# Art Gallery Problem

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# Introduction

Given the layout of a museum, what is the minimum number of guards needed to guard every point in the museum? This problem, often called the Art Gallery Problem, is an example of a problem at the intersection of several areas, including geometry, discrete math, and optimization. By working through this problem, one can explore ideas from different areas of mathematics, and see how these ideas can be combined to solve real-world problems!

# Guards

First, let's define mathematically what we mean to guard a museum. In this problem, guards must stay at fixed positions but are able to see every angle from their position by rotating. To represent this mathematically, a point P in the museum is visible to a guard if the line segment from the guard to point P lies within the museum or along the boundary.

# Finding the Number of Guards

What is the minimum number of guards needed to guard a museum whose floor plan is a polygon with n walls? Note that in order to answer this question, we need to show

- there exist positions for the guards such that every point in the museum is guarded
- no fewer guards can guard every point in the museum.

# Art Gallery Theorem

Art Gallery Theorem: Any museum with n walls can be guarded by at most $\lfloor \frac{n}{3} \rfloor guards$.

# Finding the Number of Guards

This problem was first solved by Vasek Chvatal in 1975 and below, we will give the beautiful proof due to Steve Fisk in 1978. In fact, Fisk's proof of this theorem is constructive, giving an algorithm (or sequence of steps) that tells us exactly where to place the guards. To show that bound in the theorem is tight, consider the museum with 15 walls in the shape of a comb

# Finding the Number of Guards

Then the guard for point A must be stationed within the shaded triangle with vertex A, the guard for point B must be stationed within the shaded triangle with vertex B, etc. Since these triangles do not overlap, at least 5 guards are needed. But by the Art Gallery Theorem,
$\lfloor \frac{15}{3} \rfloor = 5$ guards are also sufficient, which we can observe by placing the guards at the lower left corner of each shaded triangle.
In general, the comb museum layout gives an example of a museum with 3n walls that requires exactly
$\lfloor \frac{3n}{3} \rfloor$=n guards, which shows that the bound in the theorem is best possible.

# Generalization

It seems that the worst case example of a comb museum occurs because there are very sharp "corners" that restrict the placement of guards. What if we consider museums whose walls meet at right angles, creating
$90^\circ$ corners? These floor plans correspond to orthogonal polygons, and three proofs given by Kahn-Klawe-Kleitman, Lubiw,
and Sack-Toussaint show that there is always a configuration of
$\lfloor \frac{n}{4} \rfloor$ guards that will guard the entire museum.

# Applications

Problems from computational geometry also naturally arise in video game programming, where it is often necessary to perform computations based on a virtual world to create a realistic user experience. Think about a video game you have played involving a virtual world and how the game must solve challenges such as detecting when objects collide, representing the surface or terrain of the virtual world, detecting motion from your input to the game, and determining the appearance/visibility of objects as you move through the world. All of these problems involve elements of computational geometry, computer graphics, computer science, and algorithms.

# Other applications of computational geometry include

- route planning for GPS: determining location, speed, and direction
- integrated circuit design
- designing and building objects such as cars, ships, and aircraft
- computer vision, to determine lines of sight and designing special effects in movies

# Proof and Art Gallery Theorem

We will prove this theorem through a sequence of claims. First, a triangulation of a polygon is a decomposition of the polygon into triangles by drawing non-intersecting diagonals between pairs of vertices.

- Claim 1: Any polygon P can be triangulated.
- Claim 2: Any triangulated polygon is 3-colorable.

- For any 3-coloring of a triangulation,
  there exists a color such that the number of vertices of this color is $\leq \lfloor \frac{n}{3} \rfloor$, and placing guards on these vertices will guard the entire museum.