Honors Contract Report – MAT 420

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1 Abstract

In this project, I simulated the refraction of light inside a rotating cylinder filled with a compressible medium. Due to centrifugal forces, the matter inside the cylinder will be pushed outwards to the edges of the object, changing its density with respect to distance from axis of rotation. This, in turn, changes the index of refraction, and therefore the trajectory taken by the light as it travels within the cylinder. To model the motion of the photons, I used a pair of ordinary differential equations that I then evaluated numerically using the fourth-order Runge-Kutta method. Ultimately, my results were very consistent with my expectations – the trajectories in the various situations I modelled all behaved in accordance with Snell's law, curving outwards as they transitioned from low-density to high-density media.

2 Introduction

Physically speaking, this model is founded on the principle of refraction – when light is travelling from point A to point B, it will always follow the path that takes the least amount of time to travel (Born and Wolf). Light takes longer to travel through a more dense medium due to an increased number of interactions with the particles composing it (though, at a microscopic level, it is actually travelling a longer distance due to the constant nature of the speed of light). Consequently, the line it follows curves as index of refraction changes.

The mathematical basis for this model is the following set of linked ordinary differential equations (Brown):

$$d_t^2 x = -\frac{\partial_x n}{n}(d_t x)^2 - 2\frac{\partial_y n}{n}(d_t x)(d_t y) + \frac{\partial_x n}{n}(d_t y)^2 \tag{1}$$

$$d_t^2 y = \frac{\partial_y n}{n}(d_t x)^2 - 2\frac{\partial_x n}{n}(d_t x)(d_t y) - \frac{\partial_y n}{n}(d_t y)^2 \tag{2}$$

These two second-order ODEs describe the trajectory of light through a variable index of refraction, and they can easily be rewritten into a system of four first-order ODEs that can be solved numerically. The index of refraction and its partial derivatives are described by the following equations (Thaler):

$$n(x, y) = m_c \rho(x, y) + 1, \tag{3}$$

where m_c is the material constant of the compressible substance;

$$\partial_x n = \frac{m_c \omega^2 x}{RT}\rho(x, y), \tag{4}$$

where ω is the rotational velocity of the cylinder; and

$$\partial_y n = \frac{m_c \omega^2 y}{RT}\rho(x, y), \tag{5}$$

with

$$\rho(x, y) = \rho_0 e^{\frac{\omega^2(x^2+y^2)}{2RT}}. \tag{6}$$

ρ_0 is the specific density of the substance, i.e. its density when stationary, and $R + T$ are the standard STP constants. It is worth noting that Thaler actually gives Eq. 6 for p, rather than ρ; however, since the former is constant with respect to the latter, the behavior (which is what we are concerned with) remains the same in both cases.

The reflection transformation is described by the following equation (Glassner):

$$\vec{v}_r = \vec{v}_i - 2(\vec{n} \cdot \vec{v}_i)\vec{n}, \tag{7}$$

where $\vec{v}_r$ is the reflected vector, $\vec{v}_i$ is the incident vector, and $\vec{n}$ is the vector normal to the reflective surface.

3 Methods

I wrote my model in the programming language Python, extensively using methods from the library Numpy throughout my code. Graphs were created with Matplotlib, and I used the pillow package (PIL) to create the density gradient, as shown in the next section.

For the sake of simplicity, I made a few underlying assumptions about the cylinder in question: first, it was taken as infinitely long. As the density equation only depends on x and y (i.e. r), this means that we can treat the cylinder as two-dimensional. Furthermore, the walls of the cylinder were assumed to be perfectly reflective; i.e., all incident light is reflected and none of it is transmitted. The system of first-order ODEs describing the trajectory of the light was evaluated numerically using the fourth-order Runge-Kutta method (Zeltkevic), iterating along the path of the light until it hits the wall. After being reflected, the cycle continues again until time runs out. The simulation was modelled over a period of 50 arbitrary units of time with 5000 total steps (corresponding to $h = 0.01$). I then graphed the trajectory on top of an empty circle, with the index of refraction being converted into a color gradient to demonstrate the dependence of n on r.

4 Results

4.1 Trajectory of light within a rotating cylinder

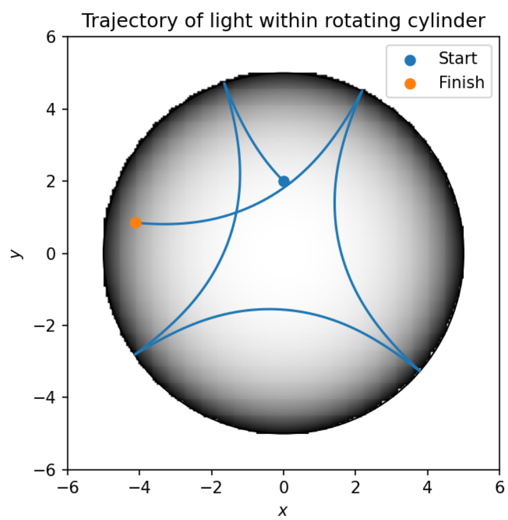

Figure 1, Trajectory within rotating cylinder.

4.2 Trajectory of light within a stationary cylinder

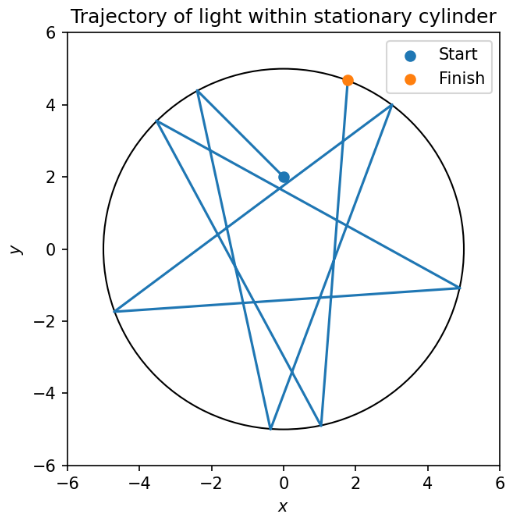

Figure 2, Trajectory within stationary cylinder.

4.3 Trajectory of light within a simple medium

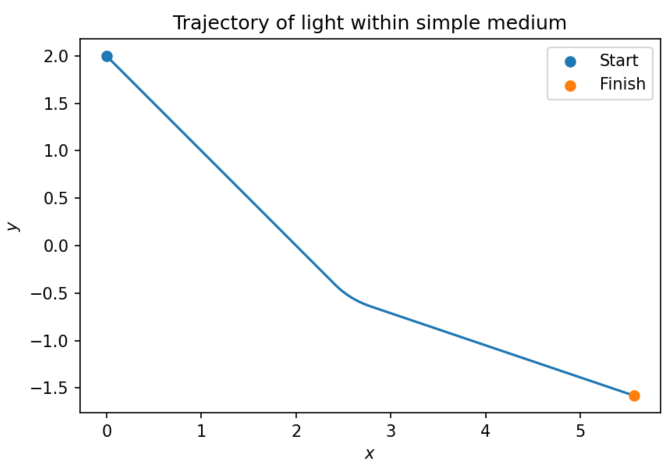

Figure 3, Trajectory within simple medium.

5 Discussion

The trajectory shown in Fig. 2, the stationary case, is straight while travelling within the cylinder and reflects sharply upon contact with its walls. As we expect from the basic principle of reflection, the incident and reflected angles are the same, thus indicating that the model's implementation of that principle is accurate. As for Fig. 3, the simple case, the model mimics the transition from a low-density to a high-density medium typically used to demonstrate the refraction of light; since it still uses a continuous density gradient, the numbers do not quite line up exactly with those given by Snell's law, but the general behavior is still the same. We can therefore be reasonably certain that the differential equations also describe reality accurately.

Fig. 1, the core result of the model, displays both of the aforementioned principles together – the trajectory both curves from refraction and bounces off the reflective cylinder walls, exactly as we expect it to. Additionally, the color gradient shown under the trajectory describing the index of refraction looks as we would expect it to. The centrifugal forces resulting from the cylinder's rotation push the material outwards, increasing its density and, by extension, the refractive index near the walls as well.

Ultimately, all three sets of results, when taken together, demonstrate that the behavior of the model is an accurate simulation of the refraction and reflection of light inside a continuous density gradient.

6 References

1. M. Zeltkevic (Apr 1998), "Runge-Kutta Methods," *Massachusetts Institute of Technology.* Retrieved from `https://web.mit.edu/10.001/Web/Course_Notes/Differential_Equations_Notes/node5.html`. Accessed Dec 2022.

2. K. Brown (Aug 2022), "Reflections on Relativity." Retrieved from `https://www.mathpages.com/rr/s8-04/8-04.htm`. Accessed Dec 2022.

3. D. Thaler (Feb 2013), "About the Atmosphere in a Rotating Cylinder." Retrieved from `https://www.foehnwall.at/meteo/rama.html`. Accessed Dec 2022.

4. M. Born and E. Wolf (1970), "Principles of Optics," *Oxford: Pergamon Press*, p. 740.

5. A. Glassner (1989), "An Introduction to Ray Tracing," *Morgan Kaufmann Publishers, Inc.*