1 Heat equation

1.1 PDE form

The (1-dimensional) heat equation in generality is a time-dependent parabolic partial differential equations of the form

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

note that the thermal diffusivity $D > 0$ and

1.2 Known analytical solutions

(1)
- **Thermal diffusivity:** Free to choose any $D > 0$
- **Domain:**
  - Spatial: $x \in [-1, 1]$
  - Time-evolution: $t \in [0, 1]$
- **Boundary conditions:**
  - $u(-1, t) = 0$
  - $u(1, t) = 0$
- **Initial conditions:** $u(x, 0) = \sin(\pi x)$
- **Solution:** $u(x, t) = \sin(\pi x)e^{-D\pi^2 t}$

Remark. An example PINN model was already developed to solve and benchmark with this case.

(2)
- **Thermal diffusivity:** Free to choose any $D > 0$
- **Domain:**
  - Spatial: $x \in [0, 1]$
  - Time-evolution: $t \in [0, 1]$
- **Boundary conditions:**
  - $u(-1, t) = e^{-\pi t}$
  - $u(1, t) = e^{-\pi t}$
- **Initial conditions:** $u(x, 0) = \cos(\pi x)$
- **Solution:** $u(x, t) = \cos(\pi x)e^{-D\pi^2 t}$

(3) Recall the hyperbolic cosine is $\cosh(x) = \frac{e^x + e^{-x}}{2}$.

- **Thermal diffusivity:** Free to choose any $D > 0$
- **Domain:**
  - Spatial: $x \in [A, B]$ for any choice of $A < B$
  - Time-evolution: Free to range $0 < t < \infty$ (but keep the domain compact meaning for computation meaning limit t to a finite number).
- **Boundary conditions:**
  - $u(A, t) = 2e^{Dt}\cosh(A)$
  - $u(B, t) = 2e^{D}\cosh(B)$
- **Initial conditions:** $u(x, 0) = 2\cosh(x)$
- **Solution:** $u(x, t) = 2e^{Dt}\cosh(x)$

(4) Here's a nice wild-card polynomial problem: Pick you favorite cubic (or degree 3) polynomial, in the most general form: $f(x) = ax^3 + bx^2 + cx + d$ for real numbers $a, b, c, d \in \mathbb{R}$. Then you can prove that $u(x, t) = f(x) + Dtf''(x)$ is the solution of the heat equation for any $D > 0$ subject to the initial condition $u(x, 0) = f(x)$.