A model-independent construction of the Gray monoidal structure for $(\infty, 2)$-categories

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Plan

1. The Gray monoidal structure for 2-categories
2. Day's extension theorem
3. The Gray monoidal structure for $(\infty, 2)$-categories
4. The comparison problem

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Monoidal categories

Definition (monoidal category)

Let $\mathcal{C}$ be a category. A **monoidal structure** on $\mathcal{C}$ consists of:

- a functor $\otimes \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$,
- an object $I \in \mathcal{C}$,
- a natural isomorphism $\alpha_{A,B,C} \colon (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$,
- a natural isomorphism $\lambda_A \colon I \otimes A \cong A$,
- a natural isomorphism $\rho_A \colon A \otimes I \cong A$,

subject to the following two coherence axioms:

$$\begin{tikzcd} ((AB)C)D \arrow[r, "\alpha"] \arrow[d, "\alpha 1"'] & (AB)(CD) \arrow[r, "\alpha"] & A(B(CD)) \\ (A(BC))D \arrow[rr, "\alpha"'] & & A((BC)D) \arrow[u, "1\alpha"'] \end{tikzcd} \qquad \begin{tikzcd} (AI)B \arrow[rr, "\alpha"] \arrow[dr, "\rho 1"'] & & A(IB) \arrow[dl, "1\lambda"] \\ & AB & \end{tikzcd}$$

A **monoidal category** is a category equipped with a monoidal structure.

A monoidal structure on a category $\mathcal{C}$ is **biclosed** if its tensor product functor $\otimes \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint in each variable:

$$\mathcal{C}(A, \underline{\mathrm{Hom}}^l(B, C)) \cong \mathcal{C}(A \otimes B, C) \cong \mathcal{C}(B, \underline{\mathrm{Hom}}^r(A, C)).$$

2-categories

Definition (2-category)

A 2-**category** $\mathcal{A}$ consists of:

- a set of objects $\mathrm{ob}\,\mathcal{A}$,
- a hom-category $\mathcal{A}(A, B)$ for each pair of objects $A, B \in \mathrm{ob}\,\mathcal{A}$,
- a composition functor $\circ \colon \mathcal{A}(B, C) \times \mathcal{A}(A, B) \longrightarrow \mathcal{A}(A, C)$ for each $A, B, C \in \mathrm{ob}\,\mathcal{A}$,
- an identity $1_A \in \mathcal{A}(A, A)$ for each $A \in \mathrm{ob}\,\mathcal{A}$,

subject to (strict) associativity and unit axioms.

Thus a 2-category consists of **objects** A, 1-**morphisms** $f \colon A \longrightarrow B$ which may be composed as in an ordinary category, and 2-**morphisms**
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$
which may be composed vertically and horizontally:

$$\begin{tikzcd} A \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, bend left, "k"] \arrow[r, bend right, "l"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \end{tikzcd}$$

The category of 2-categories

Definition (2-functor)

Let $\mathcal{A}$ and $\mathcal{B}$ be a pair of 2-categories. A 2-**functor** $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ consists of:

- a function $F\colon \operatorname{ob}\mathcal{A} \longrightarrow \operatorname{ob}\mathcal{B}$,
- a functor $F\colon \mathcal{A}(A, B) \longrightarrow \mathcal{B}(FA, FB)$ for each pair of objects $A, B \in \mathcal{A}$,

which (strictly) preserve horizontal composition and identities.

Let **2-Cat** denote the category of 2-categories and 2-functors.

The subject of the first half of this talk is the following monoidal structure on **2-Cat** due to John Gray in the early 1970s.

Theorem (Gray)

The category **2-Cat** *admits a (non-symmetric) biclosed monoidal structure whose tensor product operation* $\mathcal{A} \otimes \mathcal{B}$ *is the* ***Gray tensor product*** *of* 2*-categories, and whose internal homs are the* ***Gray hom*** 2*-**categories*** $\underline{\mathsf{Fun}}^{\mathrm{lax}}(\mathcal{A}, \mathcal{B})$ *and* $\underline{\mathsf{Fun}}^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})$.

We shall presently describe this biclosed monoidal structure on **2-Cat**, beginning with its internal homs. But first, let's draw some pictures.

The unit object of the Gray monoidal structure on **2-Cat** is the terminal 2-category **1**, which is freely generated by a single object $\bullet$.

Let **2** denote the poset $\{0 < 1\}$, which we may regard as the 2-category freely generated by a 1-morphism: $\bullet \longrightarrow \bullet$.

The Gray tensor product $\mathbf{2} \otimes \mathbf{2}$ is a "lax square":

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
$$

The triple Gray tensor product $\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2}$ is a commutative cube whose faces are lax squares:

$$
\begin{tikzcd}
 & \bullet \arrow[rr] \arrow[dr, phantom, "\Downarrow"] & & \bullet \arrow[dr] \arrow[dd, phantom, "\Downarrow"] & \\
\bullet \arrow[ur] \arrow[rr] \arrow[dr] & & \bullet \arrow[ur] \arrow[dr] & & \bullet \\
 & \bullet \arrow[rr] \arrow[ur, phantom, "\Downarrow"] & & \bullet \arrow[ur] &
\end{tikzcd}
=
\begin{tikzcd}
 & \bullet \arrow[rr] \arrow[dr] \arrow[dd, phantom, "\Downarrow"] & & \bullet \arrow[dr] \arrow[dl, phantom, "\Downarrow"] & \\
\bullet \arrow[ur] \arrow[dr] & & \bullet \arrow[rr] \arrow[dr, phantom, "\Downarrow"] & & \bullet \\
 & \bullet \arrow[rr] \arrow[ur] & & \bullet \arrow[ur] &
\end{tikzcd}
$$

Lax natural transformations

Definition ((op)lax natural transformation)

Let $F, G\colon \mathcal{A} \longrightarrow \mathcal{B}$ be 2-functors between 2-categories. A **lax natural transformation** (resp. **oplax natural transformation**) $\sigma\colon F \longrightarrow G$ consists of:

- for each object $A \in \mathcal{A}$, a 1-morphism $\sigma_A\colon FA \longrightarrow GA$ in $\mathcal{B}$,
- for each morphism $f\colon A \longrightarrow B$ in $\mathcal{A}$, a 2-morphism

$$\begin{tikzcd} FA \arrow[r, "\sigma_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\sigma_f"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\sigma_B"'] & GB \end{tikzcd} \qquad \left(\text{resp.} \quad \begin{tikzcd} FA \arrow[r, "\sigma_A"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Uparrow\sigma_f"] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\sigma_B"'] & GB \end{tikzcd} \right)$$

  in $\mathcal{B}$,

subject to axioms.

Definition (modification)

Let $\sigma, \tau\colon F \to G$ be (op)lax natural transformations between 2-functors. A **modification** $m\colon \sigma \longrightarrow \tau$ consists of a 2-cell $m_A\colon \sigma_A \longrightarrow \tau_A$ for each object $A \in \mathcal{A}$, subject to axioms.

The Gray tensor product of 2-categories

Gray hom 2-categories

For each pair of 2-categories $\mathcal{A}$ and $\mathcal{B}$, let $\underline{\mathsf{Fun}}^{\mathrm{lax}}(\mathcal{A},\mathcal{B})$ (resp. $\underline{\mathsf{Fun}}^{\mathrm{oplax}}(\mathcal{A},\mathcal{B})$) denote the 2-category with:

- objects: 2-functors $\mathcal{A} \longrightarrow \mathcal{B}$,
- 1-morphisms: lax (resp. oplax) natural transformations, and
- 2-morphisms: modifications.

The **Gray tensor product** of a pair of 2-categories may be defined by the following universal property:

Proposition (Gray)

For each pair of 2-categories $\mathcal{A}$ and $\mathcal{B}$, there exists a 2-category $\mathcal{A}\otimes\mathcal{B}$ with natural isomorphisms:

$$\mathbf{2\text{-}Cat}(\mathcal{A},\underline{\mathsf{Fun}}^{\mathrm{oplax}}(\mathcal{B},\mathcal{C})) \cong \mathbf{2\text{-}Cat}(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) \cong \mathbf{2\text{-}Cat}(B,\underline{\mathsf{Fun}}^{\mathrm{lax}}(\mathcal{A},\mathcal{C})).$$

The Gray tensor product of 2-categories (cont.)

Proposition (Gray)

Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be 2-categories. A 2-functor $F\colon \mathcal{A}\otimes\mathcal{B} \longrightarrow \mathcal{C}$ amounts to:

- *a function* $\mathrm{ob}\,F\colon\ \mathrm{ob}\,\mathcal{A}\times\mathrm{ob}\,\mathcal{B}\longrightarrow\mathrm{ob}\,\mathcal{C}$,
- *a 2-functor* $F(A,-)\colon\mathcal{B}\longrightarrow\mathcal{C}$ *agreeing with the function* $\mathrm{ob}\,F$ *on objects, for each object* $A\in\mathcal{A}$,
- *a 2-functor* $F(-,B)\colon\mathcal{A}\longrightarrow\mathcal{C}$ *agreeing with the function* $\mathrm{ob}\,F$ *on objects, for each object* $B\in\mathcal{B}$,
- *a 2-morphism*

$$\begin{tikzcd} F(A,B) \arrow[r, "F(A{,}g)"] \arrow[d, "F(f{,}B)"'] \arrow[dr, phantom, "\Downarrow F(f{,}g)"] & F(A,B') \arrow[d, "F(f{,}B')"] \\ F(A',B) \arrow[r, "F(A'{,}g)"'] & F(A',B') \end{tikzcd}$$

in $\mathcal{C}$ for each pair of 1-morphisms $f\colon A\longrightarrow A'$ in $\mathcal{A}$ and $g\colon B\longrightarrow B'$ in $\mathcal{B}$,

subject to axioms.

One can deduce from the above Proposition a presentation of the 2-category $\mathcal{A}\otimes\mathcal{B}$ by generators and relations.

The monoidal category of Gray cubes

Note that a key generator in the presentation of a Gray tensor product $\mathcal{A} \otimes \mathcal{B}$ is the 2-morphism

$$\begin{tikzcd} A \otimes B \arrow[r, "A \otimes g"] \arrow[d, "f \otimes B"'] \arrow[dr, phantom, "\Downarrow f \otimes g"] & A \otimes B' \arrow[d, "f \otimes B'"] \\ A' \otimes B \arrow[r, "A' \otimes g"'] & A' \otimes B' \end{tikzcd}$$

for each pair of 1-morphisms $f \colon A \longrightarrow A'$ in $\mathcal{A}$ and $g \colon B \longrightarrow B'$ in $\mathcal{B}$.
We may regard this diagram as the image of the 2-functor $f \otimes g \colon \mathbf{2} \otimes \mathbf{2} \longrightarrow \mathcal{A} \otimes \mathcal{B}$.

Similarly, a key relation in the presentation of a triple Gray tensor product $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$ is the commutative cube arising as the image of the 2-functor $f \otimes g \otimes h \colon \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \longrightarrow \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$ for each triple of 1-morphisms f in $\mathcal{A}$, g in $\mathcal{B}$, and h in $\mathcal{C}$.

Definition (The monoidal category of Gray cubes)

Let $\mathcal{Q}$ denote the full subcategory of **2-Cat** consisting of the Gray cubes $\mathbf{2}^{\otimes n}$ for all $n \geq 0$. Note that the Gray monoidal structure on **2-Cat** restricts to a monoidal structure on $\mathcal{Q}$, since the Gray tensor product of two Gray cubes is a Gray cube.

Plan

Dense functors

A full subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is said to be **dense in** $\mathcal{C}$ if every object of $\mathcal{C}$ is a "canonical colimit" of objects in $\mathcal{A}$. More generally:

Definition (dense functor)

A functor $J \colon \mathcal{A} \longrightarrow \mathcal{C}$ is **dense** if every object $C \in \mathcal{C}$ is the colimit of the composite functor $\mathcal{A}/C \xrightarrow{\mathrm{dom}} \mathcal{A} \xrightarrow{J} \mathcal{C}$, i.e.

$$C \cong \varinjlim_{\substack{(A, JA \to C) \\ \in \mathcal{A}/C}} JA.$$

Equivalently, a functor $J \colon \mathcal{A} \longrightarrow \mathcal{C}$ is dense iff the restricted Yoneda embedding

$$\mathcal{C} \xrightarrow{y_{\mathcal{C}}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) \xrightarrow{J^*} \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set}),$$

which sends an object C to the presheaf $A \mapsto \mathcal{C}(JA, C)$, is fully faithful.

A full subcategory $\mathcal{A}$ of a category $\mathcal{C}$ is **dense in** $\mathcal{C}$ if the full inclusion functor $\mathcal{A} \longrightarrow \mathcal{C}$ is dense.

Example (Yoneda embedding)

For every small category $\mathcal{A}$, the Yoneda embedding $y_{\mathcal{A}} \colon \mathcal{A} \longrightarrow \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ is dense.

Day's extension theorem for monoidal categories

The following theorem is a special case of the main theorem from Brian Day's 1970 PhD thesis (cf. Ross Street, arXiv:0303175, Proposition 9).

Theorem (Day's extension theorem)

Let $\mathcal{A}$ be a small monoidal category, let $\mathcal{C}$ be a complete and cocomplete category, and let $J\colon \mathcal{A} \longrightarrow \mathcal{C}$ be a fully faithful and dense functor. Then there exists a unique biclosed monoidal structure on $\mathcal{C}$ for which $J\colon \mathcal{A} \longrightarrow \mathcal{C}$ is strong monoidal iff there exist functors $H^l, H^r\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$ and isomorphisms

$$\mathcal{C}(JA, H^l(B,C)) \cong \mathcal{C}(J(A \otimes B), C) \cong \mathcal{C}(JB, H^r(A,C))$$

natural in $A, B \in \mathcal{A}$, $C \in \mathcal{C}$.

Idea of proof.

Since $\mathcal{A}$ is small and monoidal, the presheaf category $\mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ is biclosed monoidal by **Day convolution**. Since $J\colon \mathcal{A} \to \mathcal{C}$ is dense and $\mathcal{C}$ is cocomplete, $\mathcal{C}$ is equivalent to a full reflective subcategory of $\mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$. **Day's reflection theorem** then gives necessary and sufficient conditions for the Day convolution structure to reflect to a biclosed monoidal structure on $\mathcal{C}$. □

Day's extension theorem for monoidal categories (cont.)

Theorem (Day's extension theorem)

Let $\mathcal{A}$ be a small monoidal category, let $\mathcal{C}$ be a complete and cocomplete category, and let $J\colon \mathcal{A} \longrightarrow \mathcal{C}$ be a fully faithful and dense functor. Then there exists a unique biclosed monoidal structure on $\mathcal{C}$ for which $J\colon \mathcal{A} \longrightarrow \mathcal{C}$ is strong monoidal iff there exist functors $H^l, H^r\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$ and isomorphisms

$$\mathcal{C}(JA, H^l(B,C)) \cong \mathcal{C}(J(A \otimes B), C) \cong \mathcal{C}(JB, H^r(A,C))$$

natural in $A, B \in \mathcal{A}$, $C \in \mathcal{C}$.

The tensor product functor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$ is given by the left Kan extension along $\mathcal{A} \times \mathcal{A} \xrightarrow{J \times J} \mathcal{C} \times \mathcal{C}$ of the composite functor $\mathcal{A} \times \mathcal{A} \xrightarrow{\otimes} \mathcal{A} \xrightarrow{J} \mathcal{C}$, so:

$$C \otimes D \cong \varinjlim_{\substack{(A, JA \to C) \\ \in \mathcal{A}/C}} \; \varinjlim_{\substack{(B, JB \to D) \\ \in \mathcal{A}/D}} J(A \otimes B).$$

Example (Day convolution)

Let $\mathcal{A}$ be a small monoidal category. There exists a unique biclosed monoidal structure on $\mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ for which the Yoneda embedding $\mathcal{A} \longrightarrow \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ is strong monoidal, known as the **Day convolution** monoidal structure.

Street's construction of the Gray monoidal structure

In an unpublished manuscript of 1988, Ross Street gives a construction of the Gray monoidal structure on the category **2-Cat** using Day's extension theorem.
He first gives a direct combinatorial definition of the full subcategory $\mathcal{Q} \subseteq$ **2-Cat** of Gray cubes and the Gray monoidal structure thereon.
He then proves by direct calculation that:

Proposition

*The full inclusion $\mathcal{Q} \longrightarrow$ **2-Cat** is a dense functor.*

Finally he shows that the functors $H^l, H^r \colon \mathcal{Q}^{\mathrm{op}} \times \mathbf{2\text{-}Cat} \longrightarrow \mathbf{2\text{-}Cat}$ defined by the Gray homs $\underline{\mathsf{Fun}}^{\mathrm{oplax}}(\mathbf{2}^{\otimes n}, \mathcal{A})$ and $\underline{\mathsf{Fun}}^{\mathrm{lax}}(\mathbf{2}^{\otimes m}, \mathcal{A})$ enjoy the natural isomorphisms:

$$\mathbf{2\text{-}Cat}(\mathbf{2}^{\otimes m}, \underline{\mathsf{Fun}}^{\mathrm{oplax}}(\mathbf{2}^{\otimes n}, \mathcal{A})) \cong \mathbf{2\text{-}Cat}(\mathbf{2}^{\otimes (m+n)}, \mathcal{A}) \cong \mathbf{2\text{-}Cat}(\mathbf{2}^{\otimes n}, \underline{\mathsf{Fun}}^{\mathrm{lax}}(\mathbf{2}^{\otimes m}, \mathcal{A})).$$

Whence Day's extension theorem implies:

Theorem

*There exists a unique biclosed monoidal structure on **2-Cat** for which the full inclusion $\mathcal{Q} \longrightarrow$ **2-Cat** is strong monoidal.*

By uniqueness, this monoidal structure is equivalent to the Gray monoidal structure on **2-Cat** originally constructed by Gray.

Monoidal ∞-categories

We now move to the setting of ∞-category theory.
(N.B. ∞-category $= (\infty, 1)$-category.)

Definition (monoidal ∞-category)

Let $\mathcal{C}$ be an ∞-category. A **monoidal structure** on $\mathcal{C}$ is a functor (of ∞-categories) $\mathcal{C}^{\otimes} \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}_{\infty}$ such that $(\mathcal{C}^{\otimes})_1 = \mathcal{C}$ and such that the induced functor

$$(\mathcal{C}^{\otimes})_n \xrightarrow{\sim} \mathcal{C}^{\times n}$$

is an equivalence for all $n \geq 0$.
A **monoidal ∞-category** is an ∞-category equipped with a monoidal structure.

This definition gives a neat packaging of the infinite levels of coherence data of a monoidal structure on an ∞-category. For example, the morphism $\delta^1 \colon [1] \to [2]$ in Δ induces the tensor product functor $\otimes$ as the composite $\mathcal{C} \times \mathcal{C} \simeq (\mathcal{C}^{\otimes})_2 \xrightarrow{(\delta_1)^*} \mathcal{C}$, and the associativity constraint is induced by the commutative square

$$[1] \xrightarrow{\delta^1} [2] \xrightarrow{\delta^1} [3] \qquad = \qquad [1] \xrightarrow{\delta^1} [2] \xrightarrow{\delta^2} [3]$$

in Δ, and so on.
Say a monoidal structure on an ∞-category $\mathcal{C}$ is **biclosed** if the tensor product functor $\otimes \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint in each variable.

Day's extension theorem for monoidal ∞-categories

Let $\mathscr{S}$ denote the ∞-category of ∞-groupoids.
We say that a functor of ∞-categories $J\colon \mathcal{A} \longrightarrow \mathcal{C}$ is **dense** if the restricted Yoneda embedding $\widetilde{J}\colon \mathcal{C} \longrightarrow \mathsf{Fun}(\mathcal{A}^{\mathrm{op}}, \mathscr{S})$ is fully faithful.

Theorem (Day's extension theorem for monoidal ∞-categories)

Let $\mathcal{A}$ be a small monoidal ∞-category, let $\mathcal{C}$ be a complete and cocomplete ∞-category, and let $J\colon \mathcal{A} \longrightarrow \mathcal{C}$ be a fully faithful and dense functor. Then there exists a unique biclosed monoidal structure on $\mathcal{C}$ for which $J\colon \mathcal{A} \longrightarrow \mathcal{C}$ is strong monoidal iff there exist functors $H^l, H^r\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$ and equivalences

$$\mathcal{C}(JA, H^l(B, C)) \simeq \mathcal{C}(J(A \otimes B), C) \simeq \mathcal{C}(JB, H^r(A, C))$$

natural in $A, B \in \mathcal{A}$, $C \in \mathcal{C}$.

Proof.

Generalisations of Day's convolution theorem and Day's reflection theorem to monoidal ∞-categories have already been proven (e.g. in Lurie's Higher Algebra), so we can prove Day's extension theorem for monoidal ∞-categories in the same way as for ordinary monoidal categories. □

Plan

The state of the art

Heuristic Definition

An $(\infty, 2)$-**category** is an ∞-category enriched in ∞-categories.

Let $\mathbf{Cat}_{(\infty,2)}$ denote the ∞-category of $(\infty, 2)$-categories. There exist many models for this ∞-category, that is, many model categories whose ∞-categorical localisations are equivalent to $\mathbf{Cat}_{(\infty,2)}$. However, only a few of these models admit a **monoidal model structure** intended to present the Gray monoidal structure on $\mathbf{Cat}_{(\infty,2)}$. These are:

- the (pre-)complicial sets of Verity (see arXiv:0604414),
- the ∞-bicategories of Lurie, with Gray monoidal structure defined by Gagna–Harpaz–Lanari (see arXiv:2006.14495), and
- the comical sets of Campion–Kapulkin–Maehara (see arXiv:2005.07603).

These three models are very closely related, and the Gray monoidal structures defined thereon are known to be equivalent.
However, the Gray tensor products defined for these models are very difficult to calculate. In particular, it has not yet been shown that any of them agree with the classical Gray tensor product on the Gray cubes.

The state of the art (cont.)

In Yuki Maehara's 2020 PhD thesis on *The Gray tensor product for 2-quasi-categories* (see arXiv:2003.11757), he defines a Gray tensor product for 2-quasi-categories (a model for $(\infty, 2)$-categories which we shall meet again later). He shows (among other things) that this Gray tensor product does agree with the classical Gray tensor product on the Gray cubes (and on many other 2-categories).

However, this tensor product functor is not part of a monoidal model structure (instead, Yuki shows that it is part of a kind of "lax monoidal" model structure), and so we are not able to show with our current technology that it presents a monoidal structure on the ∞-category $\mathbf{Cat}_{(\infty,2)}$.

Another approach to the Gray monoidal structure on $\mathbf{Cat}_{(\infty,2)}$ is given by Gaitsgory and Rozenblyum in the Appendix to their book on Derived Algebraic Geometry. However, their account relies on many unproven assertions; in particular, they do not prove that their definitions indeed comprise a monoidal structure on the ∞-category $\mathbf{Cat}_{(\infty,2)}$.

Our goal for the remainder of this talk is to construct a biclosed monoidal structure on $\mathbf{Cat}_{(\infty,2)}$ which we show does agree with the classical Gray tensor product on the Gray cubes, and to show it is unique with this property.

Gaunt 2-categories

We shall use the following "model-independent" fact about $\mathbf{Cat}_{(\infty,2)}$ (which is one of Barwick and Schommer-Pries's axioms for $\mathbf{Cat}_{(\infty,2)}$, see arXiv:1112.0040).

Definition (gaunt 2-category)

A 2-category A is **gaunt** if its only invertible 1-morphisms and 2-morphisms are the identities.
Let $\mathbf{Gaunt}_2$ denote the full subcategory of $\mathbf{2\text{-}Cat}$ consisting of the gaunt 2-categories.

Example

Every Gray cube $\mathbf{2}^{\otimes n}$ is a gaunt 2-category.

Fact

There is a fully faithful functor of ∞-categories $\mathbf{Gaunt}_2 \longrightarrow \mathbf{Cat}_{(\infty,2)}$.

Corollary

There is a fully faithful functor of ∞-categories $\mathcal{Q} \longrightarrow \mathbf{Cat}_{(\infty,2)}$.

Joyal's category Θ_2

Recall (The simplex category Δ)

The simplex category Δ consists of the posets $\{0 < 1 < \cdots < n\}$ for each $n \geq 0$, and the order-preserving maps between them. We may also regard Δ as the full subcategory of **Cat** containing the categories freely generated by the graphs:

$$\bullet \qquad \bullet \longrightarrow \bullet \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \ldots$$

Joyal's category Θ_2 is a 2-dimensional analogue of the simplex category Δ. It may be defined as the full subcategory of **2-Cat** containing the 2-categories freely generated by the 2-**dimensional globular pasting diagrams**, e.g.:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=M, phantom, "\Downarrow"] \arrow[Rightarrow, from=M, to=D, phantom, "\Downarrow"] \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r] & \bullet \arrow[Rightarrow, from=U, to=M, phantom, "\Downarrow"] \arrow[Rightarrow, from=M, to=D, phantom, "\Downarrow"] \end{tikzcd}$$

Each 2-category belonging to Θ_2 is gaunt. Hence we have a fully faithful functor of ∞-categories $\Theta_2 \longrightarrow \mathbf{Cat}_{(\infty,2)}$.

Fact

The full inclusion $\Theta_2 \longrightarrow \mathbf{Cat}_{(\infty,2)}$ is dense.

The Gray cubes are dense in $\mathbf{Cat}_{(\infty,2)}$

Proposition

Each 2-category belonging to Θ_2 is a retract in $\mathbf{Gaunt}_2$ of a Gray cube.

Proof.

By clever combinatorics. □

E.g.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\alpha"] & b \arrow[d, "g"] \\ c \arrow[r, "k"'] & d \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} a \arrow[r, "gf"] \arrow[d, "kh"'] \arrow[dr, phantom, "\Downarrow\alpha"] & d \arrow[d, equal] \\ d \arrow[r, equal] & d \end{tikzcd}$$

Corollary

The full inclusion $\mathcal{Q} \longrightarrow \mathbf{Cat}_{(\infty,2)}$ is dense.

Proof.

The idempotent splitting completion of $\mathcal{Q}$ in $\mathbf{Cat}_{(\infty,2)}$ contains Θ_2, which is dense in $\mathbf{Cat}_{(\infty,2)}$. It now follows by standard facts about dense functors that $\mathcal{Q}$ is also dense in $\mathbf{Cat}_{(\infty,2)}$. □

To apply Day's extension theorem to the dense functor $\mathcal{Q} \longrightarrow \mathbf{Cat}_{(\infty,2)}$, it remains to prove the necessary and sufficient condition. To do this, we use some results from Yuki Maehara's PhD thesis on *The Gray tensor product for 2-quasi-categories*.

The model structure for 2-quasi-categories

There exists a model structure on $\widehat{\Theta_2} := \mathrm{Fun}(\Theta_2^{\mathrm{op}}, \mathbf{Set})$, due to Dimitri Ara, whose ∞-categorical localisation is $\mathbf{Cat}_{(\infty,2)}$, and whose fibrant objects are called 2-**quasi-categories**.

There exists a standard "nerve" embedding $N\colon \mathbf{Gaunt}_2 \longrightarrow \widehat{\Theta_2}$ whose composition with the localisation functor $\widehat{\Theta_2} \longrightarrow \mathbf{Cat}_{(\infty,2)}$ is the full inclusion $\mathbf{Gaunt}_2 \longrightarrow \mathbf{Cat}_{(\infty,2)}$.

In his PhD thesis, Maehara defined and studied (among other things) a "Gray tensor product" functor $\widehat{\Theta_2} \times \widehat{\Theta_2} \xrightarrow{\otimes_M} \widehat{\Theta_2}$ defined by:

$$X \otimes_M Y = \varinjlim_{(S,x)\in\Theta_2/X} \ \varinjlim_{(T,y)\in\Theta_2/Y} N(S \otimes T).$$

We shall use the following two facts about this functor proved by Maehara:

Maehara's Gray tensor product for 2-quasi-categories (cont.)

Theorem (Maehara)

1. *Maehara's Gray tensor product functor* $\otimes_M \colon \widehat{\Theta_2} \times \widehat{\Theta_2} \longrightarrow \widehat{\Theta_2}$ *is a left Quillen functor of two variables w.r.t. the model structure for 2-quasi-categories.*
2. *There is a natural weak equivalence* $N(\mathbf{2}^{\otimes m}) \otimes_M N(\mathbf{2}^{\otimes n}) \simeq N(\mathbf{2}^{\otimes (m+n)})$ *in the model structure for 2-quasi-categories for each* $m, n \geq 0$.

Let $\otimes_M^{\mathbb{L}} \colon \mathbf{Cat}_{(\infty,2)} \times \mathbf{Cat}_{(\infty,2)} \longrightarrow \mathbf{Cat}_{(\infty,2)}$ denote the (∞-categorical) left derived functor of $\otimes_M$.

Corollary

1. *The functor* $\otimes_M^{\mathbb{L}} \colon \mathbf{Cat}_{(\infty,2)} \times \mathbf{Cat}_{(\infty,2)} \longrightarrow \mathbf{Cat}_{(\infty,2)}$ *has a right adjoint in each variable.*
2. *There is a natural equivalence* $\mathbf{2}^{\otimes m} \otimes_M^{\mathbb{L}} \mathbf{2}^{\otimes n} \simeq \mathbf{2}^{\otimes (m+n)}$ *for each* $m, n \geq 0$.

Let us define the functors $H^l, H^r \colon \mathcal{Q}^{\mathrm{op}} \times \mathbf{Cat}_{(\infty,2)} \longrightarrow \mathbf{Cat}_{(\infty,2)}$ to be the restrictions of the right adjoint functors of part (1) of the Corollary along the full inclusion $\mathcal{Q} \longrightarrow \mathbf{Cat}_{(\infty,2)}$ in their first variables.

The Gray monoidal structure on $\mathbf{Cat}_{(\infty,2)}$

We are now ready to apply Day's extension theorem for monoidal ∞-categories to the full inclusion $\mathcal{Q} \longrightarrow \mathbf{Cat}_{(\infty,2)}$ of the monoidal category of Gray cubes into the ∞-category of $(\infty,2)$-categories, and thereby prove our main theorem.

Theorem (The Gray monoidal structure for $(\infty,2)$-categories)

There exists a unique biclosed monoidal structure on $\mathbf{Cat}_{(\infty,2)}$ for which the full inclusion $\mathcal{Q} \longrightarrow \mathbf{Cat}_{(\infty,2)}$ is strong monoidal w.r.t. the Gray monoidal structure on the category $\mathcal{Q}$ of Gray cubes.

Proof.

We apply Day's extension theorem to the full inclusion $\mathcal{Q} \longrightarrow \mathbf{Cat}_{(\infty,2)}$. We have already proved that this inclusion is dense. On the previous slide, we defined functors $H^l, H^r \colon \mathcal{Q}^{\mathrm{op}} \times \mathbf{Cat}_{(\infty,2)} \longrightarrow \mathbf{Cat}_{(\infty,2)}$. It remains to observe that we have the following natural equivalences thanks to the Corollary on the previous slide:

$$\begin{aligned}
\mathbf{Cat}_{(\infty,2)}(\mathbf{2}^{\otimes(m+n)}, \mathcal{A}) &\simeq \mathbf{Cat}_{(\infty,2)}(\mathbf{2}^{\otimes m} \otimes^{\mathbb{L}}_{M} \mathbf{2}^{\otimes n}, \mathcal{A}) \\
&\simeq \mathbf{Cat}_{(\infty,2)}(\mathbf{2}^{\otimes m}, H^l(\mathbf{2}^{\otimes n}, \mathcal{A})) \\
&\simeq \mathbf{Cat}_{(\infty,2)}(\mathbf{2}^{\otimes n}, H^r(\mathbf{2}^{\otimes m}, \mathcal{A})).
\end{aligned}$$

$\square$

Plan

The comparison problem

Theorem (The Gray monoidal structure for $(\infty, 2)$-categories)

There exists a unique biclosed monoidal structure on $\mathbf{Cat}_{(\infty,2)}$ for which the full inclusion $\mathcal{Q} \longrightarrow \mathbf{Cat}_{(\infty,2)}$ is strong monoidal w.r.t. the Gray monoidal structure on the category $\mathcal{Q}$ of Gray cubes.

By the uniqueness part of this theorem, to show that a biclosed monoidal structure on $\mathbf{Cat}_{(\infty,2)}$ (such as any of the "Gray" biclosed monoidal structures on $\mathbf{Cat}_{(\infty,2)}$ arising from the monoidal model categories of Verity, Lurie/Gagna–Harpaz–Lanari, and Campion–Kapulkin–Maehara mentioned earlier) agrees with the Gray monoidal structure on $\mathbf{Cat}_{(\infty,2)}$ constructed in this talk, it suffices to show that it agrees with the classical Gray tensor product on the full subcategory of Gray cubes.

And while we cannot yet show with current technology that the constructions from Yuki's thesis directly give a monoidal structure on $\mathbf{Cat}_{(\infty,2)}$, we are able at least to show that the Gray tensor product functor for 2-quasi-categories defined therein is equivalent to the Gray tensor product for $(\infty, 2)$-categories defined in this talk. For they both agree with the classical Gray tensor product on the dense subcategory $\mathcal{Q}$ of $\mathbf{Cat}_{(\infty,2)}$, and they both preserve colimits in each variable.

Thank you!