The model category of algebraically cofibrant 2-categories

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Three-dimensional category theory

Theorem (Gordon–Power–Street)

Every tricategory is triequivalent to a **Gray***-category.*

A **Gray**-category is a category enriched over **2-Cat** equipped with Gray's symmetric monoidal closed structure.

To a large extent, one can model the category theory of tricategories by "homotopy coherent" **Gray**-enriched category theory, i.e. category theory enriched over **2-Cat** as a **monoidal model category** (wrt Gray's symmetric monoidal structure and Lack's model structure).

A fundamental obstruction

However, there is a fundamental obstruction to the development of a *purely* **Gray**-enriched model for three-dimensional category theory:

> Not every 2-category is **cofibrant** in Lack's model structure on **2-Cat**.

In practice, the result is that certain basic constructions fail to define **Gray**-functors; they are at best "locally weak **Gray**-functors".

A new base for enrichment

This obstruction can be overcome by the introduction of a new base for enrichment: the monoidal model category $\mathbf{2\text{-}Cat}_Q$ of **algebraically cofibrant 2-categories**, which is the subject of this talk.

We will see that:

- Every object of $\mathbf{2\text{-}Cat}_Q$ is cofibrant.
- $\mathbf{2\text{-}Cat}_Q$ is monoidally Quillen equivalent to $\mathbf{2\text{-}Cat}$.

But further surprises await:

- The full subcategory of fibrant objects in $\mathbf{2\text{-}Cat}_Q$ is equivalent to the category of bicategories and normal pseudofunctors.
- $\mathbf{2\text{-}Cat}_Q$ is a cartesian closed model category.

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1. The category of algebraically cofibrant 2-categories

2. The left-induced model structure

3. Bicategories as fibrant objects

4. Monoidal structures

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The category of free categories

Definition (atomic morphism)

A morphism f in a category is **atomic** if:

- f is not an identity, and
- if $f = hg$, then g is an identity or h is an identity.

Definition (free category)

A category C is **free** if every morphism f in C can be uniquely expressed as a composite of atomic morphisms $(n \geq 0,\ f = f_n \circ \cdots f_1)$.

Definition (morphism of free categories)

A functor $C \longrightarrow D$ between free categories is a **morphism of free categories** if it sends each atomic morphism in C to an atomic morphism or an identity morphism in D.

The category of cofibrant 2-categories

Definition (cofibrant 2-category)

A 2-category is **cofibrant** if its underlying category is free.

Definition (morphism of cofibrant 2-categories)

A 2-functor between cofibrant 2-categories is a **morphism of cofibrant 2-categories** if its underlying functor is a morphism of free categories.

Cofibrant 2-categories and their morphisms form a non-full replete subcategory of $\mathbf{2\text{-}Cat}$, which we denote by $\mathbf{2\text{-}Cat}_Q$.

Cofibrant 2-categories as Q-coalgebras

The inclusion $\mathbf{2\text{-}Cat}_Q \longrightarrow \mathbf{2\text{-}Cat}$ has a right adjoint $Q\colon \mathbf{2\text{-}Cat} \longrightarrow \mathbf{2\text{-}Cat}_Q$, which sends a 2-category A to its **normal pseudofunctor classifier** QA.

$$\frac{QA \longrightarrow B \qquad \text{2-functors}}{A \rightsquigarrow B \qquad \text{normal pseudofunctors}}$$

The normal pseudofunctor classifier QA of a 2-category A may be described as follows:

- The objects of QA are the objects of A.
- The morphisms of QA are composable paths of non-identity morphisms in A.
- The 2-cells of QA between a parallel pair of such composable paths is a 2-cell in A between their composites.

Proposition

The adjunction

$$\begin{tikzcd} \mathbf{2\text{-}Cat} \arrow[r, bend right, "Q"'] \arrow[r, phantom, "\perp"] & \mathbf{2\text{-}Cat}_Q \arrow[l, bend right] \end{tikzcd}$$

is comonadic. In particular, the inclusion functor $\mathbf{2\text{-}Cat}_Q \longrightarrow \mathbf{2\text{-}Cat}$ *creates colimits. Furthermore, the category* $\mathbf{2\text{-}Cat}_Q$ *is locally finitely presentable.*

Plan

Model structures

A model structure on a category $\mathcal{E}$ enables one to "do homotopy theory" in $\mathcal{E}$.

Model structures

A **model structure** on a category $\mathcal{E}$ consists of three classes of morphisms $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ in $\mathcal{E}$ – called **cofibrations**, **weak equivalences**, and **fibrations** – such that:

- $\mathcal{W}$ satisfies the 2-out-of-3 property, and
- $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorisation systems in $\mathcal{E}$.

An object A is **cofibrant** if $0 \longrightarrow A$ is a cofibration.
An object X is **fibrant** if $X \longrightarrow 1$ is a fibration.

The morphisms in the classes $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{W} \cap \mathcal{F}$ are called **trivial cofibrations** and **trivial fibrations** respectively.

Lemma

A model structure on a category may be determined by either:

- *its cofibrations and weak equivalences, or*
- *its trivial fibrations and weak equivalences.*

Lack's model structure for 2-categories

Lack's model structure on 2-Cat

Lack constructed a model structure on **2-Cat** in which a 2-functor is:

- a **weak equivalence** iff it is a **biequivalence**, i.e. is surjective on objects up to equivalence, and is an equivalence on hom-categories;
- a **fibration** iff it is an **equivfibration**, i.e. has the equivalence lifting property, and is an isofibration on hom-categories;
- a **trivial fibration** iff it is surjective on objects, and is a surjective equivalence on hom-categories.

Every 2-category is **fibrant** in this model structure.

A 2-category is **cofibrant** in this model structure if and only if it is a cofibrant 2-category.

The left-induced model structure on $\mathbf{2}\text{-}\mathbf{Cat}_Q$

Theorem (The model category of algebraically cofibrant 2-categories)

There exists a (unique) model structure on $\mathbf{2}\text{-}\mathbf{Cat}_Q$ *in which a morphism of cofibrant 2-categories is:*

- *a* **cofibration** *iff it is a cofibration in Lack's model structure on* $\mathbf{2}\text{-}\mathbf{Cat}$*;*
- *a* **weak equivalence** *iff it is a weak equivalence in Lack's model structure on* $\mathbf{2}\text{-}\mathbf{Cat}$*.*

This model structure is combinatorial.

We say that this model structure is **left-induced** from Lack's model structure on $\mathbf{2}\text{-}\mathbf{Cat}$.

Since everything in sight is sufficiently nice (i.e. $\mathbf{2}\text{-}\mathbf{Cat}$ and $\mathbf{2}\text{-}\mathbf{Cat}_Q$ are locally finitely presentable and Lack's model structure on $\mathbf{2}\text{-}\mathbf{Cat}$ is cofibrantly generated), it suffices to prove that the **acyclicity condition** holds:

RTP: the acyclicity condition

In $\mathbf{2}\text{-}\mathbf{Cat}_Q$, any morphism which has the RLP wrt all cofibrations is a biequivalence.

Cofibrations and trivial fibrations in $\mathbf{2\text{-}Cat}_Q$

Proposition (cofibrations in $\mathbf{2\text{-}Cat}_Q$)

A morphism of cofibrant 2-categories is a **cofibration** *in Lack's model structure iff it is:*

- *injective on objects, and*
- *faithful on $\{atomic\} \cup \{identity\}$ morphisms.*

Proposition (trivial fibrations in $\mathbf{2\text{-}Cat}_Q$)

A morphism of cofibrant 2-categories has the RLP (in $\mathbf{2\text{-}Cat}_Q$) wrt to the cofibrations iff if it is:

- *surjective on objects,*
- *full on $\{atomic\} \cup \{identity\}$ morphisms, and*
- *fully faithful on 2-cells.*

Corollary

The acyclicity condition holds, i.e. every morphism in $\mathbf{2\text{-}Cat}_Q$ which has the RLP wrt all cofibrations is a biequivalence.

A Quillen equivalence

Theorem (2-categories vs algebraically cofibrant 2-categories)

The adjunction

$$\begin{tikzcd} \mathbf{2\text{-}Cat} \arrow[r, phantom, "\perp"] & \mathbf{2\text{-}Cat}_Q \arrow[l, bend right] \end{tikzcd}$$

$$\mathbf{2\text{-}Cat} \underset{Q}{\overset{\longleftarrow}{\underset{\longrightarrow}{\perp}}} \mathbf{2\text{-}Cat}_Q$$

is a Quillen equivalence between Lack's model structure on **2-Cat** *and the model structure on* $\mathbf{2\text{-}Cat}_Q$.

Proof.

By definition of the model structure on $\mathbf{2\text{-}Cat}_Q$, the left adjoint preserves cofibrations, and preserves and reflects weak equivalences.
For each 2-category A, the counit morphism $QA \longrightarrow A$ is a weak equivalence in **2-Cat**. □

Plan

Fibrant objects

The functor $Q\colon \mathbf{2\text{-}Cat} \longrightarrow \mathbf{2\text{-}Cat}_Q$ is a right Quillen functor.
Hence, for every 2-category A, QA is a fibrant object in $\mathbf{2\text{-}Cat}_Q$.

Proposition

A cofibrant 2-category is a fibrant object in the left-induced model structure on $\mathbf{2\text{-}Cat}_Q$ if and only if it is a retract in $\mathbf{2\text{-}Cat}_Q$ of the normal pseudofunctor classifier QA of some 2-category A.

Proof.

"If": A retract of a fibrant object is fibrant.
"Only if": For every cofibrant 2-category A, the unit morphism $\alpha\colon A \longrightarrow QA$ is a trivial cofibration in $\mathbf{2\text{-}Cat}_Q$.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\alpha"'] & A \\ QA \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

□

The full subcategory of fibrant objects

The full image of the functor $Q\colon \mathbf{2\text{-}Cat} \longrightarrow \mathbf{2\text{-}Cat}_Q$ is the category $\mathbf{2\text{-}Cat}_{\mathbf{nps}}$ of 2-categories and normal pseudofunctors.

$$\mathbf{2\text{-}Cat}_Q(QA, QB) \cong \mathbf{2\text{-}Cat}(QA, B) \cong \mathbf{2\text{-}Cat}_{\mathbf{nps}}(A, B)$$

So we have a functor $Q\colon \mathbf{2\text{-}Cat}_{\mathbf{nps}} \longrightarrow (\mathbf{2\text{-}Cat}_Q)_{\mathrm{fib}}$ which is

- fully faithful, and
- surjective on objects up to retracts.

Hence this functor witnesses $(\mathbf{2\text{-}Cat}_Q)_{\mathrm{fib}}$ as the **Cauchy completion** of $\mathbf{2\text{-}Cat}_{\mathbf{nps}}$. But the Cauchy completion of $\mathbf{2\text{-}Cat}_{\mathbf{nps}}$ is none other than $\mathbf{Bicat}_{\mathbf{nps}}$.

Theorem

The **normal strictification** *functor* $Q\colon \mathbf{Bicat}_{\mathbf{nps}} \longrightarrow \mathbf{2\text{-}Cat}_Q$ *is fully faithful, and its essential image consists of the fibrant objects for the left-induced model structure.*

Intrinsic characterisation of fibrant objects

Theorem

Let A be a cofibrant 2-category. Then the following are equivalent.

1. *A is a fibrant object in the left-induced model structure on $\mathbf{2\text{-}Cat}_Q$.*
2. *$A \cong QB$ for some bicategory B.*
3. *Every non-identity morphism in A is isomorphic (via an invertible 2-cell) to an atomic morphism in A.*
4. *A has the RLP in $\mathbf{2\text{-}Cat}_Q$ wrt $\mathbf{3} \longrightarrow Q\mathbf{3}$.*

Proof.

The step (iii) $\Rightarrow$ (ii) uses two-dimensional monad theory. □

Fibrations between fibrant objects

Theorem

Let $F\colon A \longrightarrow B$ be a normal pseudofunctor between bicategories. Then the following are equivalent.

1. *$QF\colon QA \longrightarrow QB$ is a fibration in the left-induced model structure on* $\mathbf{2\text{-}Cat}_Q$.
2. *$F\colon A \longrightarrow B$ is an equivfibration, i.e. has the equivalence lifting property and is an isofibration on hom-categories.*

This theorem characterises the fibrations with fibrant codomain in $\mathbf{2\text{-}Cat}_Q$.

I do not have an explicit description of the fibrations in $\mathbf{2\text{-}Cat}_Q$ with arbitrary codomain.

Remark

The left-induced model structure on $\mathbf{2\text{-}Cat}_Q$ is not right proper.

Plan

The Gray monoidal structure

Gray's symmetric monoidal structure on $\mathbf{2}\text{-}\mathbf{Cat}$ restricts to one on $\mathbf{2}\text{-}\mathbf{Cat}_Q$.

$$\mathbf{2} \otimes \mathbf{2} \quad = \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

By the adjoint functor theorem (or by direct construction), this symmetric monoidal structure on $\mathbf{2}\text{-}\mathbf{Cat}_Q$ is **closed**.

Theorem

$\mathbf{2}\text{-}\mathbf{Cat}_Q$ *is a* **monoidal model category** *wrt the Gray monoidal structure and the left-induced model structure. The adjunction* $V \dashv Q \colon \mathbf{2}\text{-}\mathbf{Cat} \longrightarrow \mathbf{2}\text{-}\mathbf{Cat}_Q$ *is a monoidal Quillen equivalence.*

If A and B are bicategories, then the Gray internal hom $[QA, QB]$ is Q of the bicategory of normal pseudofunctors, pseudonatural transformations, and modifications between them.

The cartesian closed structure

Unlike Lack's model structure on **2-Cat**, the model structure on $\mathbf{2\text{-}Cat}_Q$ is also cartesian.

Theorem

The category $\mathbf{2\text{-}Cat}_Q$ *is cartesian closed, and is a* **cartesian model category** *wrt the left-induced model structure.*

$$\mathbf{2} \otimes \mathbf{2} \quad = \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\cong"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \qquad ; \qquad \mathbf{2} \boxtimes \mathbf{2} \quad = \quad \begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \arrow[dl, phantom, near start, "\cong"] \\ \cdot \arrow[r] \arrow[ur, phantom, near start, "\cong"] & \cdot \end{tikzcd}$$

N.B. The category $\mathbf{Bicat_{nps}}$ is cartesian closed and the full embedding $Q\colon \mathbf{Bicat_{nps}} \longrightarrow \mathbf{2\text{-}Cat}_Q$ preserves cartesian internal homs.

So if A and B are bicategories, then the cartesian internal hom $\underline{\mathbf{Hom}}(QA, QB)$ is Q of the bicategory of normal pseudofunctors, "enhanced" pseudonatural transformations, and modifications between them.

An accessible ∞-cosmos of bicategories

Let $N_g : \mathbf{2\text{-}Cat}_Q \longrightarrow \mathbf{sSet}$ denote the "nerve" functor induced by the cosimplicial object $\Delta \longrightarrow \mathbf{2\text{-}Cat}_Q$ that sends $[n]$ to $Q[n]$.
(N.B. The "nerve" of a cofibrant 2-category forgets its non-invertible 2-cells.)

Theorem

The category $\mathbf{2\text{-}Cat}_Q$ *is a combinatorial* **Joyal-enriched model category** *wrt to the left-induced model structure and the simplicial enrichment obtained from the cartesian closed structure of* $\mathbf{2\text{-}Cat}_Q$ *by change of base along* $N_g : \mathbf{2\text{-}Cat}_Q \longrightarrow \mathbf{sSet}$.

Hence, by Steve's talk on Wednesday, we may deduce:

Corollary

The category of bicategories and normal pseudofunctors underlies an **accessible ∞-cosmos** *whose isofibrations are the equivfibrations and whose simplicial hom-sets are the Duskin nerves of the "piths" of the cartesian internal hom bicategories.*

Thank you!