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- **contrapositive**: $p \rightarrow q \iff \neg q \rightarrow \neg p$
- **negation**: distribute logical $\neg$ over the expression
- **DeMorgan's Law**: $\neg(p \wedge q) = \neg p \vee \neg q$, $\neg(p \vee q) = \neg p \wedge \neg q$

Example 1

For which values of p, q, and r is the following expression true? Give a succinct description (not the full truth table):

$$(\neg q \vee r) \wedge (p \rightarrow q) \wedge (\neg r \vee \neg p)$$

Example 2

Show that the following two expressions are not logically equivalent:

$$(q \to r) \vee p$$
$$q \to (r \vee p)$$

Example 3

What is the contrapositive of the statement: “If it rains, then the ground gets wet”?

Example 4

Negate the statement: “If I study and sleep well, then I will pass the exam.”

Example 5

Use DeMorgan's Law to simplify: $\neg(p \vee \neg q)$

Example 6

Which of the following are logically equivalent?

A. $\neg(p \wedge q)$

B. $\neg p \vee \neg q$

C. $\neg p \wedge \neg q$

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- Special notation: $\varnothing = \{\}$, the empty set.

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(a, b)

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- Cartesian Product: $A \times B = \{(a,b) \mid a \in A, b \in B\}$
- Powerset (set of all subsets): $\underline{\underline{\mathbb{P}(A)}} = \{S \mid S \subseteq A\}$ $\phi \in \mathbb{P}(A)$

Important Points

- Sets can contain sets! The set $A = \{1, 2, 3\}$ is distinct from the set $B = \{A\} = \{\{1, 2, 3\}\}$

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- Sets need not have the same "type" of element: $\{1, (3, \text{"no"}), f\}$ is a perfectly valid set.

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2. What is $\varnothing \times \{1, 2, 3\}$? $\rightarrow \varnothing$
3. What is $\{\varnothing\} \times \{1, 2, 3\}$? $\rightarrow \{(\varnothing, 1), \ldots\}$
4. For sets A and B, what is $|A \times B|$, in terms of $|A|$ and $|B|$?
5. What is $|A \cup B|$? $|A| + |B| - |A \cap B|$

6. What is $|\mathbb{P}(A)|$? 2^n where $n = |A|$
7. For any set A, what element is *always* in $\mathbb{P}(A)$? $\rightarrow \varnothing$
8. What is $\mathbb{P}(\{1, 2, 3\})$?

Set Proofs

$A \subseteq B$

Proving Inclusion

Let $a \in A$.

(logic...)

Thus, $a \in B$, so $A \subseteq B$.

$a = 10$

$a = 1$

a

Set Proofs

Proving Inclusion

Let $a \in A$. (logic...) Thus, $a \in B$, so $A \subseteq B$.

$A \neq B$

$A \subseteq B$

How would you prove that $\underbrace{A \not\subseteq B}$? How about $\underbrace{A \subsetneq B}$?

Practice Problems II

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2. Prove that $\{p > 2 \mid p \text{ prime}\} \subseteq \{x : 2 \nmid x \in \mathbb{N}\}$
3. Prove that $\{x^2 \mid x \in \mathbb{R}\} = \{y \geq 0 \mid y \in \mathbb{R}\}$

1. Let $a \in A$

then we have $a = 6x \Rightarrow a = 2 \cdot 3x$

$a = \underline{2y}$ where $y \in B$

$\square$

Counting Elements in Sets

- Number of ways to order n objects: $n! = n \cdot (n-1) \cdot (n-2) \dots \cdot 1$

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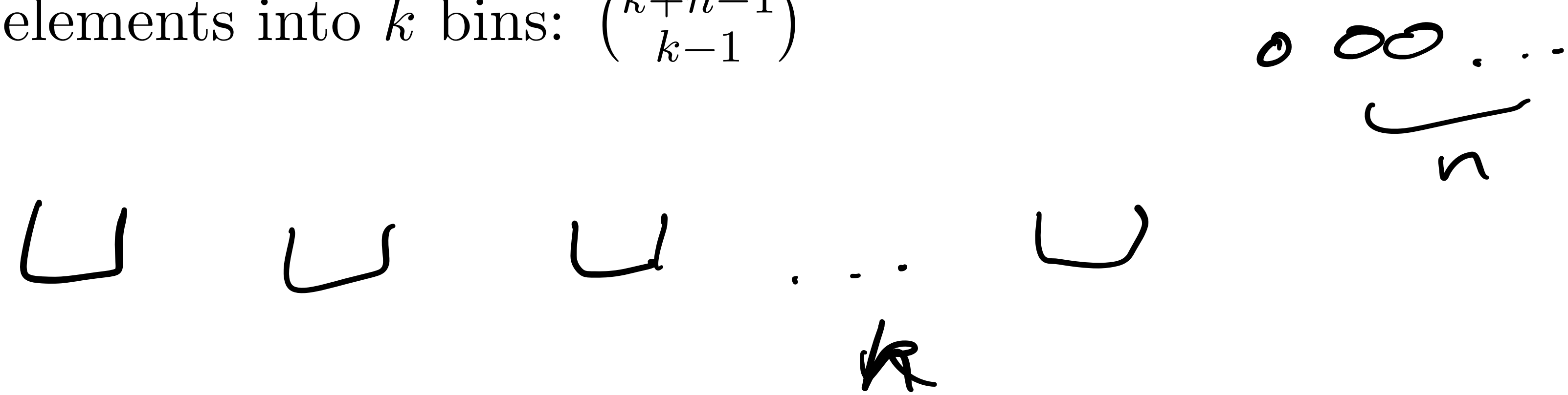
- Stars and Bars: How many ways to partition n identical elements into k bins: $\binom{k+n-1}{k-1}$



Practice Problems III

1. In a race with 6 contestants, how many possible top 3 players are there (order matters)? $P(6,3) = \frac{6!}{(6-3)!} = 120$

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1. In a race with 6 contestants, how many possible top 3 players are there (order matters)?
2. How many ways are there to divide $2n$ students into two even groups? ${}_{2n}C_n = \binom{2n}{n}$

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1. In a race with 6 contestants, how many possible top 3 players are there (order matters)?
2. How many ways are there to divide $2n$ students into two even groups?
3. How many 10-bit binary strings are there with exactly 3 0 bits? $\binom{10}{3} = 120$ _ _ _ _ _ _ _ _ _ _

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4. How many triples of nonnegative numbers (x, y, z) are there such that $x + y + z = 10$? $\Rightarrow \binom{12}{2}$

$$\underset{x}{\sqcup} + \underset{y}{\sqcup} + \underset{z}{\sqcup} = 10$$

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4. How many triples of nonnegative numbers (x, y, z) are there such that $x + y + z = 10$? $\binom{12}{2}$
5. Same question but for *positive* numbers.

$$\underbrace{\text{lo}}_{x} + \underbrace{\text{lo}}_{y} + \underbrace{\text{lo}}_{z} = \underset{7}{\cancel{10}} \text{ balls} \Rightarrow \binom{9}{2}$$

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5. Same question but for *positive* numbers.
6. (Summer 2024 Review Question) Let $A = \{(a, b) \in \mathbb{R}^2 \mid a = 3 - b^2\}, B = \{(x, y) \in \mathbb{R}^2 \mid |x| \geq 1 \text{ or } |y| \geq 1\}$. Prove that $A \subseteq B$.

Questions/Examples

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$\{2, 3, 4\}$

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  - $f : \mathbb{N} \cap [2, 4] \to \mathbb{N}$ $\quad f(2) = 3, f(3) = 5, f(4) = 7$
  - $f : \{2, 3, 4\} \to \mathbb{N}$ $\quad f(x) = 2x - 1$

Onto, One-to-one and Bijective

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- A function $f : A \to B$ is one-to-one if $f(x) = f(y)$ implies $x = y$ for all $x, y \in A$.

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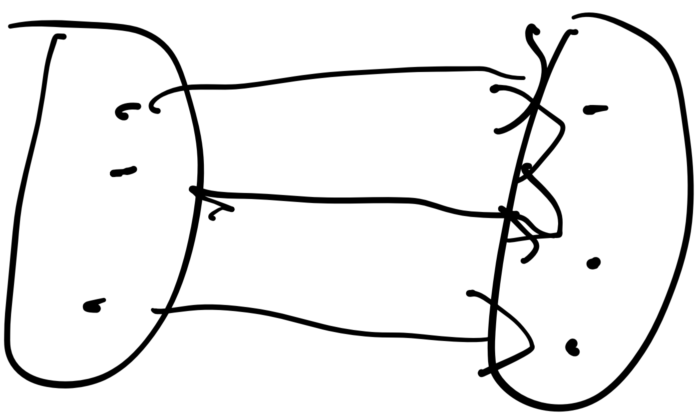
- A nice way to think about it: onto means every $b \in B$ has at *least* one preimage; one-to-one means every $b \in B$ has at *most* one preimage.



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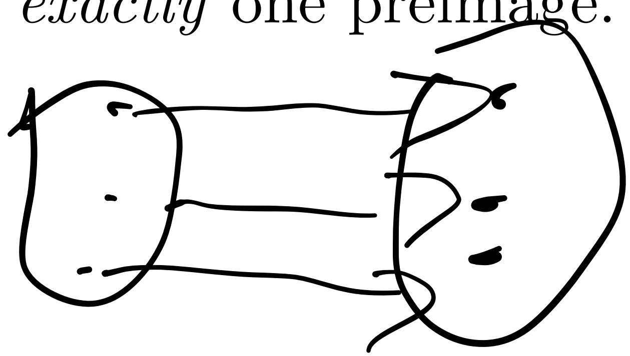
- A nice way to think about it: onto means every $b \in B$ has at *least* one preimage; one-to-one means every $b \in B$ has at *most* one preimage.
- So f being bijective (onto and one-to-one) means every $b \in B$ has *exactly* one preimage.



Practice Questions I

Which of these functions are onto, one-to-one, or both?

Signature	$f(x)$	Onto?	One-to-one?
$\mathbb{R} \to \mathbb{R}$	x^2	X	X
$[0, \infty) \to [0, \infty)$	x^2	YES	YES
$\mathbb{N} \to \mathbb{Z}$	$(-1)^x x$	X	YES
$\{2, 5, 6\} \to \{3, 6, 7, 8\}$	$x + 1$	X	YES
$\mathbb{N} \cup \{-1\} \to \mathbb{N}$	$2x + 1$	Not a	function!!

Practice Questions II

1. Let $g : \mathbb{N} \to \mathbb{N}$ be onto, and define $f : \mathbb{N}^2 \to \mathbb{Z}$ by $f(n, m) = (m-1)g(n)$. Prove f is onto.

$b \in \mathbb{Z}$

$b \geq 0$

$x \in \mathbb{N}$ $\quad g(x) = |b|$

$(y-1)g(x)$ $\quad (2-1)b = b$

$b < 0$

$g(x) = |b|$

$(0-1)|b| = -b$

let $m=0$

$b=1$ $\quad m=1$ $\quad = 0$

Practice Questions II

1. Let $g : \mathbb{N} \to \mathbb{N}$ be onto, and define $f : \mathbb{N}^2 \to \mathbb{Z}$ by $f(n, m) = (m-1)g(n)$. Prove f is onto.
2. (SU24 Review) Let $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Q}^2$, $f(x, y) = (\frac{x}{y}, x + y)$. Show that f is one-to-one.

$$f(x, y) = f(a, b)$$

$$\frac{x}{y} = \frac{a}{b}$$

$$f\left(\frac{x}{y}, x+y\right) = f\left(\frac{a}{b}, a+b\right)$$

$$x + y = a + b$$

$$\cancel{x+y = a+b}$$

$$x = \frac{ya}{b}$$

$$\boxed{(x, y) = (a, b)}$$

$$\frac{y}{b} = 1$$

$$y = b, \quad x = a$$

$$\frac{ya}{b} + y = a + b$$

$$\frac{y(a+b)}{b} = a + b$$

$$\frac{ya + yb}{b} = a + b$$

$$\frac{ya}{b} + \frac{yb}{b} = a + b$$

Practice Questions II

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3. Suppose we have finite sets A and B with $|A| = |B| = n \in \mathbb{N}$. Show that if $f : A \to B$ is one-to-one, then f is onto. How many such functions are there, for fixed A and B?

A B

Practice Questions II

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4. Suppose $A \subseteq \mathbb{R}$ and $f : A \to A$, $f(x) = \sqrt{2}x$. For what sets A is f a bijection?

$A = [0]$

$A = \mathbb{N} \to f(1) = \sqrt{2} \notin \text{codomain}(f) = \mathbb{N}$

$A = \{(\sqrt{2})^n \mid n \in \mathbb{Z}\}$

Practice Questions II

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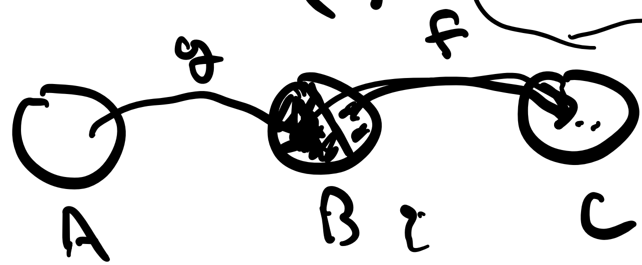
4. Suppose $A \subseteq \mathbb{R}$ and $f : A \to A$, $f(x) = \sqrt{2}x$. For what sets A is f a bijection?
5. Suppose that $f \circ g$ is one-to-one. Does f have to be one-to-one? Does g?

$a_1 \neq a_2$

$g(a_1) = g(a_2)$

$f(g(a_1))$
$= f(g(a_2))$

$f \circ g$ is one-to-one $\Leftrightarrow$ $\forall\, a_1, a_2 \in A,\ f(g(a_1)) = f(g(a_2))$
$\Rightarrow a_1 = a_2$

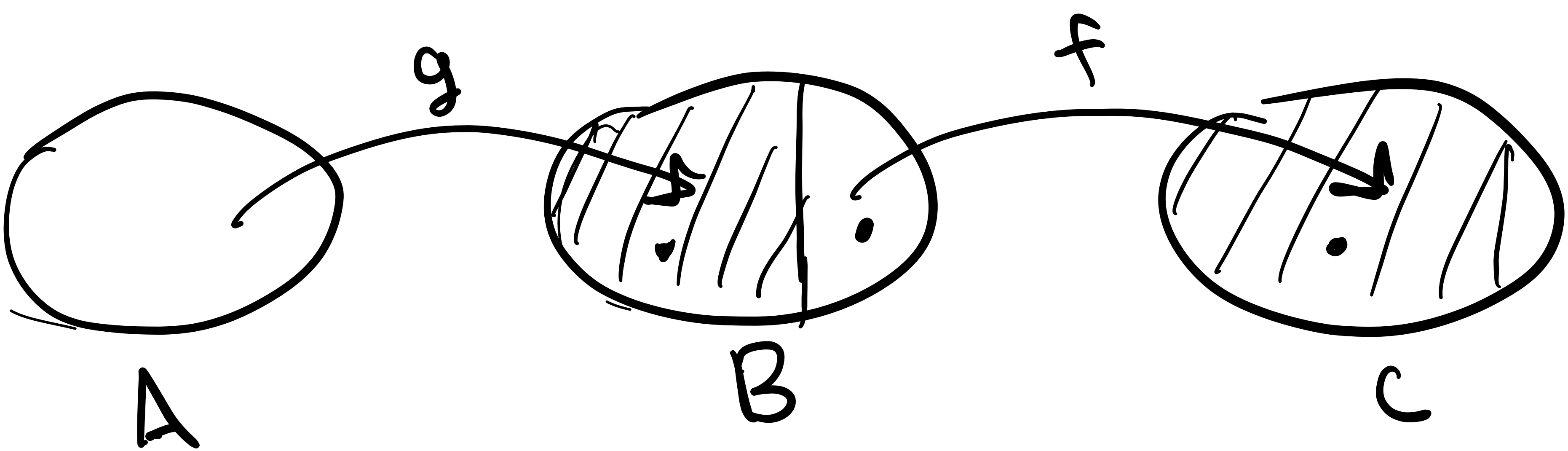


Questions/Examples

$f \circ g$ is one-to-one $\Leftrightarrow \forall\, a_1, a_2 \in A,\ f(g(a_1)) = f(g(a_2))$
$\Rightarrow a_1 = a_2$

if g is not 1-1, $\exists\, a_1, a_2 \in A$ s.t. $a_1 \neq a_2$ &
$g(a_1) = g(a_2)$
by def of function $\Downarrow$
$f(g(a_1)) = f(g(a_2))$
$f \circ g$ is not 1-1

f is 1-1 $\Leftrightarrow \forall\, b_1, b_2 \in B,$
$f(b_1) = f(b_2) \Rightarrow b_1 = b_2$



Questions/Examples

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Recurrences: Review

- We can specify functions with a recursive formula:

$$T(n) = \begin{cases} c & n \leq B \\ \langle \text{Formula with smaller arguments to } T \rangle & n > B \end{cases}$$

Recurrences: Review

- We can specify functions with a recursive formula:

$$T(n) = \begin{cases} c & n \leq B \\ \langle \text{Formula with smaller arguments to } T \rangle & n > B \end{cases}$$

- Some examples:

Recurrence	Closed Formula
$T(n) = \begin{cases} 1 & n \leq 1 \\ 2T(n-1) & n > 1 \end{cases}$	$T(n) = 2^n$
$T(n) = \begin{cases} 1 & n \leq 1 \\ 2T(n/2) + 1 & n > 1 \end{cases}$	$T(n) = 2n - 1$
$T(n) = \begin{cases} 0 & n \leq 1 \\ 2T(n/2) + n & n > 1 \end{cases}$	$T(n) = n \log(n)$

Solving A Recurrence

The general strategy for solving a recurrence is a recurrence tree.

- Each node has a value equal to the nonrecursive work done.

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- Each node has a value equal to the nonrecursive work done.
- Each node has a child for every recursive call it makes.

Example: $T(n) = \begin{cases} 0 & n \leq 1 \\ \underbrace{2T(n/2)}_{\text{recursive}} + \underbrace{n}_{\text{nonrecursive}} & n > 1 \end{cases}$

$T(n)$	n
$T(n/2)$ $T(n/2)$	$2(n/2) = n$
$T(n/4)$ $T(n/4)$ $T(n/4)$ $T(n/4)$	$4(n/4) = n$
$T(1)$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $T(1)$	$n(1) = n$

Solving A Recurrence

1. At level k...

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   - How many nodes are there?

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Solving A Recurrence

1. At level k...
   - How many nodes are there?
   - What is the input?
   - How much work is done by each node?
   - What is the total nonrecursive work? ($W(k)$)
2. What is the height of the tree (i.e. at what level h do we reach the base case input)?
3. What is the work done by the leaf nodes?
   $L = [\# \text{ of leaf nodes}] \times [\text{work done by each}]$

Solving A Recurrence

1. At level k...
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   - How much work is done by each node?
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3. What is the work done by the leaf nodes?
   $L =$ [# of leaf nodes] $\times$ [work done by each]
4. The total work is $\underbrace{\sum_{k=0}^{h-1} W(k)}_{\text{internal work}} + \underbrace{L}_{\text{leaf work}}$

Recurrence Practice I

$$T(n) = \begin{cases} c & n \leq 1 \\ 3T(n/2) + n & n > 1 \end{cases}$$

What is $T(n)$ for n a power of 2?

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What is $T(n)$ for n a power of 2?

1. At level k...
   - There are ________nodes.
   - The input is ________.
   - Each node does ________work.
   - The total nonrecursive work $W(k) =$ ________
2. $h =$ ________
3. What is the work done by the leaf nodes?
   $L = [\# \text{ of leaf nodes}] \times [\text{work done by each}] =$ ________
4. $\underbrace{\sum_{k=0}^{h-1} W(k)}_{\text{internal work}} + \underbrace{L}_{\text{leaf work}} =$ ________

Recurrence Practice I

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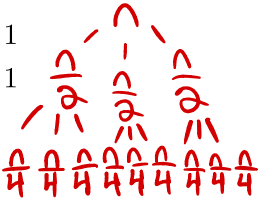


What is $T(n)$ for n a power of 2?

1. At level k...
   - There are $\underline{3^k}$ nodes.
   - The input is $\underline{n/2^k}$.
   - Each node does $\underline{n/2^k}$ work.
   - The total nonrecursive work $W(k) = \underline{3^k \frac{n}{2^k}} = n\left(\frac{3}{2}\right)^k$
2. $h = \underline{\log_2 n}$
3. What is the work done by the leaf nodes?
   $L =$ [# of leaf nodes] $\times$ [work done by each] $= \underline{3^{\log_2 n}} \cdot c$
4. $\underbrace{\sum_{k=0}^{h-1} W(k)}_{\text{internal work}} + \underbrace{L}_{\text{leaf work}} = \underline{\sum_{k=0}^{\log_2 n - 1} n\left(\frac{3}{2}\right)^k} + c\,3^{\log_2 n}$

Geometric: $\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1}$ $(r \neq 1)$

Answer: $\underline{T(n) = (c+2)n^{\log_2 3} - 2n}$

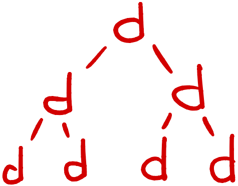
Recurrence Practice II

$$T(n) = \begin{cases} c & n \leq 2 \\ 2T(n-1) + d & n > 2 \end{cases}$$

What is $T(n)$ for $n \geq 2$?

Recurrence Practice II

$$T(n) = \begin{cases} c & n \le 2 \\ 2T(n-1) + d & n > 2 \end{cases}$$



What is $T(n)$ for $n \ge 2$?

nodes: 2^k

Prob. size: $n-k$

height: $n-2$

non-leaf work: $2^k \cdot d$

leaf work: $2^{n-2} \cdot c$

$$\sum_{k=0}^{n-3} 2^k \cdot d + 2^{n-2} c$$

$$= \frac{2^{n-2}-1}{2-1} d + 2^{n-2} c$$

$$= (2^{n-2}-1)d + 2^{n-2} c$$

Answer: $\underline{T(n) = (c+d)2^{n-2} - d}$

Recurrence Practice III

$$T(n) = \begin{cases} c & n \leq 1 \\ T(n/2) + n^2 & n > 1 \end{cases}$$

What is $T(n)$ for n a power of 2?

Recurrence Practice III

$$T(n) = \begin{cases} c & n \leq 1 \\ T(n/2) + n^2 & n > 1 \end{cases}$$

What is $T(n)$ for n a power of 2?

n^2

$\left(\frac{n}{2}\right)^2$

$\left(\frac{n}{4}\right)^2$

# nodes	1
Prob. size	$n/2^k$
height	$\log_2 n$
non-leaf	$\left(\frac{n}{2^k}\right)^2 = \left(\frac{1}{4}\right)^k n^2$
leaf	c

$$\sum_{k=0}^{\log n - 1} \left(\frac{1}{4}\right)^k n^2 + c$$

Answer: $\underline{T(n) = \frac{4}{3}(n^2 - 1) + c}$

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- In the previous slide, $T(n) = \frac{4}{3}(n^2 - 1) + c$ is always $\Theta(n^2)$, no matter what c is.

What about Θ?

- If all we care about if what T is Θ of, the specific constants don't matter.
- In the previous slide, $T(n) = \frac{4}{3}(n^2 - 1) + c$ is always $\Theta(n^2)$, no matter what c is.
- Similar for $T(n) = (c + d)2^{n-2} - d$ is $\Theta(2^n)$ and $T(n) = (c + 2)n^{\log_2 3} - 2n$ is $\Theta(n^{\log_2 3})$.

Recurrence Practice V

What is the Big-Theta running time of SHAKE?

```
SHAKE(A[1..n])
    if n ≤ 1:
        return A
    m ← ⌊n/2⌋
    A₁ ← SHAKE(A[1..m])
    A₂ ← SHAKE(A[m + 1..n])
    B ← []
    for i in 1..m:
        B.add(A₁[i])
        B.add(A₂[i])
    return B
```

Recurrence Practice V

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```
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    if n ≤ 1:
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    m ← ⌊n/2⌋
    A1 ← SHAKE(A[1..m])
    A2 ← SHAKE(A[m+1..n])
    B ← []
    for i in 1..m:
        B.add(A1[i])
        B.add(A2[i])
    return B
```

$$T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{2}$$

$$\frac{n}{2} = \frac{n}{2}$$

$$\frac{n}{4} + \frac{n}{4} = \frac{n}{2}$$

$$\frac{n}{8} + \frac{n}{8} + \frac{n}{8} + \frac{n}{8} = \frac{n}{2}$$

Work at level $k = \frac{n}{2}$

height $= \log_2 n$

Answer: $\Theta(n \log n)$

Recurrence Practice VI: HARD

$$T(n) = \begin{cases} c & n \leq 1 \\ T(n/2) + T(n/3) + T(n/6) + n & n > 1 \end{cases}$$

What is $T(n)$ Big-Theta of?

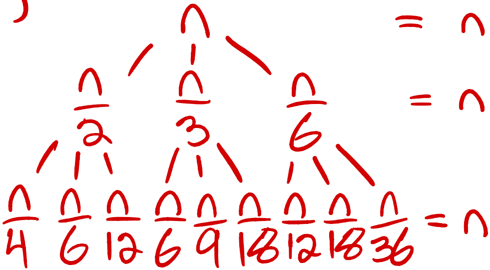
Recurrence Practice VI: HARD

$$T(n) = \begin{cases} c & n \leq 1 \\ T(n/2) + T(n/3) + T(n/6) + n & n > 1 \end{cases}$$

What is $T(n)$ Big-Theta of?

Work at level $k = \Theta(n)$

height $= \Theta(\log n)$



Answer: $\underline{\Theta(n \log n)}$

Questions/Examples

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