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- **negation**: distribute logical $\neg$ over the expression
- **DeMorgan's Law**: $\neg(p \wedge q) = \neg p \vee \neg q$, $\neg(p \vee q) = \neg p \wedge \neg q$

Example 1

For which values of p, q, and r is the following expression true? Give a succinct description (not the full truth table):

$$(\neg q \vee r) \wedge (p \to q) \wedge (\neg r \vee \neg p)$$

Example 2

Show that the following two expressions are not logically equivalent:

$$(q \to r) \vee p$$
$$q \to (r \vee p)$$

Example 3

What is the contrapositive of the statement: “If it rains, then the ground gets wet”?

Example 4

Negate the statement: “If I study and sleep well, then I will pass the exam.”

Example 5

Use DeMorgan's Law to simplify: $\neg(p \vee \neg q)$

Example 6

Which of the following are logically equivalent?

A. $\neg(p \wedge q)$
B. $\neg p \vee \neg q$
C. $\neg p \wedge \neg q$

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- Special notation: $\varnothing = \{\}$, the empty set.

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- Powerset (set of all subsets): $\mathbb{P}(A) = \{S \mid S \subseteq A\}$

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- Sets need not have the same "type" of element: $\{1, (3, \text{“no”}), f\}$ is a perfectly valid set.

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How would you prove that $A \not\subseteq B$? How about $A \subsetneq B$?

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3. Prove that $\{x^2 \mid x \in \mathbb{R}\} = \{y \geq 0 \mid y \in \mathbb{R}\}$

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- Stars and Bars: How many ways to partition n identical elements into k bins: $\binom{k+n-1}{k-1}$

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5. Same question but for *positive* numbers.
6. (Summer 2024 Review Question) Let $A = \{(a, b) \in \mathbb{R}^2 \mid a = 3 - b^2\}, B = \{(x, y) \in \mathbb{R}^2 \mid |x| \geq 1 \text{ or } |y| \geq 1\}$. Prove that $A \subseteq B$.

Questions/Examples

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  - $f : \mathbb{N} \cap [2, 4] \to \mathbb{N}$ $\quad$ $f(2) = 3, f(3) = 5, f(4) = 7$
  - $f : \{2, 3, 4\} \to \mathbb{N}$ $\quad$ $f(x) = 2x - 1$

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- A nice way to think about it: onto means every $b \in B$ has at *least* one preimage; one-to-one means every $b \in B$ has at *most* one preimage.

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- So f being bijective (onto and one-to-one) means every $b \in B$ has *exactly* one preimage.

Practice Questions I

Which of these functions are onto, one-to-one, or both?

Signature	$f(x)$	Onto?	One-to-one?
$\mathbb{R} \to \mathbb{R}$	x^2		
$[0, \infty) \to [0, \infty)$	x^2		
$\mathbb{N} \to \mathbb{Z}$	$(-1)^x x$		
$\{2, 5, 6\} \to \{3, 6, 7, 8\}$	$x + 1$		
$\mathbb{N} \cup \{-1\} \to \mathbb{N}$	$2x + 1$		

Practice Questions II

1. Let $g : \mathbb{N} \to \mathbb{N}$ be onto, and define $f : \mathbb{N}^2 \to \mathbb{Z}$ by $f(n, m) = (m - 1)g(n)$. Prove f is onto.

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5. Suppose that $f \circ g$ is one-to-one. Does f have to be one-to-one? Does g?

Questions/Examples

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Recurrences: Review

- We can specify functions with a recursive formula:

$$T(n) = \begin{cases} c & n \leq B \\ \langle \text{Formula with smaller arguments to } T \rangle & n > B \end{cases}$$

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- Some examples:

Recurrence	Closed Formula
$T(n) = \begin{cases} 1 & n \leq 1 \\ 2T(n-1) & n > 1 \end{cases}$	$T(n) = 2^n$
$T(n) = \begin{cases} 1 & n \leq 1 \\ 2T(n/2)+1 & n > 1 \end{cases}$	$T(n) = 2n - 1$
$T(n) = \begin{cases} 0 & n \leq 1 \\ 2T(n/2)+n & n > 1 \end{cases}$	$T(n) = n\log(n)$

Solving A Recurrence

The general strategy for solving a recurrence is a recurrence tree.

- Each node has a value equal to the nonrecursive work done.

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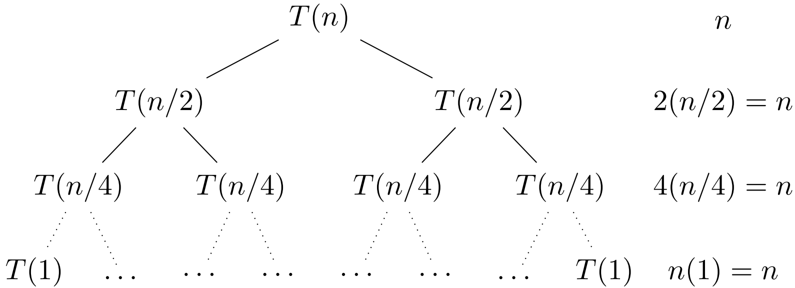
Example: $T(n) = \begin{cases} 0 & n \le 1 \\ \underbrace{2T(n/2)}_{\text{recursive}} + \underbrace{n}_{\text{nonrecursive}} & n > 1 \end{cases}$



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4. The total work is $\underbrace{\sum_{k=0}^{h-1} W(k)}_{\text{internal work}} + \underbrace{L}_{\text{leaf work}}$

Recurrence Practice I

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Answer: $T(n) = (c+2)n^{\log_2 3} - 2n$

Recurrence Practice II

$$T(n) = \begin{cases} c & n \leq 2 \\ 2T(n-1) + d & n > 2 \end{cases}$$

What is $T(n)$ for $n \geq 2$?

Recurrence Practice II

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Recurrence Practice III

$$T(n) = \begin{cases} c & n \leq 1 \\ T(n/2) + n^2 & n > 1 \end{cases}$$

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Recurrence Practice III

$$T(n) = \begin{cases} c & n \leq 1 \\ T(n/2) + n^2 & n > 1 \end{cases}$$

What is $T(n)$ for n a power of 2?

Answer: $T(n) = \frac{4}{3}(n^2 - 1) + c$

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- In the previous slide, $T(n) = \frac{4}{3}(n^2 - 1) + c$ is always $\Theta(n^2)$, no matter what c is.
- Similar for $T(n) = (c + d)2^{n-2} - d$ is $\Theta(2^n)$ and $T(n) = (c + 2)n^{\log_2 3} - 2n$ is $\Theta(n^{\log_2 3})$.

Recurrence Practice V

What is the Big-Theta running time of SHAKE?

```
SHAKE(A[1..n])
    if n ≤ 1:
        return A
    m ← ⌊n/2⌋
    A_1 ← SHAKE(A[1..m])
    A_2 ← SHAKE(A[m + 1..n])
    B ← []
    for i in 1..m:
        B.add(A_1[i])
        B.add(A_2[i])
    return B
```

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    return B
```

Answer: $\Theta(n \log n)$

Recurrence Practice VI: HARD

$$T(n) = \begin{cases} c & n \leq 1 \\ T(n/2) + T(n/3) + T(n/6) + n & n > 1 \end{cases}$$

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Recurrence Practice VI: HARD

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What is $T(n)$ Big-Theta of?

Answer: $\Theta(n \log n)$

Questions/Examples

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- **Goal:** Prove a mathematical statement is true for all natural numbers.
- Outline: base case, inductive hypothesis, inductive step

Induction Proofs

The general strategy for proving a claim by induction is to

(a) define the **base case(s)** and show the claim holds for them

(b) state the **inductive hypothesis** assuming that the claim holds true for all $n < k$ $(n \in \mathbb{N})$

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- You can also do $n \leq k$ and then $n = k + 1$.
- **Always use strong induction!** Your inductive hypothesis must hold for *all* values up to k.

How many base cases?

Make sure to justify all the base cases that are necessary to establish the claim.

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- What base cases do you need to prove this claim: $\forall n, f_n < 2^n$, where $f_{n+1} = f_n + f_{n-1}$?

Induction Example I

Prove that the following holds for all natural numbers n.

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

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- Base case(s): For _________, we have
- Inductive hypothesis: Suppose $\left[\qquad\qquad\qquad\qquad \right]$
- Inductive step: Consider $n =$ _________. We want to show that $\left[\qquad\qquad\qquad\qquad \right]$.

  $$\vdots$$

  Therefore $\left[\qquad\qquad\qquad \right]$, which is what we needed to show.

Induction Example I

- Inductive step

$$\left(\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}\right)$$

Induction Example I (Solution)

Prove that the following holds for all natural numbers n.

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

- Base case(s): For $n = 0$, we have $\sum_{i=0}^{n} x^i = x^0 = 1$ and $\frac{x^{n+1}-1}{x-1} = \frac{x^{0+1}-1}{x-1} = \frac{x-1}{x-1} = 1$. So $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$.
- Inductive hypothesis: Suppose $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$ for $n = 0, 1, \ldots, k$.
- Inductive step (next slide)

Induction Example I (Solution)

- Inductive step: Consider $n = k+1$. We want to show that $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$.
  $\sum_{i=0}^{k+1} x^i = \sum_{i=0}^{k+1} x^i = x^{k+1} + \sum_{i=0}^{k} x^i$.
  By the inductive hypothesis, $\sum_{i=0}^{k} x^i = \frac{x^{k+1}-1}{x-1}$. So:

$$
\begin{aligned}
\sum_{i=0}^{k+1} x^i = x^{k+1} + \sum_{i=0}^{k} x^i &= x^{k+1} + \frac{x^{k+1}-1}{x-1} \\
&= \frac{(x-1)x^{k+1}}{x-1} + \frac{x^{k+1}-1}{x-1} \\
&= \frac{x^{k+2} - x^{k+1} + x^{k+1} - 1}{x-1} \\
&= \frac{x^{k+2}-1}{x-1}.
\end{aligned}
$$

  Therefore $\sum_{i=0}^{k+1} x^i = \frac{x^{k+2}-1}{x-1}$, which is what we needed to show.

Inequality Induction

- Very similar to ‘equality’ induction! Don’t overthink it! Just manipulating inequalities instead of equations, still primarily algebra

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- Very similar to 'equality' induction! Don't overthink it! Just manipulating inequalities instead of equations, still primarily algebra
- The only 'trick' is that you may have to simplify or change to a term to something even smaller/bigger (resp. the inequality) to make your algebra match your 'goal'. (Confused? The next example should illuminate this more clearly.)

Inequality Induction Example I

Let f_n be the nth Fibonacci number (i.e. $f_{n+1} = f_n + f_{n-1}$). Prove that $f_n \geq (\frac{3}{2})^{n-2}$.

Inequality Induction Example I

Let f_n be the nth Fibonacci number (i.e. $f_{n+1} = f_n + f_{n-1}$ where $f_1 = f_2 = 1$). Prove that $f_n \geq (\frac{3}{2})^{n-2}$.

- Base cases: For $n = 1$, we have $f_1 = 1$, $\left(\frac{3}{2}\right)^{1-2} = \frac{2}{3}$. $1 \geq \frac{2}{3}$.
  For $n = 2$, we have $f_2 = 1$, $\left(\frac{3}{2}\right)^{2-2} = 1$. $1 \geq 1$.
- Inductive hypothesis: Suppose that $f_n \geq \left(\frac{3}{2}\right)^{n-2}$ for $n = 1, 2, \ldots, k-1$.
- Inductive step: Consider $n = k$. We want to show that $f_n \geq (\frac{3}{2})^{n-2}$.
  $f_k =$ ________
  By the inductive hypothesis, we have $f_{k-1} \geq \left(\frac{3}{2}\right)^{k-3}$ and $f_{k-2} \geq \left(\frac{3}{2}\right)^{k-4}$.
  So ________
  Therefore $f_n \geq \left(\frac{3}{2}\right)^{n-2}$, which is what we needed to show.

Inequality Induction Example I

- Inductive step: Consider $n = k$. We want to show that $f_n \geq (\frac{3}{2})^{n-2}$.
  $f_k = f_{k-1} + f_{k-2}$
  By the inductive hypothesis, we have $f_{k-1} \geq \left(\frac{3}{2}\right)^{k-3}$ and $f_{k-2} \geq \left(\frac{3}{2}\right)^{k-4}$. So

$$\begin{aligned} f_n &\geq \left(\frac{3}{2}\right)^{n-3} + \left(\frac{3}{2}\right)^{n-4} \\ &\geq \left(\frac{3}{2}\right)^{n-4} \left(\frac{3}{2} + 1\right) = \left(\frac{3}{2}\right)^{n-4} \left(\frac{5}{2}\right) \\ &\geq \left(\frac{3}{2}\right)^{n-4} \left(\frac{9}{4}\right) = \left(\frac{3}{2}\right)^{n-4} \left(\frac{3}{2}\right)^{2} \\ &\geq \left(\frac{3}{2}\right)^{n-2} \end{aligned}$$

  Therefore $f_n \geq (\frac{3}{2})^{n-2}$, which is what we needed to show.

Tree Induction

- Most important detail to remember: induct on **height of the tree**, *never* the number of nodes/leaves/etc.

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- Most important detail to remember: induct on **height of the tree**, *never* the number of nodes/leaves/etc.
- Also critical: build your tree from the 'bottom-up' (adding root nodes at the top, never by adding leaves)
- May have to do casework, e.g. if node has different values depending on number/properties of children

Important Tree Terminology

- **binary**: each node has 0, 1, or 2 children
- ***n*-ary**: each node has between 0 and n children
- **full**: each node has strictly either 0 or n children
- **complete**: every level, except possibly the last, is completely filled
- **perfect**: full and complete; all levels filled, all internal nodes have 2 children, all leaves at same depth

Tree Induction Example

Define a Filbert tree to be a binary tree containing 2D points such that:

- Each leaf node contains (3, 1), (-2, -5), or (2,2).
- An internal node with one child labeled (a, b) has label $(a + 1, b - 1)$.
- An internal node with two children labeled (x, y) and (a, b) has label $(x + a, y + b)$.

Prove that the point in the root node of any Filbert tree is on or below the line $x = y$.

Tree Induction Example

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Tree Induction Example (Solution)

Prove that the point in the root node of any Filbert tree is on or below the line $x = y$.
Proof by induction on h, where h is the height of the tree.

- Base case(s): For Filbert tree where $h = 0$, the root node is a leaf and so contains (3, 1), (-2, -5), or (2,2), all of which are on or below the line $x = y$.
- Inductive hypothesis: Suppose that the point in the root node of any Filbert tree is on or below the line $x = y$ for trees of height $h = 0, 1, \ldots, k - 1$ $(k \geq 1)$.

Tree Induction Example (Solution)

- Inductive step: Let T be a Filbert tree of height k. There are 2 cases.
  Case 1: The root of T has one child subtree, whose root contains (a, b). The root of T contains $(a+1, b-1)$. By the inductive hypothesis, (a, b) is on or below $x = y$, i.e. $b \leq a$. Since $b \leq a$, $b - 1 \leq a + 1$, so this point is on or below $x = y$.
  Case 2: The root of T has two child subtrees, whose roots contain (x, y) and (a, b). Then the root of T contains $(x + a, y + b)$. By the inductive hypothesis, $y \leq x$ and $b \leq a$. So $y + b \leq x + a \implies (x + a, y + b)$ is on or below $x = y$.
  In all cases the root node contains a point on or below $x = y$, which is what we needed to show.

(Another) Inequality Induction Example

Prove by induction that for any two lists of nonnegative numbers $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$,

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \geq \left(\sum_{i=1}^{n} x_i y_i\right)^2$$

You may use the AM-GM inequality: For any real numbers $a, b \geq 0$, $\frac{a+b}{2} \geq \sqrt{ab}$.

(Another) Inequality Induction Example

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \geq \left(\sum_{i=1}^{n} x_i y_i\right)^2,$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Questions/Examples

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Graphs: Review

- (Abstract Def) An (undirected) graph is a tuple $G = (V, E)$ where V is any set and $E \subseteq \{\{u, v\} : u, v \in V\}$.

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- (Abstract Def) An (undirected) graph is a tuple $G = (V, E)$ where V is any set and $E \subseteq \{\{u, v\} : u, v \in V\}$.
- (Usable Def) A graph is a set of vertices and edges between them.

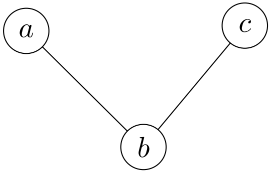


Figure: A graph with vertices $V = \{a, b, c\}$ and edges $E = \{\{a, b\}, \{b, c\}\}$.

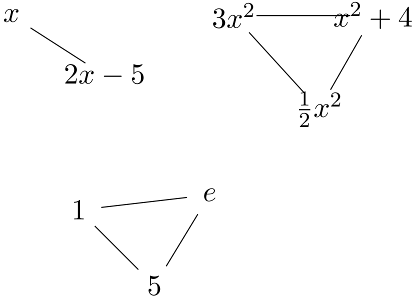
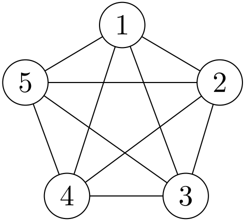


Figure: A graph with edges if nodes are Big-Theta of each other

Special Graphs

- The graph K_n has n nodes and an edge between every pair of vertices:



K_5

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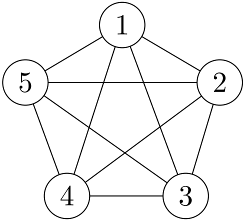
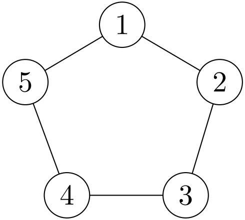


K_5

- The graph C_n has n nodes in a single cycle.

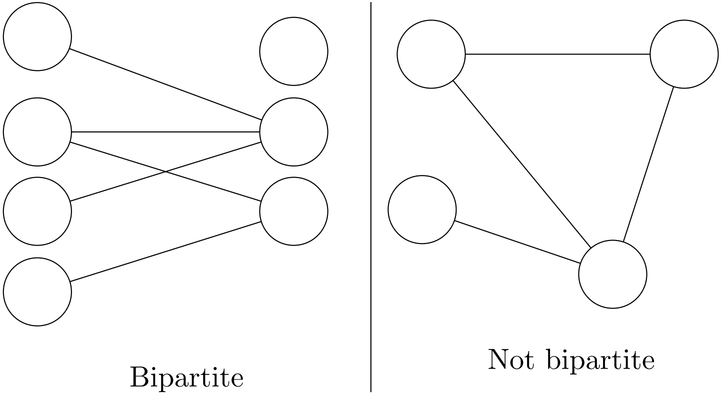


C_5

Bipartite

A graph is bipartite if its vertices can be divided into disjoint sets L and R such that every edge is between L and R.

Bipartite $\Leftrightarrow$ 2-Colorable $\Leftrightarrow$ No odd cycles



Bipartite

Not bipartite

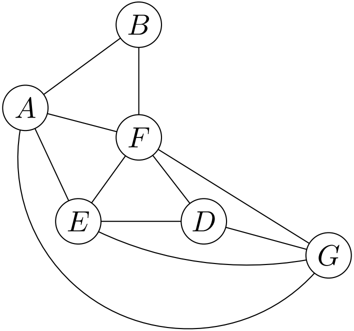
Paths, Walks, and Cycles

A walk in a graph is a sequence of vertices $(v_1, v_2, ..., v_k)$ and edges $(e_1, e_2, ..., e_{k-1})$ where each edge connects the two vertices on either end of it ($e_i = \{v_i, v_{i+1}\}$).

A path is a walk that doesn't repeat vertices.

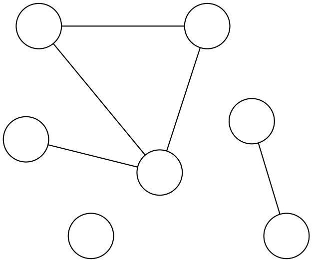
A cycle is a path where the start and end vertices are connected with an edge ($\{v_n, v_1\} \in E$).

The distance between two vertices is the length of the shortest path between them.



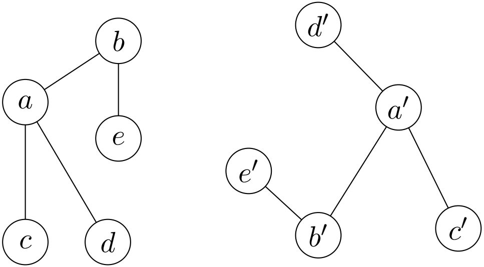
Connected Components

A connected component of a graph is a maximal set of vertices where each pair has a connecting walk. The graph below can be split into 3 connected components.



Isomorphisms

- Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there is a bijection $f : V \to V'$ such that $\{v_1, v_2\} \in E$ if and only if $\{f(v_1), f(v_2)\} \in E'$. (What type of relation is this?)



Practice Questions I

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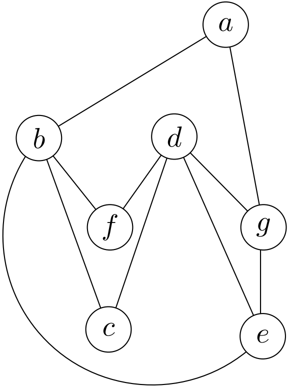
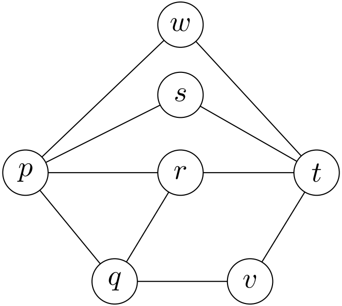
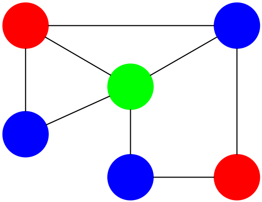
- How many edges are in K_n? C_n?
- (SU24) Are these graphs isomorphic? What is the diameter of the left graph? What is the distance between a and c?





Coloring

- A (proper) k-coloring of a graph $G = (V, E)$ is an function $f : V \to \{1, 2, ..., k\}$ such that for every edge $\{u, v\} \in E, f(u) \neq f(v)$.



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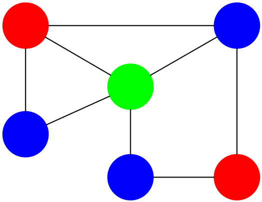
- The chromatic number of a graph G is $\chi(G)$, the smallest k such that G is k-colorable.



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- If G has K_n as a subgraph, $\chi(G) \geq n$.
- If G can be colored with k vertices, then $\chi(G) \leq k$.

Practice Questions II

- What is an upper bound for $\chi(G)$, given that G has n vertices?

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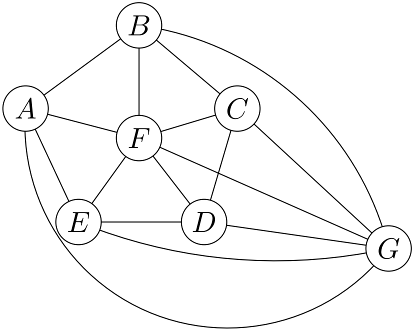
- (SU24) What is the chromatic number of the graph below? Prove it.



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- The *degree* of a vertes is the number of edges incident to is. In a graph $G = (V, E)$, what is $\sum_{v \in V} \deg(v)$?

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- (Past Examlet) In a tree of height h, which best describes the diameter of the tree?
  (a) h (b) $2h$ (c) $\leq 2h$ (d) $h + 1$ (e) $\leq h$

Tree Induction

Given a tree T and a function $w : T \to \mathbb{R}_+$, the *weighted sum* of the tree is $\sum_{v \in T} w(v)2^{-h(v)}$, where $h(v)$ is the depth of the node.

A weight function is *fair* if, for every node $v \in T$,

1. If the node has one child c, $w(u) = w(c)$
2. If the node has two children c_1 and c_2, $w(u) = w(c_1) + w(c_2)$

Let T be a tree and r be its root. Prove by (strong) induction that for any fair weight function w, the weighed sum of the tree is no more than $2w(r)$.

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