{F}_2)$ is a monomorphism,*
5. *X has no homological exponent.*

One can actually find a lot of Postnikov pieces which retract onto an Eilenberg-MacLane space. If one concentrate on some particular H-spaces with only two non-trivial homotopy groups, we have the following general result which will be proved in Section 4.2:

1.3.8. Theorem. *Consider a stable two stage Postnikov system of the form*

$$\begin{tikzcd} K\Sigma^d\mathbb{F}_2 \arrow[r, equal] \arrow[d] & K\Sigma^d\mathbb{F}_2 \arrow[d] \\ X \arrow[r] \arrow[d] & PK\Sigma^{d+1}\mathbb{F}_2 \simeq * \arrow[d] \\ K\Sigma^m U \arrow[r, "k"'] & K\Sigma^{d+1}\mathbb{F}_2, \end{tikzcd}$$

with $d > m \geqslant 1$, $U = \bigoplus_{1\leqslant i\leqslant s}\mathbb{F}_2$ *and* $k : K\Sigma^m U \to K\Sigma^{d+1}\mathbb{F}_2$ *an H-map. Moreover, consider the set*

$$S = \{\mathrm{incl}_i^*\, k^*(u_{d+1}) \in H^{d+1}(K\Sigma^m\mathbb{F}_2;\mathbb{F}_2) \mid 1 \leqslant i \leqslant s\},$$

where $\mathrm{incl}_i : K\Sigma^m\mathbb{F}_2 \to K\Sigma^m U$ *is the* i*-th obvious inclusion,* $\mathrm{incl}_i^*\, k^* : H^{d+1}(K\Sigma^{d+1}\mathbb{F}_2;\mathbb{F}_2) \to H^{d+1}(\Sigma^m\mathbb{F}_2;\mathbb{F}_2)$ *is the induced homomorphism and where* u_{d+1} *denotes the fundamental class in* $H^{d+1}(K\Sigma^{d+1}\mathbb{F}_2;\mathbb{F}_2)$*. The H-space* X *retracts onto* $K\Sigma^m\mathbb{F}_2$ *if and only if one of the two following assumptions is verified:*

- $0 \in S$ *or*
- $\mathrm{Card}(S) < s$.

The existence of a retract onto an Eilenberg-MacLane space clearly implies that the space cannot have a homology exponent. This "topological" feature is here sufficient to conclude. For spaces without such a retract, we need to develop more "algebraic" tools in order to study homology exponents. To see this, let us look at the following interesting example:

1.3.9. Example. Let X be the space given by the fibration

$$X \xrightarrow{i} K(\mathbb{Z}/2,2) \xrightarrow{k} K(\mathbb{Z}/2,4),$$

where its single non-trivial k-invariant is

$$\begin{aligned} k \in & [K(\mathbb{Z}/2,2), K(\mathbb{Z}/2,4)] \\ \cong & H^4(K(\mathbb{Z}/2,2);\mathbb{F}_2) \\ \cong & \mathbb{F}_2\{u_2^2\} \end{aligned}$$

given by $k = u_2^2$ where u_2 is the fundamental class of $K(\mathbb{Z}/2,2)$. The space X has only two non-trivial homotopy groups $\pi_2(X) \cong \mathbb{Z}/2 \cong \pi_3(X)$.

From the viewpoint of homotopy groups, this space seems "less complicated" than the space of Example 1.3.6. Nevertheless, it is really "more difficult" to study. This particular space, as well as the following theorem, are the genesis of the rest of this work.

1.3.10. Theorem. *The space X of Example 1.3.9 has the following properties:*

1. *X is not a GEM,*
2. *X is an infinite loop space,*
3. *X retracts neither onto the Eilenberg-MacLane space $K(\mathbb{Z}/2, 2)$, nor onto $K(\mathbb{Z}/2, 3)$,*
4. *However, X has no homological exponent.*

This result will be proved in Section 4.3. It is due in part to the following lemma which states that the ∞-transverse implications of an element in the cohomology of the total space of a fibration can be read in the cohomology of the fibre.

1.3.11. Lemma. *Let $i : F \to X$ be a continuous map. If $x \in H^*(X; \mathbb{F}_2)$ is such that $i^*(x) \neq 0 \in H^*(F; \mathbb{F}_2)$ is ∞-transverse, then x is ∞-transverse.*

In the last part of this work, we shall exploit further this result and set a strategy for detecting ∞-transverse implications in Postnikov pieces. Let us state our first main result (see Section 5.4):

1.3.12. Theorem. *Let X be a 1-connected 2-local H-space of finite type with only two non-trivial (finite) homotopy groups. Then X has no homology exponent.*

Finally, we generalize this kind of result to more complicated Postnikov pieces, namely certain stable two stage Postnikov systems. Before stating our second main result, let us set the following definition:

1.3.13. Definition. A space X is m**-anticonnected** if $\pi_i(X) = 0$ for all $i > m$ and **strictly** m**-anticonnected** if it is m-anticonnected and $\pi_m(X) \neq 0$.

The most general result we obtain is the following theorem which is proved in Section 5.6.

1.3.14. Theorem. *Let G and H be two 2-local finitely generated graded abelian groups. Consider a 2-stage Postnikov system*

$$\begin{tikzcd} KH \arrow[r, equal] \arrow[d] & KH \arrow[d] \\ X \arrow[r] \arrow[d] & PK\Sigma H \arrow[d] \\ KG \arrow[r, "f"'] & K\Sigma H \end{tikzcd}$$

such that KG is a 1-connected m-anticonnected GEM, KH a 1-connected strictly n-anticonnected GEM with $n \in [m+2, \infty[$ and f an H-map (i.e. the system is **stable***). Then X has no homology exponent.*

It is important to point out that this result does not imply Theorem 1.3.12 since there is a "homotopy gap" in dimension $m+1$ that seems difficult to fill. See discussion at the end of Section 5.6, p. 82.

1.4. Guesses, questions and further developments

Our results enable us to formulate the following conjecture:

1.4.1. Conjecture. Let X be a 2-local H-space of finite type. If X has a homology exponent, then either $X \simeq B\pi_1 X$, or the Postnikov tower for X has infinitely many non-trivial k-invariants and, in particular, X is not a Postnikov piece.

In order to attack this conjecture, let us first look at the following problem.

1.4.2. Question. Let X be a 2-local H-space (of finite type) and G a finitely generated 2-torsion abelian group. If X has a homology exponent, is the space $\mathrm{map}_*(K(G,2), X)$ weakly contractible?

To see that an affirmative answer to Question 1.4.2 implies Conjecture 1.4.1, suppose that X is a 2-local H-space with finitely many non-trivial k-invariants. Then $X \simeq X[m] \times \mathrm{GEM}$ for some integer m. However, the fact that X has no homology exponent implies that X should be a Postnikov piece $X \simeq X[m]$. Consider then the Postnikov tower of the space $X[m]$:

$$\begin{tikzcd}
K(\pi_m X, m) \arrow[r, "i"] & X[m] \arrow[d] & \\
& X[m-1] \arrow[r, "k^{m+1}"] \arrow[d] & K(\pi_m X, m+1) \\
& \vdots \arrow[d] & \\
& K(\pi_1 X, 1). &
\end{tikzcd}$$

The map $i : K(\pi_m X, m) \to X[m]$ induces an isomorphism on the m-th homotopy groups. Therefore $\Omega^{m-2} i : K(\pi_m X, 2) \to \Omega^{m-2} X[m]$ also induces an isomorphism on the 2nd homotopy groups and thus the adjoint map $\Sigma^{m-2} K(\pi_m X, 2) \to X[m]$ is not nullhomotopic. This contradicts the fact that $\mathrm{map}_*(K(\pi_m X, 2), X[m])$ is weakly contractible.

W. Browder proved in [**13**, Theorem 6.11, p. 46] that every H-space of finite type which has the homotopy type of a finite CW-complex and which is 1-connected is actually 2-connected. Its result relies on the notion of ∞-implications in the Bockstein spectral sequence of the H-space. One can readilly check that an element which is ∞-transverse has ∞-implications. Therefore, it is interesting to set the following question.

1.4.3. Question. Let X be a 1-connected 2-local H-space of finite type with a homology exponent. Is X always 2-connected? If it is not the case for all such H-spaces, is it true for infinite loop spaces?

And finally, we conclude with the following reasonnable guess.

1.4.4. Conjecture. We can generalize to odd primes and relax the 2-local hypotheses in all results, conjectures and questions above.

1.5. Organization of the work

Chapter 2 is devoted to the study of the (co)homology of Eilenberg-MacLane spaces. The main results of J.-P. Serre [**53**] and H. Cartan [**15**] are exposed and developped. The particular aspects of duality between mod-2 homology and cohomology are studied and a proof of Theorem 1.3.2 is given.

Chapter 3 presents the "Eilenberg-MacLane machine" which is a C++ program designed to compute explicitely integral homology groups of Eilenberg-MacLane spaces. The main algorithms are quoted and some heuristic results derived from the computations of the machine are proved, namely Propositions 1.3.4 and 1.3.5.

Chapter 4 provides examples of spaces which retract, respectively do not retract, onto an Eilenberg-MacLane space, and thus classify them. A general classification result for a wide family of H-spaces is given. One can find the proofs of Theorems 1.3.7, 1.3.8 and 1.3.10 in this chapter.

Chapter 5 exploits transverse implications provided by Eilenberg-MacLane spaces to detect transverse elements in some 2-stage Postnikov systems. The two main results of this work, Theorems 1.3.12 and 1.3.14, as well as the requisite technical stuff to prove them, are established in the last sections.

CHAPTER 2

(Co)Homology of Eilenberg-MacLane spaces

2.1. J.-P. Serre's description

2.1.1. Definition. A non-empty finite sequence of positive integers $I = (a_0, \dots, a_k)$, where k is varying, is **admissible** if $a_i \geqslant 2a_{i+1}$ for all $0 \leqslant i \leqslant k-1$. Let $\mathcal{S}$ be the set of all such admissible sequences. The **stable degree** is a map $\deg_{\mathrm{st}} : \mathcal{S} \to \mathbb{N}$ defined by $\deg_{\mathrm{st}}(I) = \sum_{i=0}^{k} a_i$ for all $I = (a_0, \dots, a_k) \in \mathcal{S}$. The stable degree induces a **grading** on the set $\mathcal{S}$ of all admissible sequences. The **excess** is a map $e : \mathcal{S} \to \mathbb{N}$ defined by $e(I) = 2a_0 - \deg_{\mathrm{st}}(I) = a_0 - \sum_{i=1}^{k} a_i$ for all $I = (a_0, \dots, a_k) \in \mathcal{S}$.

2.1.2. Convention. Let $n \geqslant 1$ and $s \geqslant 1$. Consider the fundamental class $\iota_n \in H^n(K(\mathbb{Z}/2^s, n); \mathbb{Z}/2^s)$ and its mod-2 reduction $u_n \in H^n(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$. Let $I = (a_0, \dots, a_k)$ be an admissible sequence. We will write $Sq^I_s u_n$ instead of $Sq^{a_0} \dots Sq^{a_{k-1}} \delta_s \iota_n$ (usually denoted by $Sq^{a_0, \dots, a_{k-1}} \delta_s \iota_n$) if $a_k = 1$ and instead of $Sq^{a_0} \dots Sq^{a_k} u_n$ (also denoted by $Sq^{a_0, \dots, a_k} u_n$ or $Sq^I u_n$) if $a_k \neq 1$. Here δ_s denotes the connecting homomorphism associated to $0 \longrightarrow \mathbb{Z}/2^s \longrightarrow \mathbb{Z}/2^{s+1} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$. In particular, since $\delta_1 = Sq^1$ and the reduction is the identity when $s = 1$, we have $Sq^I_1 u_n = Sq^I u_n$.

Let R be a ring and X be a space. For any $n \geqslant 0$, the cohomology group $H^n(X; R)$ is a left R-module. Therefore, the set $\{H^n(X; R) \mid n \geqslant 0\}$ of all cohomology groups is a graded R-module which we denote $H^*(X; R)$. Moreover, $H^*(X; R)$, endowed with the cup product, is a graded R-algebra.

In 1953, J.-P. Serre computed the mod-2 cohomology of Eilenberg-MacLane spaces and stated the following result:

2.1.3. Theorem. *Let $n \geqslant 1$ and $s \geqslant 1$. The graded $\mathbb{F}_2$-algebra $H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$ is isomorphic to the graded polynomial $\mathbb{F}_2$-algebra on generators $Sq^I_s u_n$, where I covers all the admissible sequences of excess $e(I) < n$ and u_n is the reduction of the fundamental class (see convention 2.1.2). The degree of a generator $Sq^I_s u_n$ is $\deg(Sq^I_s u_n) = \deg_{\mathrm{st}}(I) + n$.*

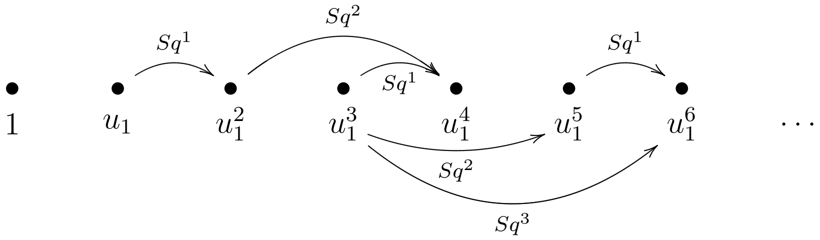
Proof. See [**53**, Théorème 2, p. 203 and Théorème 4, p. 206]. $\square$

This result explicits the $\mathcal{A}_2$-module structure of $H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$. In particular, this is a differential graded $\mathbb{F}_2$-algebra under the Bockstein homomorphism $\beta = Sq^1$.

For instance, let us determine the unstable $\mathcal{A}_2$-module structure of the algebra $H^*(K(\mathbb{Z}/2, 1); \mathbb{F}_2)$. The excess of an admissible sequence I is zero if and only if $I = (0)$ and the fundamental class $u_1 \in H^1(K(\mathbb{Z}/2, 1); \mathbb{F}_2)$ is then the only generator. Therefore we have

$$H^*(K(\mathbb{Z}/2, 1); \mathbb{F}_2) \cong \mathbb{F}_2[u_1]$$

and the $\mathcal{A}_2$-action is given pictorially as follows



In particular, this example shows the Adem relation $Sq^1Sq^2 = Sq^3$. Have a glance at Appendix A for more Adem relations.

It is well known that an Eilenberg-MacLane space associated with an abelian group has a unique H-space structure up to homotopy. The H-space structure can be seen as inherited from the loop space structure or from the addition law of the associated abelian group. Therefore, the graded $\mathbb{F}_2$-algebra $H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$ is also a differential graded Hopf algebra.

2.1.4. Theorem. *Let $n \geqslant 1$ and $s \geqslant 1$. The differential graded $\mathbb{F}_2$-algebra $H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$ is a connected, biassociative, bicommutative and primitively generated differential graded Hopf algebra.*

Proof. See [**57**, pp. 54-55]. □

It is now easy to determine the modules of primitives and indecomposables of $H^* = H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$. The modules of indecomposable elements is clearly given by

$$QH^* \cong \mathbb{F}_2\{Sq_s^I u_n \mid I \text{ admissible and } e(I) < n\},$$
$$\text{the graded } \mathbb{F}_2\text{-vector space generated by}$$
$$\text{all the } Sq_s^I u_n \text{ with } I \text{ admissible and } e(I) < n.$$

Since H^* is primitively generated, the Milnor-Moore theorem (see p. 5) gives the following short exact sequence of graded $\mathbb{F}_2$-vector spaces:

$$0 \longrightarrow P(\xi H) \longrightarrow PH \longrightarrow QH \longrightarrow 0.$$

Therefore, every indecomposable element is primitive and every primitive element which is decomposable is a square of a primitive element. Thus we have

$PH^* \cong \mathbb{F}_2\{(Sq_s^I u_n)^{2^i} \mid I \text{ admissible, } e(I) < n \text{ and } i \geqslant 0\}$,
the graded $\mathbb{F}_2$-vector space generated by
all the iterated squares of $Sq_s^I u_n$ with I admissible and $e(I) < n$.

2.1.5. Definition. Let H^* denote a (positively) graded vector space over a field k. The **Poincaré series** of H^* is the formal power series

$$P(H^*, t) = \sum_{i \geqslant 0} \dim_k H^i \cdot t^i \in \mathbb{Z}[[t]].$$

2.1.6. Proposition. *Let $P_n(t)$ denote the Poincaré series $P(H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2), t)$. We have*

$$P_n(t) = \prod_{h_1 \geqslant \cdots \geqslant h_{n-1} \geqslant 0} \frac{1}{1 - t^{1+2^{h_1}+\cdots+2^{h_{n-1}}}}.$$

Proof. See [**53**, Théorème 1 and formula (17.7), pp. 211-212] or Lemma 5.5.3, p. 74. □

2.1.7. Definition. The **Serre function** associated to $P_n(t)$ is the continuous map given by

$$\varphi_n(x) = \log_2 P_n(1 - 2^{-x})$$

for all $x \in [0, \infty[$.

2.1.8. Theorem. *We have the asymptotic growth formula*

$$\varphi_n(x) \sim \frac{x^n}{n!} \qquad i.e. \qquad \lim_{x \to +\infty} \frac{\varphi_n(x)}{\frac{x^n}{n!}} = 1.$$

Proof. See [**53**, Théorème 6, pp. 215-216]. □

2.2. H. Cartan's description

Given an associative H-space X, R. J. Milgram [**38**] gives a construction for a classifying space $\widehat{B}X$ which has the following advantages over that of A. Dold and R. Lashof [**20**]:

1. if X is abelian then $\widehat{B}X$ is also an abelian associative H-space with unit,
2. if X is a CW-complex and the multiplication is cellular then $\widehat{B}X$ is also a CW-complex and the cellular chain complex $C_*(\widehat{B}X)$ is isomorphic to the bar construction $B(C_*(X))$ on the cellular chain complex,
3. there is an explicit diagonal approximation $\widehat{B}X \to \widehat{B}X \times \widehat{B}X$ which is cellular and, in $C_*(\widehat{B}X)$, induces exactly H. Cartan's diagonal approximation for the bar construction.

Let G be an abelian group with the discrete topology. Then it is a CW-complex consisting of 0-cells and the multiplication is cellular. Therefore, one can inductively apply the R. J. Milgram's construction to construct $K(G,n)$ spaces for all $n \geqslant 0$. R. J. Milgram proved the following result.

2.2.1. Theorem. *Let G be an abelian group and $n \geqslant 0$. The Eilenberg-MacLane space $K(G,n)$ is a topological abelian group and a CW-complex with cellular multiplication. Moreover, for all $n \geqslant 1$, there is an isomorphism of DGA-algebras*

$$C_*(K(G,n)) \cong B(C_*(K(G,n-1)))$$

and, for any ring R, there are isomorphisms

$$\begin{aligned} H_*(K(G,n);R) &\cong H_*(B(C_*(K(G,n-1)));R), \\ H^*(K(G,n);R) &\cong H^*(B(C_*(K(G,n-1)));R), \end{aligned}$$

being ring homomorphisms respectively of Pontrjagin and cohomology rings.

Proof. See [**38**, Theorems 4.1 and 4.2, p. 249]. □

Thus iterating the bar construction on $C_*(K(G,0)) = \mathbb{Z}G$ and taking the (co)homology computes the (co)homology of Eilenberg-MacLane spaces associated to G. In other words we have $H_*(K(G,n);R) \cong H_*(B^n\mathbb{Z}G;R)$, where $B^n\mathbb{Z}G$ denotes the n-th iterated bar construction on $\mathbb{Z}G$. H. Cartan [**15**] studied the (co)homology of $B^n\mathbb{Z}G$ and therefore was able to compute the (co)homology of Eilenberg-MacLane spaces. Let us present and summarize his results in what follows.

2.2.2. Proposition. *Let G be a non-trivial finitely generated 2-torsion abelian group and $n \geqslant 0$. There are maps*

$$\begin{array}{ll}
\sigma : G \otimes \mathbb{Z}/2^s \to H_1(B\mathbb{Z}G; \mathbb{Z}/2^s) & \textit{for all } s \geqslant 1, \\
\sigma : H_q(B^n\mathbb{Z}G; \mathbb{Z}/2^s) \to H_{q+1}(B^{n+1}\mathbb{Z}G; \mathbb{Z}/2^s) & \textit{for all } q \geqslant 0 \textit{ and } s \geqslant 1, \\
\varphi_2 : H_{2q}(B^n\mathbb{Z}G; \mathbb{F}_2) \to H_{4q+2}(B^{n+1}\mathbb{Z}G; \mathbb{F}_2) & \textit{for all } q \geqslant 1, \\
\gamma_2 : H_q(B^n\mathbb{Z}G; \mathbb{F}_2) \to H_{2q}(B^n\mathbb{Z}G; \mathbb{F}_2) & \textit{for all } q \geqslant 0, \\
\beta_2 : H_q(B^n\mathbb{Z}G; \mathbb{F}_2) \to H_{q-1}(B^n\mathbb{Z}G; \mathbb{F}_2), & \textit{for all } q \geqslant 0,
\end{array}$$

verifying the basic relations (when it makes sense)

(R1) $\beta_2\sigma\varphi_2 = \sigma^2\gamma_2,$

(R2) $\varphi_2 = \gamma_2\sigma,$

(R3) $\beta_2\sigma = \sigma\beta_2,$

(R4) $\beta_2\varphi_2 = \sigma\gamma_2 + \beta_2\sigma \cdot \sigma,$
where $\cdot$ denotes the Pontryagin product.

The maps σ and β_2 are the **homology suspension** *and the* **Bockstein** *homomorphisms respectively. The maps φ_2 and γ_2 are called* **transpotence** *and* **divided square** *respectively.*

Proof. Existence of σ, φ_2, γ_2 and β_2 is stated in [**15**], Exposé 6 pp. 1-2, Exposé 6 Théorème 3 p. 8, Exposé 7 p. 11 and Exposé 8 p. 3 respectively. Basic relations (R2), (R3) and (R4) are stated in Exposé 8, Proposition 1 p. 1, Proposition 2 p. 3 and Théorème 1 p. 4 respectively. Basic relation (R1) is a consequence of the facts that, following (R4), $\beta_2\varphi_2 = \sigma\gamma_2$ modulo decomposables, that the suspension σ is trivial on decomposables (Exposé 6 Proposition 1 p. 2) and that, following (R3), suspension commutes with Bockstein. □

Remark. In what follows, we will use supplementary maps

$$\psi_{2^s} : {}_{2^s}G \to H_2(B\mathbb{Z}G; \mathbb{Z}/2^s) \qquad \text{for all } s \geqslant 1,$$

where ${}_{2^s}G$ is the subgroup of G containing all the elements of order $\leqslant 2^s$ (for instance, if $G = \mathbb{Z}/2^r$, $r \geqslant 2$, is generated by $u \in G$, then ψ_2 is defined on ${}_2G$ which is generated by $u' = 2^{r-1}u$). These maps are defined by H. Cartan in [**15**], Exposé 11 pp. 1-2.

We will abstract some compositions of the maps σ, φ_2, γ_2, β_2 and ψ_{2^s} in the notion of so called *admissible words.*

2.2.3. Definition. Let W be the free monoid (with unit denoted "()") generated by the symbols σ, ψ_{2^s} for all integers $s \geqslant 1$, φ_2, γ_2 and β_2. Such a symbol is called a **letter** and any element in W is called a **word**. Letters composing a word are read from the left to the right.

On W consider the equivalence relation generated by:

(R1') $\beta_2\sigma\varphi_2 \sim \sigma^2\gamma_2$,

(R2') $\varphi_2 \sim \gamma_2\sigma$,

(R3') $\beta_2\sigma \sim \sigma\beta_2$,

(R4') $\beta_2\varphi_2 \sim \sigma\gamma_2$.

Two words α and α' are **synonyms** if $\alpha \sim \alpha'$. A word α is **admissible** if it is a synonym of one of the following words:

- σ^k with $k \geqslant 1$,
- $\sigma^k\psi_{2^s}$ with $k \geqslant 0$ and $s \geqslant 1$,
- $\sigma^k\varphi_2\gamma_2^{h_{i+1}}\epsilon_i\gamma_2^{h_i}\ldots\epsilon_1\gamma_2^{h_1}\epsilon$ with $k \geqslant 0$, $i \geqslant 0$, $\epsilon_j \in \{\sigma^2, \varphi_2\}$ for all $1 \leqslant j \leqslant i$ and $\epsilon \in \{\sigma^2, \psi_2\}$,
- $\beta_2\sigma^k\varphi_2\gamma_2^{h_{i+1}}\epsilon_i\gamma_2^{h_i}\ldots\epsilon_1\gamma_2^{h_1}\epsilon$ with $k \geqslant 0$, $i \geqslant 0$, $\epsilon_j \in \{\sigma^2, \varphi_2\}$ for all $1 \leqslant j \leqslant i$ and $\epsilon \in \{\sigma^2, \psi_2\}$.

Let $\mathcal{W}$ be the set of all admissible words divided out by $\sim$. We will not distinguish admissible words and their classes of synonyms. An admissible word is of **first kind** (resp. **second kind**) if it has (a synonym with) σ (resp. ψ_2) as last letter. The sets $\mathcal{W}^I$ and $\mathcal{W}^{II}$ contain admissible words of first and second kind respectively. The **degree** is a map $\deg : W \to \mathbb{N}$ defined by induction by the following rules:

- $\deg() = 0$,
- $\deg(\beta_2\alpha) = \deg(\alpha) - 1$,
- $\deg(\sigma\alpha) = \deg(\alpha) + 1$,
- $\deg(\gamma_2\alpha) = 2\deg(\alpha)$,
- $\deg(\varphi_2\alpha) = 2\deg(\alpha) + 2$ and
- $\deg(\psi_{2^s}) = 2$ for all $s \geqslant 1$.

The degree induces a **grading** on the set $\mathcal{W}$ of all admissible words. The **height** is a map $h : W \to \mathbb{N}$ which maps every word $\alpha \in W$ to the number of its letters equals to σ, φ_2 or ψ_{2^s}, $s \geqslant 1$. Inductively we have:

- $h() = 0$,
- $h(\beta_2\alpha) = h(\alpha)$,
- $h(\sigma\alpha) = h(\alpha) + 1$,
- $h(\gamma_2\alpha) = h(\alpha)$,
- $h(\varphi_2\alpha) = h(\alpha) + 1$ and
- $h(\psi_{2^s}) = 1$ for all $s \geqslant 1$.

The height induces a map on the set $\mathcal{W}$ of all admissible words. The **stable degree** is a map $\deg_{\text{st}} : \mathcal{W} \to \mathbb{N}$ defined by $\deg_{\text{st}}(\alpha) = \deg(\alpha) - h(\alpha)$ for all $\alpha \in \mathcal{W}$. Let $\mathcal{W}_{q,n}$ be the graded subset given by all admissible words $\alpha \in \mathcal{W}$ such that $\deg_{\text{st}}(\alpha) = q$ and $h(\alpha) = n$. Let $\mathcal{W}_{*,n} = \cup_{q\geqslant 0}\mathcal{W}_{q,n}$. We also define $\mathcal{W}^I_{q,n} = \mathcal{W}_{q,n} \cap \mathcal{W}^I$, $\mathcal{W}^{II}_{q,n} = \mathcal{W}_{q,n} \cap \mathcal{W}^{II}$, $\mathcal{W}^I_{*,n} = \mathcal{W}_{*,n} \cap \mathcal{W}^I$ and $\mathcal{W}^{II}_{*,n} = \mathcal{W}_{*,n} \cap \mathcal{W}^{II}$.

It may be useful to make some remarks.

1. First of all, it is very easy to verify that the degree, height and stable degree agree on synonyms. It suffices to see that $\deg(\beta_2\sigma\varphi_2\alpha) = 2\deg(\alpha) + 2 = \deg(\sigma^2\gamma_2\alpha)$, $\deg(\varphi_2\alpha) = 2\deg(\alpha) + 2 = \deg(\gamma_2\sigma\alpha)$, $\deg(\beta_2\sigma\alpha) = \deg(\alpha) = \deg(\sigma\beta_2\alpha)$, $\deg(\beta_2\varphi_2\alpha) = 2\deg(\alpha) + 1 = \deg(\sigma\gamma_2\alpha)$, $h(\beta_2\sigma\varphi_2\alpha) = h(\alpha) + 2 = h(\sigma^2\gamma_2\alpha)$, $h(\varphi_2\alpha) = h(\alpha) + 1 = h(\gamma_2\sigma\alpha)$, $h(\beta_2\sigma\alpha) = h(\alpha) + 1 = h(\sigma\beta_2\alpha)$ and $h(\beta_2\varphi_2\alpha) = h(\alpha) + 1 = h(\sigma\gamma_2\alpha)$.

2. The height of an admissible word α cannot be zero since α is non-empty and its last letter is σ or ψ_{2^s}, $s \geqslant 1$, all of height equal to 1.

3. The equivalence given by (R4') does not agree with the basic relation (R4). The main reason is that we cannot consider any object like $\sigma\gamma_2 + \beta_2\sigma \cdot \sigma$ in the monoid W. Let us see why we persist to set the relation (R4'). Assume that the set $\{\beta_2\varphi_2\alpha, \sigma\gamma_2\alpha, \beta_2\sigma\alpha \cdot \sigma\alpha\}$ plays the role of an $\mathbb{F}_2$-generator set in the computation of the graded $\mathbb{F}_2$-vector space $H_*(K(G, n); \mathbb{F}_2)$ (this is actually the case by Cartan's results). The generator $\sigma\gamma_2\alpha$ is a linear combination of $\beta_2\varphi_2\alpha$ and $\beta_2\sigma\alpha \cdot \sigma\alpha$ since basic relation (R4) holds. Therefore the two generators $\beta_2\varphi_2\alpha$ and $\beta_2\sigma\alpha \cdot \sigma\alpha$ form a basis. Suppose now in W that we do not set any relation between the three elements $\beta_2\varphi_2\alpha$, $\beta_2\sigma\alpha \cdot \sigma\alpha$ and $\sigma\gamma_2\alpha$. In this case, we would have three basis elements. Thus it is important to increase the number of relations by one in order to decrease the dimension to two. The only possible relations we could set are $\beta_2\varphi_2 \sim \sigma\gamma_2$, $\beta_2\varphi_2 \sim \beta_2\sigma \cdot \sigma$ and $\beta_2\varphi_2 \sim \sigma\gamma_2 + \beta_2\sigma \cdot \sigma$. The only formally possible one in the monoid W is $\beta_2\varphi_2 \sim \sigma\gamma_2$, namely (R4'). Relation (R4') mimics (R4) in the sense that the number of relations between elements of given degree and height is preserved.

4. For all admissible word $\alpha \in \mathcal{W}$, one can obviously associate a map $\alpha : G \otimes \mathbb{Z}/2^s \to H_*(K(G, n); \mathbb{F}_2)$ if $\alpha \in \mathcal{W}^I$ or $\alpha : {}_{2^s}G \to H_*(K(G, n); \mathbb{F}_2)$ if $\alpha \in \mathcal{W}^{II}$ (in the case where $\alpha = \beta_2\varphi_2\alpha'$ or $\sigma\gamma_2\alpha'$, the associated map is $\beta_2\varphi_2\alpha'$). Moreover, this map is linear when G is a cyclic group. See [**15**, Exposé 9, pp. 1-2; Proposition 2, p. 2 and Dernière remarque, p.10]. We will not distinguish admissible words and their corresponding linear maps in what follows.

2.2.4. Definition. Let A be a graded algebra which is commutative (in the graded sense). We say that A is endowed with a **divided powers** system if, for each $x \in A$ of positive even degree $\deg(x)$, there is a sequence of elements $\gamma_i(x) \in A$ for all $i \geqslant 0$, in degrees $\deg(\gamma_i(x)) = i\deg(x)$, satisfying the following properties:

(1) $\gamma_0(x) = 1$,

(2) $\gamma_1(x) = x$,

(3) $\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x)$,

(4) $\gamma_i(x+y) = \sum_{i=k+l} \gamma_k(x)\gamma_l(y)$,

(5) for all $i \geqslant 0$, $\gamma_i(xy) = \begin{cases} 0 & \text{if } \deg(x), \deg(y) \text{ are odd, } i \geqslant 2 \\ x^i\gamma_i(y) & \text{if } \deg(x), \deg(y) \geqslant 2 \text{ are even,} \end{cases}$

(6) $\gamma_j(\gamma_i(x)) = \prod_{k=i}^{(j-1)i} \binom{k+i-1}{k}\gamma_{ij}(x)$.

Property (3) implies that $x^i = i!\gamma_i(x)$ for all $i \geqslant 0$, which justifies the terminology of *divided powers* for the elements $\gamma_i(x)$. We will often omit the parenthesis in the notation $\gamma_i(x)$ and simply write $\gamma_i x$.

Let us make some remarks on divided powers algebras in characteristic 2 that will be useful in the sequel. In this case, divided powers are called **divided squares**. Property (6) gives:

(6') $$\gamma_i(\gamma_2(x)) = \gamma_{2i}(x).$$

Moreover, if one consider **strictly commutative** graded algebras A in characteristic 2, i.e. commutative algebras such that $x^2 = 0$ if $\deg(x) > 0$, one can define $\gamma_i(x)$ on every element $x \in A$ such that $\deg(x) \geqslant 2$ (even or odd). In this case, properties (1) to (6) remains valid and property (5) becomes:

(5') $$\text{for all } i \geqslant 2,\ \gamma_i(xy) = \begin{cases} 0 & \text{if } \deg(x), \deg(y) \text{ are } > 0, \\ x^i\gamma_i(y) & \text{if } \deg(x) = 0. \end{cases}$$

Let us look at a very important exemple of divided powers algebra. The **divided polynomial algebra** $\Gamma_R[x]$ on one generator of even degree q is defined as a graded R-module having generators x_i, in degrees qi, for all $i \geqslant 0$, with the following properties:

$$\begin{aligned} x_0 &= 1, \\ x_1 &= x, \\ x_i x_j &= \binom{i+j}{i} x_{i+j} \qquad\qquad \text{for all } i, j \geqslant 0. \end{aligned}$$

This algebra is endowed with a divided power algebra system given by

$$\begin{aligned}
\gamma_0(x_i) &= 1,\\
\gamma_j(x_i) &= \prod_{k=i}^{(j-1)i} \binom{k+i-1}{k} x_{ij} \qquad \text{for all } i \geqslant 0 \text{ and } j \geqslant 1.
\end{aligned}$$

In particular, we have $\gamma_j(x) = x_j$ for all $j \geqslant 0$. Suppose that we are working in characteristic 2. Let $j = j_0 + j_1 2 + \cdots + j_k 2^k$ be the 2-adic developpment of j, with $j_0, \ldots, j_k \in \{0, 1\}$. Then property (6') implies that

$$\begin{aligned}
\gamma_j(x) &= \underbrace{\gamma_{j_0}(x)}_{\in\{1,x\}} \underbrace{\gamma_{j_1}(\gamma_2(x))}_{\in\{1,\gamma_2(x)\}} \ldots \underbrace{\gamma_{j_k}(\gamma_{2^k}(x))}_{\in\{1,\gamma_{2^k}(x)\}}\\
&= \gamma_{j_0}(x)\ \gamma_{j_1}(\gamma_2(x))\ \gamma_{j_2}(\gamma_2\gamma_2(x)) \ldots \gamma_{j_k}(\underbrace{\gamma_2 \ldots \gamma_2}_{k \text{ times}}(x))\\
&= \gamma_{j_0}(x)\ \gamma_{j_1}(\gamma_2(x))\ \gamma_{j_2}(\gamma_2^2(x)) \ldots \gamma_{j_k}(\gamma_2^k(x)).
\end{aligned}$$

It is then easy to see that the following isomorphisms hold:

$$\begin{aligned}
\Gamma_{\mathbb{F}_2}[x] &\cong \bigotimes_{i \geqslant 0} \Lambda_{\mathbb{F}_2}(\gamma_2^i x) && \text{as graded } \mathbb{F}_2\text{-algebras,}\\
&\cong \mathbb{F}_2[x] && \text{as graded } \mathbb{F}_2\text{-vector spaces,}
\end{aligned}$$

with $\gamma_2^i x \mapsto x^{2^i}$ for all $i \geqslant 0$.

The **exterior algebra** $\Lambda_R(x)$ on one generator of degree q is the graded quotient algebra $R[x]/(x^2)$ of the polynomial algebra $R[x]$. We will write $\Gamma[x]$ and $\Lambda(x)$ instead of $\Gamma_{\mathbb{Z}}[x]$ and $\Lambda_{\mathbb{Z}}(x)$ respectively.

2.2.5. Definition. Let (x, y) be a couple of positive bidegree $(q, q+1)$. For all $h \in \mathbb{Z}$ we define the **elementary complex** associated to (x, y) as follows:

$$EC_h(x,y) = \begin{cases} (\Gamma[x] \otimes \Lambda(y), d_h) & \text{if } q \text{ is even,}\\ (\Lambda(x) \otimes \Gamma[y], d_h) & \text{if } q \text{ is odd,} \end{cases}$$

with differential d_h given in both cases by

$$\begin{aligned}
d_h x &= 0 \text{ and}\\
d_h y &= hx.
\end{aligned}$$

2.2.6. Lemma. *Let (x, y) be a couple of positive bidegree $(q, q+1)$ and $h \in \mathbb{Z}$. Then we have*

$$H_n(EC_h(x,y);\mathbb{Z}) \cong$$

$$\begin{cases} \mathbb{Z}/|\ell h| < \gamma_\ell(x) > & \textit{if } q \textit{ is even and } n = \ell q\\ \mathbb{Z}/|h| < x\gamma_\ell(y) > & \textit{if } q \textit{ is odd and } n = (\ell+1)q + \ell\\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof is immediate from the definitions. See [**15**, Exposé 11, p. 3]. □

2.2.7. Remark. If x is of even degree $q \geqslant 2$ and $h \neq 0$, then there are homogeneous classes of **arbitrarily high order** in the integral homology of $EC_h(x, y)$, namely those of order $|\ell h|$ given by $\gamma_\ell(x)$ in dimensions ℓq for all $\ell \geqslant 1$.

2.2.8. Definition. Let G be a non-trivial finitely generated 2-torsion abelian group of type $(s_1, \ldots, s_l)$ and $n \geqslant 1$. Let $U = \{u_1, \ldots, u_l\}$ be a set of generators for G such that u_j is of order s_j for all $1 \leqslant j \leqslant l$. For all $n \geqslant 1$ we define the following complexes:

$$(X', d') = \bigotimes_{1\leqslant j\leqslant l} EC_{(-1)^{n-1}2^{s_j}}(\sigma^n u_j, \sigma^{n-1}\psi_{2^{s_j}} u_j),$$

$$(X'', d'') = \bigotimes_{\substack{1\leqslant j\leqslant l\\ 0\leqslant k\leqslant n-3\\ \alpha\in\mathcal{W}^{I}_{*,n-k-1}}} EC_{(-1)^k 2}(\beta_2\sigma^k\varphi_2\alpha u_j, \sigma^k\varphi_2\alpha u_j),$$

$$(X''', d''') = \bigotimes_{\substack{1\leqslant j\leqslant l\\ 0\leqslant k\leqslant n-2\\ \alpha\in\mathcal{W}^{II}_{*,n-k-1}}} EC_{(-1)^k 2}(\beta_2\sigma^k\varphi_2\alpha u'_j, \sigma^k\varphi_2\alpha u'_j),$$

$$\text{where } u'_j = 2^{s_j-1}u_j \text{ for all } 1 \leqslant j \leqslant l.$$

Generators of X', X'' and X''' are of **genus** 1, 2 and 3 respectively. The tensor product $(X, d) = (X', d') \otimes (X'', d'') \otimes (X''', d''') \otimes \mathbb{Z}_{(2)}$ is the **Cartan's complex** associated to G and n.

2.2.9. Remark. This definition is *ad hoc* for our computations at the prime 2. Actually, H. Cartan gives a much more general one for all primes. See [**15**, Exposé 11, pp.5-7] or [**48**, Définition 2.6, pp. 32-33] for a complete and original definiton.

2.2.10. Theorem (H. Cartan, 1955)**.** *Let G be a non-trivial finitely generated 2-torsion abelian group, $n \geqslant 1$ and X the associated Cartan's complex. There is an isomorphism of complexes*

$$B^n\mathbb{Z}G \cong X$$

and, for any ring R, isomorphisms of graded R-modules

$$H_*(B^n\mathbb{Z}G; R) \cong H_*(X; R),$$
$$H^*(B^n\mathbb{Z}G; R) \cong H^*(X; R).$$

Proof. See [**15**, Exposé 11, pp. 7-10] and, specifically when R is of characteristic 2, see [**15**, Exposé 9, Dernière remarque, p. 10]. □

2.2.11. Corollary. *Let G be a non-trivial finitely generated 2-torsion abelian group, $n \geqslant 1$ and X the associated Cartan's complex. For any ring R, there are isomorphisms of graded R-modules*

$$H_*(K(G,n);R) \cong H_*(X;R) \text{ and}$$
$$H^*(K(G,n);R) \cong H^*(X;R).$$

PROOF. We have

$$\begin{aligned} H_*(K(G,n);R) &\cong H_*(B^n(C_*(K(G,0)))) && \text{by Theorem 2.2.1} \\ &\cong H_*(B^n\mathbb{Z}G;R) \\ &\cong H_*(X;R) && \text{by Theorem 2.2.10,} \end{aligned}$$

the last isomorphism being an isomorphism of graded R-modules. □

2.2.12. Theorem. *Let $n \geqslant 1$ and $s \geqslant 1$. The graded $\mathbb{F}_2$-vector space $H_*(K(\mathbb{Z}/2^s,n);\mathbb{F}_2)$ is isomorphic to*

$$\Gamma_{\mathbb{F}_2}[\mathcal{W}^+_{*,n}] \otimes \Lambda_{\mathbb{F}_2}(\mathcal{W}^-_{*,n}),$$

where $\mathcal{W}^+_{*,n}$ and $\mathcal{W}^-_{*,n}$ denote the set of all even, respectively odd degree elements in $\mathcal{W}_{*,n}$.

PROOF. It is obvious to see that the complex $X \otimes \mathbb{Z}/2$ is acyclic. Therefore, its homology is given by $X \otimes \mathbb{Z}/2$ itself. Let us compute $EC_0(x,y) \otimes \mathbb{F}_2$. If x is of even degree, then

$$EC_0(x,y) \otimes \mathbb{F}_2 = (\Gamma[x] \otimes \Lambda(y)) \otimes \mathbb{F}_2 \cong \Gamma_{\mathbb{F}_2}[x] \otimes \Lambda_{\mathbb{F}_2}(y)$$

as graded $\mathbb{F}_2$-vector spaces. A similar result holds when x is of odd degree. To conclude, it suffices now to see that

$$\begin{aligned} \mathcal{W}_{*,n} =& \{\sigma^n, \sigma^{n-1}\psi_{2^s}\} \cup \\ & \bigcup_{\substack{0 \leqslant k \leqslant n-3 \\ \alpha \in \mathcal{W}^I_{*,n-k-1}}} \{\beta_2\sigma^k\varphi_2\alpha, \sigma^k\varphi_2\alpha\} \cup \\ & \bigcup_{\substack{0 \leqslant k \leqslant n-2 \\ \alpha \in \mathcal{W}^{II}_{*,n-k-1}}} \{\beta_2\sigma^k\varphi_2\alpha, \sigma^k\varphi_2\alpha\}. \end{aligned}$$

□

A very similar proof gives the following dual result in cohomology:

2.2.13. Theorem. *Let $n \geqslant 1$ and $s \geqslant 1$. The graded $\mathbb{F}_2$-vector space $H^*(K(\mathbb{Z}/2^s,n);\mathbb{F}_2)$ is isomorphic to*

$$\mathbb{F}_2[\mathcal{W}^+_{*,n}] \otimes \Lambda_{\mathbb{F}_2}(\mathcal{W}^-_{*,n}).$$

2.2.14. Corollary. *Let $n \geqslant 1$ and $s \geqslant 1$. The differential graded $\mathbb{F}_2$-algebra $H_*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$ is the connected, biassociative, bicommutative and coprimitive differential graded Hopf algebra given by $\Gamma_{\mathbb{F}_2}[\mathcal{W}^+_{*,n}] \otimes \Lambda_{\mathbb{F}_2}(\mathcal{W}^-_{*,n})$.*

Proof. It suffices to remark that $\Gamma_{\mathbb{F}_2}[x]^{\mathrm{dual}} \cong \mathbb{F}_2[x]$ as Hopf algebras and to consider Theorem 2.1.4. $\square$

2.2.15. Example. Since $\mathcal{W}_{*,1} = \{\sigma, \psi_2\}$ with $\deg(\sigma) = 1$ and $\deg(\psi_2) = 2$ we have

$$H^*(K(\mathbb{Z}/2, 1); \mathbb{F}_2) \cong \mathbb{F}_2[\psi_2] \otimes \Lambda_{\mathbb{F}_2}(\sigma).$$

It is easy to verify that we have an isomorphism of graded $\mathbb{F}_2$-vector spaces

$$\mathbb{F}_2[\psi_2] \otimes \Lambda_{\mathbb{F}_2}(\sigma) \cong \mathbb{F}_2[u_1],$$

where $\mathbb{F}_2[u_1]$ is the Serre's description of the cohomology algebra (see Theorem 2.1.3), given by

$$\begin{aligned} \psi_2^k \otimes 1 &\mapsto u_1^{2k}, \\ \psi_2^k \otimes \sigma &\mapsto u_1^{2k+1}, \end{aligned}$$

for all $k \geqslant 0$.

2.3. Duality

2.3.1. Theorem (duality). *Suppose (C_*, d) is a chain complex with finite type homology. Let $\{B^r_*, d^r\}$ denote its associated mod-2 homology Bockstein spectral sequence and let $\{B^*_r, d_r\}$ denote the mod-2 cohomology spectral sequence associated to the cochain complex $(\mathrm{Hom}(C_*, \mathbb{Z}), \delta)$. Then $\{B^r_*, d^r\}$ and $\{B^*_r, d_r\}$ are dual in the following sense: $B^*_r \cong \mathrm{Hom}(B^r_*, \mathbb{F}_2)$ and d^r is adjoint to d_r, i.e.*

$$< d^r x, \bar{x} >_r = < x, d_r \bar{x} >_r$$

where $< x, \bar{x} >_r$ is the evaluation on $x \in B^r_n$ by the map in $\mathrm{Hom}(B^r_n, \mathbb{F}_2)$ corresponding to $\bar{x} \in B^n_r$ via the above isomorphism. Moreover, if $f : (C_*, d) \to (C'_*, d')$ is a morphism of chain complexes, then f^r is the adjoint of f_r, i.e.

$$< f^r x, \bar{x} >_r = < x, f_r \bar{x} >_r .$$

If X is a H-space of finite type, then the mod-2 homology Bockstein spectral sequence for X, $B^r_*(X) = B^r_*(C_*(X), d)$, is a spectral sequence of Hopf algebras dual to the mod-2 cohomology Hopf algebras $B^*_r(X)$.

Proof. See [**13**, Proposition 1.4, p. 28 and Proposition 4.7, pp. 36-37] or [**36**, Theorem 10.12, p. 466]. □

2.3.2. Convention. When no confusion is possible, we will write $< -, - >$ instead of $< -, - >_r$. The notation $< -, - >_r$ will only be used when the situation requires it.

2.3.3. Lemma. *Let $n \geqslant 1$ and $s \geqslant 1$. Consider the dual mod-2 homology and cohomology Bockstein spectral sequences $\{B^r_*, d^r\}$ and $\{B^*_r, d_r\}$ associated to $K(\mathbb{Z}/2^s, n)$. Let $x \in B^r_m$ and $x' \in B^m_r$. Then we have*

$$< \gamma_2(x), (x')^2 > = \begin{cases} < x, x' > & \text{if } \gamma_2(x) \neq 0 \text{ and } (x')^2 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [**15**, Exposé 15, Théorème 3, pp. 8-9]. □

Recall that $\mathcal{S}$ denotes the set of all admissible sequences.

2.3.4. Definition. Let $(-)^- : \mathcal{S} \to \mathcal{S}$ be the map given by $I^- = (a_1, \ldots, a_k)$ for all $I = (a_0, \ldots, a_k) \in \mathcal{S}$.

2.3.5. Definition. For any integer $n \geqslant 0$ and any admissible sequence $I = (a_0, \ldots, a_k) \in \mathcal{S}$ with $e(I) \leqslant n$ we define an admissible word $g_I(n) \in \mathcal{W}$ inductively as follows:

$$
\begin{aligned}
&g_{(0)}(n) = \sigma^n, \\
&g_{(1)}(n) = \sigma^{n-1}\psi_2, \\
&g_I(n) = \begin{cases} \beta_2 \sigma^{n-e(I)-1}\varphi_2 g_{I^-}(e(I)) & \text{if } a_0 \equiv 0(2) \text{ and } e(I) < n, \\ \gamma_2 g_{I^-}(n) & \text{if } a_0 \equiv 0(2) \text{ and } e(I) = n, \\ \sigma^{n-e(I)}\varphi_2 g_{I^-}(e(I)-1) & \text{if } a_0 \equiv 1(2), \end{cases} \\
&\qquad\qquad \text{when } I \neq (0) \text{ and } (1).
\end{aligned}
$$

2.3.6. Theorem. *Let $n \geqslant 1$ and $s \geqslant 1$. Consider the dual graded $\mathbb{F}_2$-vector spaces $H_* = H_*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$ and $H^* = H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$. Let $u \in \mathbb{Z}/2^s$ be a generator, $u' = 2^{s-1}u$, and $u_n \in H^n$ be the reduction mod-2 of the fundamental class in $H^n(K(\mathbb{Z}/2^s, n); \mathbb{Z}/2^s)$. Then we have the following* **duality relations***:*

(D1) $\quad < \sigma^n u, u_n >= 1,$

(D2) $\quad < \sigma^{n-1}\psi_{2^s} u, d_s u_n >= 1,$

(D3) $\quad < g_I(n)u, Sq_s^J u_n >= \delta_{IJ} \qquad$ *if $g_I(n) \in \mathcal{W}^I$,*

(D4) $\quad < g_I(n)u', Sq_s^J u_n >= \delta_{IJ} \qquad$ *if $g_I(n) \in \mathcal{W}^{II}$,*

where $\delta_{IJ} = \begin{cases} 1 & \textit{if } I = J, \\ 0 & \textit{if } I \neq J. \end{cases}$

Proof. Let us begin with relation (D1). We have

$$
\begin{aligned}
&< \sigma^n u, u_n > \\
&\quad =< u, u_0 > && \text{by duality,} \\
&\quad = 1 && \text{by definition.}
\end{aligned}
$$

For relation (D2) we have

$$
\begin{aligned}
&< \psi_{2^s} u, d_s u_1 > \\
&\quad =< d^s \psi_{2^s} u, u_1 > && \text{by duality,} \\
&\quad =< \sigma u, u_1 > && \text{by [\textbf{15}, Exposé 11, p. 1],} \\
&\quad =< u, u_0 > && \text{by duality,} \\
&\quad = 1 && \text{by definition.}
\end{aligned}
$$

We will prove relation (D3) when $I = J$ by induction on the length of the admissible sequence $I = (a_0, \ldots, a_k)$. Suppose $a_0 \equiv 0(2)$ and

$e(I) < n$. Then we have

$$
\begin{aligned}
&< g_I(n)u, Sq_s^I u_n > \\
&\quad =< \beta_2\sigma^{n-e(I)-1}\varphi_2 g_{I^-}(e(I))u, Sq_s^I u_n > && \text{by definition,}\\
&\quad =< \varphi_2 g_{I^-}(e(I))u, Sq^1 Sq_s^I u_{e(I)+1} > && \text{by duality,}\\
&\quad =< \gamma_2\sigma g_{I^-}(e(I))u, Sq^1 Sq_s^I u_{e(I)+1} > && \text{by basic relation (R2),}\\
&\quad =< \sigma g_{I^-}(e(I))u, Sq_s^{I^-} u_{e(I)+1} > && \text{by Lemma 2.3.3,}\\
&\quad =< g_{I^-}(e(I))u, Sq_s^{I^-} u_{e(I)} > && \text{by duality,}\\
&\quad = 1 && \text{by induction.}
\end{aligned}
$$

Suppose $a_0 \equiv 0(2)$ and $e(I) = n$. Then we have

$$
\begin{aligned}
&< g_I(n)u, Sq_s^I u_n > \\
&\quad =< \gamma_2 g_{I^-}(n)u, Sq_s^I u_n > && \text{by definition,}\\
&\quad =< g_{I^-}(n)u, Sq_s^{I^-} u_n > && \text{by Lemma 2.3.3,}\\
&\quad = 1 && \text{by induction.}
\end{aligned}
$$

Suppose $a_0 \equiv 1(2)$. Then we have

$$
\begin{aligned}
&< g_I(n)u, Sq_s^I u_n > \\
&\quad =< \sigma^{n-e(I)}\varphi_2 g_{I^-}(e(I)-1)u, Sq_s^I u_n > && \text{by definition,}\\
&\quad =< \varphi_2 g_{I^-}(e(I)-1)u, Sq_s^I u_{e(I)} > && \text{by duality,}\\
&\quad =< \gamma_2\sigma g_{I^-}(e(I)-1)u, Sq_s^I u_{e(I)} > && \text{by basic relation (R2),}\\
&\quad =< \sigma g_{I^-}(e(I)-1)u, Sq_s^{I^-} u_{e(I)} > && \text{by Lemma 2.3.3,}\\
&\quad =< g_{I^-}(e(I)-1)u, Sq_s^{I^-} u_{e(I)-1} > && \text{by duality,}\\
&\quad = 1 && \text{by induction.}
\end{aligned}
$$

We conclude by remarking that the first step of the induction is given by (D1). Relation (D4) when $I = J$ is proved by substituting u' to u in the three preceeding computations and by remarking that the first step of the induction is given by $g_{(1)}(1)$, which means that it remains to compute $< \psi_2 u', Sq_s^1 u_1 >$. By [**15**, Exposé 6, p. 9], $\psi_2 u'$ is the single generator of $H_2(K(\mathbb{Z}/2^s, 1); \mathbb{F}_2)$, and by Theorem 2.1.3, $Sq_s^1 u_1 = \delta_s \iota_1$ is the single generator of $H^*(K(\mathbb{Z}/2^s, 1); \mathbb{F}_2)$. Thus the desired result $< \psi_2 u', Sq_s^1 u_1 >= 1$ is forced by duality. The duality relations (D3) and (D4) when $I \neq J$ are established by induction in a very similar way. $\square$

2.4. Transverse implications in Eilenberg-MacLane spaces

Recall from Section 1.3 the definition of an ℓ-transverse element in a mod-2 cohomology Bockstein spectral sequence.

1.3.1. Definition. Let X be a space and $\{B_r^*, d_r\}$ be its mod-2 cohomology Bockstein spectral sequence. Let n and r be two positive integers. An element $x \in B_r^n$ is said to be **ℓ-transverse** if $d_{r+l}x^{2^l} \neq 0 \in B_{r+l}^{2^l n}$ for all $0 \leqslant l \leqslant \ell$. An element $x \in B_r^n$ is said to be **∞-transverse**, or simply **transverse**, if it is ℓ-transverse for all $\ell \geqslant 0$. We will also speak of **transverse implications** of an element $x \in B_r^n$.

We will also need the notion of transverse implications in the mod-2 homology Bockstein spectral sequence.

2.4.1. Definition. Let X be a space and $\{B_*^r, d^r\}$ be its mod-2 homology Bockstein spectral sequence. Moreover, assume that B_*^1 is endowed with a divided square algebra structure. Recall that in this case $\gamma_2^l x = \underbrace{\gamma_2 \circ \cdots \circ \gamma_2(x)}_{l \text{ times}}$ for any $l \geqslant 0$. An element $x \in B_n^r$ is said to be **ℓ-transverse** if $\gamma_2^l x \neq 0 \in \operatorname{im} d^{r+l} \subset B_{2^l n}^{r+l}$ for all $0 \leqslant l \leqslant \ell$. An element $x \in B_n^r$ is said to be **∞-transverse**, or simply **transverse**, if it is ℓ-transverse for all $\ell \geqslant 0$.

2.4.2. Definition. For all $r \geqslant 1$ and $m \geqslant 0$, we formally define the following assertions:

$$
\begin{aligned}
P_m^r(x,x') :& \quad \text{“ } \gamma_2^{m+1}x \neq 0 \in \operatorname{im} d^{r+m+1} \text{ ”} \\
Q_m^r(x,x') :& \quad \text{“ } \gamma_2^{m+1+k}x \neq 0 \in B^{r+m+1} \text{ for all } k \geqslant 0 \text{ ”} \\
R_m^r(x,x') :& \quad \text{“ } < \gamma_2^{m+k}x, (x')^{2^{m+k}} >_{r+m} = 1 \text{ for all } k \geqslant 0 \text{ ”} \\
S_m^r(x,x') :& \quad \text{“ } d_{r+m}(x')^{2^m} \neq 0 \in B_{r+m} \text{ ”}
\end{aligned}
$$

2.4.3. Lemma. *Let X be a space, $\{B_*^r, d^r\}$ and $\{B_r^*, d_r\}$ be its mod-2 homology and cohomology Bockstein spectral sequences. Moreover, assume that B_1^* and B_*^1 are endowed with a polynomial and a divided square algebra structure respectively. Let $x \in B_n^r$ and $x' \in B_r^n$. If $P_m^r(x,x')$, $Q_m^r(x,x')$, $R_m^r(x,x')$ and $S_m^r(x,x')$, then $R_{m+1}^r(x,x')$ and $S_{m+1}^r(x,x')$.*

Proof. The fact that $d_{r+m}(x')^{2^m} \neq 0$ (i.e. $S_m^r(x,x')$) implies that $d_{r+m}(x')^{2^{m+1+k}} = 0$ for all $k \geqslant 0$. If we suppose that $(x')^{2^{m+1+k}} = 0 \in B_{r+m+1}$ for some $k \geqslant 0$, then it has to exist $z \in B_{r+m}$ such that

$(x')^{2^{m+1+k}} = d_{r+m}z$. In this case we have

$$\begin{aligned} 1 &= <\gamma_2^{m+1+k}x, (x')^{2^{m+1+k}}>_{r+m} && \text{since } R_m^r(x,x'),\\ &= <\gamma_2^{m+1+k}x, d_{r+m}z>_{r+m}\\ &= <d^{r+m}\gamma_2^{m+1+k}x, z>_{r+m}\\ &= 0 \end{aligned}$$

since $\gamma_2^{m+1+k}x \neq 0 \in B^{r+m+1}$ (i.e. $Q_m^r(x,x')$) implies that $d^{r+m}\gamma_2^{m+1+k} = 0$. This contradiction forces $(x')^{2^{m+1+k}} \neq 0 \in B_{r+m+1}$ for all $k \geqslant 0$. As an obvious consequence we have

$$R_{m+1}^r(x,x'):\qquad <\gamma_2^{m+1+k}x, (x')^{2^{m+1+k}}>_{r+m+1} = 1 \quad \text{for all } k \geqslant 0.$$

When $k = 0$ we have

$$\begin{aligned} 1 &= <\gamma_2^{m+1}x, (x')^{2^{m+1}}>_{r+m+1}\\ &= <d^{r+m+1}w, (x')^{2^{m+1}}>_{r+m+1} && \text{for some } w \in B^{r+m+1} \text{ since } P_m^r(x,x'),\\ &= <w, d_{r+m+1}(x')^{2^{m+1}}>_{r+m+1} && \text{by duality,} \end{aligned}$$

which implies that $d_{r+m+1}(x')^{2^{m+1}} \neq 0 \in B_{r+m+1}$ (i.e. $S_{m+1}^r(x,x')$). □

2.4.4. Theorem. *Let X be a space, $\{B_*^r, d^r\}$ and $\{B_r^*, d_r\}$ be its mod-2 homology and cohomology Bockstein spectral sequences. Moreover, assume that B_1^* and B_*^1 are endowed with a polynomial and a divided square algebra structure respectively. Let $x \in B_n^r$ and $x' \in B_r^n$. If $<\gamma_2^k x, (x')^{2^k}>_r = 1$ for all $k \geqslant 0$, i.e. $R_0^r(x,x')$, and x is ∞-transverse, then x' is ∞-transverse.*

Proof. We will prove $R_m^r(x,x')$ and $S_m^r(x,x')$ for all $m \geqslant 0$ by induction on m. We have

$$\begin{aligned} 1 &= <x, x'>_r && \text{since } R_0^r(x,x') \text{ (case } k=0),\\ &= <d^r w, x'>_r && \text{for some } w \in B^r \text{ since } x \text{ is 0-transverse,}\\ &= <w, d_r x'>_r && \text{by duality.} \end{aligned}$$

which implies that $d_r x' \neq 0 \in B_r$, i.e. $S_0^r(x,x')$. Before we start our induction process on m, let us remark that we have $P_m^r(x,x')$ and $Q_m^r(x,x')$ for all $m \geqslant 0$ since x is ∞-transverse (this is an obvious consequence of the definition). Suppose now that $R_m^r(x,x')$ and $S_m^r(x,x')$. Lemma 2.4.3 implies that $R_{m+1}^r(x,x')$ and $S_{m+1}^r(x,x')$. In particular, we just have proved that $d_{r+m}(x')^{2^m} \neq 0 \in B_{r+m}$ (i.e. $S_m^r(x,x')$) for all $m \geqslant 0$. In other words, x' is ∞-transverse. □

2.4.5. Lemma. *Let $n \geqslant 1$ and $s \geqslant 1$. Consider the dual mod-2 homology and cohomology Bockstein spectral sequences $\{B_*^r, d^r\}$ and $\{B_r^*, d_r\}$ associated to $K(\mathbb{Z}/2^s, n)$. Let $\sigma^n u \in B_n^s$ and $u_n \in B_s^n$. We have $<\gamma_2^k \sigma^n u, (u_n)^{2^k}>_s = 1$ for all $k \geqslant 0$, i.e. $R_0^s(\sigma^n u, u_n)$.*

Proof. By Lemma 2.3.3, we have $R_0^1(\sigma^n u, u_n)$. It suffices then to see that $(u_n)^{2^k} \neq 0 \in B_s$ for all $k \geqslant 0$. Suppose that there exist $k \geqslant 0$ and $1 \leqslant r \leqslant s-1$ such that $(u_n)^{2^k} = d_r z \neq 0 \in B_r$ for some $z \in B_r$. Then we have the following contradiction:

$$\begin{aligned} 1 &= < \gamma_2^k \sigma^n u, (u_n)^{2^k} >_r && \text{since } \gamma_2^k \sigma^n u \text{ and } (u_n)^{2^k} \neq 0 \\ &= < \gamma_2^k \sigma^n u, d_r z >_r && \\ &= < d^r \gamma_2^k \sigma^n u, z >_r && \text{by duality,} \\ &= 0 && \end{aligned}$$

since $\gamma_2^k \sigma^n u \neq 0 \in B^s$ implies that $d^r \gamma_2^k \sigma^n u = 0$. Moreover we have $d_s u_n \neq 0 \in B_s$. This implies that $u_n \neq 0 \in B_r$ and $d_r u_n = 0 \in B_r$ for all $1 \leqslant r \leqslant s-1$. Therefore, $(u_n)^{2^k}$ is a square in B_r for all $k \geqslant 1$ and $1 \leqslant r \leqslant s$. Thus we have $d_r (u_n)^{2^k} = 0$ for all $k \geqslant 0$ and $1 \leqslant r \leqslant s-1$. □

In order to detect ∞-transverse implications in the cohomology of $K(G,n)$, it suffices now to detect ∞-transverse implications in homology by using Cartan's methods and to use duality relations of Theorem 2.3.6.

As we have already mentioned in Remark 2.2.7, elements of arbitrarily high order in the integral homology of $K(\mathbb{Z}/2^s, n)$ are given by divided squares of even degree admissible words of the form σ^n, $\beta_2 \sigma^k \varphi_2 \alpha$ with $0 \leqslant k \leqslant n-3$, $\alpha \in \mathcal{W}^I_{*,n-k-1}$ and $\beta_2 \sigma^k \varphi_2 \alpha$ with $0 \leqslant k \leqslant n-2$, $\alpha \in \mathcal{W}^{II}_{*,n-k-1}$. Now a simple application of Lemma 2.2.6 gives the following results.

2.4.6. Lemma. *Let $n \geqslant 1$, $s \geqslant 1$ and u be a generator of $\mathbb{Z}/2^s$. If n is even then $\gamma_2^r \sigma^n u \in H_{2^r n}(K(\mathbb{Z}/2^s, n); \mathbb{Z})$ is of order 2^{r+s} for all $r \geqslant 0$. In other words, $\sigma^n u \in B_n^s$ is ∞-transverse.*

Proof. If we consider the Cartan's complex associated to $K(\mathbb{Z}/2^s, n)$ and in particular the elementary complex $EC_{(-1)^{n-1}2^s}(\sigma^n u, \sigma^{n-1}\psi_{2^s} u)$, Lemma 2.2.6 implies that $\gamma_2^r \sigma^n u$ is of order $2^r \cdot 2^s = 2^{r+s}$ since $\sigma^n u$ lies in even degree. □

2.4.7. Lemma. *Let $n \geqslant 1$, $s \geqslant 1$ and u be a generator of $\mathbb{Z}/2^s$. If $0 \leqslant k \leqslant n-3$ and $\alpha \in \mathcal{W}^I_{*,n-k-1}$ are such that $\beta_2 \sigma^k \varphi_2 \alpha u \in H_*(K(\mathbb{Z}/2^s, n); \mathbb{Z})$ lies in even degree, then $\gamma_2^r \beta_2 \sigma^k \varphi_2 \alpha u$ is of order 2^{r+1} for all $r \geqslant 0$. In other words, $\beta_2 \sigma^k \varphi_2 \alpha u \in B_*^1$ is ∞-transverse.*

Proof. If we consider the Cartan's complex associated to $K(\mathbb{Z}/2^s, n)$ and in particular the elementary complex $EC_{(-1)^k 2}(\beta_2 \sigma^k \varphi_2 \alpha u, \sigma^k \varphi_2 \alpha u)$, Lemma 2.2.6 implies that $\gamma_2^r \beta_2 \sigma^k \varphi_2 \alpha u$ is of order $2^r \cdot 2 = 2^{r+1}$ since $\beta_2 \sigma^k \varphi_2 \alpha u$ lies in even degree. □

2.4.8. Lemma. *Let $n \geqslant 1$, $s \geqslant 1$, u be a generator of $\mathbb{Z}/2^s$ and $u' = 2^{s-1}u$. If $0 \leqslant k \leqslant n-2$ and $\alpha \in \mathcal{W}^{II}_{*,n-k-1}$ are such that $\beta_2\sigma^k\varphi_2\alpha u' \in H_*(K(\mathbb{Z}/2^s, n); \mathbb{Z})$ lies in even degree, then $\gamma_2^r\beta_2\sigma^k\varphi_2\alpha u'$ is of order 2^{r+1} for all $r \geqslant 0$. In other words, $\beta_2\sigma^k\varphi_2\alpha u' \in B^1_*$ is ∞-transverse.*

PROOF. Analogous to the two previous proofs. $\square$

We are now able to establish conditions for detecting ∞-transverse elements in the cohomology of Eilenberg-MacLane spaces.

1.3.2. Theorem. *Let G be a non-trivial finitely generated 2-torsion abelian group of type $(s_1, \ldots, s_l)$ and let $n \geqslant 2$. Consider the Eilenberg-MacLane space $K(G, n)$ and its mod-2 cohomology Bockstein spectral sequence $\{B^*_r, d_r\}$. Suppose that one of the following assumptions holds:*

- *n is even and $x \in B^n_{s_j}$ is 0-transverse for any $1 \leqslant j \leqslant l$,*
- *$x \in P^{\text{even}} B^*_1$ is 0-transverse ($Sq^1 x \neq 0$).*

Then x is ∞-transverse.

PROOF. It is clearly sufficient to establish the result when $G = \mathbb{Z}/2^s$, $s \geqslant 1$.

Assume that n is even and $x \in B^n_s$ is 0-transverse. Then $x = u_n$. By Lemma 2.4.5, we have $R^s_0(\sigma^n u, u_n)$. By Lemma 2.4.6, $\sigma^n u$ is ∞-transverse. Then u_n is ∞-transverse by Theorem 2.4.4.

Assume now that $x \in P^{\text{even}} B^*_1$ is such that $Sq^1 x \neq 0$. Then we can write $x = \sum_{1 \leqslant i \leqslant j} Sq^{I_i}_s u_n$ with $\#\{I_i\} = j$. Pick $I = I_i$ such that $Sq^1 Sq^I_s u_n \neq 0$.

Suppose that $g_I(n) \in \mathcal{W}^I$. Then we have

$$\begin{aligned}
< g_I(n)u, x > &=< g_I(n)u, \sum_{1\leqslant i\leqslant j} Sq^{I_i}_s u_n > &&\text{by assumption,}\\
&= \sum_{1\leqslant i\leqslant j} < g_I(n)u, Sq^{I_i}_s u_n > &&\text{by linearity,}\\
&=< g_I(n)u, Sq^I_s u_n >= 1 &&\text{by Theorem 2.3.6 (D3).}
\end{aligned}$$

By Lemma 2.3.3, we have $R^1_0(g_I(n)u, Sq^I_s u_n)$. By Lemma 2.4.7, $g_I(n)u$ is ∞-transverse. Then x is ∞-transverse by Theorem 2.4.4.

Suppose that $g_I(n) \in \mathcal{W}^{II}$. Then we have

$$\begin{aligned}
< g_I(n)u', x > &=< g_I(n)u', \sum_{1\leqslant i\leqslant j} Sq^{I_i}_s u_n > &&\text{by assumption,}\\
&= \sum_{1\leqslant i\leqslant j} < g_I(n)u', Sq^{I_i}_s u_n > &&\text{by linearity,}\\
&=< g_I(n)u', Sq^I_s u_n >= 1 &&\text{by Theorem 2.3.6 (D4).}
\end{aligned}$$

By Lemma 2.3.3, we have $R^1_0(g_I(n)u', Sq^I_s u_n)$. By Lemma 2.4.8, $g_I(n)u'$ is ∞-transverse. Then x is ∞-transverse by Theorem 2.4.4. $\square$

1.3.3. Corollary. *Let G be a non-trivial finitely generated 2-torsion abelian group and $n \geqslant 2$. The Eilenberg-MacLane space $K(G, n)$ has no homology exponent.*

PROOF. According to the Künneth formula, it is sufficient to establish the result when $G = \mathbb{Z}/2^s$ for some $s \geqslant 1$. If n is even, consider the fundamental class $u_n \in H^n(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$. This class survives to B_s^n and is 0-transverse. Then $u_n \in B_s^n$ is ∞-transverse. If n is odd, consider the admissible sequence $(2, 1)$. Its excess is exactly 1 and therefore $Sq_s^{2,1} u_n \in P^{\text{even}} H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2)$ when $n \geqslant 3$. Moreover we have $Sq^1 Sq_s^{2,1} u_n = Sq_s^{3,1} u_n$ by Adem relations, which means that $Sq_s^{2,1} u_n$ is 0-transverse. Hence $Sq_s^{2,1} u_n \in B_1^{n+3}$ is ∞-transverse. $\square$

Chapter 3 will be devoted to the "Eilenberg-MacLane machine" which is a C++ program designed to compute the integral homology groups of the Eilenberg-MacLane spaces. The proof of Corollary 1.3.3 enables us to predict some results that the computations of the machine should confirm.

For instance, let us look at the Eilenberg-MacLane space $K(\mathbb{Z}/2, 2)$ and its integral homology and cohomology groups $H_* = H_*(K(\mathbb{Z}/2, 2); \mathbb{Z})$ and $H^* = H^*(K(\mathbb{Z}/2, 2); \mathbb{Z})$ respectively. The fundamental class $u_2 \in H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$ is ∞-transverse. This implies that the classes u_2, u_2^2, u_2^4, and more generally $(u_2)^{2^l}$ for all $l \geqslant 0$, give elements of order 2, 4, 8 and 2^{l+1} in $H^3 \cong H_2$, $H^5 \cong H_4$, $H^9 \cong H_8$ and $H^{2^{l+1}+1} \cong H_{2^{l+1}}$ respectively. A glance at Table C.1, p. 91, shows that $H_2 \cong \mathbb{Z}/2$, $H_4 \cong \mathbb{Z}/4$, $H_8 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8$, and so on. This corroborates our guesses.

Let us finally look at the Eilenberg-MacLane space $K(\mathbb{Z}/2, 3)$ and its integral homology and cohomology groups $H_* = H_*(K(\mathbb{Z}/2, 3); \mathbb{Z})$ and $H^* = H^*(K(\mathbb{Z}/2, 3); \mathbb{Z})$ respectively. The element $Sq^{2,1} u_3$, which lies in degree 6, is ∞-transverse. This implies that there are elements of order 2, 4, 8, and more generally 2^{l+1} for all $l \geqslant 0$, in $H^7 \cong H_6$, $H^{13} \cong H_{12}$, $H^{25} \cong H_{24}$ and $H^{6 \cdot 2^l + 1} \cong H_{6 \cdot 2^l}$ respectively. A glance at Table C.2, p. 98, shows that $H_6 \cong \mathbb{Z}/2$, $H_{12} \cong \mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/4$, $H_{24} \cong \mathbb{Z}/2^{\oplus 36} \oplus \mathbb{Z}/8$, and so on.

2.5. A counterexample towards transverse implications

Let us consider the mod-2 cohomology Bockstein spectral sequence of BSO, the classifying space of the special orthogonal group. It is well known that

$$H^*(BSO;\mathbb{F}_2) \cong \mathbb{F}_2[w_j \mid j \geqslant 2],$$

where $\deg(w_j) = j$ for all $j \geqslant 1$ (see [**6**] or [**36**, p. 216]). By Wu's formulae (see for instance [**42**, 8, Part I, p. 138]), it is also well known that

$$Sq^1 w_j = \begin{cases} w_{j+1} & \text{if } j \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

These informations are sufficient to deduce the entire Bockstein spectral sequence. Actually, we have

$$\begin{aligned}
B_1^* &\cong \mathbb{F}_2[w_j \mid j \geqslant 2],\\
B_2^* &\cong \mathbb{F}_2[w_{2j}^2 \mid j \geqslant 1],\\
B_3^* &\cong B_2^*, \text{ since } d_2 \text{ is trivial for dimension reasons,}\\
&\vdots\\
B_\infty^* &\cong B_2^* \cong \mathbb{F}_2[w_{2j}^2 \mid j \geqslant 1].
\end{aligned}$$

Let us consider the lowest degree element w_2 which is obviously primitive and 0-transverse since $Sq^1 w_2 = w_3$ by Wu's formulae. We have $d_2 w_2^2 = 0$ and therefore w_2 is not 1-transverse. This shows that the hypothesis of Theorem 1.3.2 are not sufficient for a generalization to all simply connected H-spaces.

CHAPTER 3

The “Eilenberg-MacLane machine”

The “Eilenberg-MacLane machine” is a program which computes the homology groups $H_*(K(\mathbb{Z}/2^s, n); \mathbb{Z})$ for all $n \geqslant 1$, $s \geqslant 1$.

A first version, written in C, is actually able to do more since it computes $H_*(K(\mathbb{Z}/p^s, n); \mathbb{Z})$ for all prime p. This version is far from being optimized and is based on the “rude force” of the computer you are working with. The aim was only to obtain a very reliable code. The program finds, in a very naive way, all admissible p-words (see [**15**, Exposé 9, p. 1] for a definition when $p \neq 2$) of given height and degree. It happens that it is a tricky, recursive problem which takes a lot of time and memory to be achieved.

A second version, written in C++, provides an optimized and reliable code at the prime 2. It is based on the results of Appendix B.1 which mainly state that admissible words are in one-to-one correspondance with admissible sequences. As it is simpler to determine all admissible sequences of given excess and degree than admissible words, the process reveals fast and parcimonious in terms of memory.

Since our interest lies only at the prime 2 in this work, we will only describe the second version of the “Eilenberg-MacLane machine” in Section 3.1.

Appendix C contains tables which were computed with the “Eilenberg-MacLane machine”. A glance at these tables reveals some heuristic results that are proved in Section 3.2.

For those who are interested in the C++ implementation, Appendix D contains some relevant header files.

3.1. Main algorithms

In this section, we state the main algorithms necessary for the Eilenberg-MacLane machine. Algorithms are given in so-called “pseudo-code”.

The next two algorithms give the list of all admissible sequences of bounded excess and degree. This is a recursive task that we split into two processes.

3.1.1. Algorithm. **Ensure:** collection $\mathcal{C}$ of all admissible sequences I such that $e(I) < n$ and $\deg_{\mathrm{st}}(I) \leqslant d$;

Require: $n \geqslant 1$ and $d \geqslant 0$;

$\mathcal{C}$ is a collection of admissible sequence for output;
set $\mathcal{C}$ as containing the single admissible sequence (0);
repeat
  call Algorithm 3.1.3 with $\mathcal{C}$ as input;
until $\mathcal{C}$ is stable;
output $\mathcal{C}$;

3.1.2. Definition. The **length** of an admissible sequence $I = (a_0, \dots, a_k)$ is given by k.

3.1.3. Algorithm. **Ensure:** $\mathcal{C}$ contains all admissible sequences of length $\leqslant \ell + 1$ such that $e(I) < n$ and $\deg_{\mathrm{st}}(I) \leqslant d$ (iterating process for Algorithm 3.1.1);
Require: a collection $\mathcal{C}$ of admissible sequences of length $\leqslant \ell$, $n \geqslant 1$ and $d \geqslant 0$;
$\mathcal{D}$ is a collection of admissible sequences;
set $\mathcal{D}$ as empty;
for all admissible sequence I of $\mathcal{C}$ **do**
  consider all the admissible sequences (a, I) such that $e(a, I) < n$ and $\deg_{\mathrm{st}}(a, I) \leqslant d$;
  put them in $\mathcal{D}$;
end for;
$\mathcal{C} \Leftarrow \mathcal{C} \cup \mathcal{D}$;
output $\mathcal{D}$;

The following algorithm implements Lemma 2.2.6.

3.1.4. Algorithm. **Ensure:** graded group $\widetilde{H}_{\leqslant d}(EC_h(x, y); \mathbb{Z}_{(2)})$;
Require: $\deg(x)$, $h \geqslant 1$ and $d \geqslant 0$;
$H_{\leqslant d}$ is the graded group for output;
set $H_{\leqslant d}$ as trivial in each degree;
for all $1 \leqslant m \leqslant d$ **do**
  if $\deg(x)$ is even **then**
    if $\deg(x)$ divides m **then**
      $H_m \Leftarrow$ cyclic group of order given by the 2-primary part of the integer $h \cdot \frac{m}{\deg(x)}$;
    end if;
  else if $\deg(x) + 1$ divides $m + 1$ **then**
    $H_m \Leftarrow$ cyclic group of order given by the 2-primary part of the integer h;
  end if;
end for;
output $H_{\leqslant d}$;

The next algorithm implements the Künneth formula for the homology of a tensor product of complexes.

3.1.5. Algorithm. <u>**Ensure:**</u> graded group $\widetilde{H}_{\leqslant d}(EC_h(x,y) \otimes_{\mathbb{Z}} C; \mathbb{Z}_{(2)})$ when C is a complex with known reduced homology graded group $\widetilde{H}_{\leqslant d}(C; \mathbb{Z}_{(2)})$;

Require: $d \geqslant 0$, $\deg(x)$, $h \geqslant 1$ and $\widetilde{H}_{\leqslant d}(C; \mathbb{Z}_{(2)})$;

1: $H_{\leqslant d}$ is the graded group for output;
2: set $H_{\leqslant d}$ as trivial in each degree;
3: call Algorithm 3.1.4 to compute $\widetilde{H}_{\leqslant d}(EC_h(x,y); \mathbb{Z}_{(2)})$
4: **for all** $1 \leqslant m \leqslant d$ **do**
5:     **for all** $p + q = m$ **do**
6:         $H_m \Leftarrow H_m \oplus \widetilde{H}_p(EC_h(x,y); \mathbb{Z}_{(2)}) \otimes \widetilde{H}_q(C; \mathbb{Z}_{(2)})$;
7:     **end for**;
8:     **for all** $p + q = m - 1$ **do**
9:         $H_m \Leftarrow H_m \oplus \mathrm{Tor}^{\mathbb{Z}}(\widetilde{H}_p(EC_h(x,y); \mathbb{Z}_{(2)}), \widetilde{H}_q(C; \mathbb{Z}_{(2)}))$;
10:     **end for**;
11: **end for**;
12: output $H_{\leqslant d}$;

The following result can be used at lines 6 and 9 when C is an elementary complex.

3.1.6. Lemma. *If G and H are two finite cyclic groups, then*

$$\mathrm{Tor}^{\mathbb{Z}}(G, H) \cong \mathbb{Z}/\gcd(\exp(G), \exp(H)) \cong G \otimes_{\mathbb{Z}} H.$$

We state now the main algorithm.

3.1.7. Algorithm. <u>**Ensure:**</u> graded group $\widetilde{H}_{\leqslant d}(K(\mathbb{Z}/2^s, n); \mathbb{Z})$;

Require: $s \geqslant 1$, $n \geqslant 1$ and $d \geqslant 0$;

call Algorithm 3.1.1 with n and d as input;
$H_{\leqslant d}$ is the graded group for output;
$H_{\leqslant d} \Leftarrow$ output of Algorithm 3.1.4 called with $n \Rightarrow \deg(x)$ and $2^s \Rightarrow h$;
for all admissible sequence $I = (a_0, \dots) \neq (0)$ of $\mathcal{C}$ such that a_0 is even **do**
    $H_{\leqslant d} \Leftarrow$ output of Algorithm 3.1.5 called with

$$\deg_{\mathrm{st}}(I) + n \Rightarrow \deg(x),$$
$$2 \Rightarrow h,$$
$$H_{\leqslant d} \Rightarrow \widetilde{H}_{\leqslant d}(C; \mathbb{Z}_{(2)});$$

end for;
output $H_{\leqslant d}$;

3.2. Heuristic results

Let us have a glance at Table C.1, p. 91. One remarks that the exponents of the integral cohomology groups of $K(\mathbb{Z}/2,2)$ behave strangely uniformously (the last column gives the $\log_2$ value of the exponent). One readilly checks in the table that the exponent of $\widetilde{H}_n(K(\mathbb{Z}/2,2);\mathbb{Z})$ is given by the exponent of $\mathbb{Z}/n\otimes\mathbb{Z}_{(2)}$. Actually, this is exactly what Proposition 1.3.4 states.

1.3.4. Proposition. *The exponent of the homology group $H_{2^rm}(K(\mathbb{Z}/2,2);\mathbb{Z})$, with m odd, is exactly 2^r. Moreover, one has the following formula:*

$$\exp \widetilde{H}_n(K(\mathbb{Z}/2,2);\mathbb{Z}) = \begin{cases} n_{(2)} & \textit{if } n \geqslant 2 \textit{ is even,} \\ 2 & \textit{if } n \geqslant 5 \textit{ is odd,} \\ 1 & \textit{if } n \in \{0,1,3\}, \end{cases}$$

where $n_{(2)}$ denotes the 2-primary part of the integer n.

Proof. It is a simple verification to see that admissible words involved in the Cartan's complex associated to $K(\mathbb{Z}/2,2)$ are given by σ^2u_2, $\sigma\psi_2u_2$, $\beta_2\gamma_2^k\psi_2$ and $\varphi_2\gamma_2^k\psi_2$ for all $k\geqslant 0$. Thus generators are the pairs $(\sigma^2u_2,\sigma\psi_2u_2)$ and $(\beta_2\varphi_2\gamma_2^k\psi_2,\varphi_2\gamma_2^k\psi_2)$ of bidegrees $(2,3)$ and $(2^{k+2}+1,2^{k+2}+2)$ for all $k\geqslant 0$. Therefore we have

$$X = \underbrace{EC_{-2}(\sigma^2u_2,\sigma\psi_2u_2)}_{X'} \otimes \underbrace{\bigotimes_{k\geqslant 0} EC_2(\beta_2\varphi_2\gamma_2^k\psi_2u_2,\varphi_2\gamma_2^k\psi_2u_2)}_{X'''}.$$

Since $\deg(\beta_2\varphi_2\gamma_2^k\psi_2u_2) = 2^{k+2}+1$ is odd for all $k\geqslant 0$, Lemma 2.2.6 implies that $2\widetilde{H}_*(X''';\mathbb{Z}_{(2)}) = 0$. Since $\deg(\sigma^2u_2) = 2$ is even, Lemma 2.2.6 implies that $\widetilde{H}_n(X';\mathbb{Z}_{(2)}) \cong \mathbb{Z}/n\otimes\mathbb{Z}_{(2)}$ for all $n\geqslant 0$. The result follows now by the Künneth formula. □

Remark. Appendix B.1 essentially states that generators are in one-to-one correspondance with admissible sequences. Therefore, to see that σ^2u_2, $\sigma\psi_2u_2$, $\beta_2\gamma_2^k\psi_2$ and $\varphi_2\gamma_2^k\psi_2$ for all $k\geqslant 0$ constitute all the admissible words involved in the Cartan's complex of $K(\mathbb{Z}/2,2)$, it suffices to see that (0) and $(2^k,2^{k-1},\dots,1)$ are the only admissible sequences of excess < 2.

A straightforward application of the Universal Coefficient Theorem in cohomology gives Proposition 1.3.5.

1.3.5. Proposition. *The exponent of the cohomology group* $H^{2^r m+1}(K(\mathbb{Z}/2,2);\mathbb{Z})$, *with* m *odd, is exactly* 2^r. *Moreover, one has the following formula:*

$$\exp \widetilde{H}^n(K(\mathbb{Z}/2,2);\mathbb{Z}) = \begin{cases} (n-1)_{(2)} & \text{if } n \geqslant 3 \text{ is odd,} \\ 2 & \text{if } n \geqslant 6 \text{ is even,} \\ 1 & \text{if } n \in \{0,1,2,4\}, \end{cases}$$

where $(n-1)_{(2)}$ *denotes the* 2*-primary part of the integer* $(n-1)$.

We conclude this chapter with some words on the cohomology Bockstein spectral sequence of $K(\mathbb{Z}/2,2)$.

Using Proposition 1.3.5, it is very easy to check that $\exp \widetilde{H}^n(K(\mathbb{Z}/2,2);\mathbb{Z}) = 2$ if and only if $n \in \{2m+6, 4m+3 \mid m \geqslant 0\}$ and that $\exp \widetilde{H}^n(K(\mathbb{Z}/2,2);\mathbb{Z}) = 2^r$, $r \geqslant 2$, if and only if $n \in \{2^r(2m+1)+1 \mid m \geqslant 0\}$. Let us set

$$\begin{aligned} N_1 &= \{2m+6, 4m+3 \mid m \geqslant 0\}, \\ N_r &= \{2^r(2m+1)+1 \mid m \geqslant 0\} \qquad \text{for all } r \geqslant 2. \end{aligned}$$

For all $r \geqslant 1$, the set N_r tells us in which dimensions elements are hit by the differential d_r in the Bockstein spectral sequence of $K(\mathbb{Z}/2,2)$. The first integers in these sets are

$$\begin{aligned} N_1 &= \{3,6,7,8,\ldots\}, \\ N_2 &= \{5,13,\ldots\}, \\ N_3 &= \{9,\ldots\}, \\ &\vdots \end{aligned}$$

which corresponds to the following pairings

$$\begin{aligned} & u_2 \mapsto d_1 u_2 = Sq^1 u_2, \\ & Sq^{2,1}u_2 \mapsto d_1 Sq^{2,1} u_2 = (Sq^1 u_2)^2, \\ & u_2^3 \mapsto d_1(u_2^3) = u_2^2 Sq^1 u_2, \\ & u_2 Sq^{2,1} u_2 \mapsto d_1(u_2 Sq^{2,1} u_2) = Sq^1 u_2 (Sq^{2,1} u_2 + u_2 Sq^1 u_2), \\ & \vdots \\ & u_2^2 \mapsto d_2 u_2^2 = u_2 Sq^1 u_2 + Sq^{2,1} u_2, \\ & u_2^6 \mapsto d_2 u_2^6 = u_2^4 (u_2 Sq^1 u_2 + Sq^{2,1} u_2), \\ & \vdots \\ & u_2^4 \mapsto d_3 u_2^4 = u_2^2 (u_2 Sq^1 u_2 + Sq^{2,1} u_2), \\ & \vdots \end{aligned}$$

It is not very interesting here to know how these computations are done (the interested reader can have a glance at [**13**, Theorem 5.4]), but one can readilly deduce from them that $(d_2u_2^2)^2 = 0 \in B_2^*$: suppose it is not the case, then $d_2(u_2^2 d_2 u_2^2) = (d_2u_2^2)^2 \neq 0$ and $d_3u_2^4 = u_2^2(u_2 Sq^1 u_2 + Sq^{2,1}u_2) = u_2^2 d_2 u_2^2 = 0 \in B_3^*$, which is false. Therefore $d_2u_2^2$ is an exterior element in B_2^*. Actually, one can see that

$$B_2^* \cong \mathbb{F}_2[u_2^2] \otimes \Lambda_{\mathbb{F}_2}(d_2u_2^2).$$

Finally, we give here, without proof, a complete determination of the Bockstein spectral sequence of $K(\mathbb{Z}/2,2)$:

$$\begin{aligned}
B_1^* &\cong \mathbb{F}_2[Sq^I u_2 \mid e(I) < 2],\\
B_2^* &\cong \mathbb{F}_2[u_2^2] \otimes \Lambda_{\mathbb{F}_2}(d_2u_2^2),\\
B_3^* &\cong \mathbb{F}_2[u_2^4] \otimes \Lambda_{\mathbb{F}_2}(d_3u_2^4),\\
&\vdots\\
B_r^* &\cong \mathbb{F}_2[u_2^{2^{r-1}}] \otimes \Lambda_{\mathbb{F}_2}(d_r u_2^{2^{r-1}}).
\end{aligned}$$

CHAPTER 4

Examples of spaces and classification

4.1. A space "with retract"

Let us consider the following example:

1.3.6. Example. Let X be the space given by the fibration

$$X \xrightarrow{i} K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,2) \xrightarrow{k} K(\mathbb{Z}/2,4),$$

where its single non-trivial k-invariant is

$$\begin{aligned} k \in & [K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,2), K(\mathbb{Z}/2,4)] \\ \cong & H^4(K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,2); \mathbb{F}_2) \\ \cong & H^4(K(\mathbb{Z}/2,2); \mathbb{F}_2) \otimes \mathbb{F}_2 \\ & \oplus H^2(K(\mathbb{Z}/2,2); \mathbb{F}_2) \otimes H^2(K(\mathbb{Z}/2,2); \mathbb{F}_2) \\ & \oplus \mathbb{F}_2 \otimes H^4(K(\mathbb{Z}/2,2); \mathbb{F}_2) \\ \cong & \mathbb{F}_2\{u_2^2 \otimes 1, u_2 \otimes v_2, 1 \otimes v_2^2\} \end{aligned}$$

given by $k = u_2 \otimes v_2$ where u_2 and v_2 are the fundamental classes of both copies of $K(\mathbb{Z}/2,2)$. The space X has only two non-trivial homotopy groups $\pi_2(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\pi_3(X) \cong \mathbb{Z}/2$.

The rest of this section is devoted to proof the following result:

1.3.7. Theorem. *The space X of Example 1.3.6 has the following properties:*

1. *X is not a GEM,*
2. *X is not an H-space,*
3. *X retracts (weakly) onto the Eilenberg-MacLane space $K(\mathbb{Z}/2,2)$, i.e. there exist maps $f : X \to K(\mathbb{Z}/2,2)$ and $g : K(\mathbb{Z}/2,2) \to X$ such that $fg \simeq \mathrm{id}_{K(\mathbb{Z}/2,2)}$,*
4. *$f^* : H^*(K(\mathbb{Z}/2,2); \mathbb{F}_2) \to H^*(X; \mathbb{F}_2)$ is a monomorphism,*
5. *X has no homological exponent.*

Proof. The space X is not a GEM since its k-invariant $u_2 \otimes v_2$ is not trivial. Moreover, $u_2 \otimes v_2$ is decomposable in $H^*(K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,2); \mathbb{F}_2)$ and therefore not primitive. Thus X is not an H-space.

Consider the following homotopy commutative diagram based on the fibration in which X is the fibre:

$$\begin{tikzcd}
 & X \arrow[d, "i"] \arrow[dr, bend left, "f"] & \\
K(\mathbb{Z}/2,2) \arrow[ur, dotted, "g"] \arrow[r, "i_1"] \arrow[dr, bend right, "*"'] & K(\mathbb{Z}/2,2)\times K(\mathbb{Z}/2,2) \arrow[r, "p_1"] \arrow[d, "k"] & K(\mathbb{Z}/2,2) \\
 & K(\mathbb{Z}/2,4), &
\end{tikzcd}$$

where i_1 denotes the inclusion into the first factor, p_1 denotes the projection onto the first factor and $f = p_1 i$. The existence of a (generally not unique) map g is a consequence of the fact that $ki_1 \simeq *$. To see that $ki_1 \simeq *$, recall first that the isomorphism $[K(\mathbb{Z}/2,2), K(\mathbb{Z}/2,4)] \cong H^4(K(\mathbb{Z}/2,2);\mathbb{F}_2)$ maps ki_1 to $(ki_1)^*(u_4)$, where $(ki_1)^* = (i_1)^*k^* : H^4(K(\mathbb{Z}/2,4);\mathbb{F}_2) \to H^4(K(\mathbb{Z}/2,2);\mathbb{F}_2)$ and $u_4 \in H^4(K(\mathbb{Z}/2,4);\mathbb{F}_2)$ is the fundamental class. Now we have

$$\begin{aligned}
(i_1)^*k^*(u_4) &= (i_1)^*(u_2 \otimes v_2) && \text{by definition of } k,\\
&= (i_1)^*(u_2 \otimes 1 \cdot 1 \otimes v_2) &&\\
&= (i_1)^*(u_2 \otimes 1)\cdot(i_1)^*(1 \otimes v_2) && \text{since } (i_1)^* \text{ is a ring map,}\\
&= 0 && \text{since } (i_1)^*(1 \otimes v_2) = 0.
\end{aligned}$$

Therefore $fg \simeq p_1 i g \simeq p_1 i_1 = \mathrm{id}$ i.e. X retracts (weakly) onto $K(\mathbb{Z}/2,2)$. Consider now the following induced commutative diagram:

$$\begin{tikzcd}
 & H^*(X;\mathbb{F}_2) \arrow[dr, bend left, "g^*"] & \\
H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2) \arrow[ur, bend left, "f^*"] \arrow[rr, equal] & & H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2).
\end{tikzcd}$$

The induced map f^* is clearly a monomorphism. Let us remark that this remains true if one look at the induced diagram via $H^*(-;R)$ for any ring R. The fact that $K(\mathbb{Z}/2,2)$ has no homology exponent then obviously implies that the same is true for X. $\square$

4.2. A generalization for some H-spaces "with retract"

Let us consider all spaces which have the same homotopy groups as those of Example 1.3.6, namely $\pi_2(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\pi_3(X) \cong \mathbb{Z}/2$. According to Postnikov theory, these spaces are classified by their unique Postnikov invariant which is an element of the cohomology group

$$H^4(K(\mathbb{Z}/2,2) \times K(\mathbb{Z}/2,2); \mathbb{F}_2) \cong \mathbb{F}_2\{u_2^2 \otimes 1, u_2 \otimes v_2, 1 \otimes v_2^2\},$$

where u_2 and v_2 are the fundamental classes of both copies of $K(\mathbb{Z}/2,2)$. Therefore, there are *a priori* $\dim_{\mathbb{F}_2} \mathbb{F}_2\{u_2^2 \otimes 1, u_2 \otimes v_2, 1 \otimes v_2^2\} = 2^3 = 8$ homotopy types corresponding to $k = 0$, $u_2^2 \otimes 1$, $1 \otimes v_2^2$, $u_2 \otimes v_2$, $u_2^2 \otimes 1 + 1 \otimes v_2^2$, $u_2^2 \otimes 1 + u_2 \otimes v_2$, $u_2 \otimes v_2 + 1 \otimes v_2^2$ and $u_2^2 \otimes 1 + u_2 \otimes v_2 + 1 \otimes v_2^2$. But only 6 are really different, and they are classified by the elements 0, $u_2^2 \otimes 1$, $u_2 \otimes v_2$, $u_2^2 \otimes 1 + u_2 \otimes v_2$, $u_2^2 \otimes 1 + 1 \otimes v_2^2$ and $u_2^2 \otimes 1 + u_2 \otimes v_2 + 1 \otimes v_2^2$. Only 3 of them are H-spaces, namely those classified by primitive elements: 0, $u_2^2 \otimes 1$ and $u_2^2 \otimes 1 + 1 \otimes v_2^2$. These three spaces are actually infinite loop spaces.

We will develop a method to determine which of these homotopy types retract onto an Eilenberg-MacLane space. This will be done in a more general way.

Consider the following particular 2-stage Postnikov system where k is an H-map (we call such a system **stable**):

$$\begin{tikzcd} KV \arrow[r, equal] \arrow[d] & KV \arrow[d] \\ X \arrow[r] \arrow[d, "i"'] & PK\Sigma V \arrow[d] \\ K\Sigma^m U \arrow[r, "k"'] & K\Sigma V, \end{tikzcd}$$

with $m \geqslant 1$, $U = \bigoplus_{1 \leqslant i \leqslant s} \mathbb{F}_2$ a finite dimensional vector space over $\mathbb{F}_2$, $V = \bigoplus_{1 \leqslant j \leqslant t} \Sigma^{d_j} \mathbb{F}_2$ a finite dimensional graded vector space over $\mathbb{F}_2$ and k an H-map. Let us remark here that, since k is an H-map, X is an H-space.

4.2.1. Definition. For all $m \geqslant 1$, $U = \bigoplus_{1 \leqslant i \leqslant s} \mathbb{F}_2$ and $U' = \bigoplus_{1 \leqslant i' \leqslant s'} \mathbb{F}_2$ we define the isomorphism Φ by the following steps:

$$
\begin{tikzcd}
\ell \arrow[d, mapsto] & {[K\Sigma^m U, K\Sigma^m U']} \arrow[d, "\cong"] \arrow[ddddd, bend left, "\Phi", "\cong"'] \\
(\underbrace{\mathrm{proj}_{i'} \ell}_{\ell_{i'}})_{1 \leqslant i' \leqslant s'} \arrow[d, mapsto] & \prod_{1 \leqslant i' \leqslant s'} [K\Sigma^m U, K\Sigma^m \mathbb{F}_2] \arrow[d, "\cong"] \\
(\ell_{i'}^*(u_m))_{1 \leqslant i' \leqslant s'} \arrow[d, mapsto] & \bigoplus_{1 \leqslant i' \leqslant s'} H^m(K\Sigma^m U; \mathbb{F}_2) \arrow[d, "\cong"] \\
(\underbrace{\mathrm{incl}_i^* \, \ell_{i'}^*(u_m)}_{\ell_{i'i} u_m})_{\substack{1 \leqslant i' \leqslant s' \\ 1 \leqslant i \leqslant s}} \arrow[d, mapsto] & \bigoplus_{\substack{1 \leqslant i' \leqslant s' \\ 1 \leqslant i \leqslant s}} H^m(K\Sigma^m \mathbb{F}_2; \mathbb{F}_2) \arrow[d, "\cong"] \\
\sum_{\substack{1 \leqslant i' \leqslant s' \\ 1 \leqslant i \leqslant s}} e_{i'} \otimes \ell_{i'i} e_i \arrow[d, mapsto] & \mathbb{F}_2^{\oplus s'} \otimes \mathbb{F}_2^{\oplus s} \arrow[d, "\cong"] \\
(\ell_{i'i})_{\substack{1 \leqslant i' \leqslant s' \\ 1 \leqslant i \leqslant s}} & M_{s' \times s}(\mathbb{F}_2)
\end{tikzcd}
$$

where the e_i's and $e_{i'}$'s denote the standard basis for $\mathbb{F}_2^{\oplus s}$ and $\mathbb{F}_2^{\oplus s'}$ respectively and where $M_{s' \times s}(\mathbb{F}_2)$ denotes the $(s' \times s)$-matrices with $\mathbb{F}_2$ coefficients.

4.2.2. Lemma. *Consider the following* 2*-stage Postnikov system*

$$
\begin{tikzcd}
KV \arrow[r, equal] \arrow[d] & KV \arrow[d] \\
X \arrow[r] \arrow[d, "i"'] & PK\Sigma V \arrow[d] \\
K\Sigma^m U \arrow[r, "k"'] & K\Sigma V,
\end{tikzcd}
$$

with $m \geqslant 1$, $U = \bigoplus_{1 \leqslant i \leqslant s} \mathbb{F}_2$ *a finite dimensional vector space over* $\mathbb{F}_2$, $V = \bigoplus_{1 \leqslant j \leqslant t} \Sigma^{d_j} \mathbb{F}_2$ *a finite dimensional graded vector space over* $\mathbb{F}_2$ *and* k *an* H*-map. If there exists a map* $\ell : K\Sigma^m U \to K\Sigma^m U$ *such that* $k\ell \simeq *$, *then* X *retracts onto* $K\Sigma^m \mathbb{F}_2^{\oplus \operatorname{rank} \Phi(\ell)}$.

Proof. We have the following homotopy commutative diagram:

$$\begin{tikzcd}
 & X \arrow[d, "i"] \\
K\Sigma^m U \arrow[ur, dotted, "\tilde{\ell}"] \arrow[r, "\ell"] \arrow[dr, bend right, "*"'] & K\Sigma^m U \arrow[d, "k"] \\
 & K\Sigma V.
\end{tikzcd}$$

The hypothesis $k\ell \simeq *$ implies the existence of a map $\tilde{\ell} : K\Sigma^m U \to X$ such that $i\tilde{\ell} \simeq \ell$. Let $s' = \operatorname{rank} \Phi(\ell)$. Consider the matrix $\Phi(\ell)$. By linear combinations of its rows we obtain a matrix with only s' non-trivial and linearly independant rows. Then, by linear combinations on the columns of $\Phi(\ell)$, we obtain a new matrix with a single non-trivial and invertible $(s' \times s')$-block. Therefore there exist matrices $P \in M_{s'\times s}(\mathbb{F}_2)$ and $J \in M_{s\times s'}(\mathbb{F}_2)$ such that $P\Phi(\ell)J = I_{s'\times s'}$ where $I_{s'\times s'}$ denotes the identity matrix. Set $p = \Phi^{-1}(P)$ and $j = \Phi^{-1}(J)$ and $U' = \mathbb{F}_2^{\oplus s'}$ in order to have $p\ell j \simeq \mathrm{id}_{K\Sigma^m U'}$ since $\Phi(p\ell j) = \Phi(p)\Phi(\ell)\Phi(j) = \Phi(\Phi^{-1}(P))\Phi(\ell)\Phi(\Phi^{-1}(J)) = P\Phi(\ell)J = I_{s'\times s'}$. Now we can complete the previous diagram with these maps:

$$\begin{tikzcd}
 & & X \arrow[d, "i"] \arrow[dr, dotted, "f"] & \\
K\Sigma^m \mathbb{F}_2^{\oplus s'} \arrow[urr, dotted, bend left, "g"] \arrow[r, "j"] & K\Sigma^m U \arrow[ur, bend left, "\tilde{\ell}"] \arrow[r, "\ell"] \arrow[dr, bend right, "*"'] & K\Sigma^m U \arrow[r, "p"] \arrow[d, "k"] & K\Sigma^m \mathbb{F}_2^{\oplus s'} \\
 & & K\Sigma V. &
\end{tikzcd}$$

This completes the proof. ☐

4.2.3. Lemma. *Consider the following* 2*-stage Postnikov system*

$$\begin{tikzcd}
KV \arrow[r, equal] \arrow[d] & KV \arrow[d] \\
X \arrow[r] \arrow[d, "i"'] & PK\Sigma V \arrow[d] \\
K\Sigma^m U \arrow[r, "k"'] & K\Sigma V,
\end{tikzcd}$$

with $m \geqslant 1$, $U = \bigoplus_{1\leqslant i\leqslant s} \mathbb{F}_2$ a finite dimensional vector space over $\mathbb{F}_2$, $V = \bigoplus_{1\leqslant j\leqslant t} \Sigma^{d_j}\mathbb{F}_2$ a finite dimensional graded vector space over $\mathbb{F}_2$ and k an H-map. If X retracts onto $K\Sigma^m\mathbb{F}_2$, then there exists $\ell : K\Sigma^m U \to K\Sigma^m U$ such that $\ell \not\simeq *$ and $k\ell \simeq *$.

Proof. Assume the existence of maps $f : X \to K\Sigma^m\mathbb{F}_2$ and $g : K\Sigma^m\mathbb{F}_2 \to X$ such that $fg \simeq \mathrm{id}$. Set $l = ig : K\Sigma^m\mathbb{F}_2 \to K\Sigma^m U$, which is clearly not nullhomotopic, and finally set $\ell = l\,\mathrm{proj}_1 : K\Sigma^m U \to K\Sigma^m U$. We then obtain $k\ell = kig\,\mathrm{proj}_1 \simeq *$ since $ki \simeq *$. $\square$

4.2.4. Definition. Let X and Y be two H-spaces. Let us define $[X,Y]_H$ as the subset of $[X,Y]$ consisting of all H-maps classes.

4.2.5. Lemma. *Let $k : K\Sigma^m U \to K\Sigma V$ be an H-map, $\ell : K\Sigma^m U \to K\Sigma^m U$ be a map (which is obviously an H-map) and consider the following composition:*

$$\begin{tikzcd}
k\ell \arrow[d, mapsto] & {[K\Sigma^m U, K\Sigma V]_H} \arrow[d, "\cong"] \\
(\underbrace{\mathrm{proj}_j\, k}_{k_j}\,\ell)_{1\leqslant j\leqslant t} \arrow[d, mapsto] & \prod\limits_{1\leqslant j\leqslant t} [K\Sigma^m U, K\Sigma\Sigma^{d_j}\mathbb{F}_2]_H \arrow[d, "\cong"] \\
(\ell^* k_j^*(u_{d_j+1}))_{1\leqslant j\leqslant t} \arrow[d, mapsto] & \bigoplus\limits_{1\leqslant j\leqslant t} P^{d_j+1}H^*(K\Sigma^m U;\mathbb{F}_2) \arrow[d, "\cong"] \\
(\mathrm{incl}_i^*\, \ell^* k_j^*(u_{d_j+1}))_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant t}} \arrow[d, equal, "\ell^* = \sum\limits_{1\leqslant i'\leqslant s} \ell_{i'}^*\,\mathrm{incl}_{i'}^*"] & \bigoplus\limits_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant t}} P^{d_j+1}H^*(K\Sigma^m\mathbb{F}_2;\mathbb{F}_2) \arrow[dd, "\cong"] \\
\Big(\sum\limits_{1\leqslant i'\leqslant s} \ell_{i'i}\,\mathrm{incl}_{i'}^*\, k_j^*(u_{d_j+1})\Big)_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant t}} \arrow[d, mapsto] & \\
\sum\limits_{\substack{1\leqslant i'\leqslant s\\ 1\leqslant i\leqslant s}} (\mathrm{incl}_{i'}^*\, k_j^*(u_{d_j+1}))_{1\leqslant j\leqslant t} \otimes \ell_{i'i}e_i & \bigoplus\limits_{1\leqslant j\leqslant t} P^{d_j+1}H^*(K\Sigma^m\mathbb{F}_2;\mathbb{F}_2)\otimes\mathbb{F}_2^{\oplus s}
\end{tikzcd}$$

where the e_i's form a basis of $\mathbb{F}_2^{\oplus s}$. Then $k\ell \simeq *$ if and only if $\sum_{1\leqslant i'\leqslant s} \mathrm{incl}_{i'}^*\, k_j^*(u_{d_j+1}) \otimes \ell_{i'i}e_i = 0$ for all $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant t$.

Proof. We have $k\ell \simeq *$ if and only if

$$\sum_{\substack{1\leqslant i'\leqslant s\\1\leqslant i\leqslant s}} (\mathrm{incl}_{i'}^*\, k_j^*(u_{d_j+1}))_{1\leqslant j\leqslant t}\otimes \ell_{i'i}e_i = 0 \quad \text{i.e.}$$

$$\sum_{1\leqslant i'\leqslant s} (\mathrm{incl}_{i'}^*\, k_j^*(u_{d_j+1}))_{1\leqslant j\leqslant t}\otimes \ell_{i'i}e_i = 0 \quad \text{for all } 1\leqslant i\leqslant s, \text{ i.e.}$$

$$\sum_{1\leqslant i'\leqslant s} \mathrm{incl}_{i'}^*\, k_j^*(u_{d_j+1})\otimes \ell_{i'i}e_i = 0 \quad \text{for all } 1\leqslant i\leqslant s \text{ and } 1\leqslant j\leqslant t.$$

□

We are now able to prove the main general result of this chapter. It involves H-spaces X with two non-trivial homotopy groups $\pi_m(X)\cong\bigoplus_{1\leqslant i\leqslant s}\mathbb{F}_2$ and $\pi_d(X)\cong\mathbb{F}_2$, where $m\geqslant 1$ and $d\geqslant 1$. In other words, we are looking at the special case where $V=\Sigma^d\mathbb{F}_2$.

1.3.8. Theorem. *Consider a stable two stage Postnikov system of the form*

$$\begin{tikzcd}
K\Sigma^d\mathbb{F}_2 \arrow[r, equal] \arrow[d] & K\Sigma^d\mathbb{F}_2 \arrow[d] \\
X \arrow[r] \arrow[d] & PK\Sigma^{d+1}\mathbb{F}_2\simeq * \arrow[d] \\
K\Sigma^m U \arrow[r, "k"'] & K\Sigma^{d+1}\mathbb{F}_2,
\end{tikzcd}$$

with $d>m\geqslant 1$, $U=\bigoplus_{1\leqslant i\leqslant s}\mathbb{F}_2$ and $k: K\Sigma^m U\to K\Sigma^{d+1}\mathbb{F}_2$ an H-map. Moreover, consider the set

$$S=\{\mathrm{incl}_i^*\,k^*(u_{d+1})\in H^{d+1}(K\Sigma^m\mathbb{F}_2;\mathbb{F}_2)\mid 1\leqslant i\leqslant s\},$$

where $\mathrm{incl}_i : K\Sigma^m\mathbb{F}_2\to K\Sigma^m U$ is the i-th obvious inclusion, $\mathrm{incl}_i^*\,k^*$: $H^{d+1}(K\Sigma^{d+1}\mathbb{F}_2;\mathbb{F}_2)\to H^{d+1}(\Sigma^m\mathbb{F}_2;\mathbb{F}_2)$ is the induced homomorphism and where u_{d+1} denotes the fundamental class in $H^{d+1}(K\Sigma^{d+1}\mathbb{F}_2;\mathbb{F}_2)$. The H-space X retracts onto $K\Sigma^m\mathbb{F}_2$ if and only if one of the two following assumptions is verified:

- $0\in S$ *or*
- $\mathrm{Card}(S)<s$.

Proof. Suppose that X retracts onto $K\Sigma^m\mathbb{F}_2$. By Lemma 4.2.3 there exists a map $\ell : K\Sigma^m U\to K\Sigma^m U$ such that $\ell\not\simeq *$ and $k\ell\simeq *$. By Lemma 4.2.5, for all $1\leqslant i\leqslant s$ we have

$$\sum_{1\leqslant i'\leqslant s}\mathrm{incl}_{i'}^*\,k^*(u_{d+1})\otimes \ell_{i'i}e_i = 0.$$

But there are integers i_0 and i'_0 such that $\ell_{i'_0 i_0} = 1$ since $\ell \not\simeq *$. Therefore we can write

$$\sum_{1 \leqslant i' \leqslant s} \mathrm{incl}^*_{i'} \, k^*(u_{d+1}) \otimes \ell_{i' i_0} e_{i_0} =$$

$$\mathrm{incl}^*_{i'_0} \, k^*(u_{d+1}) \otimes e_{i_0} + \sum_{\substack{1 \leqslant i' \leqslant s \\ i' \neq i'_0}} \mathrm{incl}^*_{i'} \, k^*(u_{d+1}) \otimes \ell_{i' i_0} e_{i_0} = 0.$$

If $\mathrm{incl}^*_{i'_0} \, k^*(u_{d+1}) \neq 0$ then there is obviously an integer $i'_1 \neq i'_0$ such that $\mathrm{incl}^*_{i'_1} \, k^*(u_{d+1}) = \mathrm{incl}^*_{i'_0} \, k^*(u_{d+1})$. We have proved that $\mathrm{Card}(S) < s$.

Conversely, suppose first that there is an integer i'_0 such that $\mathrm{incl}^*_{i'_0} \, k^*(u_{d+1}) = 0$. Set

$$\ell_{i'i} = \begin{cases} 1 & \text{if } (i', i) = (i'_0, 1), \\ 0 & \text{otherwise,} \end{cases}$$

which satisfies $\ell \not\simeq *$. Then, for all $1 \leqslant i \leqslant s$ we have

$$\sum_{1 \leqslant i' \leqslant s} \mathrm{incl}^*_{i'} \, k^*(u_{d+1}) \otimes \ell_{i'i} e_i =$$

$$\mathrm{incl}^*_{i'_0} \, k^*(u_{d+1}) \otimes \ell_{i'_0 i} e_i = 0.$$

By Lemma 4.2.5 this implies that $k\ell \simeq *$. By Lemma 4.2.2, X retracts onto $K\Sigma^m \mathbb{F}_2$. Suppose now that there are $i'_0 \neq i'_1$ such that $\mathrm{incl}^*_{i'_0} \, k^*(u_{d+1}) = \mathrm{incl}^*_{i'_1} \, k^*(u_{d+1})$. Set

$$\ell_{i'i} = \begin{cases} 1 & \text{if } (i', i) \in \{(i'_0, 1), (i'_1, 1)\}, \\ 0 & \text{otherwise,} \end{cases}$$

which satisfies $\ell \not\simeq *$. Then, for all $1 \leqslant i \leqslant s$ we have

$$\sum_{1 \leqslant i' \leqslant s} \mathrm{incl}^*_{i'} \, k^*(u_{d+1}) \otimes \ell_{i'i} e_i =$$

$$\mathrm{incl}^*_{i'_0} \, k^*(u_{d+1}) \otimes \ell_{i'_0 i} e_i + \mathrm{incl}^*_{i'_1} \, k^*(u_{d+1}) \otimes \ell_{i'_1 i} e_i = 0.$$

By Lemma 4.2.5 this implies that $k\ell \simeq *$. By Lemma 4.2.2, X retracts onto $K\Sigma^m \mathbb{F}_2$. This completes the proof. $\square$

This result is general enough to solve the problem set at the beginning of this section. Recall that among the 6 different homotopy types with two non-trivial homotopy groups $\pi_2(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\pi_3(X) \cong \mathbb{Z}/2$, only 3 are H-spaces. The following corollary asserts that these H-spaces retract onto an Eilenberg-MacLane space (and therefore do not admit a homology exponent).

4.2.6. Corollary. *Every H-space X with non-trivial homotopy groups $\pi_2(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\pi_3(X) \cong \mathbb{Z}/2$ retracts onto $K(\mathbb{Z}/2, 2)$.*

Proof. The only possible primitive k-invariants are

$$k^*(u_4) \in \{\underbrace{0}_{k_0}, \underbrace{u_2^2 \otimes 1}_{k_1}, \underbrace{1 \otimes v_2^2}_{k_2}, \underbrace{u_2^2 \otimes 1 + 1 \otimes v_2^2}_{k_3}\}.$$

For all $\alpha = 1, 2, 3, 4$ set $S_\alpha = \{\mathrm{incl}_i^* \, k_\alpha \mid 1 \leqslant i \leqslant s\}$. We have

$S_0 = \{0\}$ which contains 0 and such that $\#S_0 = 1 < 2$,

$S_1 = \{0, u_2^2\}$ which contains 0,

$S_2 = \{0, u_2^2\}$ which contains 0 and

$S_3 = \{u_2^2\}$ whose cardinality is smaller than 2.

□

The existence of a retract does actually not only rely on the values of the homotopy groups, but also depends on their relative position. To see this, let us consider all H-spaces such that, for some integer $n \geqslant 3$,

$$\pi_m(X) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } m = 2, \\ \mathbb{Z}/2 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

This is a very simple application of Theorem 1.3.8 to see that all these H-spaces retract onto $K(\mathbb{Z}/2, 2)$.

Moreover, let us consider the H-space such that,

$$\pi_m(X) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } m = 3, \\ \mathbb{Z}/2 & \text{if } m = 4, \\ 0 & \text{otherwise.} \end{cases}$$

It is also not very diffucult to see that this H-space retracts onto $K(\mathbb{Z}/2, 3)$.

The following result concerns the “next” space we would like to consider:

4.2.7. Corollary. *There exists an H-space X with non-trivial homotopy groups $\pi_3(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\pi_5(X) \cong \mathbb{Z}/2$ which does not retract onto $K(\mathbb{Z}/2, 3)$.*

Proof. It suffices to consider the following 2-stage Postnikov system:

$$\begin{tikzcd} K\Sigma^5\mathbb{F}_2 \arrow[r, equal] \arrow[d] & K\Sigma^5\mathbb{F}_2 \arrow[d] \\ X \arrow[r] \arrow[d] & PK\Sigma^6\mathbb{F}_2 \arrow[d] \\ K\Sigma^3(\mathbb{F}_2 \oplus \mathbb{F}_2) \arrow[r, "k"'] & K\Sigma^6\mathbb{F}_2, \end{tikzcd}$$

where k is such that $k^*(u_6) = u_3^2 \otimes 1 + 1 \otimes Sq^{2,1} u_3$. We have $\mathrm{incl}_1^* k^*(u_6) = u_3^2$ and $\mathrm{incl}_2^* k^*(u_6) = Sq^{2,1} u_3$. If we consider S as in the Theorem we have $0 \notin S$ and $\mathrm{Card}(S) = 2$. Therefore X does not retract onto $K(\mathbb{Z}/2, 3)$. $\square$

The space of Corollary 4.2.7 is then the “first” H-space having only two non-trivial homotopy groups with values $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\mathbb{Z}/2$ and the property that it does not retract onto an Eilenberg-MacLane space.

However, as we will see later in this work, the H-space of Corollary 4.2.7 has no homology exponent (see Theorems 1.3.12 and 1.3.14 in Sections 5.4 and 5.6 respectively).

4.3. A space “without retract”

Let us now consider the simplest possible example of a space with two non-trivial homotopy groups:

1.3.9. Example. Let X be the space given by the fibration

$$X \xrightarrow{i} K(\mathbb{Z}/2,2) \xrightarrow{k} K(\mathbb{Z}/2,4),$$

where its single non-trivial k-invariant is

$$\begin{aligned} k \in & [K(\mathbb{Z}/2,2), K(\mathbb{Z}/2,4)] \\ \cong & H^4(K(\mathbb{Z}/2,2);\mathbb{F}_2) \\ \cong & \mathbb{F}_2\{u_2^2\} \end{aligned}$$

given by $k = u_2^2$ where u_2 is the fundamental class of $K(\mathbb{Z}/2,2)$. The space X has only two non-trivial homotopy groups $\pi_2(X) \cong \mathbb{Z}/2 \cong \pi_3(X)$. Moreover, this is an H-space.

As a corollary of Theorem 1.3.8 we have the following result:

4.3.1. Corollary. *The H-space of Example 1.3.9 does not retract onto $K(\mathbb{Z}/2,2)$.*

Proof. Since $\mathrm{incl}_i^* \, k^*(u_4) = u_2^2$, we have $S = \{u_2^2\}$, S as in Theorem 1.3.8. The two conditions $0 \notin S$ and $\mathrm{Card}(S) = 1$ are satisfied and consequently the space does not retract onto $K(\mathbb{Z}/2,2)$. ☐

As we did for Example 1.3.6 in Section 4.1, we state and summarize the properties of the space X in the following theorem:

1.3.10. Theorem. *The space X of Example 1.3.9 has the following properties:*

1. *X is not a GEM,*
2. *X is an infinite loop space,*
3. *X retracts neither onto the Eilenberg-MacLane space $K(\mathbb{Z}/2,2)$, nor onto $K(\mathbb{Z}/2,3)$,*
4. *However, X has no homological exponent.*

Proof. The space X is not a GEM since its k-invariant u_2^2 is not trivial. It is an infinite loop space since $u_2^2 = Sq^2u_2 = \sigma^* Sq^2 u_3 = \sigma^{(2)} Sq^2 u_4 = \dots$, where $\sigma^{(n)}$ denotes the n-fold cohomology suspension. We have just checked that X does not retract onto $K(\mathbb{Z}/2,2)$.

In order to show that X does not retract onto $K(\mathbb{Z}/2,3)$, let us consider now the mod-2 cohomology Serre spectral sequence of the fibration $K(\mathbb{Z}/2,3) \xrightarrow{j} X \xrightarrow{i} K(\mathbb{Z}/2,2)$.

The E_2-term looks like the following:

$$
\begin{array}{c|cccccc}
\mathbf{Sq^1u_3} & 0 & * & * & * & * & * \\
\mathbf{u_3} & 0 & u_2u_3 & * & * & * & * \\
\mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\mathbf{1} & \mathbf{0} & \mathbf{u_2} & \mathbf{Sq^1u_2} & \mathbf{u_2^2} & \mathbf{Sq^{2,1}u_2} & \mathbf{u_2^3} \\
 & & & & & \mathbf{u_2Sq^1u_2} & \mathbf{(Sq^1u_2)^2}
\end{array}
$$

We have $\bigoplus_s E_\infty^{s,2-s} \cong E_\infty^{2,0}$ and $H^2(X;\mathbb{F}_2) \cong \mathbb{F}_2\{v\}$ with $u_2 \mapsto v$ via the composition isomorphism

$$i^* : \ H^2(K(\mathbb{Z}/2,2);\mathbb{F}_2) \cong E_2^{2,0} \twoheadrightarrow E_3^{2,0} \cong E_\infty^{2,0} \cong H^2(X;\mathbb{F}_2) \ .$$

The transgression on u_3 is given by the k-invariant. To see this, consider the following homotopy pullback along the path-loop fibration:

$$
\begin{tikzcd}
K(\mathbb{Z}/2,3) \arrow[r, equal] \arrow[d, "j"'] & K(\mathbb{Z}/2,3) \arrow[d] \\
X \arrow[r] \arrow[d, "i"'] & * \arrow[d] \\
K(\mathbb{Z}/2,2) \arrow[r, "k"'] & K(\mathbb{Z}/2,4)
\end{tikzcd}
$$

and the Serre spectral sequence of both columns. By naturality of the spectral sequence, we have the following commutative diagram:

$$
\begin{tikzcd}
H^3(K(\mathbb{Z}/2,3);\mathbb{F}_2) \arrow[r, "\cong"] \arrow[d, equal] & H^4(K(\mathbb{Z}/2,4);\mathbb{F}_2) \arrow[d, "k^*"] \\
H^3(K(\mathbb{Z}/2,3);\mathbb{F}_2) \arrow[r, "\text{(transgression)}"'] & H^4(K(\mathbb{Z}/2,2);\mathbb{F}_2).
\end{tikzcd}
$$

The Serre's transgression theorem (see for instance [**36**, Theorem 6.8, p. 189]) implies that d_4 coincides with the transgression. Thus we have $d_4u_3 = k^*(u_4) = u_2^2$.

Therefore $\bigoplus_s E_\infty^{s,3-s} \cong E_\infty^{3,0}$ and $H^3(X;\mathbb{F}_2) \cong \mathbb{F}_2\{w\}$ with $Sq^1u_2 \mapsto w$ via the composition

$$i^* : \ H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2) \cong E_2^{3,0} \twoheadrightarrow \cdots \twoheadrightarrow E_4^{3,0} \cong E_\infty^{3,0} \cong H^3(X;\mathbb{F}_2) \ .$$

We clearly have $Sq^1v = w$. Suppose that there are maps $f : X \to K(\mathbb{Z}/2,3)$ and $g : K(\mathbb{Z}/2,3) \to X$ with $fg \simeq \mathrm{id}_{K(\mathbb{Z}/2,3)}$.

The only non-trivial map $f : X \to K(\mathbb{Z}/2,3)$ is given by the single non-trivial element $w \in H^3(X;\mathbb{F}_2)$. Therefore we have

$$\begin{aligned} g^*f^*(u_3) &= g^*(w) \\ &= g^*(Sq^1 v) \\ &= Sq^1 g^*(v) && \text{by naturality,} \\ &= 0 && \text{since } g^*(v) \in H^2(K(\mathbb{Z}/2,3);\mathbb{F}_2) = 0. \end{aligned}$$

In other words, we always have $fg \simeq *$ and X cannot retract onto $K(\mathbb{Z}/2,3)$.

Let us show that $Sq^1 u_3 \neq 0 \in E_5^{0,4}$. For connexity reasons, it suffices to show that $d_2 Sq^1 u_3 = 0 \in E_2^{2,3} = \mathbb{F}_2\{u_2u_3\}$. Suppose that $d_2 Sq^1 u_3 = u_2u_3$. Then we would have $u_2u_3 = 0 \in E_4^{2,3}$ and $0 = d_4(u_2u_3) = u_3d_4u_2 + u_2d_4u_3 = u_2u_2^2 = u_2^3$ (this make sense since both u_2 and u_3 survive in E_4, as well as their product u_2u_3). This is absurd since $u_2^3 \neq 0 \in E_4^{6,0}$ by connexity.

Let us show now that $d_5 Sq^1 u_3 = 0$. Since cohomology operations "commute" with transgressions (see [**36**, Corollary 6.9, p. 189]), we have $d_5 Sq^1 u_3 = Sq^1 u_2^2 = 0$.

Finally, we conclude that $Sq^1 u_3 \neq 0 \in E_6^{0,4} \cong E_\infty^{0,4}$. Therefore, there exists $x' \in H^4(X;\mathbb{F}_2)$ such that $x' \mapsto Sq^1 u_3$ via the composition

$$j^* :\ H^4(X;\mathbb{F}_2) \twoheadrightarrow E_\infty^{0,4} \cong E_6^{0,4} \subset \cdots \subset E_2^{0,4} \cong H^4(K(\mathbb{Z}/2,2);\mathbb{F}_2)\ .$$

Set $x = Sq^2 x'$. We have $j^*(x) = j^*(Sq^2 x') = Sq^2 j^*(x') = Sq^{2,1}u_3$ which is ∞-transverse by Theorem 1.3.2 (see also the proof of Corollary 1.3.3, p. 40).

Suppose that x is not ∞-transverse. Then there exists an integer $r \geqslant 0$ such that $d_{r+1}x^{2^r} = 0$ in the Bockstein spectral sequence of X. Therefore we have

$$\begin{aligned} d_{r+1}j^*(x)^{2^r} &= d_{r+1}j^*(x^{2^r}) && \text{since } j^* \text{ is an algebra map,} \\ &= j^* d_{r+1} x^{2^r} && \text{by naturality,} \\ &= 0 && \text{since } d_{r+1}x^{2^r} = 0, \end{aligned}$$

which contradicts ∞-transverse implications of $j^*(x) = Sq^{2,1}u_3$. Thus x is also ∞-transverse and X cannot admit a homology exponent. $\square$

CHAPTER 5

Cohomology of some 2-stage Postnikov systems

5.1. Transverse implications coming from the fibre

Let us recall that if X is a space and $\{B_r^*, d_r\}$ is its mod-2 cohomology Bockstein spectral sequence, then an element $x \in B_r^n$ is **ℓ-transverse** if $d_{r+l}x^{2^l} \neq 0 \in B_{r+l}^{2^l n}$ for all $0 \leqslant l \leqslant \ell$. Moreover, $x \in B_r^n$ is **∞-transverse**, or simply **transverse**, if it is ℓ-transverse for all $\ell \geqslant 0$.

The argument we used in the proof of Theorem 1.3.10, p. 61, can be used in many situations. Let us formulate it as the following lemma.

1.3.11. Lemma. *Let $i : F \to X$ be a continuous map. If $x \in H^*(X; \mathbb{F}_2)$ is such that $i^*(x) \neq 0 \in H^*(F; \mathbb{F}_2)$ is ∞-transverse, then x is ∞-transverse.*

Proof. Suppose that x is not ∞-transverse. Then there exists $r \geqslant 0$ such that $d_{r+1}x^{2^r} = 0$. Therefore we have

$$\begin{aligned} d_{r+1}i^*(x)^{2^r} &= d_{r+1}i^*(x^{2^r}) && \text{since } i^* \text{ is an algebra map,} \\ &= i^* d_{r+1}x^{2^r} && \text{by naturality,} \\ &= 0 && \text{since } d_{r+1}x^{2^r} = 0, \end{aligned}$$

which contradicts ∞-transversity of $i^*(x)$. □

This simple result allows us to give a general property of spaces which admit a homology exponent.

5.1.1. Theorem. *Let X be a 2-local H-space of finite type, G a finitely generated 2-torsion abelian group and n an integer $\geqslant 2$. If X has a homology exponent, then $i^* Sq^1 P^{\,\mathrm{even}} H^*(X; \mathbb{F}_2) = 0$ for all H-maps $i : K(G, n) \to X$.*

Proof. Suppose that there exists $x \in P^{\,\mathrm{even}} H^*(X; \mathbb{F}_2)$ such that $i^* Sq^1 x = Sq^1 i^*(x) \neq 0 \in H^*(K(G, n); \mathbb{F}_2)$. Then $i^*(x) \neq 0 \in P^{\,\mathrm{even}} H^*(K(G, n); \mathbb{F}_2)$ since i^* sends primitives to primitives. By Theorem 1.3.2, $i^*(x)$ is ∞-transverse since $Sq^1 i^*(x) \neq 0$. By Lemma 1.3.11, x is then ∞-transverse and X has no homological exponent. □

We can exploit the duality of the homology and cohomology mod-2 Bockstein spectral sequences in order to give a homological analogue of this result.

5.1.2. Theorem. *Let X be a 2-local H-space of finite type, G a finitely generated 2-torsion abelian group and n an integer $\geqslant 2$. If X has a homology exponent, then $i_* P_{\text{even}} \beta H_*(K(G,n);\mathbb{F}_2) = 0$ for all H-maps $i : K(G,n) \to X$, where $\beta : H_*(K(G,n);\mathbb{F}_2) \to H_{*-1}(K(G,n);\mathbb{F}_2)$ is the Bockstein homomorphism.*

Proof. Let us first notice that $PH_*(K(G,n);\mathbb{F}_2) \subset QH_*(K(G,n);\mathbb{F}_2)$ since $H^*(K(G,n);\mathbb{F}_2)$ is primitively generated and dual. Suppose now that there exists an element $x \in H_*(K(G,n);\mathbb{F}_2)$ such that $\beta x \in P_{\text{even}} H_*(K(G,n);\mathbb{F}_2) \subset Q_{\text{even}} H_*(K(G,n);\mathbb{F}_2)$ and $i_*\beta x \neq 0$. Observe that $i_*\beta x \in Q_{\text{even}} H_*(X;\mathbb{F}_2)$ since i_* sends indecomposables to indecomposables. By duality, let $x' \in P^{\text{even}} H^*(X;\mathbb{F}_2)$ such that $< i_*\beta x, x' >= 1$. Then we have $< x, i^* Sq^1 x' >= 1$ and thus $i^* Sq^1 x' \neq 0$. The result follows from Theorem 5.1.1. $\square$

5.2. Properties of the Eilenberg-Moore spectral sequence

The results exposed here are mainly those of S. Eilenberg and J. C. Moore [**23**], L. Smith [**57**] [**56**] and J. C. Moore and L. Smith [**45**] [**46**].

5.2.1. Theorem. *Let $F \to E \to B$ be a fibration with 1-connected base space B and connected fiber F. There is a second quadrant spectral sequence $\{E_r^{*,*}, d_r\}$ such that $E_2^{p,q} \cong \mathrm{Tor}^{p,q}_{H^*(B;\mathbb{F}_2)}(\mathbb{F}_2, H^*(E;\mathbb{F}_2))$, and converging to $H^*(F;\mathbb{F}_2)$. Moreover, the spectral sequence is a spectral sequence of algebras, converging to its target as an algebra. This spectral sequence is called the* **mod-2 cohomology Eilenberg-Moore spectral sequence** *of the fibration $F \to E \to B$.*

Proof. See [**36**, Corollary 7.16, p. 252 and Corollary 7.18, p. 256]. See also [**57**, Theorem 6.2, pp. 51-52] for more properties of the spectral sequence. □

5.2.2. Definition. Let $f : E \to B$ be a fibration. If $H^*(E;\mathbb{F}_2)$ and $H^*(B;\mathbb{F}_2)$ admit Hopf algebras structures such that $f^* : H^*(B;\mathbb{F}_2) \to H^*(E;\mathbb{F}_2)$ is a morphism of Hopf algebras, then $f : E \to B$ is said to be a **WH-fibration**.

Remark. J. C. Moore and L. Smith speak more generally of *Hopf fibre squares* in [**57**, p. 56] and Hopf fibre squares *in the weak sense* in [**45**, Appendix: Generalizations, pp. 779-780]. This justifies the notation WH for "*weak*" and "*Hopf*".

5.2.3. Example. Consider a 2-stage (stable) Postnikov system with primitive k-invariant. The fibration involved in the system is a WH-fibration.

This enables us to modify slightly Theorem 5.2.1 as follows:

5.2.4. Theorem. *Let $F \to E \to B$ be a WH-fibration with 1-connected base space B and connected fiber F. There is a second quadrant spectral sequence $\{E_r^{*,*}, d_r\}$ such that $E_2^{p,q} \cong \mathrm{Tor}^{p,q}_{H^*(B;\mathbb{F}_2)}(\mathbb{F}_2, H^*(E;\mathbb{F}_2))$, and converging to $H^*(F;\mathbb{F}_2)$. Moreover, the spectral sequence is a spectral sequence of (bigraded) Hopf algebras, converging to its target as a Hopf algebra.*

Proof. See [**57**, Theorem 1.1, pp. 59-61] and [**45**, Appendix: Generalizations, pp. 779-780]. □

In what follows, we would like to identify the E_2-term of the spectral sequence as a Hopf algebra.

5.2.5. Theorem. *Let $f : E \to B$ be a WH-fibration with 1-connected base space B and connected fiber F. Consider its mod-2 cohomology Eilenberg-Moore spectral sequence $\{E_r^{*,*}, d_r\}$. As a (bigraded) Hopf algebra, we have*

$$E_2^{*,*} \cong H^*(E;\mathbb{F}_2)//f^* \otimes \mathrm{Tor}_{H^*(B;\mathbb{F}_2)\backslash\backslash f^*}(\mathbb{F}_2,\mathbb{F}_2),$$

where $H^*(E;\mathbb{F}_2)//f^*$ denotes the cokernel of $f^* : H^*(B;\mathbb{F}_2) \to H^*(E;\mathbb{F}_2)$ and $H^*(B;\mathbb{F}_2)\backslash\backslash f^*$ the kernel of f^* (in the category of connected positively graded Hopf algebra over $\mathbb{F}_2$ whose multiplication is commutative).

Proof. See [**57**, Theorem 2.4, p. 67]. □

We can specialize this result when $H^*(B;\mathbb{F}_2)$ is a free graded commutative $\mathbb{F}_2$-algebra, i.e. a graded polynomial $\mathbb{F}_2$-algebra, which is cocommutative as a Hopf algebra. For instance, this is the case if B is an Eilenberg-MacLane space.

5.2.6. Proposition. *Let H be a cocommutative Hopf algebra and $I \subset H$ a Hopf ideal. There exists a unique sub-Hopf algebra $S \subset H$ such that $I = \bar{S} \cdot H$, where $\bar{S}$ denotes the augmentation ideal of S.*

Proof. See [**56**, Proposition 1.4, pp. 311-312]. □

5.2.7. Definition. Let H be a cocommutative Hopf algebra, H' a Hopf algebra and $\varphi : H \to H'$ be a Hopf algebra map. Following Proposition 5.2.6, we denote by **sub-ker** φ the unique sub-Hopf algebra of H generating the Hopf ideal $\ker\varphi$.

5.2.8. Corollary. *Let $f : E \to B$ be a WH-fibration with 1-connected base space B and connected fiber F. Suppose further that $H^*(B;\mathbb{F}_2)$ is a free graded commutative $\mathbb{F}_2$-algebra which is cocommutative as a Hopf algebra. Consider the mod-2 cohomology Eilenberg-Moore spectral sequence $\{E_r^{*,*}, d_r\}$ associated with the fibration $f : E \to B$. As a (bigraded) Hopf algebra we have*

$$E_2^{*,*} \cong H^*(E;\mathbb{F}_2)//f^* \otimes \Lambda_{\mathbb{F}_2}(s^{-1,0}Q\,\text{sub-ker}\, f^*),$$

where $s^{-1,0}$ shifts the bidegree of the elements in $Q\,\text{sub-ker}\, f^*$ by $(-1,0)$, the decomposable elements of sub-ker f^**. Moreover, all the differentials d_r are trivial for $r \geqslant 2$ and therefore $E_2 \cong E_\infty$.*

Sketch of proof. It is easy to see that $H^*(B;\mathbb{F}_2)\backslash\backslash f^* \cong \text{sub-ker}\, f^*$. But sub-ker f^* is a sub-Hopf algebra of a polynomial algebra. Then by A. Borel's structure theorem, sub-ker f^* is also a polynomial algebra. Using a Koszul resolution, it is easy to compute that $\mathrm{Tor}_{\mathbb{F}_2[x]}(\mathbb{F}_2,\mathbb{F}_2) \cong \Lambda_{\mathbb{F}_2}(s^{-1,0}x)$ as bigraded objects. Therefore we have $\mathrm{Tor}_{\text{sub-ker}\, f^*}(\mathbb{F}_2,\mathbb{F}_2) \cong \Lambda_{\mathbb{F}_2}(s^{-1,0}Q\,\text{sub-ker}\, f^*)$ since sub-ker f^*

is generated by its indecomposables. The differential d_2 is trivial on $E_2^{0,*} \cong H^*(E; \mathbb{F}_2)$ and $E_2^{-1,*} \cong Q\,\text{sub-ker}\, f^*$ since the spectral sequence lies in the second quadrant. Therefore d_2 is trivial since elements in $E_2^{0,*}$ and $E_2^{-1,*}$ generate $E_2^{*,*}$. The same argument holds for higher differentials and $d_r = 0$ for all $r \geqslant 2$. Therefore, we have $E_2 \cong E_\infty$. See also [**57**, Theorem 3.2, pp. 75-76]. ☐

We present here a technical, but non-surprising, result which we will need in the sequel.

5.2.9. Theorem. *Let $f : E \to B$ be a WH-fibration with 1-connected base space B and connected fiber F. Suppose further that $H^*(B; \mathbb{F}_2)$ is a free graded commutative $\mathbb{F}_2$-algebra which is cocommutative as a Hopf algebra. Consider the inclusion map $i : \Omega B \to F$ and its induced map $i^* : H^*(F; \mathbb{F}_2) \to H^*(\Omega B; \mathbb{F}_2)$. If $x \in Q\,\text{sub-ker}\, f^* \subset H^*(B; \mathbb{F}_2)$, then*

$$i^*(\tilde{x}) = \sigma^*(x),$$

where $\tilde{x}$ is the image of $x \in Q\,\text{sub-ker}\, f^* \cong E_2^{-1,*} \cong E_\infty^{-1,*}$ in $H^*(F; \mathbb{F}_2)$ and σ^* denotes the cohomology suspension $H^*(B; \mathbb{F}_2) \to H^{*-1}(\Omega B; \mathbb{F}_2)$.

Proof. See [**56**, Proposition 5.4, p. 325]. ☐

To conclude this section, let us work out the Eilenberg-Moore spectral sequence of the space of Example 1.3.9: let X be the space given by the fibration

$$X \longrightarrow K(\mathbb{Z}/2, 2) \xrightarrow{f} K(\mathbb{Z}/2, 4),$$

with

$$\begin{aligned} f \in & [K(\mathbb{Z}/2, 2), K(\mathbb{Z}/2, 4)] \\ \cong & H^4(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \\ \cong & \mathbb{F}_2\{u_2^2\} \end{aligned}$$

given by $f = u_2^2$ where u_2 is the fundamental class of $K(\mathbb{Z}/2, 2)$.

The definition of the functor $\text{Tor}^{*,*}_{(-)}(-, -)$ in Theorem 5.2.1 is rather technical and involves homological algebra (see for instance [**36**, Definition 7.5, p. 240]). In order to compute $\text{Tor}_\Gamma(M, N)$, we need a proper projective resolution of the left Γ-module N. This can be done using one of the more useful explicit constructions in homological algebra, namely the **bar construction**. The bar construction can become quite large and complicated. But one of the features of homological algebra is the invariance of the derived functors with regard to the choice of resolution and so the construction of smaller and more manageable resolutions is of key interest. We will give here a method to compute $\text{Tor}_\Gamma(M, N)$ when Γ, M and N are free graded commutative objects over $\mathbb{F}_2$. This will be achieved by using **Koszul complexes**.

5.2.10. Theorem. *Let S be a graded set and L a graded commutative algebra over $\mathbb{F}_2[S]$. Then there is an isomorphism of bigraded algebras over $\mathbb{F}_2$*

$$\mathrm{Tor}^{*,*}_{\mathbb{F}_2[S]}(\mathbb{F}_2, L) \cong H(\Lambda_{\mathbb{F}_2}(s^{-1,0}S) \otimes_{\mathbb{F}_2} L, d_L)$$

where the Koszul complex $\Lambda_{\mathbb{F}_2}(s^{-1,0}S) \otimes_{\mathbb{F}_2} L$ has the differential d_L given by

$$d_L(s^{-1,0}x \otimes l) = 1 \otimes xl \textit{ for all } x \in S$$

and the bidegree given by

$$\begin{aligned} \mathrm{bideg}(1 \otimes l) &= (0, \deg(l)) \textit{ and} \\ \mathrm{bideg}(s^{-1,0}x \otimes 1) &= (-1, \deg(x)). \end{aligned}$$

Proof. See [**36**, Corollary 7.23, p. 260]. $\square$

Recall once more that the space X of Example 1.3.9 is given by the following homotopy pullback:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ K(\mathbb{Z}/2,2) \arrow[r, "f"'] & K(\mathbb{Z}/2,4), \end{tikzcd}$$

where $f^*(u_4) = Sq^2 u_2 = u_2^2$.

If one considers the graded set

$$S = \{Sq^I u_4 \mid I \text{ admissible and } e(I) < 4\},$$

then the E_1-term of the Eilenberg-Moore spectral sequence is given by

$$E_1^{*,*} \cong \Lambda_{\mathbb{F}_2}(s^{-1,0}S) \otimes_{\mathbb{F}_2} H^*(K(\mathbb{Z}/2,2); \mathbb{F}_2))$$

and the differential

$$d_1(s^{-1,0}x \otimes l) = 1 \otimes f^*(x) \cdot l \text{ for all } x \in S.$$

We can now picture the $E_1^{p,q}$-term of the spectral sequence (in the range $-1 \leqslant p \leqslant 0$ and $0 \leqslant q \leqslant 7$):

$$
\begin{array}{ccc|l}
s^{-1,0}u_4 \otimes Sq^1 u_2 & \xmapsto{d_1} & 1 \otimes u_2^2 Sq^1 u_2 & \mathbf{7} \\
\boxed{s^{-1,0}Sq^3 u_4 \otimes 1} & & \boxed{1 \otimes u_2 Sq^{2,1} u_2} & \\
\boxed{s^{-1,0}Sq^{2,1} u_4 \otimes 1} & & & \\
\boxed{s^{-1,0}Sq^1 u_4 \otimes u_2} & & & \\
s^{-1,0}Sq^2 u_4 \otimes 1 & \xmapsto{d_1} & 1 \otimes (Sq^1 u_2)^2 & \mathbf{6} \\
s^{-1,0}u_4 \otimes u_2 & \xmapsto{d_1} & 1 \otimes u_2^3 & \\
\boxed{s^{-1,0}Sq^1 u_4 \otimes 1} & & \boxed{1 \otimes Sq^{2,1} u_2} & \mathbf{5} \\
 & & \boxed{1 \otimes u_2 Sq^1 u_2} & \\
s^{-1,0}u_4 \otimes 1 & \xmapsto{d_1} & 1 \otimes u_2^2 & \mathbf{4} \\
 & & \boxed{1 \otimes Sq^1 u_2} & \mathbf{3} \\
 & & \boxed{1 \otimes u_2} & \mathbf{2} \\
 & & & \mathbf{1} \\
 & & \boxed{1} & \mathbf{0} \\
\hline
\mathbf{-1} & & \mathbf{0} & \\
\boxed{\Lambda_{\mathbb{F}_2}(s^{-1,0}Q\,\text{sub-ker}\, f^*)} & & \boxed{H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2)//f^*} &
\end{array}
$$

Boxed elements are those which survive to $E_2 \cong E_\infty$ by Corollary 5.2.8.

Therefore we have the following table for $H^*(X;\mathbb{F}_2)$:

n	$H^n(X;\mathbb{F}_2)$	$\dim_{\mathbb{F}_2}$
0	$\mathbb{F}_2$	1
1	0	0
2	$\mathbb{F}_2\{u_2\}$	1
3	$\mathbb{F}_2\{Sq^1u_2\}$	1
4	$\mathbb{F}_2\{s^{-1,0}Sq^1u_4\}$	1
5	$\mathbb{F}_2\{Sq^{2,1}u_2, u_2Sq^1u_2\}$	2
6	$\mathbb{F}_2\{s^{-1,0}Sq^3u_4, s^{-1,0}Sq^{2,1}u_4, s^{-1}Sq^1u_4 \cdot u_2\}$	3

We can also apply Theorem 5.2.9 to obtain the images of elements "coming from" $\Lambda_{\mathbb{F}_2}(s^{-1,0}Q\,\text{sub-ker}\,f^*)$ via $i^* : H^*(X;\mathbb{F}_2) \to H^*(K(\mathbb{Z}/2,3);\mathbb{F}_2)$:

$$\begin{aligned} i^*(s^{-1,0}Sq^1u_4) &= Sq^1u_3, \\ i^*(s^{-1,0}Sq^{2,1}u_4) &= Sq^{2,1}u_3, \\ i^*(s^{-1,0}Sq^3u_4) &= Sq^3u_3 = u_3^2. \end{aligned}$$

5.3. Strategy for detecting transverse implications

Let X be a 1-connected 2-local Postnikov piece of finite type and let $n \geqslant 2$ be the largest integer such that $\pi_n X \neq 0$. Then the Postnikov tower yields the following fibre square in which X is the homotopy fibre of f:

$$\begin{tikzcd}
K(\pi_n X, n) \arrow[d, "i"'] & \\
X \arrow[r] \arrow[d] & PK(\pi_n X, n+1) \simeq * \arrow[d] \\
X[n-1] \arrow[r, "f"'] & K(\pi_n X, n+1).
\end{tikzcd}$$

Moreover, if X is a homotopy associative H-space, then f and i are H-maps.

Consider an element $x \in Q\,\text{sub-ker}\, f^* \subset H^{*+1}(K(\pi_n X, n+1); \mathbb{F}_2)$. Following Corollary 5.2.8 and Theorem 5.2.9, consider also its image $\tilde{x} \in H^*(X; \mathbb{F}_2)$. Theorem 5.2.9 implies that $i^*(\tilde{x}) = \sigma^*(x)$ where $\sigma^* : H^{*+1}(K(\pi_n X, n+1); \mathbb{F}_2) \to H^*(K(\pi_n X, n); \mathbb{F}_2)$ denotes the cohomology suspension. Suppose now that $i^*(\tilde{x}) = \sigma^*(x)$ is ∞-transverse. Then, following Lemma 1.3.11, $\tilde{x}$ is ∞-transverse as well. **Therefore it suffices to find an element** $x \in Q\,\text{sub-ker}\, f^* \subset H^{*+1}(K(\pi_n X, n+1); \mathbb{F}_2)$ **such that** $\sigma^*(x)$ **is ∞-transverse in** $H^*(K(\pi_n X, n); \mathbb{F}_2)$ in order to detect an ∞-transverse element in the mod-2 cohomology Bockstein spectral sequence of X.

According to Theorem 1.3.2, recall that $\sigma^*(x) \in H^*(K(\pi_n X, n); \mathbb{F}_2)$ is ∞-transverse if $\sigma^*(x) \in P^{\,\text{even}} H^*(K(\pi_n X, n); \mathbb{F}_2)$ is such that $Sq^1 \sigma^*(x) \neq 0$. Since σ^* sends indecomposables to primitives (see [**28**, pp. 210-211]) and commutes with Steenrod squares (because they are cohomology operations), **these conditions are satisfied when** $x \in Q^{\,\text{odd}}\,\text{sub-ker}\, f^*$ **is such that** $Sq^1 x \notin \ker \sigma^*$.

5.4. Proof of the first main theorem

5.4.1. Definition. For all $l \geqslant 0$, we define the admissible sequence

$$\gamma(l) = (2^l - 1, 2^{l-1} - 1, \dots, 3, 1),$$

of stable degree $\deg_{\mathrm{st}}(\gamma(l)) = 2(2^l - 1) - l$ and excess $e(\gamma(l)) = l$.

5.4.2. Lemma. *Let G be a non-trivial finitely generated 2-torsion abelian group and m an integer $\geqslant 2$. Then*

$$Sq^{\gamma(m-1)} P^{m+2} H^*(K(G,m); \mathbb{F}_2) = 0,$$
$$Sq^{\gamma(m)} P^{\geqslant m+3} H^*(K(G,m); \mathbb{F}_2) = 0.$$

Proof. It suffices to establish the assertion for the group $G = \mathbb{Z}/2^s$ with $s \geqslant 1$. Let us recall that

$$H^*(K(G,m); \mathbb{F}_2) \cong \mathbb{F}_2[Sq_s^I u_m \mid I \text{ admissible and } e(I) < m].$$

It is then clear that we have

$$H^{m+2}(K(G,m); \mathbb{F}_2) \cong \mathbb{F}_2\{Sq_s^2 u_m\},$$

the $\mathbb{F}_2$-vector space spanned by $Sq_s^2 u_m$, the unique non-trivial element of degree $m+2$. This element is primitive and therefore

$$P^{m+2} H^*(K(G,m); \mathbb{F}_2) = \mathbb{F}_2\{Sq^2 u_m\}.$$

Let us show by induction that $Sq^{\gamma(m-2)} Sq^2 u_m$ is a square. If $m = 2$ we have $Sq^2 u_2 = (u_2)^2$. Suppose now that $m \geqslant 3$. Then we have

$$\begin{aligned}
\sigma^* Sq^{\gamma(m-2)} Sq^2 u_m &= \sigma^* Sq^{2^{m-2}-1} Sq^{\gamma(m-3)} Sq^2 u_m && \text{by definition of } \gamma,\\
&= Sq^{2^{m-2}-1} Sq^{\gamma(m-3)} Sq^2 \sigma^* u_m \\
&= Sq^{2^{m-2}-1} Sq^{\gamma(m-3)} Sq^2 u_{m-1} \\
&= Sq^{2^{m-2}-1}(\text{square}) && \text{by induction,}\\
&= 0 && \text{by Cartan's formula.}
\end{aligned}$$

Therefore, $Sq^{\gamma(m-2)} Sq^2 u_m$ is decomposable since it is well known that $\sigma^* : QH^*(K(G,m)\mathbb{F}_2) \to PH^*(K(G,m-1); \mathbb{F}_2)$ is a monomorphism (see [**15**, Exposé 15, Proposition 3, p. 8]). By the Milnor-Moore theorem, $Sq^{\gamma(m-2)} Sq^2 u_m$ is a square since it is primitive and decomposable (see p. 5). Finally, $Sq^{\gamma(m-1)} Sq^2 u_m = 0$ by Cartan's formula. This proves the first statement.

An $\mathbb{F}_2$-basis for $P^{\geqslant m+3} H^*(K(G,m); \mathbb{F}_2)$ is given by $\{Sq_s^I u_m \mid e(I) < m \text{ and } \deg_{\mathrm{st}}(I) \geqslant 3\}$. Let us show by induction that $Sq^{\gamma(m-1)} Sq_s^I u_m$ is a square. If $m = 2$ then the basis contains elements of the form $Sq_s^I u_2$ with $I = (2^l, 2^{l-1}, \dots, 1)$ for all $l \geqslant 1$. We have $Sq^1 Sq_s^I u_2 = Sq^{2^l+1} Sq_s^{I^-} u_2 = (Sq_s^{I^-} u_2)^2$ since

$\deg(Sq_s^{I^-}u_2) = (2^{l-1}+\cdots+1)+2 = (2^l-1)+2 = 2^l+1$ (see Definition 2.3.4 for I^-). Suppose now that $m \geqslant 3$. Then we have

$$\begin{aligned}
\sigma^* Sq^{\gamma(m-1)} Sq_s^I u_m &= \sigma^* Sq^{2^{m-1}-1} Sq^{\gamma(m-2)} Sq_s^I u_m && \text{by definition of } \gamma,\\
&= Sq^{2^{m-1}-1} Sq^{\gamma(m-2)} Sq_s^I \sigma^* u_m &&\\
&= Sq^{2^{m-1}-1} Sq^{\gamma(m-2)} Sq_s^I u_{m-1} &&\\
&= Sq^{2^{m-1}-1}(\text{square}) && \text{by induction,}\\
&= 0 && \text{by Cartan's formula.}
\end{aligned}$$

This proves the second statement. □

We are now able to prove the main theorem of this work.

1.3.12. Theorem. *Let X be a 1-connected 2-local H-space of finite type with only two non-trivial (finite) homotopy groups. Then X has no homology exponent.*

Proof. Let us consider the 2-stage Postnikov system of the space X:

$$\begin{tikzcd}
K(\pi_n X, n) \arrow[d, "i"'] & \\
X \arrow[r] \arrow[d] & PK(\pi_n X, n+1) \arrow[d] \\
K(\pi_m X, m) \arrow[r, "f"'] & K(\pi_n X, n+1)
\end{tikzcd}$$

where $n > m \geqslant 2$. If f is trivial, then X is a product of two Eilenberg-MacLane spaces and has clearly no homological exponent. Suppose that f is non-trivial and consider $u \in P^{n+1}H^*(K(\pi_n X, n+1); \mathbb{F}_2) = H^{n+1}(K(\pi_n X, n+1); \mathbb{F}_2)$, a generator of a direct factor $\mathbb{Z}/2^s$ in $\pi_n X$, such that $f^*(u) \neq 0$. Now set

$$I = \begin{cases} (2^m-2, \gamma(m-1)) & \text{if } n = m+1,\\ (2^{m+1}-2, \gamma(m)) & \text{if } n \geqslant m+2 \text{ and } m+n \equiv 1(2),\\ (2^{m+2}-2, \gamma(m+1)) & \text{if } n \geqslant m+2 \text{ and } m+n \equiv 0(2) \end{cases}$$

and set $x = Sq_s^I u$. By Lemma 5.4.2 we have $x \in \ker f^*$. Since I is admissible of excess $< n$, x is indecomposable in $H^*(K(\pi_n X, n+1); \mathbb{F}_2)$ and *a fortiori* in sub-ker f^*. It is also easy to check that $\deg(x)$ is odd. Then $x \in Q^{\mathrm{odd}}$ sub-ker f^*. Moreover $\sigma^* Sq^1 x = Sq^1 Sq_s^I \sigma^*(u) \neq 0 \in H^*(K(\pi_n X, n); \mathbb{F}_2)$ since $e(I) < n$, $\deg(\sigma^*(u)) = n$ and I begins with an even integer. We conclude with results of Section 5.3. □

5.5. Asymptotic behaviour of Poincaré series

5.5.1. Definition. Let G be a finitely generated 2-torsion abelian group of type $(s_1, \dots, s_l)$. For all $0 \leqslant \delta < n$ we define the free graded commutative algebra

$$H^*_{[\delta]}(K\Sigma^n G) = \bigotimes_{j=1}^{l} \mathbb{F}_2[Sq^I_{s_j} u_j \mid I \text{ admissible and } e(I) < n - \delta]$$

where $u_j \in H^n(K(\mathbb{Z}/2^{s_j}, n); \mathbb{Z}/2)$ is the mod-2 reduction of the fundamental class for all $1 \leqslant j \leqslant l$. Moreover we set $P_{[\delta]}(K\Sigma^n G)(t) = P(H^*_{[\delta]}(K\Sigma^n G), t)$, the Poincaré series.

5.5.2. Remark. The polynomial algebra $H^*_{[\delta]}(K\Sigma^n G)$ is a sub-Hopf algebra of the differential, connected, bicommutative and primitive Hopf algebra $H^*(K(G,n); \mathbb{F}_2)$. But this is not a differential sub-Hopf algebra. To see this, suppose that $n \geqslant 2$ is even. Consider $Sq^{n-2}u_n \in H^*_{[1]}(K\Sigma^n\mathbb{Z}/2)$. We have $Sq^1 Sq^{n-2} u_n = Sq^{n-1} u_n \notin H^*_{[1]}(K\Sigma^n \mathbb{Z}/2)$.

5.5.3. Lemma. *Let G be a finitely generated 2-torsion abelian group. For all $0 \leqslant \delta < n$, we have the Poincaré series*

$$P_{[\delta]}(K\Sigma^n G)(t) = \prod_{h_1 \geqslant \cdots \geqslant h_{n-\delta-1} \geqslant 0} \left(\frac{1}{1 - t^{\delta+1+2^{h_1}+\cdots+2^{h_{n-\delta-1}}}} \right)^{\operatorname{rank}(G)}.$$

Proof. For any admissible sequence I of excess $e(I) < n - \delta$, the Poincaré series of $\mathbb{F}_2[Sq^I u_n]$ is given by

$$\begin{aligned} P(\mathbb{F}_2[Sq^I u_n], t) &= \sum_{i \geqslant 0} \dim_{\mathbb{F}_2} \mathbb{F}_2[Sq^I u_n] \cdot t^i \\ &= \sum_{i \geqslant 0} t^{i \deg(Sq^I u_n)} \\ &= \frac{1}{1 - t^{deg(Sq^I u_n)}} \\ &= \frac{1}{1 - t^{\deg_{\mathrm{st}}(I) + n}}. \end{aligned}$$

Let us assume that $G \cong \bigoplus_{j=1}^{l} \mathbb{Z}/2^{s_j}$, where $l = \operatorname{rank}(G)$. We then have

$$\begin{aligned}
H^*_{[\delta]}(K\Sigma^n G) &\cong H^*_{[\delta]}(K\Sigma^n \bigoplus_{j=1}^{l} \mathbb{Z}/2^{s_j}) \\
&\cong \bigotimes_{j=1}^{l} H^*_{[\delta]}(K\Sigma^n \mathbb{Z}/2^{s_j}) \\
&\cong \bigotimes_{j=1}^{l} H^*_{[\delta]}(K\Sigma^n \mathbb{Z}/2) \\
&\cong H^*_{[\delta]}(K\Sigma^n \mathbb{Z}/2)^{\otimes \operatorname{rank}(G)} \\
&\cong \mathbb{F}_2[Sq^I u_n \mid I \text{ admissible and } e(I) < n-\delta]^{\otimes \operatorname{rank}(G)} \\
&\cong (\bigotimes_{e(I)<n-\delta} \mathbb{F}_2[Sq^I u_n])^{\otimes \operatorname{rank}(G)}
\end{aligned}$$

It is now easy to compute the Poincaré series of $H^*_{[\delta]}(K\Sigma^n G)$. The two above computations give

$$\begin{aligned}
P(KH^*_{[\delta]}(\Sigma^n G), t) &= P\left(\left(\bigotimes_{e(I)<n-\delta} \mathbb{F}_2[Sq^I u_n]\right)^{\otimes \operatorname{rank}(G)}, t\right) \\
&= P\left(\bigotimes_{e(I)<n-\delta} \mathbb{F}_2[Sq^I u_n], t\right)^{\operatorname{rank}(G)} \\
&= \left(\prod_{e(I)<n-\delta} P(\mathbb{F}_2[Sq^I u_n], t)\right)^{\operatorname{rank}(G)} \\
&= \left(\prod_{e(I)<n-\delta} \frac{1}{1 - t^{\deg_{\mathrm{st}}(I)+n}}\right)^{\operatorname{rank}(G)} \\
&= \prod_{e(I)<n-\delta} \left(\frac{1}{1 - t^{\deg_{\mathrm{st}}(I)+n}}\right)^{\operatorname{rank}(G)}.
\end{aligned}$$

The next step in proving the Lemma consists in computing the number of admissible sequences I such that $e(I) < n-\delta$ and $\deg_{\mathrm{st}}(I)+n = N$ for a given integer N. Let $I = (a_0, \ldots, a_k)$ be an admissible sequence. Set

$$\begin{aligned}
\alpha_1 &= a_0 - 2a_1,\\
\alpha_2 &= a_1 - 2a_2,\\
&\dots\\
\alpha_k &= a_{k-1} - 2a_k,\\
\alpha_{k+1} &= a_k \text{ and}\\
\alpha_0 &= n - \delta - 1 - \sum_{i=1}^{k+1} \alpha_1.
\end{aligned}$$

It is then very easy to verify that

$$\begin{aligned}
&\sum_{i=0}^{k+1} \alpha_i = n - \delta - 1 \text{ and}\\
&N = \delta + 1 + \sum_{i=0}^{k+1} 2^i \alpha_i.
\end{aligned}$$

Since there are $n - \delta - 1$ increasing powers of 2 in $N - (\delta + 1)$, we can write $N = \delta + 1 + 2^{h_1} + \cdots + 2^{h_{n-\delta-1}}$ with $h_1 \geqslant \cdots \geqslant h_{n-\delta-1}$ and the desired result follows. This also proves Proposition 2.1.6, if one set $\delta = 0$. □

5.5.4. Definition. Let $G = \bigoplus_{d\geqslant 1} \Sigma^d G_d$ be a locally finite 2-torsion graded abelian group. For all $\delta \geqslant 0$ we define the free graded commutative algebra

$$H^*_{[\delta]}(KG) = \bigotimes_{d\geqslant \delta+1} H^*_{[\delta]}(K\Sigma^d G_d).$$

Moreover we set

$$P_{[\delta]}(KG)(t) = \prod_{d\geqslant \delta+1} P_{[\delta]}(K\Sigma^d G_d)(t),$$

where the product is the Cauchy's product of power series.

5.5.5. Definition. Let $G = \bigoplus_{d\geqslant 1} \Sigma^d G_d$ be a locally finite 2-torsion graded abelian group. For all $\delta \geqslant 0$, the **δ-derived Serre function** associated to KG is the continuous map given by

$$\varphi_{[\delta]}(KG)(x) = \log_2 P_{[\delta]}(KG)(1 - 2^{-x})$$

for all $x \in [0, +\infty[$.

5.5.6. Lemma. *Let G be a finitely generated 2-torsion abelian group. For all $0 \leqslant \delta < n$, we have the asymptotic growth formula*

$$\varphi_{[\delta]}(\Sigma^n G)(x) \sim \operatorname{rank}(G) \cdot \frac{x^{n-\delta}}{(n-\delta)!}$$
$$\textit{i.e.} \lim_{x \to \infty} \frac{\varphi_{[\delta]}(\Sigma^n G)(x)}{\operatorname{rank}(G) \cdot \frac{x^{n-\delta}}{(n-\delta)!}} = 1.$$

We need two technical results in order to prove this Lemma by induction. For all $n \geqslant q \geqslant 1$, let $R_{n,q}$ denote the assertion "$\varphi_{[n-q]}(\Sigma^n G)(x) \sim \operatorname{rank}(G) \cdot \frac{x^q}{q!}$".

5.5.7. Lemma. *The assertion $R_{n,1}$ is true for all $n \geqslant 1$.*

Proof. Suppose $q = 1$ and $n \geqslant 1$. The sequence $h_1 \geqslant \cdots \geqslant h_{q-1} \geqslant 0$ is empty and then, by Lemma 5.5.3, we have

$$P_{[n-1]}(K\Sigma^n G)(t) = \left(\frac{1}{1-t^n}\right)^{\operatorname{rank}(G)} = \left(\frac{1}{(1-t)f(t)}\right)^{\operatorname{rank}(G)}$$

with $f(t) \in \mathbb{Z}[t]$ of degree $n-1$. The $(n-1)$-derived Serre function becomes

$$\begin{aligned}\varphi_{[n-1]}(K\Sigma^n G)(x) &= \log_2 P_{[n-1]}(K\Sigma^n G)(1-2^{-x})\\ &= \operatorname{rank}(G)(x - \log_2 f(1-2^{-x})).\end{aligned}$$

Then we have clearly

$$\lim_{x \to +\infty} \frac{\varphi_{[n-1]}(K\Sigma^n G)(x)}{\operatorname{rank}(G) x} = \lim_{x \to +\infty} (1 - \frac{\log_2 f(1-2^{-x})}{x}) = 1.$$

□

5.5.8. Lemma. *If $R_{n,q}$ and $R_{n,q+1}$ are true, then $R_{n+1,q+1}$ is true.*

Proof. Let us consider the formal power serie

$$\widetilde{P}_{[\delta]}(K\Sigma^n G)(t) = \prod_{h_1 \geqslant \cdots \geqslant h_{n-\delta-1} \geqslant 0} \left(\frac{1}{1 - t^{\delta + 2^{h_1+1} + \cdots + 2^{h_{n-\delta-1}+1}}}\right)^{\operatorname{rank}(G)}.$$

It is easy to verify that

$$\widetilde{P}_{[n-q]}(K\Sigma^n G)(t) = P_{[n-q])}(K\Sigma^{n-1} G)(t) / P_{[n-q+1]}(K\Sigma^{n-1} G)(t).$$

For all $0 \leqslant t < 1$, we have

$$\widetilde{P}_{[n-q]}(K\Sigma^n G)(t) \leqslant P_{[n-q]}(K\Sigma^n G)(t) \leqslant P_{[n-q]}(K\Sigma^{n-1} G)(t)$$

since $n-q+2^{h_1+1}+\cdots+2^{h_{q-1}+1} \geqslant n-q+1+2^{h_1}+\cdots+2^{h_{q-1}} \geqslant n-q+2^{h_1}+\cdots+2^{h_{q-1}}$. Then for all $0 \leqslant t < 1$, we have

$$\begin{aligned} &P_{[n-q]}(K\Sigma^{n-1}G)(t)/P_{[n-q+1]}(K\Sigma^{n-1}G)(t) \\ &\qquad \leqslant P_{[n-q]}(K\Sigma^{n}G)(t) \\ &\qquad\qquad \leqslant P_{[n-q]}(K\Sigma^{n-1}G)(t) \end{aligned}$$

and therefore, for all $x \in [0,\infty[$, we have

$$\begin{aligned} &\varphi_{[n-q]}(K\Sigma^{n-1}G)(x) - \varphi_{[n-q+1]}(K\Sigma^{n-1}G)(x) \\ &\qquad \leqslant \varphi_{[n-q]}(K\Sigma^{n}G)(x) \\ &\qquad\qquad \leqslant \varphi_{[n-q]}(K\Sigma^{n-1}G)(x). \end{aligned}$$

By hyphothesis

$$\varphi_{[n-q]}(K\Sigma^{n-1}G)(x) \sim \operatorname{rank}(G)\frac{x^q}{q!} \quad \text{and}$$
$$\varphi_{[n-q+1]}(K\Sigma^{n-1}G)(x) \sim \operatorname{rank}(G)\frac{x^{q-1}}{(q-1)!}.$$

Thus

$$\begin{aligned} &\varphi_{[n-q]}(K\Sigma^{n-1}G)(x) - \varphi_{[n-q+1]}(K\Sigma^{n-1}G)(x) \\ &\qquad \sim \operatorname{rank}(G)\frac{x^q}{q!} - \operatorname{rank}(G)\frac{x^{q-1}}{(q-1)!} \\ &\qquad \sim \operatorname{rank}(G)\frac{x^q}{q!} \end{aligned}$$

and the result follows. □

Proof of Lemma 5.5.6. The problem can then be represented pictorially as follows:

$$\begin{array}{c|ccccc}
 & & & & & \mathbf{R_{5,5}} \\
 & & & & \mathbf{R_{4,4}} & R_{5,4} \\
q & & & \mathbf{R_{3,3}} & R_{4,3} & R_{5,3} \\
 & & \mathbf{R_{2,2}} & R_{3,2} & R_{4,2} & R_{5,2} \\
 & \mathbf{R_{1,1}} & \mathbf{R_{2,1}} & \mathbf{R_{3,1}} & \mathbf{R_{4,1}} & \mathbf{R_{5,1}} \\
\hline
 & & & n & &
\end{array}$$

where the sequences $R_{n,n}$ and $R_{n,1}$ are true assertions for all $n \geqslant 1$ by the results of Serre (Theorem 2.1.8) and Lemma 5.5.7 respectively. Applying Lemma 5.5.8 to $R_{2,1}$ and $R_{2,2}$ proves $R_{3,2}$. Applying Lemma 5.5.8 to $R_{3,1}$ and $R_{3,2}$ proves $R_{4,2}$. By induction, $R_{n,2}$ is true for all $n \geqslant 1$. By induction on the rows, it is easy to establish the result. □

5.5.9. Lemma. *Let $G = \bigoplus_{d=1}^{m} \Sigma^d G_d$ be a finitely generated 2-torsion graded abelian group. Assume that KG is strictly m-anticonnected i.e. G_m is non-trivial. For all $0 \leqslant \delta < m$, we have the asymptotic growth formula*

$$\varphi_{[\delta]}(KG)(x) \sim \operatorname{rank}(G_m) \cdot \frac{x^{m-\delta}}{(m-\delta)!}$$
$$i.e.\ \lim_{x\to\infty} \frac{\varphi_{[\delta]}(KG)(x)}{\operatorname{rank}(G_m)\cdot\frac{x^{m-\delta}}{(m-\delta)!}} = 1.$$

Proof. We have

$$\begin{aligned}
\varphi_{[\delta]}(KG)(x) &= \log_2 P_{[\delta]}(KG)(1-2^{-x}) && \text{by definition,}\\
&= \log_2 \prod_{d=\delta+1}^{m} P_{[\delta]}(K\Sigma^d G_d)(1-2^{-x}) && \text{by definition,}\\
&= \sum_{d=\delta+1}^{m} \log_2 P_{[\delta]}(K\Sigma^d G_d)(1-2^{-x}) \\
&= \sum_{d=\delta+1}^{m} \varphi_{[\delta]}(K\Sigma^d G_d)(x) && \text{by definition,}\\
&\sim \sum_{d=\delta+1}^{m} \operatorname{rank}(G_d)\cdot\frac{x^{d-\delta}}{(d-\delta)!} && \text{by Lemma 5.5.6,}\\
&\sim \operatorname{rank}(G_m)\cdot\frac{x^{m-\delta}}{(m-\delta)!}.
\end{aligned}$$

□

Remark. This lemma implies obviously that $\varphi_{[\delta]}(KG)(x) \sim \frac{d^\delta}{dx^\delta}\varphi_{[0]}(KG)(x)$. This justifies the terminology of δ-*derived* function.

5.6. Proof of the second main theorem

5.6.1. Lemma. *Let H be a primitive Hopf algebra. If H' is a sub-Hopf algebra, then H' is primitive.*

Proof. For a direct proof, see [**57**, Proposition 2.6, p. 69]. For a better result, see [**41**, Proposition 6.3, p. 247]. □

5.6.2. Lemma. *Let H be a cocommutative primitive Hopf algebra, H' be a polynomial Hopf algebra and $\varphi : H \to H'$ be a Hopf algebra map. Then $Q\,\text{sub-ker}\,\varphi = \text{sub-ker}\,\varphi \cap QH$.*

Proof. If $x \neq 0 \in \text{sub-ker}\,\varphi$ is indecomposable, then it is primitive in H since $\text{sub-ker}\,\varphi$ is a primitively generated sub-Hopf algebra by Lemma 5.6.1. If it is decomposable in H, then it is an iterated square of a primitive by Milnor-Moore theorem, say $x = w^{2^n}$, and $\varphi(x) = \varphi(w^{2^n}) = \varphi(w)^{2^n} = 0$. This implies that $w \in \text{sub-ker}\,\varphi$ since H' is polynomial. Therefore, x is decomposable in $\text{sub-ker}\,\varphi$, which is a contradiction. Thus x is indecomposable in H. The other inclusion is obvious. □

5.6.3. Lemma. *Let $1 \leqslant m \leqslant n$, KG be a strictly m-anticonnected GEM, KH be a strictly n-anticonnected GEM and $f : KG \to K\Sigma H$ a continuous map. Consider the induced map of graded vector spaces $f^* : H^*(K\Sigma H; \mathbb{F}_2) \to H^*(KG; \mathbb{F}_2)$. For all $0 < \delta \leqslant n - m$ we have*

$$\text{sub-ker}\, f^* \cap QH^*_{[\delta]}(K\Sigma H) \neq 0.$$

Proof. Suppose that $\ker f^* \cap H^*_{[\delta]}(K\Sigma H) = 0$. Then we have

$$P_{[\delta]}(K\Sigma H)(t) \leqslant P_{[0]}(KG)(t)$$

and thus by Lemma 5.5.9

$$\begin{aligned}\text{rank}(H_n) \cdot \frac{x^{n+1-\delta}}{(n+1-\delta)!} &\sim \varphi_{[\delta]}(K\Sigma H)(x) \\ &\leqslant \varphi_{[0]}(KG)(x) \sim \text{rank}(G_m) \cdot \frac{x^m}{m!}.\end{aligned}$$

This contradicts the fact that $\delta \leqslant n - m$. Therefore

$$\ker f^* \cap H^*_{[\delta]}(K\Sigma H) \neq 0.$$

Let $x \neq 0 \in \ker f^* \cap H^*_{[\delta]}(K\Sigma H)$. Then $x = a \cdot k$ with $a \in H^*_{[\delta]}(K\Sigma H)$ and $k \neq 0 \in \overline{\text{sub-ker}\, f^*} \cap H^*_{[\delta]}(K\Sigma H)$. Since $\text{sub-ker}\, f^*$ is polynomial by Borel's results, it is generated (as an algebra) by its indecomposables and therefore $Q\,\text{sub-ker}\, f^* \cap H^*_{[\delta]}(K\Sigma H) \neq 0$. Thus by Lemma 5.6.2 we have $Q\,\text{sub-ker}\, f^* \cap H^*_{[\delta]}(K\Sigma H) = \text{sub-ker}\, f^* \cap QH^*_{[\delta]}(K\Sigma H) \neq 0$. □

Our second main result is:

1.3.14. Theorem. *Let G and H be two 2-local finitely generated graded abelian groups. Consider a 2-stage Postnikov system*

$$\begin{tikzcd} KH \arrow[r, equal] \arrow[d] & KH \arrow[d] \\ X \arrow[r] \arrow[d] & PK\Sigma H \arrow[d] \\ KG \arrow[r, "f"'] & K\Sigma H \end{tikzcd}$$

such that KG is a 1-connected m-anticonnected GEM, KH a 1-connected strictly n-anticonnected GEM with $n \in [m+2, \infty[$ and f an H-map (i.e. the system is **stable***). Then X has no homology exponent.*

Proof. By hypothesis, KG is strictly m'-anticonnected, with $m' \leqslant m$. By Lemma 5.6.3 we have sub-ker $f^* \cap QH^*_{[2]}(K\Sigma H) \neq 0$ since $n \geqslant m+2 \geqslant m'+2$. Let $\sum_{1\leqslant i\leqslant j} Sq^{I_i}_{s_i} u_i \neq 0 \in$ sub-ker $f^* \cap QH^*_{[2]}(K\Sigma H)$ with all the $Sq^{I_i}_{s_i} u_i$'s distinct. We have $e(I_i) < \deg(u_i) - 2$ for all $1 \leqslant i \leqslant j$. Renumber the I_i's such that I_1 can be written $I_1 = (a_0, \ldots, a_k)$ with a_0 maximal. Let $d = \deg(\sum_{1\leqslant i\leqslant j} Sq^{I_i}_{s_i} u_i)$ and set

$$I' = \begin{cases} (2a_0) & \text{if } d \equiv 1(2), \\ (4a_0+2, 2a_0+1) & \text{if } d \equiv 0(2), \end{cases}$$

$$I'' = \begin{cases} (2a_0+1) & \text{if } d \equiv 1(2), \\ (4a_0+3, 2a_0+1) & \text{if } d \equiv 0(2). \end{cases}$$

Let us remark that the sequences (I', I_i) and (I'', I_i) are all distinct admissible sequences since a_0 is maximal. Consider now $x = Sq^{I'} \sum_{1\leqslant i\leqslant j} Sq^{I_i}_{s_i} u_i = \sum_{1\leqslant i\leqslant j} Sq^{I', I_i}_{s_i} u_i$. One readily check that $x \in P^{\text{odd}} H^*(K\Sigma H; \mathbb{F}_2) = Q^{\text{odd}} H^*(K\Sigma H; \mathbb{F}_2)$. We have

$$\begin{aligned} \sigma^* Sq^1 x &= \sigma^* Sq^1 \sum_{1\leqslant i\leqslant j} Sq^{I', I_i}_{s_i} u_i \\ &= \sigma^* \sum_{1\leqslant i\leqslant j} Sq^1 Sq^{I', I_i}_{s_i} u_i \\ &= \sigma^* \sum_{1\leqslant i\leqslant j} Sq^{I'', I_i}_{s_i} u_i. \end{aligned}$$

Renumber the I_i's again such that $e(I'', I_i) < \deg(u_i)$ for all $1 \leqslant i \leqslant j_0$ and $e(I'', I_i) \geqslant \deg(u_i)$ for all $j_0 + 1 \leqslant i \leqslant j$. We clearly have $j_0 \geqslant 1$

since $e(I'', I_1) \leqslant e(I_1) + 2 < \deg(u_i)$. Then the equality becomes

$$\begin{aligned}\sigma^* Sq^1 x &= \sigma^* \sum_{1 \leqslant i \leqslant j_0} \underbrace{Sq_{s_i}^{I'',I_i}}_{e<\deg(u_i)} u_i + \sigma^* \sum_{j_0+1 \leqslant i \leqslant j} \underbrace{Sq_{s_i}^{I'',I_i}}_{e \geqslant \deg(u_i)} u_i \\ &= \sigma^* \sum_{1 \leqslant i \leqslant j_0} \underbrace{Sq_{s_i}^{I'',I_i}}_{e<\deg(u_i)} u_i + \sigma^* \sum \text{squares} \\ &= \sigma^* \sum_{1 \leqslant i \leqslant j_0} \underbrace{Sq_{s_i}^{I'',I_i}}_{e<\deg(u_i)} u_i\end{aligned}$$

since σ^* vanishes on decomposables. Consequently $\sigma^* Sq^1 x \neq 0$ because all the $Sq_{s_i}^{I'',I_i} u_i$ are distinct indecomposables and $\sigma^* : QH^*(GEM; \mathbb{F}_2) \to PH^*(\Omega GEM; \mathbb{F}_2)$ is in general a monomorphism (see [**15**, Exposé 15, Proposition 3, p. 8]). Thus we have proved that $Sq^1 x \notin \ker \sigma^*$. We conclude the proof by using results of Section 5.3. □

Let us make some closing comments on the methods that were used here.

The hypothesis of Theorem 1.3.14 force KH to be at least $(m+2)$-anticonnected when KG is m-anticonnected. Suppose it is not the case and, for instance, KH is strictly $(m+1)$-anticonnected as KG is strictly m-anticonnected. Let us now reconsider the proof of the theorem. By Lemma 5.6.3 we have sub-ker $f^* \cap QH^*_{[1]}(K\Sigma H) \neq 0$. Thus if $Sq_s^I u \neq 0 \in$ sub-ker $f^* \cap QH^*_{[1]}(K\Sigma H)$, then $e(I) < \deg(u) - 1$ and $e(I'', I) \leqslant \deg(u)$. Suppose that $e(I'', I) = \deg(u)$. Then $\sigma^* Sq_s^{I'',I} u = 0$ since $Sq_s^{I'',I} u$ is a square. In these conditions, it is not possible to ensure the existence of an element $x \in Q^{\,\text{odd}}\, H^*(K\Sigma H; \mathbb{F}_2)$ such that $Sq^1 x \notin \ker \sigma^*$ and the strategy developped in Section 5.3 cannot be utilized.

It is possible to show that if $X[m]$ is a strictly m-anticonnected 2-local Postnikov piece of finite type, then there is a generalized Serre function verifying the following asymptotic property:

$$\varphi(X)(x) \overset{\leqslant}{\sim} \operatorname{rank}(\pi_m X) \cdot \frac{x^m}{m!}.$$

Therefore, one might want to replace KG by $X[m]$ in the Theorem and ask if it is still true. An adaptation of Lemma 5.6.3 to this situation gives

$$Q\,\text{sub-ker}\, f^* \cap H^*_{[\delta]}(K\Sigma H) \neq 0.$$

But Lemma 5.6.2 is not valid anymore since $H^*(X[m]; \mathbb{F}_2)$ is not polynomial in general. Thus we can only conclude that

$$\text{sub-ker}\, f^* \cap PH^*_{[\delta]}(K\Sigma H) \neq 0.$$

Suppose that sub-ker $f^* \cap PH^*_{[\delta]}(K\Sigma H)$ contains only squares. Then the method for proving Theorem 1.3.14 is compromised since cohomology suspension vanishes on decomposables.

5.7. Conclusion and conjectures

The purpose of this last section is to summarize our contribution to the investigation of the relationships between homotopy and homology exponents.

First, we have seen in Chapter 2 that the Eilenberg-MacLane spaces which admit a homotopy exponent do not have a homology exponent.

We were able to extend this assertion to spaces with two non-trivial homotopy groups. Our main theorem (Theorem 1.3.12) asserts that every 2-local Postnikov piece with two non-trivial homotopy groups do not have a homology exponent. The same is also true for spaces which satisfy the hypothesis of Theorem 1.3.14.

This leads us to the following reasonable conjecture:

1.4.1. Conjecture. *Let X be a 2-local H-space of finite type. If X has a homology exponent, then either $X \simeq B\pi_1 X$, or the Postnikov tower for X has infinitely many non-trivial k-invariants and, in particular, X is not a Postnikov piece.*

Section 1.4 provides a short discussion of this conjecture..

Remark. In the present work, we decided to prove all our results for 2-local H-spaces. Of course, we could have done the same for any odd prime instead of 2.

APPENDIX A

Adem relations among Steenrod Squares

$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-k-1}{i-2j} Sq^{i+j-k} Sq^k$$
$$\text{for all } 0 < i < 2j.$$

$$\begin{aligned}
Sq^1Sq^1 = 0, Sq^1Sq^3 = 0, \ldots; &\quad Sq^1Sq^{2n+1} = 0 \\
Sq^1Sq^2 = Sq^3, Sq^1Sq^4 = Sq^5, \ldots; &\quad Sq^1Sq^{2n} = Sq^{2n+1} \\
Sq^2Sq^2 = Sq^3Sq^1, Sq^2Sq^6 = Sq^7Sq^1, \ldots; &\quad Sq^2Sq^{4n-2} = Sq^{4n-1}Sq^1 \\
Sq^2Sq^3 = Sq^5 + Sq^4Sq^1, \ldots; &\quad Sq^2Sq^{4n-1} = Sq^{4n+1} + Sq^{4n}Sq^1 \\
Sq^2Sq^4 = Sq^6 + Sq^5Sq^1, \ldots; &\quad Sq^2Sq^{4n} = Sq^{4n+2} + Sq^{4n+1}Sq^1 \\
Sq^2Sq^5 = Sq^6Sq^1, \ldots; &\quad Sq^2Sq^{4n+1} = Sq^{4n+2}Sq^1 \\
Sq^3Sq^2 = 0, Sq^3Sq^6 = 0, \ldots; &\quad Sq^3Sq^{4n+2} = 0 \\
Sq^3Sq^3 = Sq^5Sq^1, \ldots; &\quad \ldots \\
\ldots; &\quad Sq^{2n-1}Sq^n = 0 \\
\vdots &
\end{aligned}$$

APPENDIX B

A universal table for admissible words

The purpose of this Appendix is to establish a universal table in which we can read all the admissible words in given stable degree and height range. This is an extension of results obtained in Chapter 2. Results are stated without proof.

B.1. Elementary theoretical results

B.1.1. Definition. The **height** is a map $h : \mathcal{S} \to \mathbb{N}$ defined for all $I = (a_0, \ldots, a_k) \in \mathcal{S}$ by

$$h(I) = \begin{cases} e(I) + 1 & \text{if } a_0 \text{ is even and} \\ e(I) & \text{if } a_0 \text{ is odd.} \end{cases}$$

Let $\mathcal{S}_{q,n}$ be the graded subset given by all admissible sequences $I \in \mathcal{S}$ such that $\deg_{\mathrm{st}}(I) = q$ and $h(I) \leqslant n$. Let $\mathcal{S}_{*,n} = \cup_{q \geqslant 0} \mathcal{S}_{q,n}$.

B.1.2. Definition. Let $n \geqslant 0$. For all admissible sequence $I = (a_0, \ldots, a_k) \in \mathcal{S}$ with $e(I) \leqslant n$ we define an admissible word $g_I(n) \in \mathcal{W}$ inductively as follows:

$$g_{(0)}(n) = \sigma^n,$$

$$g_{(1)}(n) = \sigma^{n-1}\psi_2,$$

$$g_I(n) = \begin{cases} \beta_2 \sigma^{n-e(I)-1} \varphi_2 g_{I^-}(e(I)) & \text{if } a_0 \equiv 0(2) \text{ and } e(I) < n, \\ \gamma_2 g_{I^-}(n) & \text{if } a_0 \equiv 0(2) \text{ and } e(I) = n, \\ \sigma^{n-e(I)} \varphi_2 g_{I^-}(e(I) - 1) & \text{if } a_0 \equiv 1(2), \end{cases}$$

when $I \neq (0)$ and (1).

Moreover we set $g_I = g_I(h(I))$.

B.1.3. Proposition. *The map $g_{(-)} : \mathcal{S} \to \mathcal{W}$ preserves heights and stable degrees.*

B.1.4. Proposition. *We have $g_I(n) = \sigma^{n-h(I)} g_I$.*

B.1.5. Proposition. *We have $\mathcal{W} - \{\sigma^n \psi_{2^s} \mid n \geqslant 0$ and $s \geqslant 2\} = \{g_I(n) \mid I \in \mathcal{S}$ and $n \geqslant 0\}$.*

B.2. Universal table in range $1 \leqslant h \leqslant 5$, $0 \leqslant \deg_{\mathrm{st}} \leqslant 10$

To obtain all admissible words $\mathcal{W}$, it suffices to apply Proposition B.1.5 and therefore to consider Proposition B.1.4 and the set $\{g_I \mid I \in \mathcal{S}\}$. Let us put this set in a table where we have substituted the only admissible word of height 1 and stable degree 1, ψ_2, by ψ_{2^s}.

$\deg_{\mathrm{st}}$ \ h	1	2	3	4	5
0	(0) σ				
1	ψ_{2^s} (1)				
2			$\beta_2\varphi_2\sigma^2$ (2)		
3		$\beta_2\varphi_2\psi_2$ (2,1)	$\varphi_2\sigma^2$ (3)		
4		$\varphi_2\psi_2$ (3,1)			$\beta_2\varphi_2\sigma^4$ (4)
5				$\beta_2\varphi_2\sigma^2\psi_2$ (4,1)	$\varphi_2\sigma^4$ (5)
6			$\beta_2\varphi_2\gamma_2\sigma^2$ (4,2)	$\varphi_2\sigma^2\psi_2$ (5,1)	
7		$\beta_2\varphi_2\gamma_2\psi_2$ (4,2,1)	$\varphi_2\gamma_2\sigma^2$ (5,2)		
8		$\varphi_2\gamma_2\psi_2$ (5,2,1)			$\beta_2\varphi_2\sigma^2\gamma_2\sigma^2$ (6,2)
9				$\beta_2\varphi_2\sigma^2\gamma_2\psi_2$ (6,2,1); $\beta_2\varphi_2^2\sigma^2$ (6,3)	$\varphi_2\sigma^2\gamma_2\sigma^2$ (7,2)
10				$\varphi_2\sigma^2\gamma_2\psi_2$ (7,2,1); $\varphi_2^2\sigma^2$ (7,3)	

Vertical lines show that admissible words are paired off by a Bockstein operation (eventually "higher" Bockstein operation in the case (σ, ψ_{2^s}) when $s \geqslant 2$). Near all admissible words stand the correpond-ing admissible sequences.

It may be interesting to remark that more than one admissible word can occur in given stable degree and height. For instance, both $\beta_2\varphi_2\sigma^2\gamma_2\psi_2$ and $\beta_2\varphi_2^2\sigma^2$ have stable degrees equal to 9 and heights equal to 4.

B.3. User's guide and examples

Suppose that you want to determine all admissible words of height h and degree d. Then consider in the line $\deg_{\mathrm{st}} = d-h$ all the admissible words of height $\leqslant h$. For each of these words, add σ^k on the left with $k \geqslant 0$ such that the degree of the new admissible word is exactly d.

Example (Admissible words of height 3 and degree 6)**.** In the line $\deg_{\mathrm{st}} = 6-3 = 3$, we must consider the words of height $\leqslant 3$, namely $\beta_2\varphi_2\psi_2$ and $\varphi_2\sigma^2$, which are of degree 5 and 6 respectively. Thus admissible words of height 3 and degree 6 are $\sigma\beta_2\varphi_2\psi_2 \sim \beta_2\sigma\varphi_2\psi_2$ and $\varphi_2\sigma^2$.

A glance at the table of admissible words involved in the calculus of $H_*(K(\mathbb{Z}/2,3);\mathbb{Z})$ at p. 105 also shows that these words $\beta_2\sigma\varphi_2\psi_2$ and $\varphi_2\sigma^2$ are the only two words of height 3 and degree 6.

Suppose now that you want to determine $H_*(K(\mathbb{Z}/2^s,n);\mathbb{F}_2)$ as a graded $\mathbb{F}_2$-vector space. Recall that following Theorem 2.2.12, we must determine all admissible words of height $\leqslant n$.

Example ($H_{\leqslant 10}(K(\mathbb{Z}/2,2);\mathbb{F}_2)$)**.** Admissible words of height 2 and stable degree $\leqslant 8$ (i.e. degree $\leqslant 10$) are σ^2 ($d=2$), $\sigma\psi_2$ ($d=3$), $\beta_2\varphi_2\psi_2$ ($d=5$), $\varphi_2\psi_2$ ($d=6$), $\beta_2\varphi_2\gamma_2\psi_2$ ($d=9$) and $\varphi_2\gamma_2\psi_2$ ($d=10$). Let us determine $\mathbb{F}_2[\sigma^2,\varphi_2\psi_2,\varphi_2\gamma_2\psi_2] \otimes \Lambda_{\mathbb{F}_2}(\sigma\psi_2,\beta_2\varphi_2\psi_2,\beta_2\varphi_2\gamma_2\psi_2)$ as graded $\mathbb{F}_2$-vector space in degrees $\leqslant 10$ (multiplication in polynomial and exterior algebras is denoted by $\cdot$).

Degree	$\mathbb{F}_2$-basis for $H_*(K(\mathbb{Z}/2,2);\mathbb{F}_2)$	$\dim_{\mathbb{F}_2}$
0	1	1
1	0	0
2	σ^2	1
3	$\sigma\psi_2$	1
4	$\gamma_2\sigma^2$	1
5	$\sigma^2 \otimes \sigma\psi_2$, $\beta_2\varphi_2\psi_2$	2
6	$\sigma^2 \cdot \gamma_2\sigma^2$, $\varphi_2\psi_2$	2
7	$\gamma_2\sigma^2 \otimes \sigma\psi_2$, $\sigma^2 \otimes \beta_2\varphi_2\psi_2$	2
8	$\gamma_2^2\sigma^2$, $\sigma\psi_2 \cdot \beta_2\varphi_2\psi_2$, $\sigma^2 \cdot \varphi_2\psi_2$	3
9	$\gamma_2\sigma^2 \otimes \beta_2\varphi_2\psi_2$, $\varphi_2\psi_2 \otimes \sigma\psi_2$, $\beta_2\varphi_2\gamma_2\psi_2$	3
10	$\gamma_2\sigma^2 \cdot \varphi_2\psi_2$, $\varphi_2\gamma_2\psi_2$	2

Of course, this result agrees with table C.1, p. 91.

APPENDIX C

Tables for Eilenberg-MacLane spaces homology

The **genus** of a generator (x, y) (see Definition 2.2.8, p. 30) is given by

$$\begin{cases} 1 & \text{if } x \text{ is of the form } \sigma^n, \\ 2 & \text{if } x \text{ is not of the form } \sigma^n \text{ and } x \text{ (or } y\text{) ends with } \sigma, \\ 3 & \text{if } x \text{ (or } y\text{) ends with } \psi_2. \end{cases}$$

C.1. Integral homology and cohomology of $K(\mathbb{Z}/2, 2)$

n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	(0)	(0)
2	$\mathbb{Z}/2$	(0)
3	(0)	$\mathbb{Z}/2$
4	$\mathbb{Z}/2^2$	(0)
5	$\mathbb{Z}/2$	$\mathbb{Z}/2^2$
6	$\mathbb{Z}/2$	$\mathbb{Z}/2$
7	$\mathbb{Z}/2$	$\mathbb{Z}/2$
8	$\mathbb{Z}/2 \oplus \mathbb{Z}/2^3$	$\mathbb{Z}/2$
9	$(\mathbb{Z}/2)^{\oplus 2}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2^3$
10	$(\mathbb{Z}/2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 2}$
11	$(\mathbb{Z}/2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 2}$
12	$(\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 3}$
13	$(\mathbb{Z}/2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z}/2^2$
14	$(\mathbb{Z}/2)^{\oplus 5}$	$(\mathbb{Z}/2)^{\oplus 3}$
15	$(\mathbb{Z}/2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 5}$
16	$(\mathbb{Z}/2)^{\oplus 4} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 4}$
17	$(\mathbb{Z}/2)^{\oplus 7}$	$(\mathbb{Z}/2)^{\oplus 4} \oplus \mathbb{Z}/2^4$
18	$(\mathbb{Z}/2)^{\oplus 6}$	$(\mathbb{Z}/2)^{\oplus 7}$
19	$(\mathbb{Z}/2)^{\oplus 8}$	$(\mathbb{Z}/2)^{\oplus 6}$
20	$(\mathbb{Z}/2)^{\oplus 8} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 8}$
21	$(\mathbb{Z}/2)^{\oplus 9}$	$(\mathbb{Z}/2)^{\oplus 8} \oplus \mathbb{Z}/2^2$
22	$(\mathbb{Z}/2)^{\oplus 11}$	$(\mathbb{Z}/2)^{\oplus 9}$
23	$(\mathbb{Z}/2)^{\oplus 12}$	$(\mathbb{Z}/2)^{\oplus 11}$
24	$(\mathbb{Z}/2)^{\oplus 12} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 12}$

to be continued on the next page

n	$H_n(-, \mathbb{Z})$	$H^n(-, \mathbb{Z})$
25	$(\mathbb{Z}/2)^{\oplus 14}$	$(\mathbb{Z}/2)^{\oplus 12} \oplus \mathbb{Z}/2^3$
26	$(\mathbb{Z}/2)^{\oplus 17}$	$(\mathbb{Z}/2)^{\oplus 14}$
27	$(\mathbb{Z}/2)^{\oplus 17}$	$(\mathbb{Z}/2)^{\oplus 17}$
28	$(\mathbb{Z}/2)^{\oplus 18} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 17}$
29	$(\mathbb{Z}/2)^{\oplus 22}$	$(\mathbb{Z}/2)^{\oplus 18} \oplus \mathbb{Z}/2^2$
30	$(\mathbb{Z}/2)^{\oplus 22}$	$(\mathbb{Z}/2)^{\oplus 22}$
31	$(\mathbb{Z}/2)^{\oplus 25}$	$(\mathbb{Z}/2)^{\oplus 22}$
32	$(\mathbb{Z}/2)^{\oplus 27} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 25}$
33	$(\mathbb{Z}/2)^{\oplus 29}$	$(\mathbb{Z}/2)^{\oplus 27} \oplus \mathbb{Z}/2^5$
34	$(\mathbb{Z}/2)^{\oplus 32}$	$(\mathbb{Z}/2)^{\oplus 29}$
35	$(\mathbb{Z}/2)^{\oplus 36}$	$(\mathbb{Z}/2)^{\oplus 32}$
36	$(\mathbb{Z}/2)^{\oplus 36} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 36}$
37	$(\mathbb{Z}/2)^{\oplus 41}$	$(\mathbb{Z}/2)^{\oplus 36} \oplus \mathbb{Z}/2^2$
38	$(\mathbb{Z}/2)^{\oplus 45}$	$(\mathbb{Z}/2)^{\oplus 41}$
39	$(\mathbb{Z}/2)^{\oplus 47}$	$(\mathbb{Z}/2)^{\oplus 45}$
40	$(\mathbb{Z}/2)^{\oplus 50} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 47}$
41	$(\mathbb{Z}/2)^{\oplus 56}$	$(\mathbb{Z}/2)^{\oplus 50} \oplus \mathbb{Z}/2^3$
42	$(\mathbb{Z}/2)^{\oplus 59}$	$(\mathbb{Z}/2)^{\oplus 56}$
43	$(\mathbb{Z}/2)^{\oplus 63}$	$(\mathbb{Z}/2)^{\oplus 59}$
44	$(\mathbb{Z}/2)^{\oplus 69} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 63}$
45	$(\mathbb{Z}/2)^{\oplus 72}$	$(\mathbb{Z}/2)^{\oplus 69} \oplus \mathbb{Z}/2^2$
46	$(\mathbb{Z}/2)^{\oplus 78}$	$(\mathbb{Z}/2)^{\oplus 72}$
47	$(\mathbb{Z}/2)^{\oplus 85}$	$(\mathbb{Z}/2)^{\oplus 78}$
48	$(\mathbb{Z}/2)^{\oplus 87} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 85}$
49	$(\mathbb{Z}/2)^{\oplus 95}$	$(\mathbb{Z}/2)^{\oplus 87} \oplus \mathbb{Z}/2^4$
50	$(\mathbb{Z}/2)^{\oplus 103}$	$(\mathbb{Z}/2)^{\oplus 95}$
51	$(\mathbb{Z}/2)^{\oplus 107}$	$(\mathbb{Z}/2)^{\oplus 103}$
52	$(\mathbb{Z}/2)^{\oplus 114} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 107}$
53	$(\mathbb{Z}/2)^{\oplus 124}$	$(\mathbb{Z}/2)^{\oplus 114} \oplus \mathbb{Z}/2^2$
54	$(\mathbb{Z}/2)^{\oplus 129}$	$(\mathbb{Z}/2)^{\oplus 124}$
55	$(\mathbb{Z}/2)^{\oplus 138}$	$(\mathbb{Z}/2)^{\oplus 129}$
56	$(\mathbb{Z}/2)^{\oplus 147} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 138}$
57	$(\mathbb{Z}/2)^{\oplus 154}$	$(\mathbb{Z}/2)^{\oplus 147} \oplus \mathbb{Z}/2^3$
58	$(\mathbb{Z}/2)^{\oplus 164}$	$(\mathbb{Z}/2)^{\oplus 154}$
59	$(\mathbb{Z}/2)^{\oplus 176}$	$(\mathbb{Z}/2)^{\oplus 164}$
60	$(\mathbb{Z}/2)^{\oplus 182} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 176}$
61	$(\mathbb{Z}/2)^{\oplus 194}$	$(\mathbb{Z}/2)^{\oplus 182} \oplus \mathbb{Z}/2^2$
62	$(\mathbb{Z}/2)^{\oplus 208}$	$(\mathbb{Z}/2)^{\oplus 194}$
63	$(\mathbb{Z}/2)^{\oplus 215}$	$(\mathbb{Z}/2)^{\oplus 208}$
64	$(\mathbb{Z}/2)^{\oplus 228} \oplus \mathbb{Z}/2^6$	$(\mathbb{Z}/2)^{\oplus 215}$

to be continued on the next page

n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
65	$(\mathbb{Z}/2)^{\oplus 244}$	$(\mathbb{Z}/2)^{\oplus 228} \oplus \mathbb{Z}/2^6$
66	$(\mathbb{Z}/2)^{\oplus 252}$	$(\mathbb{Z}/2)^{\oplus 244}$
67	$(\mathbb{Z}/2)^{\oplus 269}$	$(\mathbb{Z}/2)^{\oplus 252}$
68	$(\mathbb{Z}/2)^{\oplus 283} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 269}$
69	$(\mathbb{Z}/2)^{\oplus 296}$	$(\mathbb{Z}/2)^{\oplus 283} \oplus \mathbb{Z}/2^2$
70	$(\mathbb{Z}/2)^{\oplus 313}$	$(\mathbb{Z}/2)^{\oplus 296}$
71	$(\mathbb{Z}/2)^{\oplus 331}$	$(\mathbb{Z}/2)^{\oplus 313}$
72	$(\mathbb{Z}/2)^{\oplus 343} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 331}$
73	$(\mathbb{Z}/2)^{\oplus 363}$	$(\mathbb{Z}/2)^{\oplus 343} \oplus \mathbb{Z}/2^3$
74	$(\mathbb{Z}/2)^{\oplus 384}$	$(\mathbb{Z}/2)^{\oplus 363}$
75	$(\mathbb{Z}/2)^{\oplus 398}$	$(\mathbb{Z}/2)^{\oplus 384}$
76	$(\mathbb{Z}/2)^{\oplus 419} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 398}$
77	$(\mathbb{Z}/2)^{\oplus 443}$	$(\mathbb{Z}/2)^{\oplus 419} \oplus \mathbb{Z}/2^2$
78	$(\mathbb{Z}/2)^{\oplus 459}$	$(\mathbb{Z}/2)^{\oplus 443}$
79	$(\mathbb{Z}/2)^{\oplus 484}$	$(\mathbb{Z}/2)^{\oplus 459}$
80	$(\mathbb{Z}/2)^{\oplus 508} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 484}$
81	$(\mathbb{Z}/2)^{\oplus 527}$	$(\mathbb{Z}/2)^{\oplus 508} \oplus \mathbb{Z}/2^4$
82	$(\mathbb{Z}/2)^{\oplus 556}$	$(\mathbb{Z}/2)^{\oplus 527}$
83	$(\mathbb{Z}/2)^{\oplus 582}$	$(\mathbb{Z}/2)^{\oplus 556}$
84	$(\mathbb{Z}/2)^{\oplus 604} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 582}$
85	$(\mathbb{Z}/2)^{\oplus 635}$	$(\mathbb{Z}/2)^{\oplus 604} \oplus \mathbb{Z}/2^2$
86	$(\mathbb{Z}/2)^{\oplus 665}$	$(\mathbb{Z}/2)^{\oplus 635}$
87	$(\mathbb{Z}/2)^{\oplus 691}$	$(\mathbb{Z}/2)^{\oplus 665}$
88	$(\mathbb{Z}/2)^{\oplus 722} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 691}$
89	$(\mathbb{Z}/2)^{\oplus 758}$	$(\mathbb{Z}/2)^{\oplus 722} \oplus \mathbb{Z}/2^3$
90	$(\mathbb{Z}/2)^{\oplus 785}$	$(\mathbb{Z}/2)^{\oplus 758}$
91	$(\mathbb{Z}/2)^{\oplus 822}$	$(\mathbb{Z}/2)^{\oplus 785}$
92	$(\mathbb{Z}/2)^{\oplus 859} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 822}$
93	$(\mathbb{Z}/2)^{\oplus 890}$	$(\mathbb{Z}/2)^{\oplus 859} \oplus \mathbb{Z}/2^2$
94	$(\mathbb{Z}/2)^{\oplus 932}$	$(\mathbb{Z}/2)^{\oplus 890}$
95	$(\mathbb{Z}/2)^{\oplus 972}$	$(\mathbb{Z}/2)^{\oplus 932}$
96	$(\mathbb{Z}/2)^{\oplus 1006} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 972}$
97	$(\mathbb{Z}/2)^{\oplus 1052}$	$(\mathbb{Z}/2)^{\oplus 1006} \oplus \mathbb{Z}/2^5$
98	$(\mathbb{Z}/2)^{\oplus 1097}$	$(\mathbb{Z}/2)^{\oplus 1052}$
99	$(\mathbb{Z}/2)^{\oplus 1136}$	$(\mathbb{Z}/2)^{\oplus 1097}$
100	$(\mathbb{Z}/2)^{\oplus 1184} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 1136}$
101	$(\mathbb{Z}/2)^{\oplus 1235}$	$(\mathbb{Z}/2)^{\oplus 1184} \oplus \mathbb{Z}/2^2$
102	$(\mathbb{Z}/2)^{\oplus 1278}$	$(\mathbb{Z}/2)^{\oplus 1235}$
103	$(\mathbb{Z}/2)^{\oplus 1332}$	$(\mathbb{Z}/2)^{\oplus 1278}$
104	$(\mathbb{Z}/2)^{\oplus 1386} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 1332}$

to be continued on the next page

n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
105	$(\mathbb{Z}/2)^{\oplus 1434}$	$(\mathbb{Z}/2)^{\oplus 1386} \oplus \mathbb{Z}/2^3$
106	$(\mathbb{Z}/2)^{\oplus 1493}$	$(\mathbb{Z}/2)^{\oplus 1434}$
107	$(\mathbb{Z}/2)^{\oplus 1554}$	$(\mathbb{Z}/2)^{\oplus 1493}$
108	$(\mathbb{Z}/2)^{\oplus 1604} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 1554}$
109	$(\mathbb{Z}/2)^{\oplus 1671}$	$(\mathbb{Z}/2)^{\oplus 1604} \oplus \mathbb{Z}/2^2$
110	$(\mathbb{Z}/2)^{\oplus 1736}$	$(\mathbb{Z}/2)^{\oplus 1671}$
111	$(\mathbb{Z}/2)^{\oplus 1793}$	$(\mathbb{Z}/2)^{\oplus 1736}$
112	$(\mathbb{Z}/2)^{\oplus 1864} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 1793}$
113	$(\mathbb{Z}/2)^{\oplus 1935}$	$(\mathbb{Z}/2)^{\oplus 1864} \oplus \mathbb{Z}/2^4$
114	$(\mathbb{Z}/2)^{\oplus 1999}$	$(\mathbb{Z}/2)^{\oplus 1935}$
115	$(\mathbb{Z}/2)^{\oplus 2076}$	$(\mathbb{Z}/2)^{\oplus 1999}$
116	$(\mathbb{Z}/2)^{\oplus 2153} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 2076}$
117	$(\mathbb{Z}/2)^{\oplus 2223}$	$(\mathbb{Z}/2)^{\oplus 2153} \oplus \mathbb{Z}/2^2$
118	$(\mathbb{Z}/2)^{\oplus 2308}$	$(\mathbb{Z}/2)^{\oplus 2223}$
119	$(\mathbb{Z}/2)^{\oplus 2392}$	$(\mathbb{Z}/2)^{\oplus 2308}$
120	$(\mathbb{Z}/2)^{\oplus 2467} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 2392}$
121	$(\mathbb{Z}/2)^{\oplus 2560}$	$(\mathbb{Z}/2)^{\oplus 2467} \oplus \mathbb{Z}/2^3$
122	$(\mathbb{Z}/2)^{\oplus 2651}$	$(\mathbb{Z}/2)^{\oplus 2560}$
123	$(\mathbb{Z}/2)^{\oplus 2734}$	$(\mathbb{Z}/2)^{\oplus 2651}$
124	$(\mathbb{Z}/2)^{\oplus 2833} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 2734}$
125	$(\mathbb{Z}/2)^{\oplus 2933}$	$(\mathbb{Z}/2)^{\oplus 2833} \oplus \mathbb{Z}/2^2$
126	$(\mathbb{Z}/2)^{\oplus 3023}$	$(\mathbb{Z}/2)^{\oplus 2933}$
127	$(\mathbb{Z}/2)^{\oplus 3132}$	$(\mathbb{Z}/2)^{\oplus 3023}$
128	$(\mathbb{Z}/2)^{\oplus 3237} \oplus \mathbb{Z}/2^7$	$(\mathbb{Z}/2)^{\oplus 3132}$
129	$(\mathbb{Z}/2)^{\oplus 3338}$	$(\mathbb{Z}/2)^{\oplus 3237} \oplus \mathbb{Z}/2^7$
130	$(\mathbb{Z}/2)^{\oplus 3454}$	$(\mathbb{Z}/2)^{\oplus 3338}$
131	$(\mathbb{Z}/2)^{\oplus 3570}$	$(\mathbb{Z}/2)^{\oplus 3454}$
132	$(\mathbb{Z}/2)^{\oplus 3677} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 3570}$
133	$(\mathbb{Z}/2)^{\oplus 3804}$	$(\mathbb{Z}/2)^{\oplus 3677} \oplus \mathbb{Z}/2^2$
134	$(\mathbb{Z}/2)^{\oplus 3930}$	$(\mathbb{Z}/2)^{\oplus 3804}$
135	$(\mathbb{Z}/2)^{\oplus 4047}$	$(\mathbb{Z}/2)^{\oplus 3930}$
136	$(\mathbb{Z}/2)^{\oplus 4182} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 4047}$
137	$(\mathbb{Z}/2)^{\oplus 4319}$	$(\mathbb{Z}/2)^{\oplus 4182} \oplus \mathbb{Z}/2^3$
138	$(\mathbb{Z}/2)^{\oplus 4446}$	$(\mathbb{Z}/2)^{\oplus 4319}$
139	$(\mathbb{Z}/2)^{\oplus 4593}$	$(\mathbb{Z}/2)^{\oplus 4446}$
140	$(\mathbb{Z}/2)^{\oplus 4738} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 4593}$
141	$(\mathbb{Z}/2)^{\oplus 4877}$	$(\mathbb{Z}/2)^{\oplus 4738} \oplus \mathbb{Z}/2^2$
142	$(\mathbb{Z}/2)^{\oplus 5035}$	$(\mathbb{Z}/2)^{\oplus 4877}$
143	$(\mathbb{Z}/2)^{\oplus 5193}$	$(\mathbb{Z}/2)^{\oplus 5035}$
144	$(\mathbb{Z}/2)^{\oplus 5341} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 5193}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
145	$(\mathbb{Z}/2)^{\oplus 5512}$	$(\mathbb{Z}/2)^{\oplus 5341} \oplus \mathbb{Z}/2^4$
146	$(\mathbb{Z}/2)^{\oplus 5682}$	$(\mathbb{Z}/2)^{\oplus 5512}$
147	$(\mathbb{Z}/2)^{\oplus 5843}$	$(\mathbb{Z}/2)^{\oplus 5682}$
148	$(\mathbb{Z}/2)^{\oplus 6025} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 5843}$
149	$(\mathbb{Z}/2)^{\oplus 6209}$	$(\mathbb{Z}/2)^{\oplus 6025} \oplus \mathbb{Z}/2^2$
150	$(\mathbb{Z}/2)^{\oplus 6383}$	$(\mathbb{Z}/2)^{\oplus 6209}$
151	$(\mathbb{Z}/2)^{\oplus 6579}$	$(\mathbb{Z}/2)^{\oplus 6383}$
152	$(\mathbb{Z}/2)^{\oplus 6776} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 6579}$
153	$(\mathbb{Z}/2)^{\oplus 6963}$	$(\mathbb{Z}/2)^{\oplus 6776} \oplus \mathbb{Z}/2^3$
154	$(\mathbb{Z}/2)^{\oplus 7175}$	$(\mathbb{Z}/2)^{\oplus 6963}$
155	$(\mathbb{Z}/2)^{\oplus 7386}$	$(\mathbb{Z}/2)^{\oplus 7175}$
156	$(\mathbb{Z}/2)^{\oplus 7586} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 7386}$
157	$(\mathbb{Z}/2)^{\oplus 7814}$	$(\mathbb{Z}/2)^{\oplus 7586} \oplus \mathbb{Z}/2^2$
158	$(\mathbb{Z}/2)^{\oplus 8040}$	$(\mathbb{Z}/2)^{\oplus 7814}$
159	$(\mathbb{Z}/2)^{\oplus 8257}$	$(\mathbb{Z}/2)^{\oplus 8040}$
160	$(\mathbb{Z}/2)^{\oplus 8498} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 8257}$
161	$(\mathbb{Z}/2)^{\oplus 8742}$	$(\mathbb{Z}/2)^{\oplus 8498} \oplus \mathbb{Z}/2^5$
162	$(\mathbb{Z}/2)^{\oplus 8975}$	$(\mathbb{Z}/2)^{\oplus 8742}$
163	$(\mathbb{Z}/2)^{\oplus 9234}$	$(\mathbb{Z}/2)^{\oplus 8975}$
164	$(\mathbb{Z}/2)^{\oplus 9494} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 9234}$
165	$(\mathbb{Z}/2)^{\oplus 9744}$	$(\mathbb{Z}/2)^{\oplus 9494} \oplus \mathbb{Z}/2^2$
166	$(\mathbb{Z}/2)^{\oplus 10022}$	$(\mathbb{Z}/2)^{\oplus 9744}$
167	$(\mathbb{Z}/2)^{\oplus 10301}$	$(\mathbb{Z}/2)^{\oplus 10022}$
168	$(\mathbb{Z}/2)^{\oplus 10567} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 10301}$
169	$(\mathbb{Z}/2)^{\oplus 10866}$	$(\mathbb{Z}/2)^{\oplus 10567} \oplus \mathbb{Z}/2^3$
170	$(\mathbb{Z}/2)^{\oplus 11163}$	$(\mathbb{Z}/2)^{\oplus 10866}$
171	$(\mathbb{Z}/2)^{\oplus 11450}$	$(\mathbb{Z}/2)^{\oplus 11163}$
172	$(\mathbb{Z}/2)^{\oplus 11767} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 11450}$
173	$(\mathbb{Z}/2)^{\oplus 12085}$	$(\mathbb{Z}/2)^{\oplus 11767} \oplus \mathbb{Z}/2^2$
174	$(\mathbb{Z}/2)^{\oplus 12393}$	$(\mathbb{Z}/2)^{\oplus 12085}$
175	$(\mathbb{Z}/2)^{\oplus 12731}$	$(\mathbb{Z}/2)^{\oplus 12393}$
176	$(\mathbb{Z}/2)^{\oplus 13069} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 12731}$
177	$(\mathbb{Z}/2)^{\oplus 13399}$	$(\mathbb{Z}/2)^{\oplus 13069} \oplus \mathbb{Z}/2^4$
178	$(\mathbb{Z}/2)^{\oplus 13759}$	$(\mathbb{Z}/2)^{\oplus 13399}$
179	$(\mathbb{Z}/2)^{\oplus 14122}$	$(\mathbb{Z}/2)^{\oplus 13759}$
180	$(\mathbb{Z}/2)^{\oplus 14471} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 14122}$
181	$(\mathbb{Z}/2)^{\oplus 14857}$	$(\mathbb{Z}/2)^{\oplus 14471} \oplus \mathbb{Z}/2^2$
182	$(\mathbb{Z}/2)^{\oplus 15243}$	$(\mathbb{Z}/2)^{\oplus 14857}$
183	$(\mathbb{Z}/2)^{\oplus 15617}$	$(\mathbb{Z}/2)^{\oplus 15243}$
184	$(\mathbb{Z}/2)^{\oplus 16027} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 15617}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
185	$(\mathbb{Z}/2)^{\oplus 16437}$	$(\mathbb{Z}/2)^{\oplus 16027} \oplus \mathbb{Z}/2^3$
186	$(\mathbb{Z}/2)^{\oplus 16838}$	$(\mathbb{Z}/2)^{\oplus 16437}$
187	$(\mathbb{Z}/2)^{\oplus 17273}$	$(\mathbb{Z}/2)^{\oplus 16838}$
188	$(\mathbb{Z}/2)^{\oplus 17709} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 17273}$
189	$(\mathbb{Z}/2)^{\oplus 18137}$	$(\mathbb{Z}/2)^{\oplus 17709} \oplus \mathbb{Z}/2^2$
190	$(\mathbb{Z}/2)^{\oplus 18599}$	$(\mathbb{Z}/2)^{\oplus 18137}$
191	$(\mathbb{Z}/2)^{\oplus 19064}$	$(\mathbb{Z}/2)^{\oplus 18599}$
192	$(\mathbb{Z}/2)^{\oplus 19517} \oplus \mathbb{Z}/2^6$	$(\mathbb{Z}/2)^{\oplus 19064}$
193	$(\mathbb{Z}/2)^{\oplus 20009}$	$(\mathbb{Z}/2)^{\oplus 19517} \oplus \mathbb{Z}/2^6$
194	$(\mathbb{Z}/2)^{\oplus 20504}$	$(\mathbb{Z}/2)^{\oplus 20009}$
195	$(\mathbb{Z}/2)^{\oplus 20986}$	$(\mathbb{Z}/2)^{\oplus 20504}$
196	$(\mathbb{Z}/2)^{\oplus 21507} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 20986}$
197	$(\mathbb{Z}/2)^{\oplus 22034}$	$(\mathbb{Z}/2)^{\oplus 21507} \oplus \mathbb{Z}/2^2$
198	$(\mathbb{Z}/2)^{\oplus 22546}$	$(\mathbb{Z}/2)^{\oplus 22034}$
199	$(\mathbb{Z}/2)^{\oplus 23102}$	$(\mathbb{Z}/2)^{\oplus 22546}$
200	$(\mathbb{Z}/2)^{\oplus 23657} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 23102}$

Generators involved in the calculus.

Degree	Genus	Generator
2	1	$(\sigma^2, \sigma\psi_2)$
5	3	$(\beta_2\varphi_2\psi_2, \varphi_2\psi_2)$
9	3	$(\beta_2\varphi_2\gamma_2\psi_2, \varphi_2\gamma_2\psi_2)$
17	3	$(\beta_2\varphi_2\gamma_2^2\psi_2, \varphi_2\gamma_2^2\psi_2)$
33	3	$(\beta_2\varphi_2\gamma_2^3\psi_2, \varphi_2\gamma_2^3\psi_2)$
65	3	$(\beta_2\varphi_2\gamma_2^4\psi_2, \varphi_2\gamma_2^4\psi_2)$
129	3	$(\beta_2\varphi_2\gamma_2^5\psi_2, \varphi_2\gamma_2^5\psi_2)$

C.2. Integral homology and cohomology of $K(\mathbb{Z}/2,3)$.

n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	(0)	(0)
2	(0)	(0)
3	$\mathbb{Z}/2$	(0)
4	(0)	$\mathbb{Z}/2$
5	$\mathbb{Z}/2$	(0)
6	$\mathbb{Z}/2$	$\mathbb{Z}/2$
7	$\mathbb{Z}/2$	$\mathbb{Z}/2$
8	$\mathbb{Z}/2$	$\mathbb{Z}/2$
9	$(\mathbb{Z}/2)^{\oplus 3}$	$\mathbb{Z}/2$
10	$(\mathbb{Z}/2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 3}$
11	$(\mathbb{Z}/2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 2}$
12	$(\mathbb{Z}/2)^{\oplus 3} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 3}$
13	$(\mathbb{Z}/2)^{\oplus 5}$	$(\mathbb{Z}/2)^{\oplus 3} \oplus \mathbb{Z}/2^2$
14	$(\mathbb{Z}/2)^{\oplus 5}$	$(\mathbb{Z}/2)^{\oplus 5}$
15	$(\mathbb{Z}/2)^{\oplus 8}$	$(\mathbb{Z}/2)^{\oplus 5}$
16	$(\mathbb{Z}/2)^{\oplus 8}$	$(\mathbb{Z}/2)^{\oplus 8}$
17	$(\mathbb{Z}/2)^{\oplus 11}$	$(\mathbb{Z}/2)^{\oplus 8}$
18	$(\mathbb{Z}/2)^{\oplus 13}$	$(\mathbb{Z}/2)^{\oplus 11}$
19	$(\mathbb{Z}/2)^{\oplus 15}$	$(\mathbb{Z}/2)^{\oplus 13}$
20	$(\mathbb{Z}/2)^{\oplus 17} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 15}$
21	$(\mathbb{Z}/2)^{\oplus 23}$	$(\mathbb{Z}/2)^{\oplus 17} \oplus \mathbb{Z}/2^2$
22	$(\mathbb{Z}/2)^{\oplus 25}$	$(\mathbb{Z}/2)^{\oplus 23}$
23	$(\mathbb{Z}/2)^{\oplus 31}$	$(\mathbb{Z}/2)^{\oplus 25}$
24	$(\mathbb{Z}/2)^{\oplus 36} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 31}$
25	$(\mathbb{Z}/2)^{\oplus 43}$	$(\mathbb{Z}/2)^{\oplus 36} \oplus \mathbb{Z}/2^3$
26	$(\mathbb{Z}/2)^{\oplus 49}$	$(\mathbb{Z}/2)^{\oplus 43}$
27	$(\mathbb{Z}/2)^{\oplus 61}$	$(\mathbb{Z}/2)^{\oplus 49}$
28	$(\mathbb{Z}/2)^{\oplus 68}$	$(\mathbb{Z}/2)^{\oplus 61}$
29	$(\mathbb{Z}/2)^{\oplus 80}$	$(\mathbb{Z}/2)^{\oplus 68}$
30	$(\mathbb{Z}/2)^{\oplus 95}$	$(\mathbb{Z}/2)^{\oplus 80}$
31	$(\mathbb{Z}/2)^{\oplus 108}$	$(\mathbb{Z}/2)^{\oplus 95}$
32	$(\mathbb{Z}/2)^{\oplus 123} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 108}$
33	$(\mathbb{Z}/2)^{\oplus 146} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 123} \oplus \mathbb{Z}/2^2$
34	$(\mathbb{Z}/2)^{\oplus 167}$	$(\mathbb{Z}/2)^{\oplus 146} \oplus \mathbb{Z}/2^2$
35	$(\mathbb{Z}/2)^{\oplus 192}$	$(\mathbb{Z}/2)^{\oplus 167}$
36	$(\mathbb{Z}/2)^{\oplus 220} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 192}$
37	$(\mathbb{Z}/2)^{\oplus 254}$	$(\mathbb{Z}/2)^{\oplus 220} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
38	$(\mathbb{Z}/2)^{\oplus 289}$	$(\mathbb{Z}/2)^{\oplus 254}$
39	$(\mathbb{Z}/2)^{\oplus 334}$	$(\mathbb{Z}/2)^{\oplus 289}$
40	$(\mathbb{Z}/2)^{\oplus 377} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 334}$
41	$(\mathbb{Z}/2)^{\oplus 431}$	$(\mathbb{Z}/2)^{\oplus 377} \oplus \mathbb{Z}/2^3$
42	$(\mathbb{Z}/2)^{\oplus 492}$	$(\mathbb{Z}/2)^{\oplus 431}$
43	$(\mathbb{Z}/2)^{\oplus 558}$	$(\mathbb{Z}/2)^{\oplus 492}$
44	$(\mathbb{Z}/2)^{\oplus 630} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 558}$
45	$(\mathbb{Z}/2)^{\oplus 718} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 630} \oplus \mathbb{Z}/2^2$
46	$(\mathbb{Z}/2)^{\oplus 810}$	$(\mathbb{Z}/2)^{\oplus 718} \oplus \mathbb{Z}/2^2$
47	$(\mathbb{Z}/2)^{\oplus 915}$	$(\mathbb{Z}/2)^{\oplus 810}$
48	$(\mathbb{Z}/2)^{\oplus 1033} \oplus \mathbb{Z}/2^2 \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 915}$
49	$(\mathbb{Z}/2)^{\oplus 1164} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 1033} \oplus \mathbb{Z}/2^2 \oplus \mathbb{Z}/2^4$
50	$(\mathbb{Z}/2)^{\oplus 1309}$	$(\mathbb{Z}/2)^{\oplus 1164} \oplus \mathbb{Z}/2^2$
51	$(\mathbb{Z}/2)^{\oplus 1477}$	$(\mathbb{Z}/2)^{\oplus 1309}$
52	$(\mathbb{Z}/2)^{\oplus 1654} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 1477}$
53	$(\mathbb{Z}/2)^{\oplus 1855} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 1654} \oplus \mathbb{Z}/2^2$
54	$(\mathbb{Z}/2)^{\oplus 2084}$	$(\mathbb{Z}/2)^{\oplus 1855} \oplus \mathbb{Z}/2^2$
55	$(\mathbb{Z}/2)^{\oplus 2331}$	$(\mathbb{Z}/2)^{\oplus 2084}$
56	$(\mathbb{Z}/2)^{\oplus 2601} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 2331}$
57	$(\mathbb{Z}/2)^{\oplus 2912} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 2601} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
58	$(\mathbb{Z}/2)^{\oplus 3250}$	$(\mathbb{Z}/2)^{\oplus 2912} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
59	$(\mathbb{Z}/2)^{\oplus 3622}$	$(\mathbb{Z}/2)^{\oplus 3250}$
60	$(\mathbb{Z}/2)^{\oplus 4036} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 3622}$
61	$(\mathbb{Z}/2)^{\oplus 4492} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 4036} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$
62	$(\mathbb{Z}/2)^{\oplus 4994}$	$(\mathbb{Z}/2)^{\oplus 4492} \oplus \mathbb{Z}/2^2$
63	$(\mathbb{Z}/2)^{\oplus 5554}$	$(\mathbb{Z}/2)^{\oplus 4994}$
64	$(\mathbb{Z}/2)^{\oplus 6161} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 5554}$
65	$(\mathbb{Z}/2)^{\oplus 6833} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 6161} \oplus \mathbb{Z}/2^3$
66	$(\mathbb{Z}/2)^{\oplus 7577}$	$(\mathbb{Z}/2)^{\oplus 6833} \oplus \mathbb{Z}/2^3$
67	$(\mathbb{Z}/2)^{\oplus 8389}$	$(\mathbb{Z}/2)^{\oplus 7577}$
68	$(\mathbb{Z}/2)^{\oplus 9276} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 8389}$
69	$(\mathbb{Z}/2)^{\oplus 10264} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 9276} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$
70	$(\mathbb{Z}/2)^{\oplus 11337} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 10264} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$
71	$(\mathbb{Z}/2)^{\oplus 12516}$	$(\mathbb{Z}/2)^{\oplus 11337} \oplus \mathbb{Z}/2^2$
72	$(\mathbb{Z}/2)^{\oplus 13810} \oplus (\mathbb{Z}/2^2)^{\oplus 2} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 12516}$
73	$(\mathbb{Z}/2)^{\oplus 15223} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 13810} \oplus (\mathbb{Z}/2^2)^{\oplus 2} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
74	$(\mathbb{Z}/2)^{\oplus 16769}$	$(\mathbb{Z}/2)^{\oplus 15223} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
75	$(\mathbb{Z}/2)^{\oplus 18470}$	$(\mathbb{Z}/2)^{\oplus 16769}$
76	$(\mathbb{Z}/2)^{\oplus 20313} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 18470}$
77	$(\mathbb{Z}/2)^{\oplus 22330} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 20313} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
78	$(\mathbb{Z}/2)^{\oplus 24544}$	$(\mathbb{Z}/2)^{\oplus 22330} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
79	$(\mathbb{Z}/2)^{\oplus 26948}$	$(\mathbb{Z}/2)^{\oplus 24544}$
80	$(\mathbb{Z}/2)^{\oplus 29561} \oplus (\mathbb{Z}/2^2)^{\oplus 3} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 26948}$
81	$(\mathbb{Z}/2)^{\oplus 32428} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 29561} \oplus (\mathbb{Z}/2^2)^{\oplus 3} \oplus \mathbb{Z}/2^4$
82	$(\mathbb{Z}/2)^{\oplus 35540} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 32428} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
83	$(\mathbb{Z}/2)^{\oplus 38928}$	$(\mathbb{Z}/2)^{\oplus 35540} \oplus \mathbb{Z}/2^2$
84	$(\mathbb{Z}/2)^{\oplus 42618} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 38928}$
85	$(\mathbb{Z}/2)^{\oplus 46626} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 42618} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
86	$(\mathbb{Z}/2)^{\oplus 50982}$	$(\mathbb{Z}/2)^{\oplus 46626} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$
87	$(\mathbb{Z}/2)^{\oplus 55727}$	$(\mathbb{Z}/2)^{\oplus 50982}$
88	$(\mathbb{Z}/2)^{\oplus 60860} \oplus (\mathbb{Z}/2^2)^{\oplus 2} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 55727}$
89	$(\mathbb{Z}/2)^{\oplus 66435} \oplus (\mathbb{Z}/2^2)^{\oplus 3} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 60860} \oplus (\mathbb{Z}/2^2)^{\oplus 2} \oplus \mathbb{Z}/2^3$
90	$(\mathbb{Z}/2)^{\oplus 72502} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 66435} \oplus (\mathbb{Z}/2^2)^{\oplus 3} \oplus \mathbb{Z}/2^3$
91	$(\mathbb{Z}/2)^{\oplus 79060}$	$(\mathbb{Z}/2)^{\oplus 72502} \oplus \mathbb{Z}/2^2$
92	$(\mathbb{Z}/2)^{\oplus 86163} \oplus (\mathbb{Z}/2^2)^{\oplus 5}$	$(\mathbb{Z}/2)^{\oplus 79060}$
93	$(\mathbb{Z}/2)^{\oplus 93881} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$	$(\mathbb{Z}/2)^{\oplus 86163} \oplus (\mathbb{Z}/2^2)^{\oplus 5}$
94	$(\mathbb{Z}/2)^{\oplus 102224} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 93881} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$
95	$(\mathbb{Z}/2)^{\oplus 111251}$	$(\mathbb{Z}/2)^{\oplus 102224} \oplus \mathbb{Z}/2^2$
96	$(\mathbb{Z}/2)^{\oplus 121028} \oplus (\mathbb{Z}/2^2)^{\oplus 3} \oplus \mathbb{Z}/2^3 \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 111251}$
97	$(\mathbb{Z}/2)^{\oplus 131592} \oplus (\mathbb{Z}/2^2)^{\oplus 3} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 121028} \oplus (\mathbb{Z}/2^2)^{\oplus 3} \oplus \mathbb{Z}/2^3 \oplus \mathbb{Z}/2^5$
98	$(\mathbb{Z}/2)^{\oplus 143008}$	$(\mathbb{Z}/2)^{\oplus 131592} \oplus (\mathbb{Z}/2^2)^{\oplus 3} \oplus \mathbb{Z}/2^3$
99	$(\mathbb{Z}/2)^{\oplus 155365}$	$(\mathbb{Z}/2)^{\oplus 143008}$
100	$(\mathbb{Z}/2)^{\oplus 168685} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 155365}$
101	$(\mathbb{Z}/2)^{\oplus 183067} \oplus (\mathbb{Z}/2^2)^{\oplus 5}$	$(\mathbb{Z}/2)^{\oplus 168685} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
102	$(\mathbb{Z}/2)^{\oplus 198614} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 183067} \oplus (\mathbb{Z}/2^2)^{\oplus 5}$
103	$(\mathbb{Z}/2)^{\oplus 215367}$	$(\mathbb{Z}/2)^{\oplus 198614} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
104	$(\mathbb{Z}/2)^{\oplus 233419} \oplus (\mathbb{Z}/2^2)^{\oplus 5} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 215367}$
105	$(\mathbb{Z}/2)^{\oplus 252907} \oplus (\mathbb{Z}/2^2)^{\oplus 7} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 233419} \oplus (\mathbb{Z}/2^2)^{\oplus 5} \oplus \mathbb{Z}/2^3$
106	$(\mathbb{Z}/2)^{\oplus 273893} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 252907} \oplus (\mathbb{Z}/2^2)^{\oplus 7} \oplus \mathbb{Z}/2^3$
107	$(\mathbb{Z}/2)^{\oplus 296492}$	$(\mathbb{Z}/2)^{\oplus 273893} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
108	$(\mathbb{Z}/2)^{\oplus 320839} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$	$(\mathbb{Z}/2)^{\oplus 296492}$
109	$(\mathbb{Z}/2)^{\oplus 347033} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$	$(\mathbb{Z}/2)^{\oplus 320839} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$
110	$(\mathbb{Z}/2)^{\oplus 375215} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 347033} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$
111	$(\mathbb{Z}/2)^{\oplus 405552}$	$(\mathbb{Z}/2)^{\oplus 375215} \oplus \mathbb{Z}/2^2$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
112	$(\mathbb{Z}/2)^{\oplus 438139} \oplus (\mathbb{Z}/2^2)^{\oplus 3} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 405552}$
113	$(\mathbb{Z}/2)^{\oplus 473165} \oplus (\mathbb{Z}/2^2)^{\oplus 5} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 438139} \oplus (\mathbb{Z}/2^2)^{\oplus 3} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
114	$(\mathbb{Z}/2)^{\oplus 510836} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 473165} \oplus (\mathbb{Z}/2^2)^{\oplus 5} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
115	$(\mathbb{Z}/2)^{\oplus 551270}$	$(\mathbb{Z}/2)^{\oplus 510836} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
116	$(\mathbb{Z}/2)^{\oplus 594669} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$	$(\mathbb{Z}/2)^{\oplus 551270}$
117	$(\mathbb{Z}/2)^{\oplus 641294} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$	$(\mathbb{Z}/2)^{\oplus 594669} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$
118	$(\mathbb{Z}/2)^{\oplus 691307} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 641294} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$
119	$(\mathbb{Z}/2)^{\oplus 744945}$	$(\mathbb{Z}/2)^{\oplus 691307} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$
120	$(\mathbb{Z}/2)^{\oplus 802488} \oplus (\mathbb{Z}/2^2)^{\oplus 5} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 744945}$
121	$(\mathbb{Z}/2)^{\oplus 864157} \oplus (\mathbb{Z}/2^2)^{\oplus 7} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 802488} \oplus (\mathbb{Z}/2^2)^{\oplus 5} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$
122	$(\mathbb{Z}/2)^{\oplus 930243} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 864157} \oplus (\mathbb{Z}/2^2)^{\oplus 7} \oplus \mathbb{Z}/2^3$
123	$(\mathbb{Z}/2)^{\oplus 1001081}$	$(\mathbb{Z}/2)^{\oplus 930243} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
124	$(\mathbb{Z}/2)^{\oplus 1076915} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$	$(\mathbb{Z}/2)^{\oplus 1001081}$
125	$(\mathbb{Z}/2)^{\oplus 1158106} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$	$(\mathbb{Z}/2)^{\oplus 1076915} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$
126	$(\mathbb{Z}/2)^{\oplus 1245057} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 1158106} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$
127	$(\mathbb{Z}/2)^{\oplus 1338079}$	$(\mathbb{Z}/2)^{\oplus 1245057} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
128	$(\mathbb{Z}/2)^{\oplus 1437564} \oplus (\mathbb{Z}/2^2)^{\oplus 8} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 1338079}$
129	$(\mathbb{Z}/2)^{\oplus 1544009} \oplus (\mathbb{Z}/2^2)^{\oplus 12} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 1437564} \oplus (\mathbb{Z}/2^2)^{\oplus 8} \oplus \mathbb{Z}/2^4$
130	$(\mathbb{Z}/2)^{\oplus 1657799} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 1544009} \oplus (\mathbb{Z}/2^2)^{\oplus 12} \oplus \mathbb{Z}/2^4$
131	$(\mathbb{Z}/2)^{\oplus 1779411}$	$(\mathbb{Z}/2)^{\oplus 1657799} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
132	$(\mathbb{Z}/2)^{\oplus 1909392} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$	$(\mathbb{Z}/2)^{\oplus 1779411}$
133	$(\mathbb{Z}/2)^{\oplus 2048227} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$	$(\mathbb{Z}/2)^{\oplus 1909392} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$
134	$(\mathbb{Z}/2)^{\oplus 2196498} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 2048227} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$
135	$(\mathbb{Z}/2)^{\oplus 2354851}$	$(\mathbb{Z}/2)^{\oplus 2196498} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
136	$(\mathbb{Z}/2)^{\oplus 2523833} \oplus (\mathbb{Z}/2^2)^{\oplus 5} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 2354851}$
137	$(\mathbb{Z}/2)^{\oplus 2704156} \oplus (\mathbb{Z}/2^2)^{\oplus 9} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 2523833} \oplus (\mathbb{Z}/2^2)^{\oplus 5} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$
138	$(\mathbb{Z}/2)^{\oplus 2896595} \oplus (\mathbb{Z}/2^2)^{\oplus 5} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 2704156} \oplus (\mathbb{Z}/2^2)^{\oplus 9} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$
139	$(\mathbb{Z}/2)^{\oplus 3101816} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 2896595} \oplus (\mathbb{Z}/2^2)^{\oplus 5} \oplus \mathbb{Z}/2^3$
140	$(\mathbb{Z}/2)^{\oplus 3320612} \oplus (\mathbb{Z}/2^2)^{\oplus 11}$	$(\mathbb{Z}/2)^{\oplus 3101816} \oplus \mathbb{Z}/2^2$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
141	$(\mathbb{Z}/2)^{\oplus 3553922} \oplus (\mathbb{Z}/2^2)^{\oplus 16}$	$(\mathbb{Z}/2)^{\oplus 3320612} \oplus (\mathbb{Z}/2^2)^{\oplus 11}$
142	$(\mathbb{Z}/2)^{\oplus 3802561} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$	$(\mathbb{Z}/2)^{\oplus 3553922} \oplus (\mathbb{Z}/2^2)^{\oplus 16}$
143	$(\mathbb{Z}/2)^{\oplus 4067468}$	$(\mathbb{Z}/2)^{\oplus 3802561} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$
144	$(\mathbb{Z}/2)^{\oplus 4349701} \oplus (\mathbb{Z}/2^2)^{\oplus 8} \oplus (\mathbb{Z}/2^3)^{\oplus 2} \oplus (\mathbb{Z}/2^4)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 4067468}$
145	$(\mathbb{Z}/2)^{\oplus 4650258} \oplus (\mathbb{Z}/2^2)^{\oplus 12} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 4349701} \oplus (\mathbb{Z}/2^2)^{\oplus 8} \oplus (\mathbb{Z}/2^3)^{\oplus 2} \oplus (\mathbb{Z}/2^4)^{\oplus 2}$
146	$(\mathbb{Z}/2)^{\oplus 4970264} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 4650258} \oplus (\mathbb{Z}/2^2)^{\oplus 12} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
147	$(\mathbb{Z}/2)^{\oplus 5310963}$	$(\mathbb{Z}/2)^{\oplus 4970264} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
148	$(\mathbb{Z}/2)^{\oplus 5673500} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$	$(\mathbb{Z}/2)^{\oplus 5310963}$
149	$(\mathbb{Z}/2)^{\oplus 6059225} \oplus (\mathbb{Z}/2^2)^{\oplus 16}$	$(\mathbb{Z}/2)^{\oplus 5673500} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$
150	$(\mathbb{Z}/2)^{\oplus 6469613} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$	$(\mathbb{Z}/2)^{\oplus 6059225} \oplus (\mathbb{Z}/2^2)^{\oplus 16}$
151	$(\mathbb{Z}/2)^{\oplus 6906024} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 6469613} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$
152	$(\mathbb{Z}/2)^{\oplus 7370006} \oplus (\mathbb{Z}/2^2)^{\oplus 11} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 6906024} \oplus \mathbb{Z}/2^2$
153	$(\mathbb{Z}/2)^{\oplus 7863307} \oplus (\mathbb{Z}/2^2)^{\oplus 18} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 7370006} \oplus (\mathbb{Z}/2^2)^{\oplus 11} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
154	$(\mathbb{Z}/2)^{\oplus 8387558} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$	$(\mathbb{Z}/2)^{\oplus 7863307} \oplus (\mathbb{Z}/2^2)^{\oplus 18} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
155	$(\mathbb{Z}/2)^{\oplus 8944581}$	$(\mathbb{Z}/2)^{\oplus 8387558} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$
156	$(\mathbb{Z}/2)^{\oplus 9536367} \oplus (\mathbb{Z}/2^2)^{\oplus 14}$	$(\mathbb{Z}/2)^{\oplus 8944581}$
157	$(\mathbb{Z}/2)^{\oplus 10164873} \oplus (\mathbb{Z}/2^2)^{\oplus 18}$	$(\mathbb{Z}/2)^{\oplus 9536367} \oplus (\mathbb{Z}/2^2)^{\oplus 14}$
158	$(\mathbb{Z}/2)^{\oplus 10832252} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$	$(\mathbb{Z}/2)^{\oplus 10164873} \oplus (\mathbb{Z}/2^2)^{\oplus 18}$
159	$(\mathbb{Z}/2)^{\oplus 11540843} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 10832252} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$
160	$(\mathbb{Z}/2)^{\oplus 12292897} \oplus (\mathbb{Z}/2^2)^{\oplus 8} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 11540843} \oplus \mathbb{Z}/2^2$
161	$(\mathbb{Z}/2)^{\oplus 13090948} \oplus (\mathbb{Z}/2^2)^{\oplus 16} \oplus (\mathbb{Z}/2^3)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 12292897} \oplus (\mathbb{Z}/2^2)^{\oplus 8} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^5$
162	$(\mathbb{Z}/2)^{\oplus 13937766} \oplus (\mathbb{Z}/2^2)^{\oplus 9} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 13090948} \oplus (\mathbb{Z}/2^2)^{\oplus 16} \oplus (\mathbb{Z}/2^3)^{\oplus 4}$
163	$(\mathbb{Z}/2)^{\oplus 14835997} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 13937766} \oplus (\mathbb{Z}/2^2)^{\oplus 9} \oplus \mathbb{Z}/2^3$
164	$(\mathbb{Z}/2)^{\oplus 15788556} \oplus (\mathbb{Z}/2^2)^{\oplus 15}$	$(\mathbb{Z}/2)^{\oplus 14835997} \oplus \mathbb{Z}/2^2$
165	$(\mathbb{Z}/2)^{\oplus 16798678} \oplus (\mathbb{Z}/2^2)^{\oplus 25}$	$(\mathbb{Z}/2)^{\oplus 15788556} \oplus (\mathbb{Z}/2^2)^{\oplus 15}$
166	$(\mathbb{Z}/2)^{\oplus 17869514} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$	$(\mathbb{Z}/2)^{\oplus 16798678} \oplus (\mathbb{Z}/2^2)^{\oplus 25}$
167	$(\mathbb{Z}/2)^{\oplus 19004472}$	$(\mathbb{Z}/2)^{\oplus 17869514} \oplus (\mathbb{Z}/2^2)^{\oplus 10}$
168	$(\mathbb{Z}/2)^{\oplus 20207248} \oplus (\mathbb{Z}/2^2)^{\oplus 13} \oplus (\mathbb{Z}/2^3)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 19004472}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
169	$(\mathbb{Z}/2)^{\oplus 21481565} \oplus (\mathbb{Z}/2^2)^{\oplus 20} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 20207248} \oplus (\mathbb{Z}/2^2)^{\oplus 13} \oplus (\mathbb{Z}/2^3)^{\oplus 4}$
170	$(\mathbb{Z}/2)^{\oplus 22831428} \oplus (\mathbb{Z}/2^2)^{\oplus 8}$	$(\mathbb{Z}/2)^{\oplus 21481565} \oplus (\mathbb{Z}/2^2)^{\oplus 20} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$
171	$(\mathbb{Z}/2)^{\oplus 24261139} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 22831428} \oplus (\mathbb{Z}/2^2)^{\oplus 8}$
172	$(\mathbb{Z}/2)^{\oplus 25774987} \oplus (\mathbb{Z}/2^2)^{\oplus 15}$	$(\mathbb{Z}/2)^{\oplus 24261139} \oplus \mathbb{Z}/2^2$
173	$(\mathbb{Z}/2)^{\oplus 27377661} \oplus (\mathbb{Z}/2^2)^{\oplus 25}$	$(\mathbb{Z}/2)^{\oplus 25774987} \oplus (\mathbb{Z}/2^2)^{\oplus 15}$
174	$(\mathbb{Z}/2)^{\oplus 29074205} \oplus (\mathbb{Z}/2^2)^{\oplus 12}$	$(\mathbb{Z}/2)^{\oplus 27377661} \oplus (\mathbb{Z}/2^2)^{\oplus 25}$
175	$(\mathbb{Z}/2)^{\oplus 30869609} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 29074205} \oplus (\mathbb{Z}/2^2)^{\oplus 12}$
176	$(\mathbb{Z}/2)^{\oplus 32769275} \oplus (\mathbb{Z}/2^2)^{\oplus 15} \oplus (\mathbb{Z}/2^3)^{\oplus 2} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 30869609} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
177	$(\mathbb{Z}/2)^{\oplus 34779072} \oplus (\mathbb{Z}/2^2)^{\oplus 26} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 32769275} \oplus (\mathbb{Z}/2^2)^{\oplus 15} \oplus (\mathbb{Z}/2^3)^{\oplus 2} \oplus \mathbb{Z}/2^4$
178	$(\mathbb{Z}/2)^{\oplus 36904863} \oplus (\mathbb{Z}/2^2)^{\oplus 12} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 34779072} \oplus (\mathbb{Z}/2^2)^{\oplus 26} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4$
179	$(\mathbb{Z}/2)^{\oplus 39152919} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 36904863} \oplus (\mathbb{Z}/2^2)^{\oplus 12} \oplus \mathbb{Z}/2^3$
180	$(\mathbb{Z}/2)^{\oplus 41529973} \oplus (\mathbb{Z}/2^2)^{\oplus 20}$	$(\mathbb{Z}/2)^{\oplus 39152919} \oplus \mathbb{Z}/2^2$
181	$(\mathbb{Z}/2)^{\oplus 44042870} \oplus (\mathbb{Z}/2^2)^{\oplus 28}$	$(\mathbb{Z}/2)^{\oplus 41529973} \oplus (\mathbb{Z}/2^2)^{\oplus 20}$
182	$(\mathbb{Z}/2)^{\oplus 46698924} \oplus (\mathbb{Z}/2^2)^{\oplus 12}$	$(\mathbb{Z}/2)^{\oplus 44042870} \oplus (\mathbb{Z}/2^2)^{\oplus 28}$
183	$(\mathbb{Z}/2)^{\oplus 49505941} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 46698924} \oplus (\mathbb{Z}/2^2)^{\oplus 12}$
184	$(\mathbb{Z}/2)^{\oplus 52471799} \oplus (\mathbb{Z}/2^2)^{\oplus 12} \oplus (\mathbb{Z}/2^3)^{\oplus 5}$	$(\mathbb{Z}/2)^{\oplus 49505941} \oplus \mathbb{Z}/2^2$
185	$(\mathbb{Z}/2)^{\oplus 55604984} \oplus (\mathbb{Z}/2^2)^{\oplus 25} \oplus (\mathbb{Z}/2^3)^{\oplus 6}$	$(\mathbb{Z}/2)^{\oplus 52471799} \oplus (\mathbb{Z}/2^2)^{\oplus 12} \oplus (\mathbb{Z}/2^3)^{\oplus 5}$
186	$(\mathbb{Z}/2)^{\oplus 58914572} \oplus (\mathbb{Z}/2^2)^{\oplus 15} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 55604984} \oplus (\mathbb{Z}/2^2)^{\oplus 25} \oplus (\mathbb{Z}/2^3)^{\oplus 6}$
187	$(\mathbb{Z}/2)^{\oplus 62409714} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 58914572} \oplus (\mathbb{Z}/2^2)^{\oplus 15} \oplus \mathbb{Z}/2^3$
188	$(\mathbb{Z}/2)^{\oplus 66100154} \oplus (\mathbb{Z}/2^2)^{\oplus 21}$	$(\mathbb{Z}/2)^{\oplus 62409714} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
189	$(\mathbb{Z}/2)^{\oplus 69996381} \oplus (\mathbb{Z}/2^2)^{\oplus 37}$	$(\mathbb{Z}/2)^{\oplus 66100154} \oplus (\mathbb{Z}/2^2)^{\oplus 21}$
190	$(\mathbb{Z}/2)^{\oplus 74109056} \oplus (\mathbb{Z}/2^2)^{\oplus 17}$	$(\mathbb{Z}/2)^{\oplus 69996381} \oplus (\mathbb{Z}/2^2)^{\oplus 37}$
191	$(\mathbb{Z}/2)^{\oplus 78449447} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 74109056} \oplus (\mathbb{Z}/2^2)^{\oplus 17}$
192	$(\mathbb{Z}/2)^{\oplus 83029570} \oplus (\mathbb{Z}/2^2)^{\oplus 18} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4 \oplus \mathbb{Z}/2^6$	$(\mathbb{Z}/2)^{\oplus 78449447} \oplus \mathbb{Z}/2^2$
193	$(\mathbb{Z}/2)^{\oplus 87861769} \oplus (\mathbb{Z}/2^2)^{\oplus 31} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 83029570} \oplus (\mathbb{Z}/2^2)^{\oplus 18} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4 \oplus \mathbb{Z}/2^6$
194	$(\mathbb{Z}/2)^{\oplus 92959120} \oplus (\mathbb{Z}/2^2)^{\oplus 15}$	$(\mathbb{Z}/2)^{\oplus 87861769} \oplus (\mathbb{Z}/2^2)^{\oplus 31} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
195	$(\mathbb{Z}/2)^{\oplus 98335475} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 92959120} \oplus (\mathbb{Z}/2^2)^{\oplus 15}$
196	$(\mathbb{Z}/2)^{\oplus 104004993} \oplus (\mathbb{Z}/2^2)^{\oplus 20}$	$(\mathbb{Z}/2)^{\oplus 98335475} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
197	$(\mathbb{Z}/2)^{\oplus 109982781} \oplus (\mathbb{Z}/2^2)^{\oplus 37}$	$(\mathbb{Z}/2)^{\oplus 104004993} \oplus (\mathbb{Z}/2^2)^{\oplus 20}$
198	$(\mathbb{Z}/2)^{\oplus 116284848} \oplus (\mathbb{Z}/2^2)^{\oplus 20}$	$(\mathbb{Z}/2)^{\oplus 109982781} \oplus (\mathbb{Z}/2^2)^{\oplus 37}$
199	$(\mathbb{Z}/2)^{\oplus 122927533} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 116284848} \oplus (\mathbb{Z}/2^2)^{\oplus 20}$
200	$(\mathbb{Z}/2)^{\oplus 129920555} \oplus (\mathbb{Z}/2^2)^{\oplus 21} \oplus (\mathbb{Z}/2^3)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 122927533} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$

Generators involved in the calculus.

Degree	Genus	Generator
3	1	$(\sigma^3, \sigma^2\psi_2)$
5	2	$(\beta_2\varphi_2\sigma^2, \varphi_2\sigma^2)$
6	3	$(\beta_2\sigma\varphi_2\psi_2, \sigma\varphi_2\psi_2)$
9	2	$(\beta_2\varphi_2\gamma_2\sigma^2, \varphi_2\gamma_2\sigma^2)$
10	3	$(\beta_2\sigma\varphi_2\gamma_2\psi_2, \sigma\varphi_2\gamma_2\psi_2)$
13	3	$(\beta_2\varphi_2^2\psi_2, \varphi_2^2\psi_2)$
17	2	$(\beta_2\varphi_2\gamma_2^2\sigma^2, \varphi_2\gamma_2^2\sigma^2)$
18	3	$(\beta_2\sigma\varphi_2\gamma_2^2\psi_2, \sigma\varphi_2\gamma_2^2\psi_2)$
21	3	$(\beta_2\varphi_2^2\gamma_2\psi_2, \varphi_2^2\gamma_2\psi_2)$
25	3	$(\beta_2\varphi_2\gamma_2\varphi_2\psi_2, \varphi_2\gamma_2\varphi_2\psi_2)$
33	2	$(\beta_2\varphi_2\gamma_2^3\sigma^2, \varphi_2\gamma_2^3\sigma^2)$
34	3	$(\beta_2\sigma\varphi_2\gamma_2^3\psi_2, \sigma\varphi_2\gamma_2^3\psi_2)$
37	3	$(\beta_2\varphi_2^2\gamma_2^2\psi_2, \varphi_2^2\gamma_2^2\psi_2)$
41	3	$(\beta_2\varphi_2\gamma_2\varphi_2\gamma_2\psi_2, \varphi_2\gamma_2\varphi_2\gamma_2\psi_2)$
49	3	$(\beta_2\varphi_2\gamma_2^2\varphi_2\psi_2, \varphi_2\gamma_2^2\varphi_2\psi_2)$
65	2	$(\beta_2\varphi_2\gamma_2^4\sigma^2, \varphi_2\gamma_2^4\sigma^2)$
66	3	$(\beta_2\sigma\varphi_2\gamma_2^4\psi_2, \sigma\varphi_2\gamma_2^4\psi_2)$
69	3	$(\beta_2\varphi_2^2\gamma_2^3\psi_2, \varphi_2^2\gamma_2^3\psi_2)$
73	3	$(\beta_2\varphi_2\gamma_2\varphi_2\gamma_2^2\psi_2, \varphi_2\gamma_2\varphi_2\gamma_2^2\psi_2)$
81	3	$(\beta_2\varphi_2\gamma_2^2\varphi_2\gamma_2\psi_2, \varphi_2\gamma_2^2\varphi_2\gamma_2\psi_2)$
97	3	$(\beta_2\varphi_2\gamma_2^3\varphi_2\psi_2, \varphi_2\gamma_2^3\varphi_2\psi_2)$
129	2	$(\beta_2\varphi_2\gamma_2^5\sigma^2, \varphi_2\gamma_2^5\sigma^2)$
130	3	$(\beta_2\sigma\varphi_2\gamma_2^5\psi_2, \sigma\varphi_2\gamma_2^5\psi_2)$
133	3	$(\beta_2\varphi_2^2\gamma_2^4\psi_2, \varphi_2^2\gamma_2^4\psi_2)$
137	3	$(\beta_2\varphi_2\gamma_2\varphi_2\gamma_2^3\psi_2, \varphi_2\gamma_2\varphi_2\gamma_2^3\psi_2)$
145	3	$(\beta_2\varphi_2\gamma_2^2\varphi_2\gamma_2^2\psi_2, \varphi_2\gamma_2^2\varphi_2\gamma_2^2\psi_2)$
161	3	$(\beta_2\varphi_2\gamma_2^3\varphi_2\gamma_2\psi_2, \varphi_2\gamma_2^3\varphi_2\gamma_2\psi_2)$
193	3	$(\beta_2\varphi_2\gamma_2^4\varphi_2\psi_2, \varphi_2\gamma_2^4\varphi_2\psi_2)$

C.3. Integral homology and cohomology of $K(\mathbb{Z}/2^2, 2)$.

n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	(0)	(0)
2	$\mathbb{Z}/2^2$	(0)
3	(0)	$\mathbb{Z}/2^2$
4	$\mathbb{Z}/2^3$	(0)
5	$\mathbb{Z}/2$	$\mathbb{Z}/2^3$
6	$\mathbb{Z}/2^2$	$\mathbb{Z}/2$
7	$\mathbb{Z}/2$	$\mathbb{Z}/2^2$
8	$\mathbb{Z}/2 \oplus \mathbb{Z}/2^4$	$\mathbb{Z}/2$
9	$(\mathbb{Z}/2)^{\oplus 2}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2^4$
10	$\mathbb{Z}/2 \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 2}$
11	$(\mathbb{Z}/2)^{\oplus 3}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2^2$
12	$(\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 3}$
13	$(\mathbb{Z}/2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z}/2^3$
14	$(\mathbb{Z}/2)^{\oplus 4} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 3}$
15	$(\mathbb{Z}/2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 4} \oplus \mathbb{Z}/2^2$
16	$(\mathbb{Z}/2)^{\oplus 4} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 4}$
17	$(\mathbb{Z}/2)^{\oplus 7}$	$(\mathbb{Z}/2)^{\oplus 4} \oplus \mathbb{Z}/2^5$
18	$(\mathbb{Z}/2)^{\oplus 5} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 7}$
19	$(\mathbb{Z}/2)^{\oplus 8}$	$(\mathbb{Z}/2)^{\oplus 5} \oplus \mathbb{Z}/2^2$
20	$(\mathbb{Z}/2)^{\oplus 8} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 8}$
21	$(\mathbb{Z}/2)^{\oplus 9}$	$(\mathbb{Z}/2)^{\oplus 8} \oplus \mathbb{Z}/2^3$
22	$(\mathbb{Z}/2)^{\oplus 10} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 9}$
23	$(\mathbb{Z}/2)^{\oplus 12}$	$(\mathbb{Z}/2)^{\oplus 10} \oplus \mathbb{Z}/2^2$
24	$(\mathbb{Z}/2)^{\oplus 12} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 12}$
25	$(\mathbb{Z}/2)^{\oplus 14}$	$(\mathbb{Z}/2)^{\oplus 12} \oplus \mathbb{Z}/2^4$
26	$(\mathbb{Z}/2)^{\oplus 16} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 14}$
27	$(\mathbb{Z}/2)^{\oplus 17}$	$(\mathbb{Z}/2)^{\oplus 16} \oplus \mathbb{Z}/2^2$
28	$(\mathbb{Z}/2)^{\oplus 18} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 17}$
29	$(\mathbb{Z}/2)^{\oplus 22}$	$(\mathbb{Z}/2)^{\oplus 18} \oplus \mathbb{Z}/2^3$
30	$(\mathbb{Z}/2)^{\oplus 21} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 22}$
31	$(\mathbb{Z}/2)^{\oplus 25}$	$(\mathbb{Z}/2)^{\oplus 21} \oplus \mathbb{Z}/2^2$
32	$(\mathbb{Z}/2)^{\oplus 27} \oplus \mathbb{Z}/2^6$	$(\mathbb{Z}/2)^{\oplus 25}$
33	$(\mathbb{Z}/2)^{\oplus 29}$	$(\mathbb{Z}/2)^{\oplus 27} \oplus \mathbb{Z}/2^6$
34	$(\mathbb{Z}/2)^{\oplus 31} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 29}$
35	$(\mathbb{Z}/2)^{\oplus 36}$	$(\mathbb{Z}/2)^{\oplus 31} \oplus \mathbb{Z}/2^2$
36	$(\mathbb{Z}/2)^{\oplus 36} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 36}$
37	$(\mathbb{Z}/2)^{\oplus 41}$	$(\mathbb{Z}/2)^{\oplus 36} \oplus \mathbb{Z}/2^3$
38	$(\mathbb{Z}/2)^{\oplus 44} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 41}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
39	$(\mathbb{Z}/2)^{\oplus 47}$	$(\mathbb{Z}/2)^{\oplus 44} \oplus \mathbb{Z}/2^2$
40	$(\mathbb{Z}/2)^{\oplus 50} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 47}$
41	$(\mathbb{Z}/2)^{\oplus 56}$	$(\mathbb{Z}/2)^{\oplus 50} \oplus \mathbb{Z}/2^4$
42	$(\mathbb{Z}/2)^{\oplus 58} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 56}$
43	$(\mathbb{Z}/2)^{\oplus 63}$	$(\mathbb{Z}/2)^{\oplus 58} \oplus \mathbb{Z}/2^2$
44	$(\mathbb{Z}/2)^{\oplus 69} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 63}$
45	$(\mathbb{Z}/2)^{\oplus 72}$	$(\mathbb{Z}/2)^{\oplus 69} \oplus \mathbb{Z}/2^3$
46	$(\mathbb{Z}/2)^{\oplus 77} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 72}$
47	$(\mathbb{Z}/2)^{\oplus 85}$	$(\mathbb{Z}/2)^{\oplus 77} \oplus \mathbb{Z}/2^2$
48	$(\mathbb{Z}/2)^{\oplus 87} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 85}$
49	$(\mathbb{Z}/2)^{\oplus 95}$	$(\mathbb{Z}/2)^{\oplus 87} \oplus \mathbb{Z}/2^5$
50	$(\mathbb{Z}/2)^{\oplus 102} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 95}$
51	$(\mathbb{Z}/2)^{\oplus 107}$	$(\mathbb{Z}/2)^{\oplus 102} \oplus \mathbb{Z}/2^2$
52	$(\mathbb{Z}/2)^{\oplus 114} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 107}$
53	$(\mathbb{Z}/2)^{\oplus 124}$	$(\mathbb{Z}/2)^{\oplus 114} \oplus \mathbb{Z}/2^3$
54	$(\mathbb{Z}/2)^{\oplus 128} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 124}$
55	$(\mathbb{Z}/2)^{\oplus 138}$	$(\mathbb{Z}/2)^{\oplus 128} \oplus \mathbb{Z}/2^2$
56	$(\mathbb{Z}/2)^{\oplus 147} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 138}$
57	$(\mathbb{Z}/2)^{\oplus 154}$	$(\mathbb{Z}/2)^{\oplus 147} \oplus \mathbb{Z}/2^4$
58	$(\mathbb{Z}/2)^{\oplus 163} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 154}$
59	$(\mathbb{Z}/2)^{\oplus 176}$	$(\mathbb{Z}/2)^{\oplus 163} \oplus \mathbb{Z}/2^2$
60	$(\mathbb{Z}/2)^{\oplus 182} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 176}$
61	$(\mathbb{Z}/2)^{\oplus 194}$	$(\mathbb{Z}/2)^{\oplus 182} \oplus \mathbb{Z}/2^3$
62	$(\mathbb{Z}/2)^{\oplus 207} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 194}$
63	$(\mathbb{Z}/2)^{\oplus 215}$	$(\mathbb{Z}/2)^{\oplus 207} \oplus \mathbb{Z}/2^2$
64	$(\mathbb{Z}/2)^{\oplus 228} \oplus \mathbb{Z}/2^7$	$(\mathbb{Z}/2)^{\oplus 215}$
65	$(\mathbb{Z}/2)^{\oplus 244}$	$(\mathbb{Z}/2)^{\oplus 228} \oplus \mathbb{Z}/2^7$
66	$(\mathbb{Z}/2)^{\oplus 251} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 244}$
67	$(\mathbb{Z}/2)^{\oplus 269}$	$(\mathbb{Z}/2)^{\oplus 251} \oplus \mathbb{Z}/2^2$
68	$(\mathbb{Z}/2)^{\oplus 283} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 269}$
69	$(\mathbb{Z}/2)^{\oplus 296}$	$(\mathbb{Z}/2)^{\oplus 283} \oplus \mathbb{Z}/2^3$
70	$(\mathbb{Z}/2)^{\oplus 312} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 296}$
71	$(\mathbb{Z}/2)^{\oplus 331}$	$(\mathbb{Z}/2)^{\oplus 312} \oplus \mathbb{Z}/2^2$
72	$(\mathbb{Z}/2)^{\oplus 343} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 331}$
73	$(\mathbb{Z}/2)^{\oplus 363}$	$(\mathbb{Z}/2)^{\oplus 343} \oplus \mathbb{Z}/2^4$
74	$(\mathbb{Z}/2)^{\oplus 383} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 363}$
75	$(\mathbb{Z}/2)^{\oplus 398}$	$(\mathbb{Z}/2)^{\oplus 383} \oplus \mathbb{Z}/2^2$
76	$(\mathbb{Z}/2)^{\oplus 419} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 398}$
77	$(\mathbb{Z}/2)^{\oplus 443}$	$(\mathbb{Z}/2)^{\oplus 419} \oplus \mathbb{Z}/2^3$
78	$(\mathbb{Z}/2)^{\oplus 458} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 443}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
79	$(\mathbb{Z}/2)^{\oplus 484}$	$(\mathbb{Z}/2)^{\oplus 458} \oplus \mathbb{Z}/2^2$
80	$(\mathbb{Z}/2)^{\oplus 508} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 484}$
81	$(\mathbb{Z}/2)^{\oplus 527}$	$(\mathbb{Z}/2)^{\oplus 508} \oplus \mathbb{Z}/2^5$
82	$(\mathbb{Z}/2)^{\oplus 555} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 527}$
83	$(\mathbb{Z}/2)^{\oplus 582}$	$(\mathbb{Z}/2)^{\oplus 555} \oplus \mathbb{Z}/2^2$
84	$(\mathbb{Z}/2)^{\oplus 604} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 582}$
85	$(\mathbb{Z}/2)^{\oplus 635}$	$(\mathbb{Z}/2)^{\oplus 604} \oplus \mathbb{Z}/2^3$
86	$(\mathbb{Z}/2)^{\oplus 664} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 635}$
87	$(\mathbb{Z}/2)^{\oplus 691}$	$(\mathbb{Z}/2)^{\oplus 664} \oplus \mathbb{Z}/2^2$
88	$(\mathbb{Z}/2)^{\oplus 722} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 691}$
89	$(\mathbb{Z}/2)^{\oplus 758}$	$(\mathbb{Z}/2)^{\oplus 722} \oplus \mathbb{Z}/2^4$
90	$(\mathbb{Z}/2)^{\oplus 784} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 758}$
91	$(\mathbb{Z}/2)^{\oplus 822}$	$(\mathbb{Z}/2)^{\oplus 784} \oplus \mathbb{Z}/2^2$
92	$(\mathbb{Z}/2)^{\oplus 859} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 822}$
93	$(\mathbb{Z}/2)^{\oplus 890}$	$(\mathbb{Z}/2)^{\oplus 859} \oplus \mathbb{Z}/2^3$
94	$(\mathbb{Z}/2)^{\oplus 931} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 890}$
95	$(\mathbb{Z}/2)^{\oplus 972}$	$(\mathbb{Z}/2)^{\oplus 931} \oplus \mathbb{Z}/2^2$
96	$(\mathbb{Z}/2)^{\oplus 1006} \oplus \mathbb{Z}/2^6$	$(\mathbb{Z}/2)^{\oplus 972}$
97	$(\mathbb{Z}/2)^{\oplus 1052}$	$(\mathbb{Z}/2)^{\oplus 1006} \oplus \mathbb{Z}/2^6$
98	$(\mathbb{Z}/2)^{\oplus 1096} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 1052}$
99	$(\mathbb{Z}/2)^{\oplus 1136}$	$(\mathbb{Z}/2)^{\oplus 1096} \oplus \mathbb{Z}/2^2$
100	$(\mathbb{Z}/2)^{\oplus 1184} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 1136}$
101	$(\mathbb{Z}/2)^{\oplus 1235}$	$(\mathbb{Z}/2)^{\oplus 1184} \oplus \mathbb{Z}/2^3$
102	$(\mathbb{Z}/2)^{\oplus 1277} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 1235}$
103	$(\mathbb{Z}/2)^{\oplus 1332}$	$(\mathbb{Z}/2)^{\oplus 1277} \oplus \mathbb{Z}/2^2$
104	$(\mathbb{Z}/2)^{\oplus 1386} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 1332}$
105	$(\mathbb{Z}/2)^{\oplus 1434}$	$(\mathbb{Z}/2)^{\oplus 1386} \oplus \mathbb{Z}/2^4$
106	$(\mathbb{Z}/2)^{\oplus 1492} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 1434}$
107	$(\mathbb{Z}/2)^{\oplus 1554}$	$(\mathbb{Z}/2)^{\oplus 1492} \oplus \mathbb{Z}/2^2$
108	$(\mathbb{Z}/2)^{\oplus 1604} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 1554}$
109	$(\mathbb{Z}/2)^{\oplus 1671}$	$(\mathbb{Z}/2)^{\oplus 1604} \oplus \mathbb{Z}/2^3$
110	$(\mathbb{Z}/2)^{\oplus 1735} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 1671}$
111	$(\mathbb{Z}/2)^{\oplus 1793}$	$(\mathbb{Z}/2)^{\oplus 1735} \oplus \mathbb{Z}/2^2$
112	$(\mathbb{Z}/2)^{\oplus 1864} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 1793}$
113	$(\mathbb{Z}/2)^{\oplus 1935}$	$(\mathbb{Z}/2)^{\oplus 1864} \oplus \mathbb{Z}/2^5$
114	$(\mathbb{Z}/2)^{\oplus 1998} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 1935}$
115	$(\mathbb{Z}/2)^{\oplus 2076}$	$(\mathbb{Z}/2)^{\oplus 1998} \oplus \mathbb{Z}/2^2$
116	$(\mathbb{Z}/2)^{\oplus 2153} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 2076}$
117	$(\mathbb{Z}/2)^{\oplus 2223}$	$(\mathbb{Z}/2)^{\oplus 2153} \oplus \mathbb{Z}/2^3$
118	$(\mathbb{Z}/2)^{\oplus 2307} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 2223}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
119	$(\mathbb{Z}/2)^{\oplus 2392}$	$(\mathbb{Z}/2)^{\oplus 2307} \oplus \mathbb{Z}/2^2$
120	$(\mathbb{Z}/2)^{\oplus 2467} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 2392}$
121	$(\mathbb{Z}/2)^{\oplus 2560}$	$(\mathbb{Z}/2)^{\oplus 2467} \oplus \mathbb{Z}/2^4$
122	$(\mathbb{Z}/2)^{\oplus 2650} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 2560}$
123	$(\mathbb{Z}/2)^{\oplus 2734}$	$(\mathbb{Z}/2)^{\oplus 2650} \oplus \mathbb{Z}/2^2$
124	$(\mathbb{Z}/2)^{\oplus 2833} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 2734}$
125	$(\mathbb{Z}/2)^{\oplus 2933}$	$(\mathbb{Z}/2)^{\oplus 2833} \oplus \mathbb{Z}/2^3$
126	$(\mathbb{Z}/2)^{\oplus 3022} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 2933}$
127	$(\mathbb{Z}/2)^{\oplus 3132}$	$(\mathbb{Z}/2)^{\oplus 3022} \oplus \mathbb{Z}/2^2$
128	$(\mathbb{Z}/2)^{\oplus 3237} \oplus \mathbb{Z}/2^8$	$(\mathbb{Z}/2)^{\oplus 3132}$
129	$(\mathbb{Z}/2)^{\oplus 3338}$	$(\mathbb{Z}/2)^{\oplus 3237} \oplus \mathbb{Z}/2^8$
130	$(\mathbb{Z}/2)^{\oplus 3453} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 3338}$
131	$(\mathbb{Z}/2)^{\oplus 3570}$	$(\mathbb{Z}/2)^{\oplus 3453} \oplus \mathbb{Z}/2^2$
132	$(\mathbb{Z}/2)^{\oplus 3677} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 3570}$
133	$(\mathbb{Z}/2)^{\oplus 3804}$	$(\mathbb{Z}/2)^{\oplus 3677} \oplus \mathbb{Z}/2^3$
134	$(\mathbb{Z}/2)^{\oplus 3929} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 3804}$
135	$(\mathbb{Z}/2)^{\oplus 4047}$	$(\mathbb{Z}/2)^{\oplus 3929} \oplus \mathbb{Z}/2^2$
136	$(\mathbb{Z}/2)^{\oplus 4182} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 4047}$
137	$(\mathbb{Z}/2)^{\oplus 4319}$	$(\mathbb{Z}/2)^{\oplus 4182} \oplus \mathbb{Z}/2^4$
138	$(\mathbb{Z}/2)^{\oplus 4445} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 4319}$
139	$(\mathbb{Z}/2)^{\oplus 4593}$	$(\mathbb{Z}/2)^{\oplus 4445} \oplus \mathbb{Z}/2^2$
140	$(\mathbb{Z}/2)^{\oplus 4738} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 4593}$
141	$(\mathbb{Z}/2)^{\oplus 4877}$	$(\mathbb{Z}/2)^{\oplus 4738} \oplus \mathbb{Z}/2^3$
142	$(\mathbb{Z}/2)^{\oplus 5034} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 4877}$
143	$(\mathbb{Z}/2)^{\oplus 5193}$	$(\mathbb{Z}/2)^{\oplus 5034} \oplus \mathbb{Z}/2^2$
144	$(\mathbb{Z}/2)^{\oplus 5341} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 5193}$
145	$(\mathbb{Z}/2)^{\oplus 5512}$	$(\mathbb{Z}/2)^{\oplus 5341} \oplus \mathbb{Z}/2^5$
146	$(\mathbb{Z}/2)^{\oplus 5681} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 5512}$
147	$(\mathbb{Z}/2)^{\oplus 5843}$	$(\mathbb{Z}/2)^{\oplus 5681} \oplus \mathbb{Z}/2^2$
148	$(\mathbb{Z}/2)^{\oplus 6025} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 5843}$
149	$(\mathbb{Z}/2)^{\oplus 6209}$	$(\mathbb{Z}/2)^{\oplus 6025} \oplus \mathbb{Z}/2^3$
150	$(\mathbb{Z}/2)^{\oplus 6382} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 6209}$
151	$(\mathbb{Z}/2)^{\oplus 6579}$	$(\mathbb{Z}/2)^{\oplus 6382} \oplus \mathbb{Z}/2^2$
152	$(\mathbb{Z}/2)^{\oplus 6776} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 6579}$
153	$(\mathbb{Z}/2)^{\oplus 6963}$	$(\mathbb{Z}/2)^{\oplus 6776} \oplus \mathbb{Z}/2^4$
154	$(\mathbb{Z}/2)^{\oplus 7174} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 6963}$
155	$(\mathbb{Z}/2)^{\oplus 7386}$	$(\mathbb{Z}/2)^{\oplus 7174} \oplus \mathbb{Z}/2^2$
156	$(\mathbb{Z}/2)^{\oplus 7586} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 7386}$
157	$(\mathbb{Z}/2)^{\oplus 7814}$	$(\mathbb{Z}/2)^{\oplus 7586} \oplus \mathbb{Z}/2^3$
158	$(\mathbb{Z}/2)^{\oplus 8039} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 7814}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
159	$(\mathbb{Z}/2)^{\oplus 8257}$	$(\mathbb{Z}/2)^{\oplus 8039} \oplus \mathbb{Z}/2^2$
160	$(\mathbb{Z}/2)^{\oplus 8498} \oplus \mathbb{Z}/2^6$	$(\mathbb{Z}/2)^{\oplus 8257}$
161	$(\mathbb{Z}/2)^{\oplus 8742}$	$(\mathbb{Z}/2)^{\oplus 8498} \oplus \mathbb{Z}/2^6$
162	$(\mathbb{Z}/2)^{\oplus 8974} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 8742}$
163	$(\mathbb{Z}/2)^{\oplus 9234}$	$(\mathbb{Z}/2)^{\oplus 8974} \oplus \mathbb{Z}/2^2$
164	$(\mathbb{Z}/2)^{\oplus 9494} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 9234}$
165	$(\mathbb{Z}/2)^{\oplus 9744}$	$(\mathbb{Z}/2)^{\oplus 9494} \oplus \mathbb{Z}/2^3$
166	$(\mathbb{Z}/2)^{\oplus 10021} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 9744}$
167	$(\mathbb{Z}/2)^{\oplus 10301}$	$(\mathbb{Z}/2)^{\oplus 10021} \oplus \mathbb{Z}/2^2$
168	$(\mathbb{Z}/2)^{\oplus 10567} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 10301}$
169	$(\mathbb{Z}/2)^{\oplus 10866}$	$(\mathbb{Z}/2)^{\oplus 10567} \oplus \mathbb{Z}/2^4$
170	$(\mathbb{Z}/2)^{\oplus 11162} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 10866}$
171	$(\mathbb{Z}/2)^{\oplus 11450}$	$(\mathbb{Z}/2)^{\oplus 11162} \oplus \mathbb{Z}/2^2$
172	$(\mathbb{Z}/2)^{\oplus 11767} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 11450}$
173	$(\mathbb{Z}/2)^{\oplus 12085}$	$(\mathbb{Z}/2)^{\oplus 11767} \oplus \mathbb{Z}/2^3$
174	$(\mathbb{Z}/2)^{\oplus 12392} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 12085}$
175	$(\mathbb{Z}/2)^{\oplus 12731}$	$(\mathbb{Z}/2)^{\oplus 12392} \oplus \mathbb{Z}/2^2$
176	$(\mathbb{Z}/2)^{\oplus 13069} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 12731}$
177	$(\mathbb{Z}/2)^{\oplus 13399}$	$(\mathbb{Z}/2)^{\oplus 13069} \oplus \mathbb{Z}/2^5$
178	$(\mathbb{Z}/2)^{\oplus 13758} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 13399}$
179	$(\mathbb{Z}/2)^{\oplus 14122}$	$(\mathbb{Z}/2)^{\oplus 13758} \oplus \mathbb{Z}/2^2$
180	$(\mathbb{Z}/2)^{\oplus 14471} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 14122}$
181	$(\mathbb{Z}/2)^{\oplus 14857}$	$(\mathbb{Z}/2)^{\oplus 14471} \oplus \mathbb{Z}/2^3$
182	$(\mathbb{Z}/2)^{\oplus 15242} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 14857}$
183	$(\mathbb{Z}/2)^{\oplus 15617}$	$(\mathbb{Z}/2)^{\oplus 15242} \oplus \mathbb{Z}/2^2$
184	$(\mathbb{Z}/2)^{\oplus 16027} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 15617}$
185	$(\mathbb{Z}/2)^{\oplus 16437}$	$(\mathbb{Z}/2)^{\oplus 16027} \oplus \mathbb{Z}/2^4$
186	$(\mathbb{Z}/2)^{\oplus 16837} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 16437}$
187	$(\mathbb{Z}/2)^{\oplus 17273}$	$(\mathbb{Z}/2)^{\oplus 16837} \oplus \mathbb{Z}/2^2$
188	$(\mathbb{Z}/2)^{\oplus 17709} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 17273}$
189	$(\mathbb{Z}/2)^{\oplus 18137}$	$(\mathbb{Z}/2)^{\oplus 17709} \oplus \mathbb{Z}/2^3$
190	$(\mathbb{Z}/2)^{\oplus 18598} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 18137}$
191	$(\mathbb{Z}/2)^{\oplus 19064}$	$(\mathbb{Z}/2)^{\oplus 18598} \oplus \mathbb{Z}/2^2$
192	$(\mathbb{Z}/2)^{\oplus 19517} \oplus \mathbb{Z}/2^7$	$(\mathbb{Z}/2)^{\oplus 19064}$
193	$(\mathbb{Z}/2)^{\oplus 20009}$	$(\mathbb{Z}/2)^{\oplus 19517} \oplus \mathbb{Z}/2^7$
194	$(\mathbb{Z}/2)^{\oplus 20503} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 20009}$
195	$(\mathbb{Z}/2)^{\oplus 20986}$	$(\mathbb{Z}/2)^{\oplus 20503} \oplus \mathbb{Z}/2^2$
196	$(\mathbb{Z}/2)^{\oplus 21507} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 20986}$
197	$(\mathbb{Z}/2)^{\oplus 22034}$	$(\mathbb{Z}/2)^{\oplus 21507} \oplus \mathbb{Z}/2^3$
198	$(\mathbb{Z}/2)^{\oplus 22545} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 22034}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
199	$(\mathbb{Z}/2)^{\oplus 23102}$	$(\mathbb{Z}/2)^{\oplus 22545} \oplus \mathbb{Z}/2^2$
200	$(\mathbb{Z}/2)^{\oplus 23657} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 23102}$

Generators involved in the calculus.

Degree	Genus	Generator
2	1	$(\sigma^2, \sigma\psi_{2^2})$
5	3	$(\beta_2\varphi_2\psi_2, \varphi_2\psi_2)$
9	3	$(\beta_2\varphi_2\gamma_2\psi_2, \varphi_2\gamma_2\psi_2)$
17	3	$(\beta_2\varphi_2\gamma_2^2\psi_2, \varphi_2\gamma_2^2\psi_2)$
33	3	$(\beta_2\varphi_2\gamma_2^3\psi_2, \varphi_2\gamma_2^3\psi_2)$
65	3	$(\beta_2\varphi_2\gamma_2^4\psi_2, \varphi_2\gamma_2^4\psi_2)$
129	3	$(\beta_2\varphi_2\gamma_2^5\psi_2, \varphi_2\gamma_2^5\psi_2)$

C.4. Integral homology and cohomology of $K(\mathbb{Z}/2^2, 3)$.

n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	(0)	(0)
2	(0)	(0)
3	$\mathbb{Z}/2^2$	(0)
4	(0)	$\mathbb{Z}/2^2$
5	$\mathbb{Z}/2$	(0)
6	$\mathbb{Z}/2$	$\mathbb{Z}/2$
7	$\mathbb{Z}/2^2$	$\mathbb{Z}/2$
8	$\mathbb{Z}/2$	$\mathbb{Z}/2^2$
9	$(\mathbb{Z}/2)^{\oplus 3}$	$\mathbb{Z}/2$
10	$(\mathbb{Z}/2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 3}$
11	$(\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 2}$
12	$(\mathbb{Z}/2)^{\oplus 3} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 2} \oplus \mathbb{Z}/2^2$
13	$(\mathbb{Z}/2)^{\oplus 5}$	$(\mathbb{Z}/2)^{\oplus 3} \oplus \mathbb{Z}/2^2$
14	$(\mathbb{Z}/2)^{\oplus 5}$	$(\mathbb{Z}/2)^{\oplus 5}$
15	$(\mathbb{Z}/2)^{\oplus 6} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 5}$
16	$(\mathbb{Z}/2)^{\oplus 7} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 6} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
17	$(\mathbb{Z}/2)^{\oplus 11}$	$(\mathbb{Z}/2)^{\oplus 7} \oplus \mathbb{Z}/2^2$
18	$(\mathbb{Z}/2)^{\oplus 13}$	$(\mathbb{Z}/2)^{\oplus 11}$
19	$(\mathbb{Z}/2)^{\oplus 13} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 13}$
20	$(\mathbb{Z}/2)^{\oplus 16} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 13} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
21	$(\mathbb{Z}/2)^{\oplus 23}$	$(\mathbb{Z}/2)^{\oplus 16} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
22	$(\mathbb{Z}/2)^{\oplus 25}$	$(\mathbb{Z}/2)^{\oplus 23}$
23	$(\mathbb{Z}/2)^{\oplus 28} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 25}$
24	$(\mathbb{Z}/2)^{\oplus 34} \oplus (\mathbb{Z}/2^2)^{\oplus 2} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 28} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$
25	$(\mathbb{Z}/2)^{\oplus 43}$	$(\mathbb{Z}/2)^{\oplus 34} \oplus (\mathbb{Z}/2^2)^{\oplus 2} \oplus \mathbb{Z}/2^3$
26	$(\mathbb{Z}/2)^{\oplus 49}$	$(\mathbb{Z}/2)^{\oplus 43}$
27	$(\mathbb{Z}/2)^{\oplus 57} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 49}$
28	$(\mathbb{Z}/2)^{\oplus 65} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 57} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
29	$(\mathbb{Z}/2)^{\oplus 80}$	$(\mathbb{Z}/2)^{\oplus 65} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$
30	$(\mathbb{Z}/2)^{\oplus 95}$	$(\mathbb{Z}/2)^{\oplus 80}$
31	$(\mathbb{Z}/2)^{\oplus 104} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 95}$
32	$(\mathbb{Z}/2)^{\oplus 120} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 104} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
33	$(\mathbb{Z}/2)^{\oplus 146} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 120} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
34	$(\mathbb{Z}/2)^{\oplus 167}$	$(\mathbb{Z}/2)^{\oplus 146} \oplus \mathbb{Z}/2^2$
35	$(\mathbb{Z}/2)^{\oplus 187} \oplus (\mathbb{Z}/2^2)^{\oplus 5}$	$(\mathbb{Z}/2)^{\oplus 167}$
36	$(\mathbb{Z}/2)^{\oplus 215} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$	$(\mathbb{Z}/2)^{\oplus 187} \oplus (\mathbb{Z}/2^2)^{\oplus 5}$
37	$(\mathbb{Z}/2)^{\oplus 253} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 215} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$
38	$(\mathbb{Z}/2)^{\oplus 289}$	$(\mathbb{Z}/2)^{\oplus 253} \oplus \mathbb{Z}/2^2$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
39	$(\mathbb{Z}/2)^{\oplus 327} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$	$(\mathbb{Z}/2)^{\oplus 289}$
40	$(\mathbb{Z}/2)^{\oplus 370} \oplus (\mathbb{Z}/2^2)^{\oplus 7} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 327} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$
41	$(\mathbb{Z}/2)^{\oplus 430} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 370} \oplus (\mathbb{Z}/2^2)^{\oplus 7} \oplus \mathbb{Z}/2^3$
42	$(\mathbb{Z}/2)^{\oplus 492}$	$(\mathbb{Z}/2)^{\oplus 430} \oplus \mathbb{Z}/2^2$
43	$(\mathbb{Z}/2)^{\oplus 550} \oplus (\mathbb{Z}/2^2)^{\oplus 8}$	$(\mathbb{Z}/2)^{\oplus 492}$
44	$(\mathbb{Z}/2)^{\oplus 622} \oplus (\mathbb{Z}/2^2)^{\oplus 9}$	$(\mathbb{Z}/2)^{\oplus 550} \oplus (\mathbb{Z}/2^2)^{\oplus 8}$
45	$(\mathbb{Z}/2)^{\oplus 717} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 622} \oplus (\mathbb{Z}/2^2)^{\oplus 9}$
46	$(\mathbb{Z}/2)^{\oplus 810}$	$(\mathbb{Z}/2)^{\oplus 717} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
47	$(\mathbb{Z}/2)^{\oplus 906} \oplus (\mathbb{Z}/2^2)^{\oplus 9}$	$(\mathbb{Z}/2)^{\oplus 810}$
48	$(\mathbb{Z}/2)^{\oplus 1023} \oplus (\mathbb{Z}/2^2)^{\oplus 11} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 906} \oplus (\mathbb{Z}/2^2)^{\oplus 9}$
49	$(\mathbb{Z}/2)^{\oplus 1162} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 1023} \oplus (\mathbb{Z}/2^2)^{\oplus 11} \oplus \mathbb{Z}/2^4$
50	$(\mathbb{Z}/2)^{\oplus 1309}$	$(\mathbb{Z}/2)^{\oplus 1162} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$
51	$(\mathbb{Z}/2)^{\oplus 1466} \oplus (\mathbb{Z}/2^2)^{\oplus 11}$	$(\mathbb{Z}/2)^{\oplus 1309}$
52	$(\mathbb{Z}/2)^{\oplus 1641} \oplus (\mathbb{Z}/2^2)^{\oplus 14}$	$(\mathbb{Z}/2)^{\oplus 1466} \oplus (\mathbb{Z}/2^2)^{\oplus 11}$
53	$(\mathbb{Z}/2)^{\oplus 1852} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 1641} \oplus (\mathbb{Z}/2^2)^{\oplus 14}$
54	$(\mathbb{Z}/2)^{\oplus 2084}$	$(\mathbb{Z}/2)^{\oplus 1852} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
55	$(\mathbb{Z}/2)^{\oplus 2319} \oplus (\mathbb{Z}/2^2)^{\oplus 12}$	$(\mathbb{Z}/2)^{\oplus 2084}$
56	$(\mathbb{Z}/2)^{\oplus 2586} \oplus (\mathbb{Z}/2^2)^{\oplus 17}$	$(\mathbb{Z}/2)^{\oplus 2319} \oplus (\mathbb{Z}/2^2)^{\oplus 12}$
57	$(\mathbb{Z}/2)^{\oplus 2908} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$	$(\mathbb{Z}/2)^{\oplus 2586} \oplus (\mathbb{Z}/2^2)^{\oplus 17}$
58	$(\mathbb{Z}/2)^{\oplus 3250}$	$(\mathbb{Z}/2)^{\oplus 2908} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$
59	$(\mathbb{Z}/2)^{\oplus 3608} \oplus (\mathbb{Z}/2^2)^{\oplus 14}$	$(\mathbb{Z}/2)^{\oplus 3250}$
60	$(\mathbb{Z}/2)^{\oplus 4017} \oplus (\mathbb{Z}/2^2)^{\oplus 22}$	$(\mathbb{Z}/2)^{\oplus 3608} \oplus (\mathbb{Z}/2^2)^{\oplus 14}$
61	$(\mathbb{Z}/2)^{\oplus 4486} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$	$(\mathbb{Z}/2)^{\oplus 4017} \oplus (\mathbb{Z}/2^2)^{\oplus 22}$
62	$(\mathbb{Z}/2)^{\oplus 4994}$	$(\mathbb{Z}/2)^{\oplus 4486} \oplus (\mathbb{Z}/2^2)^{\oplus 7}$
63	$(\mathbb{Z}/2)^{\oplus 5537} \oplus (\mathbb{Z}/2^2)^{\oplus 17}$	$(\mathbb{Z}/2)^{\oplus 4994}$
64	$(\mathbb{Z}/2)^{\oplus 6138} \oplus (\mathbb{Z}/2^2)^{\oplus 23} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 5537} \oplus (\mathbb{Z}/2^2)^{\oplus 17}$
65	$(\mathbb{Z}/2)^{\oplus 6826} \oplus (\mathbb{Z}/2^2)^{\oplus 7} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 6138} \oplus (\mathbb{Z}/2^2)^{\oplus 23} \oplus \mathbb{Z}/2^3$
66	$(\mathbb{Z}/2)^{\oplus 7577}$	$(\mathbb{Z}/2)^{\oplus 6826} \oplus (\mathbb{Z}/2^2)^{\oplus 7} \oplus \mathbb{Z}/2^3$
67	$(\mathbb{Z}/2)^{\oplus 8371} \oplus (\mathbb{Z}/2^2)^{\oplus 18}$	$(\mathbb{Z}/2)^{\oplus 7577}$
68	$(\mathbb{Z}/2)^{\oplus 9251} \oplus (\mathbb{Z}/2^2)^{\oplus 28}$	$(\mathbb{Z}/2)^{\oplus 8371} \oplus (\mathbb{Z}/2^2)^{\oplus 18}$
69	$(\mathbb{Z}/2)^{\oplus 10256} \oplus (\mathbb{Z}/2^2)^{\oplus 11}$	$(\mathbb{Z}/2)^{\oplus 9251} \oplus (\mathbb{Z}/2^2)^{\oplus 28}$
70	$(\mathbb{Z}/2)^{\oplus 11337} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 10256} \oplus (\mathbb{Z}/2^2)^{\oplus 11}$
71	$(\mathbb{Z}/2)^{\oplus 12495} \oplus (\mathbb{Z}/2^2)^{\oplus 21}$	$(\mathbb{Z}/2)^{\oplus 11337} \oplus \mathbb{Z}/2^2$
72	$(\mathbb{Z}/2)^{\oplus 13779} \oplus (\mathbb{Z}/2^2)^{\oplus 33} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 12495} \oplus (\mathbb{Z}/2^2)^{\oplus 21}$
73	$(\mathbb{Z}/2)^{\oplus 15211} \oplus (\mathbb{Z}/2^2)^{\oplus 14}$	$(\mathbb{Z}/2)^{\oplus 13779} \oplus (\mathbb{Z}/2^2)^{\oplus 33} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
74	$(\mathbb{Z}/2)^{\oplus 16768} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 15211} \oplus (\mathbb{Z}/2^2)^{\oplus 14}$
75	$(\mathbb{Z}/2)^{\oplus 18445} \oplus (\mathbb{Z}/2^2)^{\oplus 25}$	$(\mathbb{Z}/2)^{\oplus 16768} \oplus \mathbb{Z}/2^2$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
76	$(\mathbb{Z}/2)^{\oplus 20276} \oplus (\mathbb{Z}/2^2)^{\oplus 39}$	$(\mathbb{Z}/2)^{\oplus 18445} \oplus (\mathbb{Z}/2^2)^{\oplus 25}$
77	$(\mathbb{Z}/2)^{\oplus 22316} \oplus (\mathbb{Z}/2^2)^{\oplus 16}$	$(\mathbb{Z}/2)^{\oplus 20276} \oplus (\mathbb{Z}/2^2)^{\oplus 39}$
78	$(\mathbb{Z}/2)^{\oplus 24543} \oplus \mathbb{Z}/2^2$	$(\mathbb{Z}/2)^{\oplus 22316} \oplus (\mathbb{Z}/2^2)^{\oplus 16}$
79	$(\mathbb{Z}/2)^{\oplus 26921} \oplus (\mathbb{Z}/2^2)^{\oplus 27}$	$(\mathbb{Z}/2)^{\oplus 24543} \oplus \mathbb{Z}/2^2$
80	$(\mathbb{Z}/2)^{\oplus 29520} \oplus (\mathbb{Z}/2^2)^{\oplus 44} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 26921} \oplus (\mathbb{Z}/2^2)^{\oplus 27}$
81	$(\mathbb{Z}/2)^{\oplus 32412} \oplus (\mathbb{Z}/2^2)^{\oplus 20}$	$(\mathbb{Z}/2)^{\oplus 29520} \oplus (\mathbb{Z}/2^2)^{\oplus 44} \oplus \mathbb{Z}/2^4$
82	$(\mathbb{Z}/2)^{\oplus 35539} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 32412} \oplus (\mathbb{Z}/2^2)^{\oplus 20}$
83	$(\mathbb{Z}/2)^{\oplus 38897} \oplus (\mathbb{Z}/2^2)^{\oplus 31}$	$(\mathbb{Z}/2)^{\oplus 35539} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
84	$(\mathbb{Z}/2)^{\oplus 42569} \oplus (\mathbb{Z}/2^2)^{\oplus 53}$	$(\mathbb{Z}/2)^{\oplus 38897} \oplus (\mathbb{Z}/2^2)^{\oplus 31}$
85	$(\mathbb{Z}/2)^{\oplus 46605} \oplus (\mathbb{Z}/2^2)^{\oplus 24}$	$(\mathbb{Z}/2)^{\oplus 42569} \oplus (\mathbb{Z}/2^2)^{\oplus 53}$
86	$(\mathbb{Z}/2)^{\oplus 50980} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 46605} \oplus (\mathbb{Z}/2^2)^{\oplus 24}$
87	$(\mathbb{Z}/2)^{\oplus 55692} \oplus (\mathbb{Z}/2^2)^{\oplus 35}$	$(\mathbb{Z}/2)^{\oplus 50980} \oplus (\mathbb{Z}/2^2)^{\oplus 2}$
88	$(\mathbb{Z}/2)^{\oplus 60804} \oplus (\mathbb{Z}/2^2)^{\oplus 58} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 55692} \oplus (\mathbb{Z}/2^2)^{\oplus 35}$
89	$(\mathbb{Z}/2)^{\oplus 66411} \oplus (\mathbb{Z}/2^2)^{\oplus 27} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 60804} \oplus (\mathbb{Z}/2^2)^{\oplus 58} \oplus \mathbb{Z}/2^3$
90	$(\mathbb{Z}/2)^{\oplus 72500} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 66411} \oplus (\mathbb{Z}/2^2)^{\oplus 27} \oplus \mathbb{Z}/2^3$
91	$(\mathbb{Z}/2)^{\oplus 79022} \oplus (\mathbb{Z}/2^2)^{\oplus 38}$	$(\mathbb{Z}/2)^{\oplus 72500} \oplus (\mathbb{Z}/2^2)^{\oplus 3}$
92	$(\mathbb{Z}/2)^{\oplus 86100} \oplus (\mathbb{Z}/2^2)^{\oplus 68}$	$(\mathbb{Z}/2)^{\oplus 79022} \oplus (\mathbb{Z}/2^2)^{\oplus 38}$
93	$(\mathbb{Z}/2)^{\oplus 93852} \oplus (\mathbb{Z}/2^2)^{\oplus 35}$	$(\mathbb{Z}/2)^{\oplus 86100} \oplus (\mathbb{Z}/2^2)^{\oplus 68}$
94	$(\mathbb{Z}/2)^{\oplus 102221} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 93852} \oplus (\mathbb{Z}/2^2)^{\oplus 35}$
95	$(\mathbb{Z}/2)^{\oplus 111208} \oplus (\mathbb{Z}/2^2)^{\oplus 43}$	$(\mathbb{Z}/2)^{\oplus 102221} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
96	$(\mathbb{Z}/2)^{\oplus 120954} \oplus (\mathbb{Z}/2^2)^{\oplus 77} \oplus \mathbb{Z}/2^3 \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 111208} \oplus (\mathbb{Z}/2^2)^{\oplus 43}$
97	$(\mathbb{Z}/2)^{\oplus 131556} \oplus (\mathbb{Z}/2^2)^{\oplus 39} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 120954} \oplus (\mathbb{Z}/2^2)^{\oplus 77} \oplus \mathbb{Z}/2^3 \oplus \mathbb{Z}/2^5$
98	$(\mathbb{Z}/2)^{\oplus 143004} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 131556} \oplus (\mathbb{Z}/2^2)^{\oplus 39} \oplus \mathbb{Z}/2^3$
99	$(\mathbb{Z}/2)^{\oplus 155317} \oplus (\mathbb{Z}/2^2)^{\oplus 48}$	$(\mathbb{Z}/2)^{\oplus 143004} \oplus (\mathbb{Z}/2^2)^{\oplus 4}$
100	$(\mathbb{Z}/2)^{\oplus 168602} \oplus (\mathbb{Z}/2^2)^{\oplus 87}$	$(\mathbb{Z}/2)^{\oplus 155317} \oplus (\mathbb{Z}/2^2)^{\oplus 48}$
101	$(\mathbb{Z}/2)^{\oplus 183027} \oplus (\mathbb{Z}/2^2)^{\oplus 45}$	$(\mathbb{Z}/2)^{\oplus 168602} \oplus (\mathbb{Z}/2^2)^{\oplus 87}$
102	$(\mathbb{Z}/2)^{\oplus 198610} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$	$(\mathbb{Z}/2)^{\oplus 183027} \oplus (\mathbb{Z}/2^2)^{\oplus 45}$
103	$(\mathbb{Z}/2)^{\oplus 215315} \oplus (\mathbb{Z}/2^2)^{\oplus 52}$	$(\mathbb{Z}/2)^{\oplus 198610} \oplus (\mathbb{Z}/2^2)^{\oplus 6}$
104	$(\mathbb{Z}/2)^{\oplus 233327} \oplus (\mathbb{Z}/2^2)^{\oplus 97} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 215315} \oplus (\mathbb{Z}/2^2)^{\oplus 52}$
105	$(\mathbb{Z}/2)^{\oplus 252860} \oplus (\mathbb{Z}/2^2)^{\oplus 54} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 233327} \oplus (\mathbb{Z}/2^2)^{\oplus 97} \oplus \mathbb{Z}/2^3$
106	$(\mathbb{Z}/2)^{\oplus 273887} \oplus (\mathbb{Z}/2^2)^{\oplus 8}$	$(\mathbb{Z}/2)^{\oplus 252860} \oplus (\mathbb{Z}/2^2)^{\oplus 54} \oplus \mathbb{Z}/2^3$
107	$(\mathbb{Z}/2)^{\oplus 296434} \oplus (\mathbb{Z}/2^2)^{\oplus 58}$	$(\mathbb{Z}/2)^{\oplus 273887} \oplus (\mathbb{Z}/2^2)^{\oplus 8}$
108	$(\mathbb{Z}/2)^{\oplus 320733} \oplus (\mathbb{Z}/2^2)^{\oplus 113}$	$(\mathbb{Z}/2)^{\oplus 296434} \oplus (\mathbb{Z}/2^2)^{\oplus 58}$
109	$(\mathbb{Z}/2)^{\oplus 346976} \oplus (\mathbb{Z}/2^2)^{\oplus 63}$	$(\mathbb{Z}/2)^{\oplus 320733} \oplus (\mathbb{Z}/2^2)^{\oplus 113}$
110	$(\mathbb{Z}/2)^{\oplus 375207} \oplus (\mathbb{Z}/2^2)^{\oplus 9}$	$(\mathbb{Z}/2)^{\oplus 346976} \oplus (\mathbb{Z}/2^2)^{\oplus 63}$
111	$(\mathbb{Z}/2)^{\oplus 405487} \oplus (\mathbb{Z}/2^2)^{\oplus 65}$	$(\mathbb{Z}/2)^{\oplus 375207} \oplus (\mathbb{Z}/2^2)^{\oplus 9}$
112	$(\mathbb{Z}/2)^{\oplus 438020} \oplus (\mathbb{Z}/2^2)^{\oplus 122} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 405487} \oplus (\mathbb{Z}/2^2)^{\oplus 65}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
113	$(\mathbb{Z}/2)^{\oplus 473101} \oplus (\mathbb{Z}/2^2)^{\oplus 69} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 438020} \oplus (\mathbb{Z}/2^2)^{\oplus 122} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
114	$(\mathbb{Z}/2)^{\oplus 510827} \oplus (\mathbb{Z}/2^2)^{\oplus 11}$	$(\mathbb{Z}/2)^{\oplus 473101} \oplus (\mathbb{Z}/2^2)^{\oplus 69} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
115	$(\mathbb{Z}/2)^{\oplus 551200} \oplus (\mathbb{Z}/2^2)^{\oplus 70}$	$(\mathbb{Z}/2)^{\oplus 510827} \oplus (\mathbb{Z}/2^2)^{\oplus 11}$
116	$(\mathbb{Z}/2)^{\oplus 594538} \oplus (\mathbb{Z}/2^2)^{\oplus 138}$	$(\mathbb{Z}/2)^{\oplus 551200} \oplus (\mathbb{Z}/2^2)^{\oplus 70}$
117	$(\mathbb{Z}/2)^{\oplus 641221} \oplus (\mathbb{Z}/2^2)^{\oplus 83}$	$(\mathbb{Z}/2)^{\oplus 594538} \oplus (\mathbb{Z}/2^2)^{\oplus 138}$
118	$(\mathbb{Z}/2)^{\oplus 691296} \oplus (\mathbb{Z}/2^2)^{\oplus 14}$	$(\mathbb{Z}/2)^{\oplus 641221} \oplus (\mathbb{Z}/2^2)^{\oplus 83}$
119	$(\mathbb{Z}/2)^{\oplus 744868} \oplus (\mathbb{Z}/2^2)^{\oplus 77}$	$(\mathbb{Z}/2)^{\oplus 691296} \oplus (\mathbb{Z}/2^2)^{\oplus 14}$
120	$(\mathbb{Z}/2)^{\oplus 802340} \oplus (\mathbb{Z}/2^2)^{\oplus 153} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 744868} \oplus (\mathbb{Z}/2^2)^{\oplus 77}$
121	$(\mathbb{Z}/2)^{\oplus 864071} \oplus (\mathbb{Z}/2^2)^{\oplus 93} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 802340} \oplus (\mathbb{Z}/2^2)^{\oplus 153} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$
122	$(\mathbb{Z}/2)^{\oplus 930229} \oplus (\mathbb{Z}/2^2)^{\oplus 16}$	$(\mathbb{Z}/2)^{\oplus 864071} \oplus (\mathbb{Z}/2^2)^{\oplus 93} \oplus \mathbb{Z}/2^3$
123	$(\mathbb{Z}/2)^{\oplus 1000996} \oplus (\mathbb{Z}/2^2)^{\oplus 85}$	$(\mathbb{Z}/2)^{\oplus 930229} \oplus (\mathbb{Z}/2^2)^{\oplus 16}$
124	$(\mathbb{Z}/2)^{\oplus 1076751} \oplus (\mathbb{Z}/2^2)^{\oplus 170}$	$(\mathbb{Z}/2)^{\oplus 1000996} \oplus (\mathbb{Z}/2^2)^{\oplus 85}$
125	$(\mathbb{Z}/2)^{\oplus 1158010} \oplus (\mathbb{Z}/2^2)^{\oplus 106}$	$(\mathbb{Z}/2)^{\oplus 1076751} \oplus (\mathbb{Z}/2^2)^{\oplus 170}$
126	$(\mathbb{Z}/2)^{\oplus 1245041} \oplus (\mathbb{Z}/2^2)^{\oplus 20}$	$(\mathbb{Z}/2)^{\oplus 1158010} \oplus (\mathbb{Z}/2^2)^{\oplus 106}$
127	$(\mathbb{Z}/2)^{\oplus 1337988} \oplus (\mathbb{Z}/2^2)^{\oplus 91}$	$(\mathbb{Z}/2)^{\oplus 1245041} \oplus (\mathbb{Z}/2^2)^{\oplus 20}$
128	$(\mathbb{Z}/2)^{\oplus 1437384} \oplus (\mathbb{Z}/2^2)^{\oplus 188} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 1337988} \oplus (\mathbb{Z}/2^2)^{\oplus 91}$
129	$(\mathbb{Z}/2)^{\oplus 1543899} \oplus (\mathbb{Z}/2^2)^{\oplus 122} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 1437384} \oplus (\mathbb{Z}/2^2)^{\oplus 188} \oplus \mathbb{Z}/2^4$
130	$(\mathbb{Z}/2)^{\oplus 1657779} \oplus (\mathbb{Z}/2^2)^{\oplus 24}$	$(\mathbb{Z}/2)^{\oplus 1543899} \oplus (\mathbb{Z}/2^2)^{\oplus 122} \oplus \mathbb{Z}/2^4$
131	$(\mathbb{Z}/2)^{\oplus 1779311} \oplus (\mathbb{Z}/2^2)^{\oplus 100}$	$(\mathbb{Z}/2)^{\oplus 1657779} \oplus (\mathbb{Z}/2^2)^{\oplus 24}$
132	$(\mathbb{Z}/2)^{\oplus 1909190} \oplus (\mathbb{Z}/2^2)^{\oplus 212}$	$(\mathbb{Z}/2)^{\oplus 1779311} \oplus (\mathbb{Z}/2^2)^{\oplus 100}$
133	$(\mathbb{Z}/2)^{\oplus 2048100} \oplus (\mathbb{Z}/2^2)^{\oplus 137}$	$(\mathbb{Z}/2)^{\oplus 1909190} \oplus (\mathbb{Z}/2^2)^{\oplus 212}$
134	$(\mathbb{Z}/2)^{\oplus 2196474} \oplus (\mathbb{Z}/2^2)^{\oplus 26}$	$(\mathbb{Z}/2)^{\oplus 2048100} \oplus (\mathbb{Z}/2^2)^{\oplus 137}$
135	$(\mathbb{Z}/2)^{\oplus 2354741} \oplus (\mathbb{Z}/2^2)^{\oplus 110}$	$(\mathbb{Z}/2)^{\oplus 2196474} \oplus (\mathbb{Z}/2^2)^{\oplus 26}$
136	$(\mathbb{Z}/2)^{\oplus 2523611} \oplus (\mathbb{Z}/2^2)^{\oplus 227} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 2354741} \oplus (\mathbb{Z}/2^2)^{\oplus 110}$
137	$(\mathbb{Z}/2)^{\oplus 2704017} \oplus (\mathbb{Z}/2^2)^{\oplus 148} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 2523611} \oplus (\mathbb{Z}/2^2)^{\oplus 227} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$
138	$(\mathbb{Z}/2)^{\oplus 2896569} \oplus (\mathbb{Z}/2^2)^{\oplus 31} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 2704017} \oplus (\mathbb{Z}/2^2)^{\oplus 148} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$
139	$(\mathbb{Z}/2)^{\oplus 3101698} \oplus (\mathbb{Z}/2^2)^{\oplus 119}$	$(\mathbb{Z}/2)^{\oplus 2896569} \oplus (\mathbb{Z}/2^2)^{\oplus 31} \oplus \mathbb{Z}/2^3$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
140	$(\mathbb{Z}/2)^{\oplus 3320370} \oplus (\mathbb{Z}/2^2)^{\oplus 253}$	$(\mathbb{Z}/2)^{\oplus 3101698} \oplus (\mathbb{Z}/2^2)^{\oplus 119}$
141	$(\mathbb{Z}/2)^{\oplus 3553765} \oplus (\mathbb{Z}/2^2)^{\oplus 173}$	$(\mathbb{Z}/2)^{\oplus 3320370} \oplus (\mathbb{Z}/2^2)^{\oplus 253}$
142	$(\mathbb{Z}/2)^{\oplus 3802528} \oplus (\mathbb{Z}/2^2)^{\oplus 39}$	$(\mathbb{Z}/2)^{\oplus 3553765} \oplus (\mathbb{Z}/2^2)^{\oplus 173}$
143	$(\mathbb{Z}/2)^{\oplus 4067338} \oplus (\mathbb{Z}/2^2)^{\oplus 130}$	$(\mathbb{Z}/2)^{\oplus 3802528} \oplus (\mathbb{Z}/2^2)^{\oplus 39}$
144	$(\mathbb{Z}/2)^{\oplus 4349432} \oplus (\mathbb{Z}/2^2)^{\oplus 277} \oplus (\mathbb{Z}/2^3)^{\oplus 2} \oplus (\mathbb{Z}/2^4)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 4067338} \oplus (\mathbb{Z}/2^2)^{\oplus 130}$
145	$(\mathbb{Z}/2)^{\oplus 4650079} \oplus (\mathbb{Z}/2^2)^{\oplus 191} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 4349432} \oplus (\mathbb{Z}/2^2)^{\oplus 277} \oplus (\mathbb{Z}/2^3)^{\oplus 2} \oplus (\mathbb{Z}/2^4)^{\oplus 2}$
146	$(\mathbb{Z}/2)^{\oplus 4970225} \oplus (\mathbb{Z}/2^2)^{\oplus 43}$	$(\mathbb{Z}/2)^{\oplus 4650079} \oplus (\mathbb{Z}/2^2)^{\oplus 191} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
147	$(\mathbb{Z}/2)^{\oplus 5310821} \oplus (\mathbb{Z}/2^2)^{\oplus 142}$	$(\mathbb{Z}/2)^{\oplus 4970225} \oplus (\mathbb{Z}/2^2)^{\oplus 43}$
148	$(\mathbb{Z}/2)^{\oplus 5673205} \oplus (\mathbb{Z}/2^2)^{\oplus 305}$	$(\mathbb{Z}/2)^{\oplus 5310821} \oplus (\mathbb{Z}/2^2)^{\oplus 142}$
149	$(\mathbb{Z}/2)^{\oplus 6059028} \oplus (\mathbb{Z}/2^2)^{\oplus 213}$	$(\mathbb{Z}/2)^{\oplus 5673205} \oplus (\mathbb{Z}/2^2)^{\oplus 305}$
150	$(\mathbb{Z}/2)^{\oplus 6469570} \oplus (\mathbb{Z}/2^2)^{\oplus 50}$	$(\mathbb{Z}/2)^{\oplus 6059028} \oplus (\mathbb{Z}/2^2)^{\oplus 213}$
151	$(\mathbb{Z}/2)^{\oplus 6905872} \oplus (\mathbb{Z}/2^2)^{\oplus 153}$	$(\mathbb{Z}/2)^{\oplus 6469570} \oplus (\mathbb{Z}/2^2)^{\oplus 50}$
152	$(\mathbb{Z}/2)^{\oplus 7369685} \oplus (\mathbb{Z}/2^2)^{\oplus 332} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 6905872} \oplus (\mathbb{Z}/2^2)^{\oplus 153}$
153	$(\mathbb{Z}/2)^{\oplus 7863087} \oplus (\mathbb{Z}/2^2)^{\oplus 238} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 7369685} \oplus (\mathbb{Z}/2^2)^{\oplus 332} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
154	$(\mathbb{Z}/2)^{\oplus 8387507} \oplus (\mathbb{Z}/2^2)^{\oplus 58}$	$(\mathbb{Z}/2)^{\oplus 7863087} \oplus (\mathbb{Z}/2^2)^{\oplus 238} \oplus (\mathbb{Z}/2^3)^{\oplus 2}$
155	$(\mathbb{Z}/2)^{\oplus 8944415} \oplus (\mathbb{Z}/2^2)^{\oplus 166}$	$(\mathbb{Z}/2)^{\oplus 8387507} \oplus (\mathbb{Z}/2^2)^{\oplus 58}$
156	$(\mathbb{Z}/2)^{\oplus 9536013} \oplus (\mathbb{Z}/2^2)^{\oplus 368}$	$(\mathbb{Z}/2)^{\oplus 8944415} \oplus (\mathbb{Z}/2^2)^{\oplus 166}$
157	$(\mathbb{Z}/2)^{\oplus 10164626} \oplus (\mathbb{Z}/2^2)^{\oplus 265}$	$(\mathbb{Z}/2)^{\oplus 9536013} \oplus (\mathbb{Z}/2^2)^{\oplus 368}$
158	$(\mathbb{Z}/2)^{\oplus 10832194} \oplus (\mathbb{Z}/2^2)^{\oplus 64}$	$(\mathbb{Z}/2)^{\oplus 10164626} \oplus (\mathbb{Z}/2^2)^{\oplus 265}$
159	$(\mathbb{Z}/2)^{\oplus 11540663} \oplus (\mathbb{Z}/2^2)^{\oplus 181}$	$(\mathbb{Z}/2)^{\oplus 10832194} \oplus (\mathbb{Z}/2^2)^{\oplus 64}$
160	$(\mathbb{Z}/2)^{\oplus 12292511} \oplus (\mathbb{Z}/2^2)^{\oplus 394} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^5$	$(\mathbb{Z}/2)^{\oplus 11540663} \oplus (\mathbb{Z}/2^2)^{\oplus 181}$
161	$(\mathbb{Z}/2)^{\oplus 13090677} \oplus (\mathbb{Z}/2^2)^{\oplus 287} \oplus (\mathbb{Z}/2^3)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 12292511} \oplus (\mathbb{Z}/2^2)^{\oplus 394} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^5$
162	$(\mathbb{Z}/2)^{\oplus 13937701} \oplus (\mathbb{Z}/2^2)^{\oplus 74} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 13090677} \oplus (\mathbb{Z}/2^2)^{\oplus 287} \oplus (\mathbb{Z}/2^3)^{\oplus 4}$
163	$(\mathbb{Z}/2)^{\oplus 14835804} \oplus (\mathbb{Z}/2^2)^{\oplus 194}$	$(\mathbb{Z}/2)^{\oplus 13937701} \oplus (\mathbb{Z}/2^2)^{\oplus 74} \oplus \mathbb{Z}/2^3$
164	$(\mathbb{Z}/2)^{\oplus 15788138} \oplus (\mathbb{Z}/2^2)^{\oplus 433}$	$(\mathbb{Z}/2)^{\oplus 14835804} \oplus (\mathbb{Z}/2^2)^{\oplus 194}$
165	$(\mathbb{Z}/2)^{\oplus 16798377} \oplus (\mathbb{Z}/2^2)^{\oplus 326}$	$(\mathbb{Z}/2)^{\oplus 15788138} \oplus (\mathbb{Z}/2^2)^{\oplus 433}$
166	$(\mathbb{Z}/2)^{\oplus 17869438} \oplus (\mathbb{Z}/2^2)^{\oplus 86}$	$(\mathbb{Z}/2)^{\oplus 16798377} \oplus (\mathbb{Z}/2^2)^{\oplus 326}$
167	$(\mathbb{Z}/2)^{\oplus 19004263} \oplus (\mathbb{Z}/2^2)^{\oplus 209}$	$(\mathbb{Z}/2)^{\oplus 17869438} \oplus (\mathbb{Z}/2^2)^{\oplus 86}$
168	$(\mathbb{Z}/2)^{\oplus 20206790} \oplus (\mathbb{Z}/2^2)^{\oplus 471} \oplus (\mathbb{Z}/2^3)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 19004263} \oplus (\mathbb{Z}/2^2)^{\oplus 209}$

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n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
169	$(\mathbb{Z}/2)^{\oplus 21481229} \oplus (\mathbb{Z}/2^2)^{\oplus 356} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$	$(\mathbb{Z}/2)^{\oplus 20206790} \oplus (\mathbb{Z}/2^2)^{\oplus 471} \oplus (\mathbb{Z}/2^3)^{\oplus 4}$
170	$(\mathbb{Z}/2)^{\oplus 22831342} \oplus (\mathbb{Z}/2^2)^{\oplus 94}$	$(\mathbb{Z}/2)^{\oplus 21481229} \oplus (\mathbb{Z}/2^2)^{\oplus 356} \oplus (\mathbb{Z}/2^3)^{\oplus 3}$
171	$(\mathbb{Z}/2)^{\oplus 24260913} \oplus (\mathbb{Z}/2^2)^{\oplus 227}$	$(\mathbb{Z}/2)^{\oplus 22831342} \oplus (\mathbb{Z}/2^2)^{\oplus 94}$
172	$(\mathbb{Z}/2)^{\oplus 25774489} \oplus (\mathbb{Z}/2^2)^{\oplus 513}$	$(\mathbb{Z}/2)^{\oplus 24260913} \oplus (\mathbb{Z}/2^2)^{\oplus 227}$
173	$(\mathbb{Z}/2)^{\oplus 27377294} \oplus (\mathbb{Z}/2^2)^{\oplus 392}$	$(\mathbb{Z}/2)^{\oplus 25774489} \oplus (\mathbb{Z}/2^2)^{\oplus 513}$
174	$(\mathbb{Z}/2)^{\oplus 29074110} \oplus (\mathbb{Z}/2^2)^{\oplus 107}$	$(\mathbb{Z}/2)^{\oplus 27377294} \oplus (\mathbb{Z}/2^2)^{\oplus 392}$
175	$(\mathbb{Z}/2)^{\oplus 30869367} \oplus (\mathbb{Z}/2^2)^{\oplus 244}$	$(\mathbb{Z}/2)^{\oplus 29074110} \oplus (\mathbb{Z}/2^2)^{\oplus 107}$
176	$(\mathbb{Z}/2)^{\oplus 32768737} \oplus (\mathbb{Z}/2^2)^{\oplus 553} \oplus (\mathbb{Z}/2^3)^{\oplus 2} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 30869367} \oplus (\mathbb{Z}/2^2)^{\oplus 244}$
177	$(\mathbb{Z}/2)^{\oplus 34778668} \oplus (\mathbb{Z}/2^2)^{\oplus 430} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 32768737} \oplus (\mathbb{Z}/2^2)^{\oplus 553} \oplus (\mathbb{Z}/2^3)^{\oplus 2} \oplus \mathbb{Z}/2^4$
178	$(\mathbb{Z}/2)^{\oplus 36904754} \oplus (\mathbb{Z}/2^2)^{\oplus 121} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 34778668} \oplus (\mathbb{Z}/2^2)^{\oplus 430} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4$
179	$(\mathbb{Z}/2)^{\oplus 39152657} \oplus (\mathbb{Z}/2^2)^{\oplus 263}$	$(\mathbb{Z}/2)^{\oplus 36904754} \oplus (\mathbb{Z}/2^2)^{\oplus 121} \oplus \mathbb{Z}/2^3$
180	$(\mathbb{Z}/2)^{\oplus 41529387} \oplus (\mathbb{Z}/2^2)^{\oplus 606}$	$(\mathbb{Z}/2)^{\oplus 39152657} \oplus (\mathbb{Z}/2^2)^{\oplus 263}$
181	$(\mathbb{Z}/2)^{\oplus 44042423} \oplus (\mathbb{Z}/2^2)^{\oplus 475}$	$(\mathbb{Z}/2)^{\oplus 41529387} \oplus (\mathbb{Z}/2^2)^{\oplus 606}$
182	$(\mathbb{Z}/2)^{\oplus 46698801} \oplus (\mathbb{Z}/2^2)^{\oplus 135}$	$(\mathbb{Z}/2)^{\oplus 44042423} \oplus (\mathbb{Z}/2^2)^{\oplus 475}$
183	$(\mathbb{Z}/2)^{\oplus 49505658} \oplus (\mathbb{Z}/2^2)^{\oplus 284}$	$(\mathbb{Z}/2)^{\oplus 46698801} \oplus (\mathbb{Z}/2^2)^{\oplus 135}$
184	$(\mathbb{Z}/2)^{\oplus 52471165} \oplus (\mathbb{Z}/2^2)^{\oplus 646} \oplus (\mathbb{Z}/2^3)^{\oplus 5}$	$(\mathbb{Z}/2)^{\oplus 49505658} \oplus (\mathbb{Z}/2^2)^{\oplus 284}$
185	$(\mathbb{Z}/2)^{\oplus 55604497} \oplus (\mathbb{Z}/2^2)^{\oplus 512} \oplus (\mathbb{Z}/2^3)^{\oplus 6}$	$(\mathbb{Z}/2)^{\oplus 52471165} \oplus (\mathbb{Z}/2^2)^{\oplus 646} \oplus (\mathbb{Z}/2^3)^{\oplus 5}$
186	$(\mathbb{Z}/2)^{\oplus 58914436} \oplus (\mathbb{Z}/2^2)^{\oplus 151} \oplus \mathbb{Z}/2^3$	$(\mathbb{Z}/2)^{\oplus 55604497} \oplus (\mathbb{Z}/2^2)^{\oplus 512} \oplus (\mathbb{Z}/2^3)^{\oplus 6}$
187	$(\mathbb{Z}/2)^{\oplus 62409413} \oplus (\mathbb{Z}/2^2)^{\oplus 303}$	$(\mathbb{Z}/2)^{\oplus 58914436} \oplus (\mathbb{Z}/2^2)^{\oplus 151} \oplus \mathbb{Z}/2^3$
188	$(\mathbb{Z}/2)^{\oplus 66099472} \oplus (\mathbb{Z}/2^2)^{\oplus 703}$	$(\mathbb{Z}/2)^{\oplus 62409413} \oplus (\mathbb{Z}/2^2)^{\oplus 303}$
189	$(\mathbb{Z}/2)^{\oplus 69995847} \oplus (\mathbb{Z}/2^2)^{\oplus 571}$	$(\mathbb{Z}/2)^{\oplus 66099472} \oplus (\mathbb{Z}/2^2)^{\oplus 703}$
190	$(\mathbb{Z}/2)^{\oplus 74108902} \oplus (\mathbb{Z}/2^2)^{\oplus 171}$	$(\mathbb{Z}/2)^{\oplus 69995847} \oplus (\mathbb{Z}/2^2)^{\oplus 571}$
191	$(\mathbb{Z}/2)^{\oplus 78449123} \oplus (\mathbb{Z}/2^2)^{\oplus 325}$	$(\mathbb{Z}/2)^{\oplus 74108902} \oplus (\mathbb{Z}/2^2)^{\oplus 171}$
192	$(\mathbb{Z}/2)^{\oplus 83028830} \oplus (\mathbb{Z}/2^2)^{\oplus 758} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4 \oplus \mathbb{Z}/2^6$	$(\mathbb{Z}/2)^{\oplus 78449123} \oplus (\mathbb{Z}/2^2)^{\oplus 325}$
193	$(\mathbb{Z}/2)^{\oplus 87861181} \oplus (\mathbb{Z}/2^2)^{\oplus 619} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4$	$(\mathbb{Z}/2)^{\oplus 83028830} \oplus (\mathbb{Z}/2^2)^{\oplus 758} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4 \oplus \mathbb{Z}/2^6$
194	$(\mathbb{Z}/2)^{\oplus 92958948} \oplus (\mathbb{Z}/2^2)^{\oplus 187}$	$(\mathbb{Z}/2)^{\oplus 87861181} \oplus (\mathbb{Z}/2^2)^{\oplus 619} \oplus (\mathbb{Z}/2^3)^{\oplus 3} \oplus \mathbb{Z}/2^4$

to be continued on the next page

n	$H_n(-,\mathbb{Z})$	$H^n(-,\mathbb{Z})$
195	$(\mathbb{Z}/2)^{\oplus 98335127} \oplus (\mathbb{Z}/2^2)^{\oplus 350}$	$(\mathbb{Z}/2)^{\oplus 92958948} \oplus (\mathbb{Z}/2^2)^{\oplus 187}$
196	$(\mathbb{Z}/2)^{\oplus 104004195} \oplus (\mathbb{Z}/2^2)^{\oplus 818}$	$(\mathbb{Z}/2)^{\oplus 98335127} \oplus (\mathbb{Z}/2^2)^{\oplus 350}$
197	$(\mathbb{Z}/2)^{\oplus 109982143} \oplus (\mathbb{Z}/2^2)^{\oplus 675}$	$(\mathbb{Z}/2)^{\oplus 104004195} \oplus (\mathbb{Z}/2^2)^{\oplus 818}$
198	$(\mathbb{Z}/2)^{\oplus 116284659} \oplus (\mathbb{Z}/2^2)^{\oplus 209}$	$(\mathbb{Z}/2)^{\oplus 109982143} \oplus (\mathbb{Z}/2^2)^{\oplus 675}$
199	$(\mathbb{Z}/2)^{\oplus 122927163} \oplus (\mathbb{Z}/2^2)^{\oplus 373}$	$(\mathbb{Z}/2)^{\oplus 116284659} \oplus (\mathbb{Z}/2^2)^{\oplus 209}$
200	$(\mathbb{Z}/2)^{\oplus 129919716} \oplus (\mathbb{Z}/2^2)^{\oplus 860} \oplus (\mathbb{Z}/2^3)^{\oplus 4}$	$(\mathbb{Z}/2)^{\oplus 122927163} \oplus (\mathbb{Z}/2^2)^{\oplus 373}$

Generators involved in the calculus.

Degree	Genus	Generator
3	1	$(\sigma^3, \sigma^2\psi_{2^2})$
5	2	$(\beta_2\varphi_2\sigma^2, \varphi_2\sigma^2)$
6	3	$(\beta_2\sigma\varphi_2\psi_2, \sigma\varphi_2\psi_2)$
9	2	$(\beta_2\varphi_2\gamma_2\sigma^2, \varphi_2\gamma_2\sigma^2)$
10	3	$(\beta_2\sigma\varphi_2\gamma_2\psi_2, \sigma\varphi_2\gamma_2\psi_2)$
13	3	$(\beta_2\varphi_2^2\psi_2, \varphi_2^2\psi_2)$
17	2	$(\beta_2\varphi_2\gamma_2^2\sigma^2, \varphi_2\gamma_2^2\sigma^2)$
18	3	$(\beta_2\sigma\varphi_2\gamma_2^2\psi_2, \sigma\varphi_2\gamma_2^2\psi_2)$
21	3	$(\beta_2\varphi_2^2\gamma_2\psi_2, \varphi_2^2\gamma_2\psi_2)$
25	3	$(\beta_2\varphi_2\gamma_2\varphi_2\psi_2, \varphi_2\gamma_2\varphi_2\psi_2)$
33	2	$(\beta_2\varphi_2\gamma_2^3\sigma^2, \varphi_2\gamma_2^3\sigma^2)$
34	3	$(\beta_2\sigma\varphi_2\gamma_2^3\psi_2, \sigma\varphi_2\gamma_2^3\psi_2)$
37	3	$(\beta_2\varphi_2^2\gamma_2^2\psi_2, \varphi_2^2\gamma_2^2\psi_2)$
41	3	$(\beta_2\varphi_2\gamma_2\varphi_2\gamma_2\psi_2, \varphi_2\gamma_2\varphi_2\gamma_2\psi_2)$
49	3	$(\beta_2\varphi_2\gamma_2^2\varphi_2\psi_2, \varphi_2\gamma_2^2\varphi_2\psi_2)$
65	2	$(\beta_2\varphi_2\gamma_2^4\sigma^2, \varphi_2\gamma_2^4\sigma^2)$
66	3	$(\beta_2\sigma\varphi_2\gamma_2^4\psi_2, \sigma\varphi_2\gamma_2^4\psi_2)$
69	3	$(\beta_2\varphi_2^2\gamma_2^3\psi_2, \varphi_2^2\gamma_2^3\psi_2)$
73	3	$(\beta_2\varphi_2\gamma_2\varphi_2\gamma_2^2\psi_2, \varphi_2\gamma_2\varphi_2\gamma_2^2\psi_2)$
81	3	$(\beta_2\varphi_2\gamma_2^2\varphi_2\gamma_2\psi_2, \varphi_2\gamma_2^2\varphi_2\gamma_2\psi_2)$
97	3	$(\beta_2\varphi_2\gamma_2^3\varphi_2\psi_2, \varphi_2\gamma_2^3\varphi_2\psi_2)$
129	2	$(\beta_2\varphi_2\gamma_2^5\sigma^2, \varphi_2\gamma_2^5\sigma^2)$
130	3	$(\beta_2\sigma\varphi_2\gamma_2^5\psi_2, \sigma\varphi_2\gamma_2^5\psi_2)$
133	3	$(\beta_2\varphi_2^2\gamma_2^4\psi_2, \varphi_2^2\gamma_2^4\psi_2)$
137	3	$(\beta_2\varphi_2\gamma_2\varphi_2\gamma_2^3\psi_2, \varphi_2\gamma_2\varphi_2\gamma_2^3\psi_2)$
145	3	$(\beta_2\varphi_2\gamma_2^2\varphi_2\gamma_2^2\psi_2, \varphi_2\gamma_2^2\varphi_2\gamma_2^2\psi_2)$
161	3	$(\beta_2\varphi_2\gamma_2^3\varphi_2\gamma_2\psi_2, \varphi_2\gamma_2^3\varphi_2\gamma_2\psi_2)$
193	3	$(\beta_2\varphi_2\gamma_2^4\varphi_2\psi_2, \varphi_2\gamma_2^4\varphi_2\psi_2)$

APPENDIX D

C++ implementation of the "Eilenberg-MacLane machine"

D.1. Header file Mod2GradedGroups.hpp

```
/*
 *
 *      2-Local graded groups
 *
 *      Mod2GradedGroups.hpp
 *      Version 2.1, 31-Jan-2001, 02:27
 *      Alain Clement <alain.clement@ima.unil.ch>
 *
 *      Tools for graded groups.
 *
 */


#ifndef _Mod2GradedGroups
#define _Mod2GradedGroups

#include <vector>

using namespace std;

class PoweredCyclicGroup
{
    public:
        int logorder;
        long int power;
        void TeXShow(ostream&);
        int exp();
};
PoweredCyclicGroup tensor(
    PoweredCyclicGroup,
    PoweredCyclicGroup);
PoweredCyclicGroup Tor(
    PoweredCyclicGroup,
    PoweredCyclicGroup);
PoweredCyclicGroup Hom(
    PoweredCyclicGroup,
    PoweredCyclicGroup);
```

```
PoweredCyclicGroup Ext(
    PoweredCyclicGroup,
    PoweredCyclicGroup);

class Group
{
    public:
        vector<PoweredCyclicGroup> PCGroups;
        void show();
        void TeXShow(ostream&);
        void add(Group);
        void add(PoweredCyclicGroup);
        void add(int logorder, long int power);
};
Group tensor(Group,Group);
Group Tor(Group,Group);
Group Hom(Group,Group);
Group Ext(Group,Group);

class SuspendedGroup : virtual public Group
{
    public:
        int degree;
};

class GradedGroup
{
    public:
        vector<SuspendedGroup> groups;
        void add(SuspendedGroup);
        void add(Group, int degree);
        void add(int logorder, long int power, int degree);
        void Show();
        Group GroupIn(int degree);
        int anticonnexity();
};
GradedGroup Kunneth(GradedGroup,GradedGroup);
GradedGroup UniversalCoefficients(GradedGroup);

#endif
```

D.2. Header file Mod2AdmissibleSequence.hpp

```
/*
 *
 *      Mod 2 admissible sequences
 *
 *      Mod2AdmissibleSequences.hpp
 *      Version 2.1, 31-Jan-2001, 02:27
 *      Alain Clement <alain.clement@ima.unil.ch>
 *
 *      Computes mod 2 admissible sequences of
 *      given stable degree and excess.
 *
 */


#ifndef _Mod2AdmissibleSequences
#define _Mod2AdmissibleSequences

#include <vector>

using namespace std;

class IntSequence
{
    public:
        vector<int> sequence;
        void show();
        void TeXShow(ostream&);
        void append(int);
        int first();
        int last();
};

class Mod2AdmissibleSequence : virtual public IntSequence
{
    public:
        bool admissible();
        int stableDegree();
        int excess();
        int genus();
        void show();
        void TeXShow(ostream&);
        void TeXShowAsWord(ostream&,int n, int s=1);
        void TeXShowAsWordButFirst(
            ostream&,int n, int s=1);
};

class Mod2AdmissibleSequences
```

```
{
    public:
        vector<Mod2AdmissibleSequence> sequences;
        Mod2AdmissibleSequences();
        Mod2AdmissibleSequences(int stableDegree,
                                int excess);
        Mod2AdmissibleSequences(int stableDegree,
                                int minExcess,
                                int maxExcess);
        void filterByHeight(int height);
        void filterByEvenFirst();
        void show();
        void TeXShow(ostream&);
        void TeXShowAsWords(ostream&,int n, int s=1);
        void append(Mod2AdmissibleSequence);

    private:
        void iterate(
            Mod2AdmissibleSequence*,
            int stableDegree,
            int excess);
};

#endif
```

D.3. Header file Mod2Words.hpp

```
/*
 *
 *      Mod 2 words
 *
 *      Mod2Words.hpp
 *      Version 2.1, 31-Jan-2001, 02:27
 *      Alain Clement <alain.clement@ima.unil.ch>
 *
 *      Provides a framework to deal with mod-2 words.
 *
 */


#ifndef _Mod2Words
#define _Mod2Words

#include <vector>

using namespace std;

enum Mod2Letter {empty,beta_2,sigma,gamma_2,phi_2,psi_2_to_s};

class Mod2PoweredLetter
{
    public:
        Mod2Letter letter;
        int power;
};

const Mod2PoweredLetter emptyMod2PoweredLetter = {empty,0};

class Mod2Word
{
    public:
        vector<Mod2PoweredLetter> word;
        void TeXShow(ostream&,int s=1);
        void concat(Mod2Letter, int power=1);
        void concat(Mod2PoweredLetter);
        Mod2PoweredLetter first();
        Mod2PoweredLetter last();
        int genus();
};
Mod2Word firstCech(Mod2Word);

#endif
```

D.4. Header file Mod2ElementaryComplexHomology.hpp

```
/*
 *
 *      Mod-2 homology of elementary complexes
 *
 *      Mod2ElementaryComplexHomology.hpp
 *      Version 2.1, 31-Jan-2001, 02:27
 *      Alain Clement <alain.clement@ima.unil.ch>
 *
 *      Provides mod-2 homology of elementary complexes.
 *
 */


#ifndef _Mod2ElementaryComplexHomology
#define _Mod2ElementaryComplexHomology

#include <vector>
#include "Mod2GradedGroups.hpp"

using namespace std;

GradedGroup ECHomology(
    int degree,
    int h,
    int anticonnexity);
GradedGroup ECCohomology(
    int degree,
    int h,
    int anticonnexity);

#endif
```

D.5. Header file Mod2EilenbergMacLaneHomology.hpp

```
/*
 *
 *      Mod-2 Eilenberg-MacLane spaces homology
 *
 *      Mod2EilenbergMacLaneHomology.hpp
 *      Version 2.1, 31-Jan-2001, 02:27
 *      Alain Clement <alain.clement@ima.unil.ch>
 *
 *      Computes the mod-2 homology of
 *      Eilenberg-MacLane spaces.
 */


#ifndef _Mod2EilenbergMacLaneHomology
#define _Mod2EilenbergMacLaneHomology

#include <vector>
#include "Mod2AdmissibleSequences.hpp"
#include "Mod2GradedGroups.hpp"
#include "Mod2ElementaryComplexHomology.hpp"

using namespace std;

GradedGroup EMHomology(
    int logorder,
    int connexityPlusOne,
    int anticonnexity);

GradedGroup EMHomology(
    int logorder,
    int connexityPlusOne,
    int anticonnexity,
    Mod2AdmissibleSequences*);

GradedGroup EMCohomology(
    int logorder,
    int connexityPlusOne,
    int anticonnexity);

#endif
```

D.6. Header file LaTeXLayout.hpp

```
/*
 *
 *      Layout for LaTeX
 *
 *      LaTeXLayout.hpp
 *      Version 2.1, 30-Jan-2002, 17:15
 *      Alain Clement <alain.clement@ima.unil.ch>
 *
 *      Provides a layout environment for
 *      outputing results of the EMM.
 *
 */


#ifndef _LaTeXLayout
#define _LaTeXLayout

#include <iostream>
#include "Mod2GradedGroups.hpp"
#include "Mod2AdmissibleSequences.hpp"

using namespace std;

void LaTeXLayoutTopMatter(ostream&);
void LaTeXLayoutTwoColumns(ostream&,
    GradedGroup,
    GradedGroup);
void LaTeXLayoutGenerators(
    ostream&,
    Mod2AdmissibleSequences*,
    int,
    int);
void LaTeXLayoutBackMatter(ostream&);

#endif
```

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