Asymptotic Convergence of Gradient Descent for Linear Regression Least Squares Optimization

July 4, 2017

1 Supplementary Materials

This code accompanies the paper *Asymptotic Convergence of Gradient Descent for Linear Regression Least Squares Optimization* (Lipshitz, 2017)

1.1 Initialization

```
In [1]: from pylab import *
        from numpy import random as random
        random.seed(1)
        N=1000.
        w = array([14., 30.]);
        x = zeros((2, int(N))).astype(float32)
        x[0,:] = arange(N).astype(float32)
        x[1,:] = 1
        y = w.dot(x) + random.normal(size=int(N), scale=100.)
```

1.2 Defining Regression

```
In [2]: yh = lambda xs, ws: \
            ws.dot(xs)

        grad = lambda ys, yhs, xs: \
            (1./xs.shape[1])*sum((yhs-ys)*xs).astype(float32)

        delta = lambda gs, a: \
            a*gs

        def regress(y, x, alpha, T=1000, wh=None, **kwargs):

            wh = random.normal(2, size=2)
            whs = zeros((T, 2))
            whs[0,:] = wh
            for i in xrange(1,T):
                wh+=delta(grad(y,yh(x,wh), x), alpha)
```

```
            whs[i,:] = wh.copy()
        return wh, whs
```

```
In [3]: def regrSample(y, x, alpha, T=1000, N=10, **kwargs):
            out = map(
                lambda a: \
                regress(y,x, alpha, T=T), xrange(N)
            )
            trains = array([o[1] for o in out])
            wDist = array([o[0] for o in out])

            return wDist, trains

        def statsRegr(*args, **kwargs):
            wDist, trains = regrSample(*args, **kwargs)
            return np.mean(trains, axis=0), np.std(trains, axis=0)
```

1.3 Running Regression above and Below the Upper Bound on α

The theoretically derived bounds on α are

$$\alpha \in \left(-2\frac{N}{|\mathbf{x}|^2}, 0\right]$$

Other α values diverge

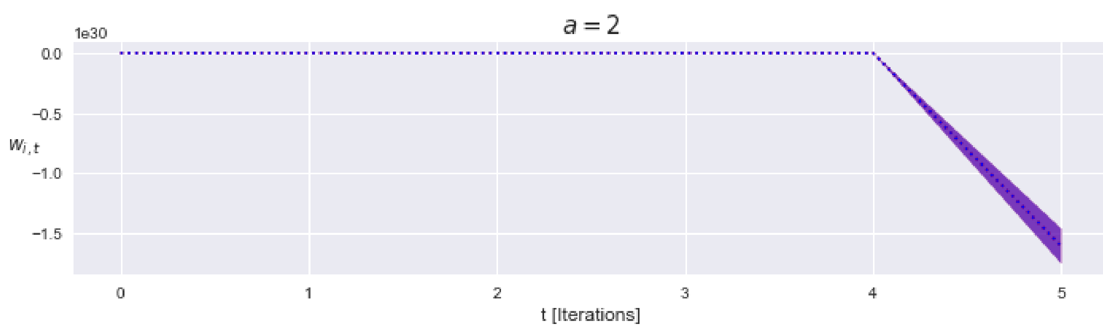
```
In [4]: def plotDynamicsForAlpha(alpha, axTitle, T=1000, N=10):
            t = np.arange(T)
            mu, sig = statsRegr(y, x, alpha, T=T, N=N)
            plot(mu[:,0], 'r:', label='$w_1$')
            plot(mu[:,1], 'b:', label='$w_2$')
            fill_between(t, \
                         mu[:,0]+sig[:,0], \
                         mu[:,0]-sig[:,0], \
                         facecolor='red', alpha=0.5)
            fill_between(t,\
                         mu[:,1]+sig[:,1], \
                         mu[:,1]-sig[:,1], \
                         facecolor='blue', alpha=0.5)
            xlabel("t [Iterations]", fontdict={'fontsize':fs*.8})
            yl = ylabel("$w_{i,t}$",fontdict={'fontsize':fs*.8})
            yl.set_rotation('horizontal')
            title(axTitle, fontdict={'fontsize':fs})
            tight_layout()
            return mu, sig
```

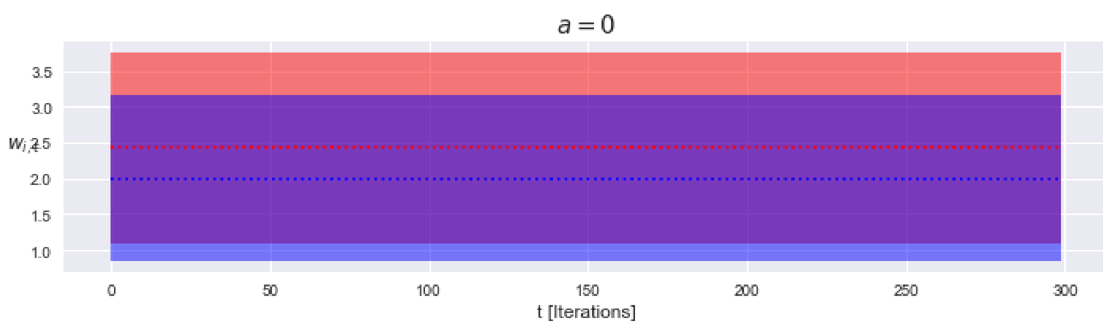
```
In [11]: alphaData = [
            ("$a=2$", 2),
            ("$a=0$",0.),
```

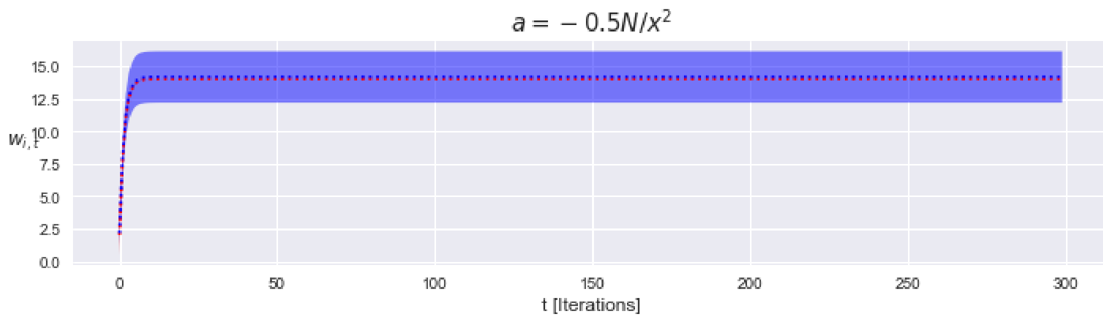
```
        ("$a=-0.5N/x^2$",-0.5*N/linalg.norm(x[0,:])**2),
        ("$a=-N/x^2$", -N/linalg.norm(x[0,:])**2),
        ("$a=-1.3N/x^2$", -1.3*N/linalg.norm(x[0,:])**2),
        ("$a=-1.6N/x^2$", -1.6*N/linalg.norm(x[0,:])**2),
        ("$a=-1.99N/x^2$", -1.99*N/linalg.norm(x[0,:])**2),
        ("$a=-2N/x^2$", -2.0*N/linalg.norm(x[0,:])**2)
    ]
```

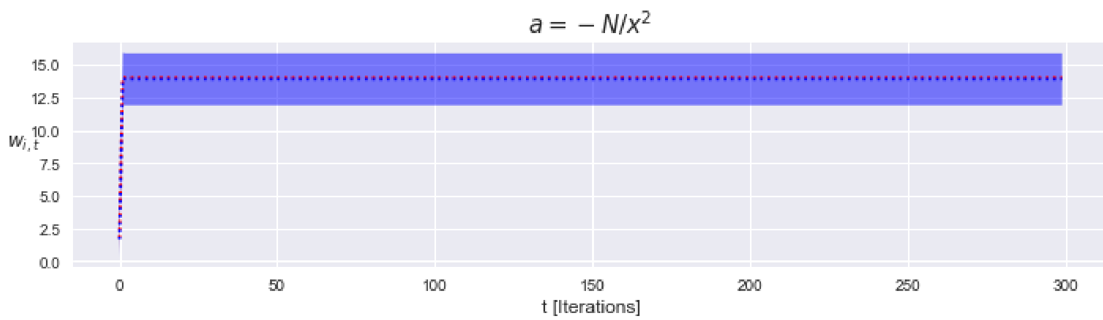
```
In [13]: %matplotlib inline
         from scipy.stats import norm
         import seaborn as sns
         fs = 15
         figure(figsize=(10,3*len(alphaData)))
         outs = []
         for i, d in enumerate(alphaData):
             k, v = d
         #     subplot(len(alphaData),1, i+1)
             figure(figsize=(10,3))
             outs.append(plotDynamicsForAlpha(v, k, T=300 ))

         tight_layout()
         # suptitle("Dynamical Learning Trajectories for Significant Alpha Values",
```

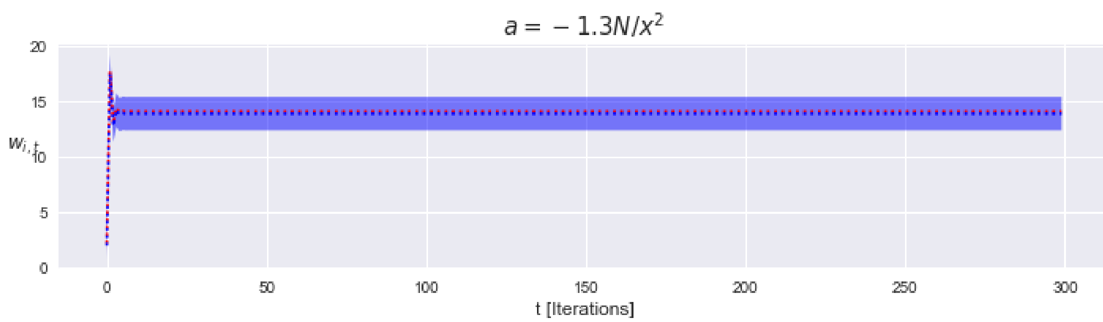
```
<matplotlib.figure.Figure at 0x116c92b10>
```

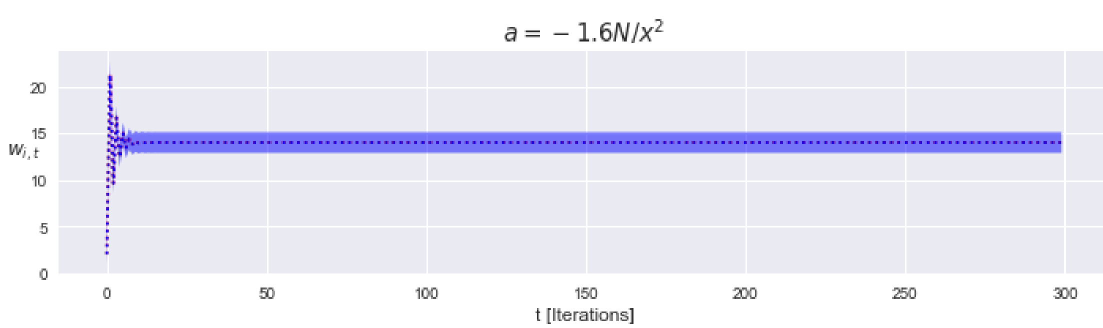


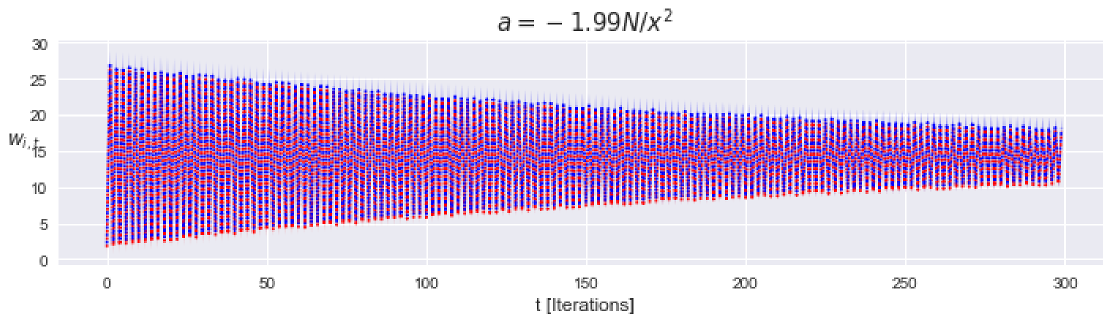


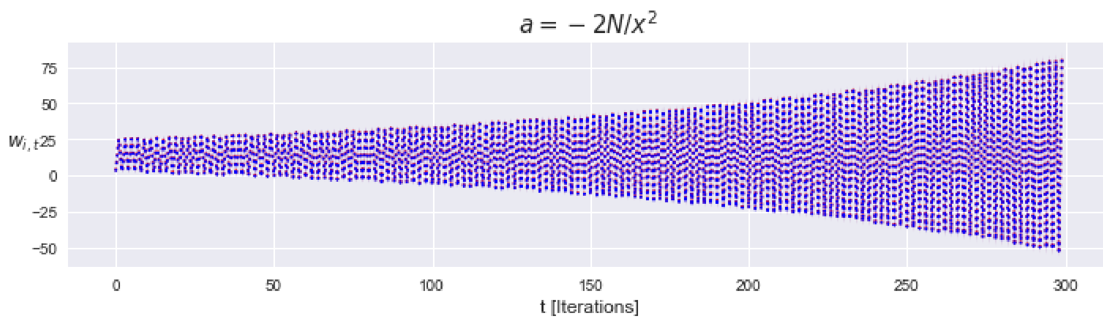

$a = -0.5N/x^2$
$w_{i,t}$
15.0
12.5
10.0
7.5
5.0
2.5
0.0
0
50
100
150
200
250
300
t [Iterations]


$a = -N/x^2$
$w_{i,t}$
15.0
12.5
10.0
7.5
5.0
2.5
0.0
0
50
100
150
200
250
300
t [Iterations]


$a = -1.3N/x^2$
$w_{i,t}$
20
15
10
5
0
0
50
100
150
200
250
300
t [Iterations]

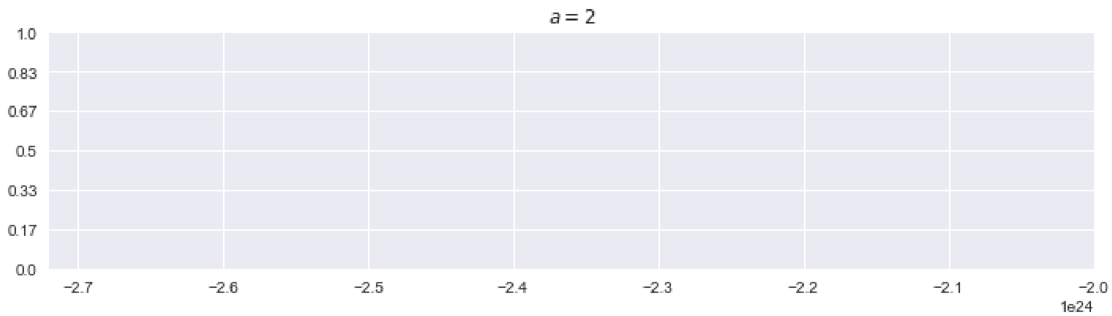

$a = -1.6N/x^2$
$w_{i,t}$
20
15
10
5
0
0
50
100
150
200
250
300
t [Iterations]

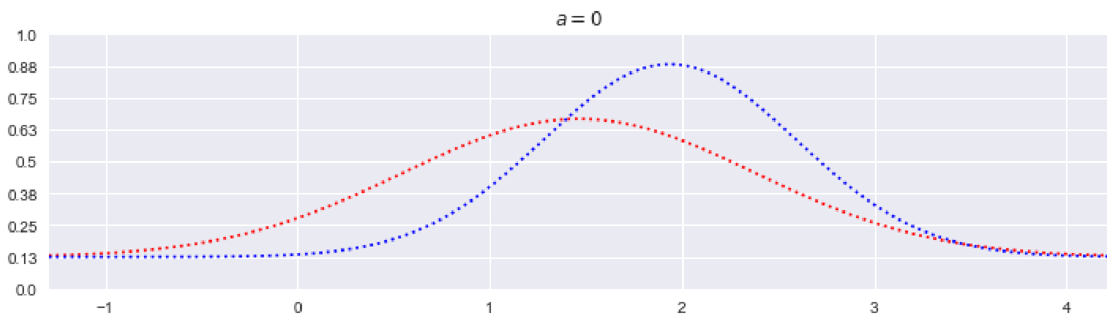


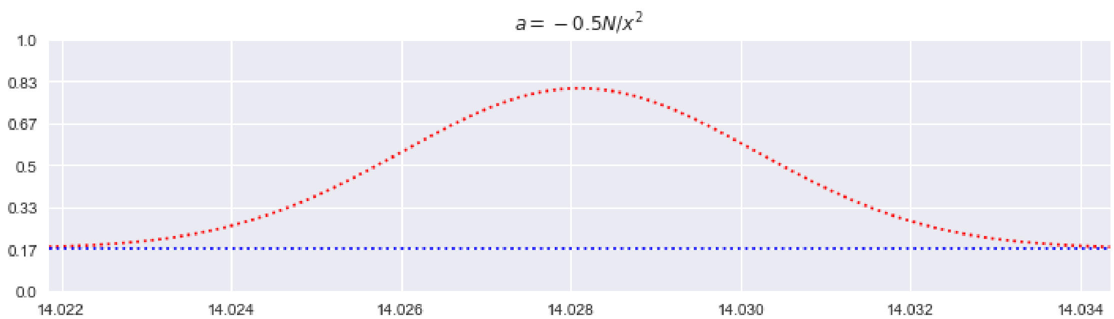


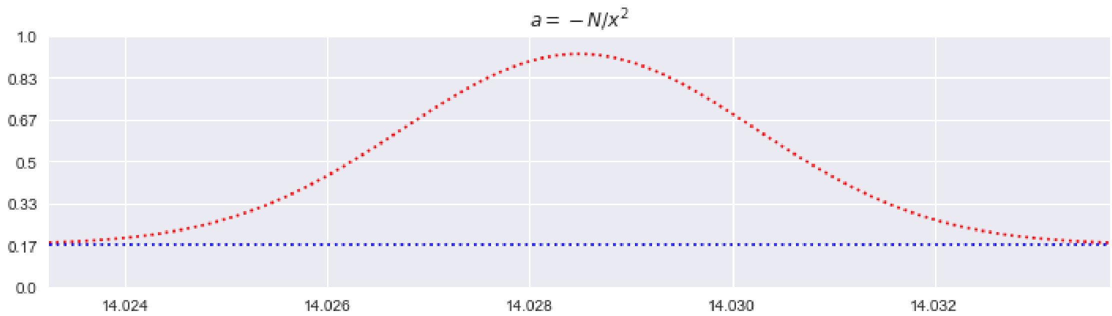
In [245]:
```
for i, axtitle in enumerate(alphaData):
    axtitle, axnum = axtitle
    mu, sig = outs[i]
    figure(figsize=(10,3))

    if np.sum(np.isnan(mu)) > 0:
        k=2
        idx0=argwhere(~np.isnan(mu[:,0]))[-1]-1
        idx1=argwhere(~np.isnan(sig[:,0]))[-1]-1
        idx = min(idx0, idx1)
        xmin = max(mu[idx,0]-k*sig[idx,0], mu[idx,0]-k*sig[idx,0])
        xmax = min(mu[idx,0]+k*sig[idx,0], mu[idx,0]+k*sig[idx,0])
        x_axis = np.linspace(xmin,xmax, num=300);
    else:
        xmin = max(mu[-1,0]-3*sig[-1,0], mu[-1,0]-3*sig[-1,0])
        xmax = min(mu[-1,0]+3*sig[-1,0], mu[-1,0]+3*sig[-1,0])
        x_axis = np.linspace(xmin,xmax, num=300);

    plt.plot(x_axis, norm.pdf(x_axis,mu[-1,0],sig[-1,0]),'r:');
```

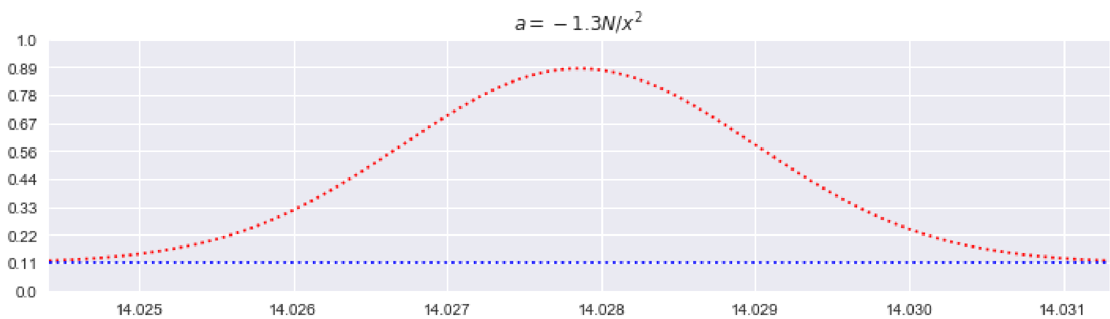
```
plt.plot(x_axis, norm.pdf(x_axis,mu[-1,1],sig[-1,1]), 'b:');
xlim(xmin = xmin, xmax=xmax)
p, v = yticks()
plt.yticks(p,map(lambda w: round(w, 2),linspace(0, 1, num=len(p))))
title(axtitle)
tight_layout()
```

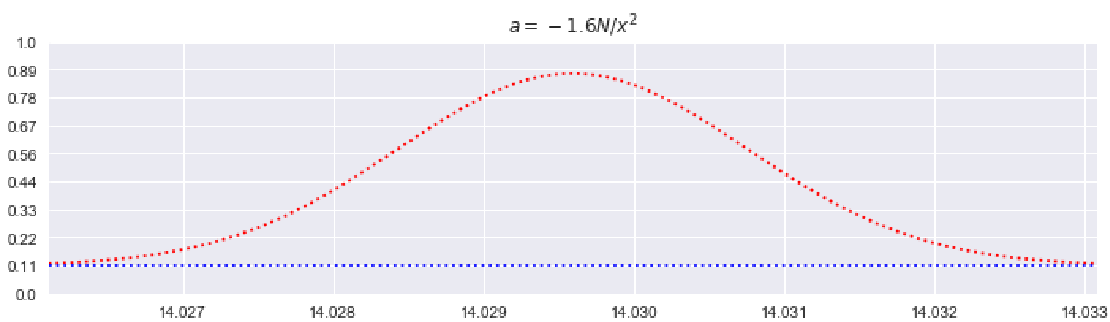


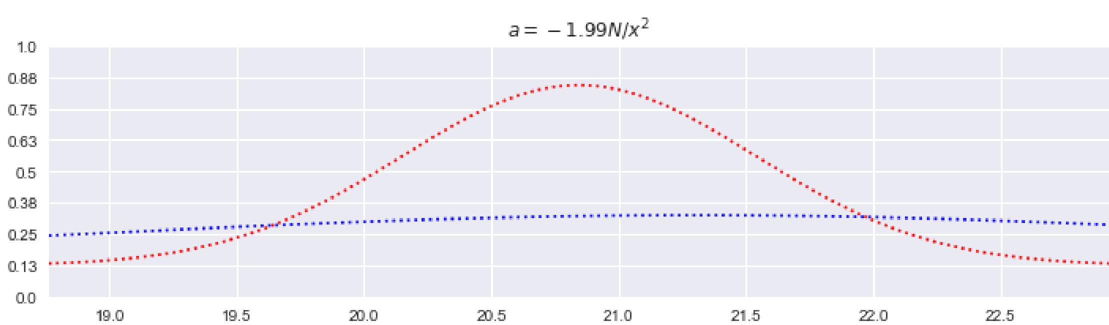


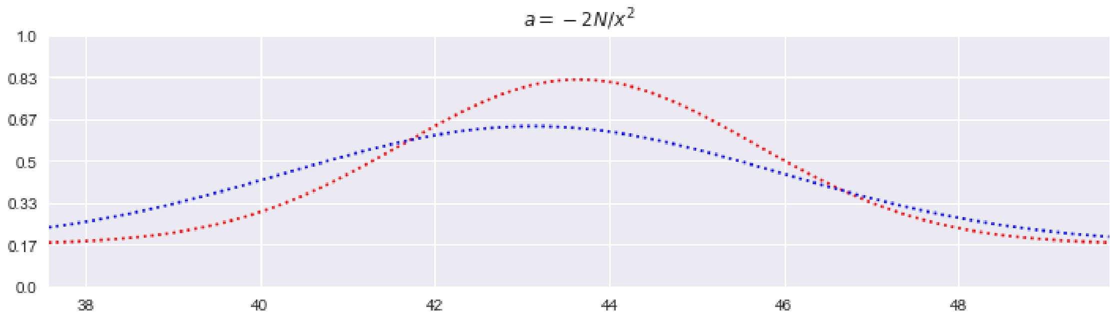



$a = -N/x^2$
1.0
0.83
0.67
0.5
0.33
0.17
0.0
14.024
14.026
14.028
14.030
14.032


$a = -1.3N/x^2$
1.0
0.89
0.78
0.67
0.56
0.44
0.33
0.22
0.11
0.0
14.025
14.026
14.027
14.028
14.029
14.030
14.031


$a = -1.6N/x^2$
1.0
0.89
0.78
0.67
0.56
0.44
0.33
0.22
0.11
0.0
14.027
14.028
14.029
14.030
14.031
14.032
14.033


$a = -1.99N/x^2$
1.0
0.88
0.75
0.63
0.5
0.38
0.25
0.13
0.0
19.0
19.5
20.0
20.5
21.0
21.5
22.0
22.5

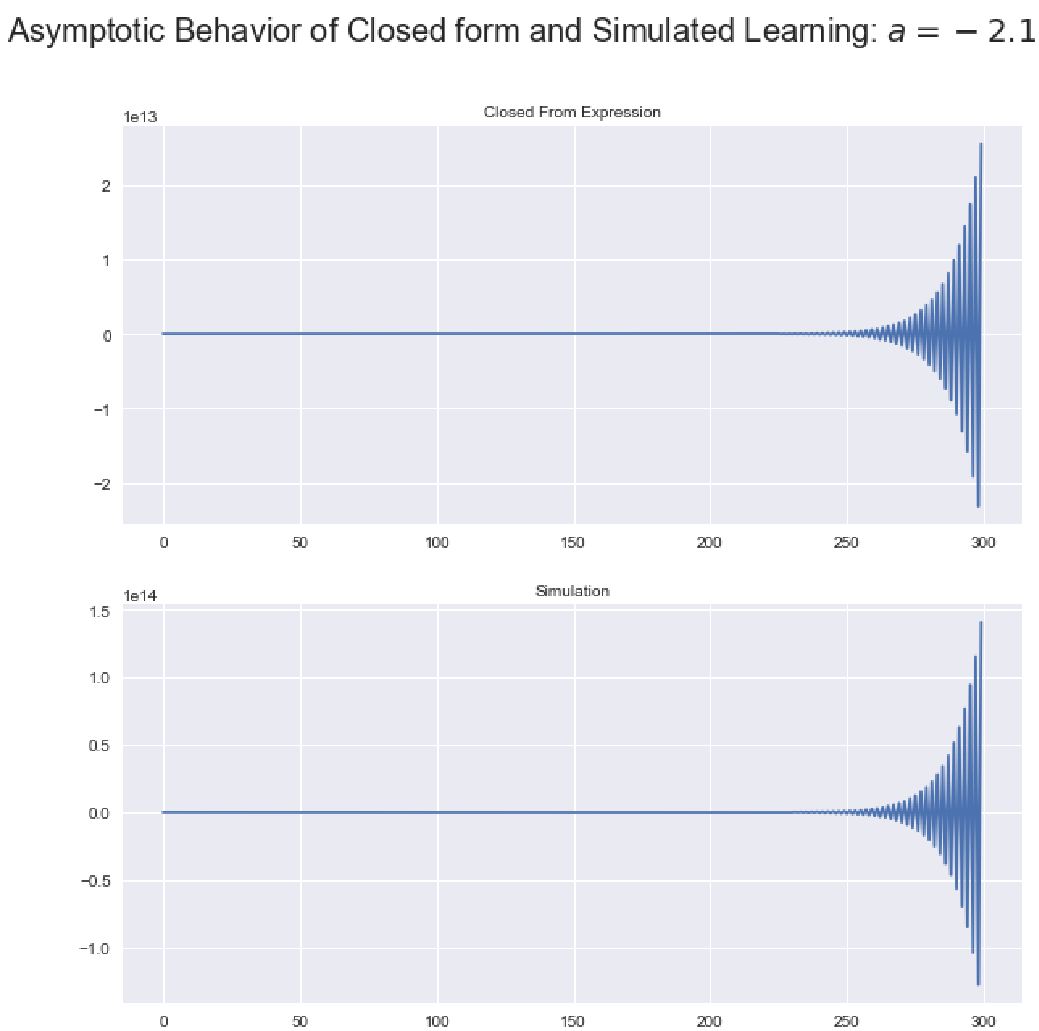


In [280]: x.shape

Out[280]: (2, 1000)

In [18]:
```
figure(figsize=(10,10))
subplot(2,1,1)
title("Closed From Expression", fontdict={'fontsize':10})
T = 300
w0 = random.normal(2, size=2)
t = np.arange(T)

a = -2.1*N/linalg.norm(x[0,:])**2
beta2 = (1/N)*a*x[0,:].dot(x[0,:])
beta1 = -(1/N)*a*x[0,:].dot(y)
ws = w0[0]*(beta2+1)**t - beta1*(1-(beta2+1)**t)/beta2
# ws = w0[0]*(-1)**t + ((-1)**t -1)*x[0,:].dot(y)/linalg.norm(x[0,:])**2
plot(ws)

subplot(2,1,2)
title("Simulation", fontdict={'fontsize':10})
wh = w0
whs = zeros((T, 2))
whs[0,:] = wh
for i in xrange(1,T):
    wh+=delta(grad(y,yh(x,wh), x), a)
    whs[i,:] = wh.copy()
plot(whs[:,0])
suptitle(("Asymptotic Behavior "
         "of Closed form and Simulated Learning: $a = -2.1N/x^2$"), fontdi
```

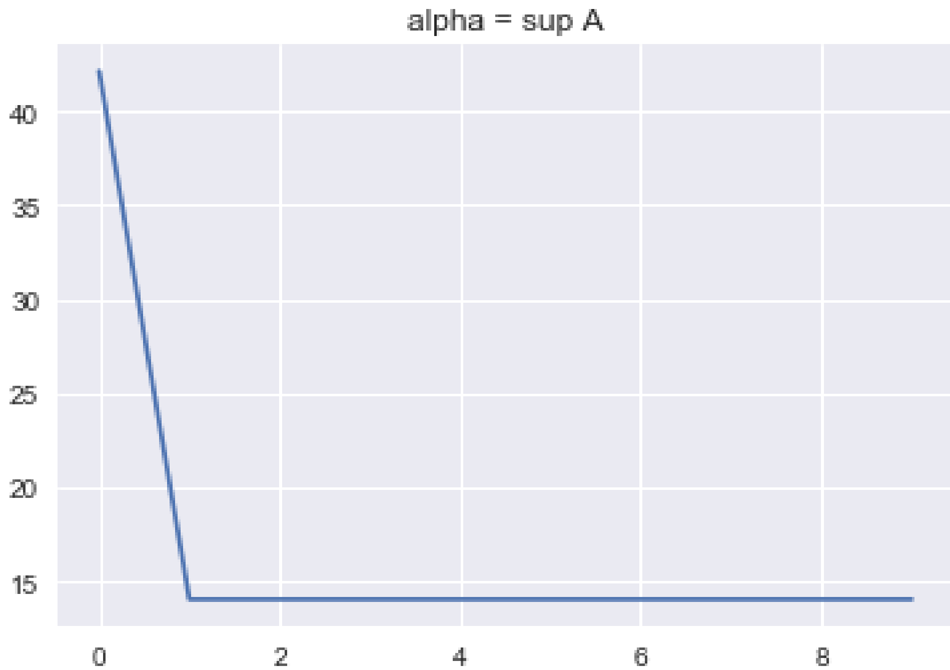
Out[18]: <matplotlib.text.Text at 0x116e90490>



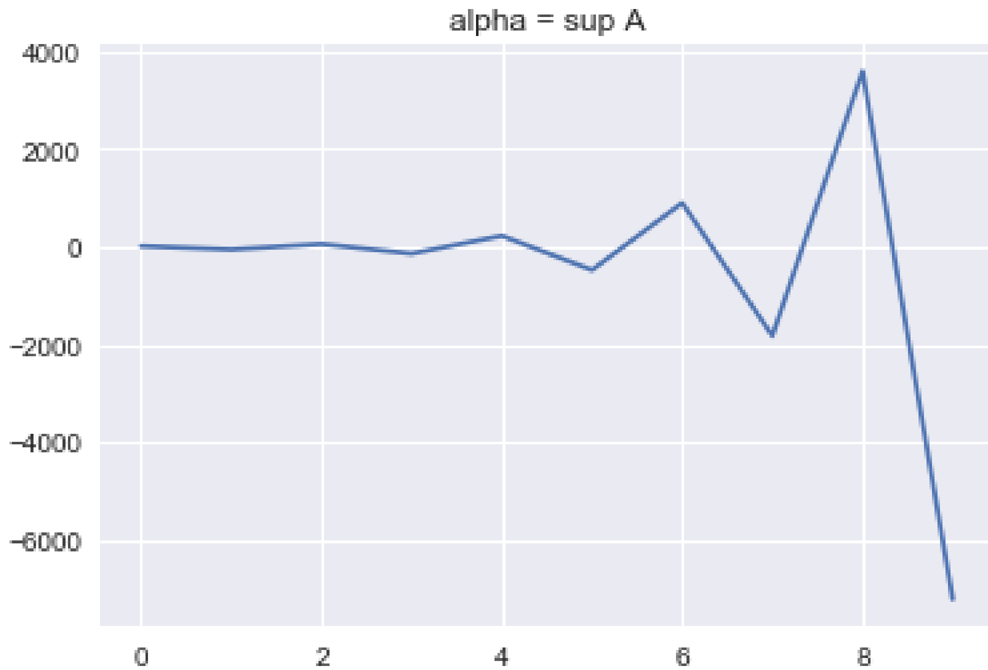
1.4 $\alpha = \sup A$

```
In [82]: t = arange(0,10)
         ws = (0**t)*(w0[0]+x[0,:].dot(y)/linalg.norm(x[0,:])**2) + x[0,:].dot(y)/l
         figure()
         ax = subplot(111)
         ax.set_title("alpha = sup A")
         ax.plot(ws)
```

```
Out[82]: [<matplotlib.lines.Line2D at 0x1158e3b50>]
```




```
In [89]: t = arange(0,10)
         ws = ((-1)**t)*w0[0] - (x[0,:].dot(y)/linalg.norm(x[0,:])**2) + (-2)**t*x[
         figure()
         ax = subplot(111)
         ax.set_title("alpha = sup A")
         ax.plot(ws)

Out[89]: [<matplotlib.lines.Line2D at 0x119789fd0>]
```




```
In [ ]:
```