JOYAL'S CYLINDER CONJECTURE

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Abstract. For each pair of simplicial sets A and B, the category $\mathbf{Cyl}(A,B)$ of cylinders (also called correspondences) from A to B admits a model structure induced from Joyal's model structure for quasi-categories. In this paper, we prove Joyal's conjecture that a cylinder $X \in \mathbf{Cyl}(A,B)$ is fibrant if and only if the canonical morphism $X \longrightarrow A \star B$ is an inner fibration, and that a morphism between fibrant cylinders in $\mathbf{Cyl}(A,B)$ is a fibration if and only if it is an inner fibration. We use this result to give a new proof of a characterisation of covariant equivalences due to Lurie, which avoids the use of the straightening theorem. In an appendix, we introduce a new family of model structures on the slice categories $\mathbf{sSet}/B$ whose fibrant objects are the inner fibrations with codomain B, which we use to prove some new results about inner anodyne extensions and inner fibrations.

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1. Introduction

Recall that the *collage*[1] of a profunctor $M\colon A \longrightarrow B$ (i.e. a functor $M\colon A^{\mathrm{op}} \times B \longrightarrow \mathbf{Set}$) is the category $C(M)$ whose set of objects is the disjoint union $\mathrm{ob}\, C(M) = \mathrm{ob}\, A + \mathrm{ob}\, B$, whose hom-sets are given by

$$C(M)(x,y) = \begin{cases} A(x,y) & \text{if } x,y \in A \\ B(x,y) & \text{if } x,y \in B \\ M(x,y) & \text{if } x \in A,\ y \in B \\ \emptyset & \text{if } x \in B,\ y \in A, \end{cases}$$

and whose identities and composition are defined in the evident way by those of the categories A and B, and by the action of M on morphisms. There is a unique functor $C(M) \longrightarrow \mathbf{2} = \{0 < 1\}$ whose fibres above 0 and 1 are A and B respectively. Bénabou observed that the collage construction defines an equivalence between the category of profunctors (between arbitrary categories) and the slice category $\mathbf{Cat}/\mathbf{2}$ (see [Str01]).

In quasi-category theory, the category of *cylinders* (or *correspondences*) is defined to be the slice category $\mathbf{sSet}/\Delta[1]$. By analogy with the previous paragraph, a cylinder $p\colon X \longrightarrow \Delta[1]$ may be thought of as a model for the collage of a quasi-categorical profunctor from $\partial_0 X := p^{-1}(0)$ to

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[1]This construction is due to Bénabou [Bén72, Bén73]; its name is due to Walters (see [Str04]).

$\partial_1 X := p^{-1}(1)$. (See [Joy08b, Chapter 7], [Joy08a, §14], [Lur09, §2.3.1], and [Ste18a] for further details and intuition concerning cylinders/correspondences.)

For each pair of simplicial sets A and B, the category $\mathbf{Cyl}(A, B)$ of *cylinders from A to B*, or *(A, B)-cylinders*, is defined to be the fibre of the functor

$$(\partial_0, \partial_1)\colon \mathbf{sSet}/\Delta[1] \longrightarrow \mathbf{sSet} \times \mathbf{sSet}$$

over the object (A, B). Thus an object of $\mathbf{Cyl}(A, B)$ is a simplicial set X (the *underlying simplicial set* of the cylinder) equipped with a map $X \longrightarrow \Delta[1]$ whose fibres above 0 and 1 are A and B respectively, as displayed below.

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d] & B \arrow[l] \arrow[d] \arrow[dl, phantom, very near start, "\llcorner"] \\ \{0\} \arrow[r] & \Delta[1] & \{1\} \arrow[l] \end{tikzcd}$$

Note that the initial and terminal objects of $\mathbf{Cyl}(A, B)$ are the disjoint union $A + B$ and join $A \star B$ respectively, equipped with the manifest structure maps.

In [Joy08a, §14.6], Joyal described a model structure on $\mathbf{Cyl}(A, B)$ – which we call the *Joyal model structure* on $\mathbf{Cyl}(A, B)$ – created[2] by the forgetful functor $\mathbf{Cyl}(A, B) \longrightarrow \mathbf{sSet}$ from the Joyal model structure for quasi-categories on $\mathbf{sSet}$, about which he made the following conjecture. (Note that, by definition, an object $X \in \mathbf{Cyl}(A, B)$ is fibrant in this model structure if and only if the canonical morphism $X \longrightarrow A \star B$ is a fibration in the Joyal model structure for quasi-categories on $\mathbf{sSet}$.)

1.1. **Conjecture** (Joyal). *A cylinder $X \in \mathbf{Cyl}(A, B)$ is fibrant if and only if the canonical morphism $X \longrightarrow A \star B$ is an inner fibration, and a morphism between fibrant cylinders in $\mathbf{Cyl}(A, B)$ is a fibration if and only if it is an inner fibration.*

The main goal of this paper is to prove this conjecture (see Theorem 5.4; see §1.3 below for an outline of our proof). Note that the special case of this conjecture in which A and B are quasi-categories is easily proven (see [Ste18a, Lemmas 3.7 and 3.8] or Lemma 5.1); we prove it for *every* pair of simplicial sets A and B.

1.2. *Remark.* For many years, it was an open question whether every monic bijective-on-0-simplices weak categorical equivalence is inner anodyne (see [Joy08a, §2.10]). Were this so, the general case of Joyal's conjecture would be as easy to prove as the special case in which A and B are quasi-categories. However, the author recently proved [Cam20] that this is not so; hence a different argument is required to prove Joyal's conjecture.

Our proof of Joyal's conjecture is contained in §§2–5 of this paper. (The contents of each section may be gleaned from its opening paragraph.) In the final section §6, we use this result to give a new proof of Lurie's characterisation of covariant equivalences (see Theorem 6.5), which avoids the use of the straightening theorem [Lur09, Theorem 2.2.1.2]. In an appendix (Appendix A), we introduce a new family of model structures on the slice categories $\mathbf{sSet}/B$ (which we call the *parametrised Joyal model structures*) whose fibrant objects are the inner fibrations with codomain B, using which we prove some new results on inner anodyne extensions and inner fibrations.

1.3. **Outline of proof.** Our proof of Joyal's conjecture may be outlined as follows.

(1) We construct (in Theorem 2.11) a model structure on $\mathbf{Cyl}(A, B)$, which we call the *ambivariant model structure*, which has the same cofibrations as the Joyal model structure (i.e. the monomorphisms), but whose (fibrations between) fibrant objects are precisely as described in Joyal's conjecture.

[2] A model structure on a category $\mathcal{A}$ is said to be *created* by a functor $F\colon \mathcal{A} \to \mathcal{C}$ from a model structure on $\mathcal{C}$ if a morphism f of $\mathcal{A}$ is a cofibration, weak equivalence, or fibration in the model structure on $\mathcal{A}$ if and only if the morphism Ff is a cofibration, weak equivalence, or fibration respectively in the model structure on $\mathcal{C}$.

(2) We observe that Joyal's conjecture is therefore equivalent to the statement that, on the category $\mathbf{Cyl}(A,B)$, the Joyal model structure and the ambivariant model structure coincide. In particular, we know that these two model structures do coincide if A and B are quasi-categories.
(3) Since every fibration in the Joyal model structure is in particular an inner fibration, it follows that every ambivariant equivalence in $\mathbf{Cyl}(A,B)$ is a weak categorical equivalence. It remains to prove the converse.
(4) For each pair of weak categorical equivalences $u\colon A \longrightarrow A'$ and $v\colon B \longrightarrow B'$, we prove that the pushforward functor

$$(u,v)_!\colon \mathbf{Cyl}(A,B) \longrightarrow \mathbf{Cyl}(A',B')$$

   (i) preserves weak categorical equivalences (easy! - see Proposition 4.2),
   (ii) reflects ambivariant equivalences (hard! - see Theorem 4.7).
(5) It then remains to argue as follows (see Theorem 5.3). Let $u\colon A \longrightarrow A'$ and $v\colon B \longrightarrow B'$ be weak categorical equivalences in **sSet** such that A' and B' are quasi-categories. For any morphism f in $\mathbf{Cyl}(A,B)$, f a weak categorical equivalence $\implies$ $(u,v)_!(f)$ a weak categorical equivalence (by (4)(i)) $\implies$ $(u,v)_!(f)$ an ambivariant equivalence (by (2), since A' and B' are quasi-categories) $\implies$ f an ambivariant equivalence (by (4)(ii)). This completes the proof.

2. Model structures for cylinders

The goal of this section is to carry out step (1) of the proof of Joyal's conjecture outlined in §1.3. For each pair of simplicial sets A and B, we construct both the *Joyal model structure* (Theorem 2.10) and the *ambivariant model structure* (Theorem 2.11) on the category $\mathbf{Cyl}(A,B)$ described in §1. (Note that the construction of the latter model structure involves the parametrised Joyal model structures introduced in Appendix A.) We shall construct both of these model structures by the following general technique.

2.1. **Restricting model structures.** We say that a model structure on a category $\mathcal{C}$ *restricts* to a model structure on a full subcategory $\mathcal{A} \subseteq \mathcal{C}$ if the full inclusion $\mathcal{A} \longrightarrow \mathcal{C}$ creates a model structure on $\mathcal{A}$ from the model structure on $\mathcal{C}$, that is, if the classes consisting of those morphisms in $\mathcal{A}$ which are cofibrations, weak equivalences, and fibrations respectively in the model structure on $\mathcal{C}$ form a model structure on $\mathcal{A}$.

2.2. **Proposition.** *Let $\mathcal{C}$ be a cofibrantly generated model category, and let $\mathcal{A}$ be a full subcategory of $\mathcal{C}$ which is both reflective and coreflective via an adjoint triple $L \dashv I \dashv R$. Then the model structure on $\mathcal{C}$ restricts to one on $\mathcal{A}$ if and only if the adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "IR"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend right, "IL"'] \end{tikzcd}$$

is a Quillen adjunction.

Proof. If the model structure on $\mathcal{C}$ restricts to one on $\mathcal{A}$, then the adjunctions $L \dashv I$ and $I \dashv R$ are Quillen adjunctions, and hence so is their composite $IL \dashv IR$.

Conversely, suppose that the adjunction $IL \dashv IR$ is a Quillen adjunction. Note that the category $\mathcal{A}$ is complete and cocomplete, since it is a (co)reflective subcategory of the complete and cocomplete category $\mathcal{C}$. By [DCH19, Theorem 2.3], the category $\mathcal{A}$ admits a model structure in which a morphism f is a fibration (resp. weak equivalence) if and only if If is a fibration (resp. weak equivalence) in the model category $\mathcal{C}$; furthermore, the adjunctions $L \dashv I$ and $I \dashv R$ are Quillen adjunctions with respect to these model structures.

It remains to show that a morphism f in $\mathcal{A}$ is a cofibration in this model structure on $\mathcal{A}$ if and only if If is a cofibration in $\mathcal{C}$. Necessity follows from the foregoing fact that the functor I is left Quillen, while sufficiency follows from the fact that L is left Quillen and the assumption that I is fully faithful. $\square$

2.3. *Remark.* One can show (by using [GKR18, Corollary 2.7] and arguing as in the proof above) that the conclusion of Proposition 2.2 also holds under the alternative hypothesis that $\mathcal{C}$ is an *accessible* model category (in the sense of [Ros17]).

2.4. **Two model structures on the factorisation category** $(A+B)/\mathbf{sSet}/(A\star B)$**.** Let A and B be a pair of simplicial sets. We shall use Proposition 2.2 to induce the "Joyal" and "ambivariant" model structures on $\mathbf{Cyl}(A,B)$ from the following two model structures on the category $(A+B)/\mathbf{sSet}/(A\star B)$ of factorisations[3] in $\mathbf{sSet}$ of the canonical inclusion $A+B \longrightarrow A\star B$.

The first of these model structures on the factorisation category $(A+B)/\mathbf{sSet}/(A\star B)$ is the one created by the forgetful functor

$$(A+B)/\mathbf{sSet}/(A\star B) \longrightarrow \mathbf{sSet}$$

from the Joyal model structure for quasi-categories on $\mathbf{sSet}$. The existence of this created model structure follows from [Hir18, §2] (which corrects [Hir03, Theorem 7.6.5(3)]).

The second model structure on $(A+B)/\mathbf{sSet}/(A\star B)$ is induced from the parametrised Joyal model structure on $\mathbf{sSet}/(A\star B)$ (introduced in Appendix A, see Theorem A.6) as follows. By [Hir03, Theorem 7.6.5(1)] – applied to the inclusion $A+B \longrightarrow A\star B$ as an object of the category $\mathbf{sSet}/(A\star B)$ equipped with the parametrised Joyal model structure – there exists a model structure on $(A+B)/\mathbf{sSet}/(A\star B)$ created by the forgetful functor

$$(A+B)/\mathbf{sSet}/(A\star B) \longrightarrow \mathbf{sSet}/(A\star B)$$

from the parametrised Joyal model structure on $\mathbf{sSet}/(A\star B)$.

Since both the Joyal model structure on $\mathbf{sSet}$ and the parametrised Joyal model structure on $\mathbf{sSet}/(A\star B)$ are cofibrantly generated, it follows by [Hir05] that both of the above model structures on the factorisation category $(A+B)/\mathbf{sSet}/(A\star B)$ are cofibrantly generated.

2.5. $\mathbf{Cyl}(A,B)$ **as a reflective and coreflective subcategory of** $(A+B)/\mathbf{sSet}/(A\star B)$**.** Recall that the disjoint union $A+B$ and join $A\star B$ are the initial and terminal objects respectively of the category $\mathbf{Cyl}(A,B)$. As observed in [Joy08a, §14.6], the forgetful functor $\mathbf{Cyl}(A,B) \longrightarrow \mathbf{sSet}$ lifts to a fully faithful functor $\mathbf{Cyl}(A,B) \longrightarrow (A+B)/\mathbf{sSet}/(A\star B)$. This full embedding has both a left adjoint, given by the composite

$$(A+B)/\mathbf{sSet}/(A\star B) \longrightarrow \mathbf{sSet}/(A\star B) \xrightarrow{L} \mathbf{Cyl}(A,B)$$

of the forgetful functor and the reflection L described in §2.6 below, and a right adjoint, given by the composite

$$(A+B)/\mathbf{sSet}/(A\star B) \longrightarrow (A+B)/\mathbf{sSet} \xrightarrow{R} \mathbf{Cyl}(A,B)$$

of the (other) forgetful functor and the coreflection R described in §2.7 below.

2.6. $\mathbf{Cyl}(A,B)$ **as a reflective subcategory of** $\mathbf{sSet}/(A\star B)$**.** As described in [Ste18a, Remark 3.5], the fully faithful functor $\mathbf{Cyl}(A,B) \longrightarrow \mathbf{sSet}/(A\star B)$ has a left adjoint L, which sends an object $X \longrightarrow A\star B$ of $\mathbf{sSet}/(A\star B)$ to the (A,B)-cylinder $L(X)$ defined by the pushout below,

$$\begin{tikzcd}
\partial_0 X + \partial_1 X \arrow[r] \arrow[d] & A+B \arrow[d] \\
X \arrow[r] & L(X) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

with the induced structure map $L(X) \longrightarrow A\star B \longrightarrow \Delta[1]$. It follows that a morphism of (A,B)-cylinders is a monomorphism in $\mathbf{Cyl}(A,B)$ if and only if its underlying morphism of simplicial sets is a monomorphism. The essential image of the full embedding $\mathbf{Cyl}(A,B) \longrightarrow \mathbf{sSet}/(A\star B)$ consists of those morphisms $X \longrightarrow A\star B$ whose pullback along the inclusion $A+B \longrightarrow A\star B$ is an isomorphism.

[3]Beware that an object of the factorisation category $(A+B)/\mathbf{sSet}/(A\star B)$ is not an arbitrary composable pair of morphisms $A+B \longrightarrow X \longrightarrow A\star B$, but rather a pair whose composite is the canonical inclusion $A+B \longrightarrow A\star B$.

2.7. $\mathbf{Cyl}(A, B)$ **as a coreflective subcategory of** $(A + B)/\mathbf{sSet}$**.** The fully faithful functor $\mathbf{Cyl}(A, B) \longrightarrow (A + B)/\mathbf{sSet}$ has a right adjoint R given by the "quasi-categorical collage construction" described in [RV19, Appendix F]. (We shall not need to know anything about this right adjoint beyond its existence.) The essential image of the full embedding $\mathbf{Cyl}(A, B) \longrightarrow (A + B)/\mathbf{sSet}$ consists of those cospans of simplicial sets

$$A \longrightarrow C \longleftarrow B$$

whose left leg $A \longrightarrow C$ is a sieve inclusion and whose right leg $B \longrightarrow C$ is the complementary cosieve inclusion (in the sense of [Joy08b, §7.2] and [Joy08a, §14.5]).

2.8. **Restricted model structures on** $\mathbf{Cyl}(A, B)$**.** The following proposition verifies the necessary and sufficient condition of Proposition 2.2 in our two cases of interest.

2.9. **Proposition.** *The reflection* $L\colon \mathbf{sSet}/(A \star B) \longrightarrow \mathbf{Cyl}(A, B)$ *preserves monomorphisms, inner anodyne extensions, and weak categorical equivalences, and inverts any morphism between objects of* $\mathbf{sSet}/(A \star B)$ *whose structure maps factor through the inclusion* $A + B \longrightarrow A \star B$*.*

Proof. By definition (see §2.6), the functor L sends a morphism $f\colon X \longrightarrow Y$ in $\mathbf{sSet}/(A \star B)$ to the morphism $L(X) \longrightarrow L(Y)$ induced by pushout from the diagram below, in which, we note, the left-pointing maps are monomorphisms and the left-hand square is a pullback.

$$\begin{tikzcd} X \arrow[d, "f"'] & \partial_0 X + \partial_1 X \arrow[l] \arrow[r] \arrow[d, "\partial_0 f + \partial_1 f"] \arrow[dl, phantom, "\llcorner"] & A + B \arrow[d, equal] \\ Y & \partial_0 Y + \partial_1 Y \arrow[l] \arrow[r] & A + B \end{tikzcd}$$

It follows from the exactness of pushouts of monomorphisms in the presheaf category $\mathbf{sSet}$ that L preserves monomorphisms. If f is inner anodyne, then [Joy08b, Lemma 3.21] implies that the morphism $\partial_0 f + \partial_1 f$ is also inner anodyne, whence [Ste18b, Lemma 2.5] implies that L preserves inner anodyne extensions. It follows similarly from [Joy08b, Corollary 7.10] and the gluing lemma (see [Ree74]) that L preserves weak categorical equivalences.

By construction, any object (X, p) of $\mathbf{sSet}/(A \star B)$ whose structure map $p\colon X \longrightarrow A \star B$ factors through the inclusion $A + B \longrightarrow A \star B$ is sent by L to the initial (A, B)-cylinder $A + B$. Hence L inverts any morphism between two such objects. □

We are now ready to construct the two model structures on $\mathbf{Cyl}(A, B)$ described in §1, and thus complete step (1) of the proof of Joyal's conjecture outlined in §1.3. The existence of the first model structure was stated by Joyal [Joy08a, §14.6]; an alternative proof of its existence is given in [Ste18a, Theorem 3.9].

2.10. **Theorem** (the Joyal model structure on $\mathbf{Cyl}(A, B)$)**.** *There exists a model structure on* $\mathbf{Cyl}(A, B)$ *created by the forgetful functor* $\mathbf{Cyl}(A, B) \longrightarrow \mathbf{sSet}$ *from the Joyal model structure for quasi-categories on* $\mathbf{sSet}$*.*

Proof. The existence of this model structure is proven by an application of Proposition 2.2 to the first model structure on the factorisation category $(A+B)/\mathbf{sSet}/(A\star B)$ described in §2.4 and the adjoint triple described in §2.5. The necessary and sufficient condition of Proposition 2.2 follows from the fact, proved in Proposition 2.9, that the reflection $L\colon \mathbf{sSet}/(A \star B) \longrightarrow \mathbf{Cyl}(A, B)$ preserves monomorphisms and weak categorical equivalences. □

We call the model structure of Theorem 2.10 the *Joyal model structure* on $\mathbf{Cyl}(A, B)$. Note that (by the remark on monomorphisms in §2.6) a morphism in $\mathbf{Cyl}(A, B)$ is a cofibration in this model structure if and only if it is a monomorphism in $\mathbf{Cyl}(A, B)$. Since, for any cylinder $X \in \mathbf{Cyl}(A, B)$, the canonical morphism $A + B \longrightarrow X$ is a monomorphism, we see that every object of $\mathbf{Cyl}(A, B)$ is cofibrant in this model structure; the same is true of the following model structure.

2.11. **Theorem** (the ambivariant model structure on $\mathbf{Cyl}(A, B)$)**.** *There exists a model structure on* $\mathbf{Cyl}(A, B)$ *whose cofibrations are the monomorphisms and whose fibrant objects are those* *cylinders $X \in \mathbf{Cyl}(A,B)$ for which the canonical morphism $X \longrightarrow A \star B$ is an inner fibration. A morphism between fibrant objects in $\mathbf{Cyl}(A,B)$ is a fibration if and only if it is an inner fibration.*

Proof. The first statement is proven by an application of Proposition 2.2 to the second model structure on $(A+B)/\mathbf{sSet}/(A \star B)$ described in §2.4 and the adjoint triple described in §2.5, which implies that there exists a model structure on $\mathbf{Cyl}(A,B)$ created by the forgetful functor $\mathbf{Cyl}(A,B) \longrightarrow \mathbf{sSet}/(A\star B)$ from the parametrised Joyal model structure on $\mathbf{sSet}/(A\star B)$. The necessary and sufficient condition of Proposition 2.2 follows from Lemma A.8 and Proposition 2.9, the latter of which implies that the reflection L preserves monomorphisms and inner anodyne extensions, and inverts the morphism $(J, x!) \longrightarrow (\Delta[0], x)$ in $\mathbf{sSet}/(A \star B)$ for every 0-simplex x of $A \star B$.

By the construction of this model structure and by Theorem A.6, a morphism between fibrant objects in $\mathbf{Cyl}(A,B)$ is a fibration if and only if it is an inner fibration and a fibrewise isofibration over $A \star B$. But this latter property is trivially satisfied, since if an object (X,p) of $\mathbf{sSet}/(A\star B)$ is an (A,B)-cylinder, then the fibre of $p \colon X \longrightarrow A \star B$ over any vertex of $A \star B$ is the terminal simplicial set. This proves the second statement. ☐

We call the model structure of Theorem 2.11 the *ambivariant model structure* on $\mathbf{Cyl}(A,B)$. We say that an object of $\mathbf{Cyl}(A,B)$ is *ambifibrant* if it is a fibrant object in this model structure, and that a morphism in $\mathbf{Cyl}(A,B)$ is an *ambivariant equivalence* if it is a weak equivalence in this model structure.

The following proposition gives a recognition principle for left Quillen functors from the ambivariant model structure on $\mathbf{Cyl}(A,B)$.

2.12. **Proposition.** *Let $\mathcal{C}$ be a model category, and let $F \colon \mathbf{Cyl}(A,B) \longrightarrow \mathcal{C}$ be a cocontinuous functor that sends monomorphisms to cofibrations. Then F sends every ambivariant equivalence in $\mathbf{Cyl}(A,B)$ to a weak equivalence in $\mathcal{C}$ if and only if it sends every inner anodyne extension in $\mathbf{Cyl}(A,B)$ to a weak equivalence in $\mathcal{C}$.*

Proof. This follows by a standard argument from Proposition 2.9, Theorem 2.11, [JT07, Lemma 7.14], and [JT07, Proposition 7.15]. ☐

We conclude this section with an observation about duality, which will help to simplify the exposition of the following two sections.

2.13. *Observation* (duality and the ambivariant model structure). Recall that the category of simplicial sets bears an involution $(-)^{\mathrm{op}} \colon \mathbf{sSet} \longrightarrow \mathbf{sSet}$ (induced from the non-trivial involution on the category Δ), which sends a simplicial set X to its *opposite* X^{op}. For each pair of simplicial sets A and B, this involution defines an isomorphism

$$\mathbf{Cyl}(A,B) \cong \mathbf{Cyl}(B^{\mathrm{op}}, A^{\mathrm{op}})$$

between the category of (A,B)-cylinders and the category of $(B^{\mathrm{op}}, A^{\mathrm{op}})$-cylinders. Since the involution $(-)^{\mathrm{op}} \colon \mathbf{sSet} \longrightarrow \mathbf{sSet}$ preserves inner fibrations, it follows that this isomorphism respects the ambivariant model structures on these two categories.

3. Cylinders as presheaves

In this section, we take Joyal's observation (see §3.1) that the category $\mathbf{Cyl}(A,B)$ of (A,B)-cylinders is equivalent to the category of presheaves over $\Delta/A \times \Delta/B$ as the basis for a deeper analysis of the ambivariant model structure on $\mathbf{Cyl}(A,B)$ introduced in §2. We use the related equivalence of categories

$$\mathbf{Cyl}(A,B) \simeq [(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B]$$

to construct (Proposition 3.16) a *Reedy model structure* on $\mathbf{Cyl}(A,B)$, induced from the covariant model structure on $\mathbf{sSet}/B$. We prove (Theorem 3.20) that the ambivariant model structure on $\mathbf{Cyl}(A,B)$ is a Bousfield localisation of this Reedy model structure, for which the *local objects* are those (A,B)-cylinders whose corresponding functor $(\Delta/A)^{\mathrm{op}} \longrightarrow \mathbf{sSet}/B$ sends the *final vertex maps* in Δ/A to covariant equivalences in $\mathbf{sSet}/B$. (A dual result relates the ambivariant model structure on $\mathbf{Cyl}(A,B)$ to the *contravariant* model structure on $\mathbf{sSet}/A$ and the *initial* vertex maps in Δ/B; see Remark 3.22.) We will use this result to prove the main theorem (Theorem 4.7) of §4.

3.1. **$\mathbf{Cyl}(A,B)$ as a presheaf category.** We begin with Joyal's observation that, for each pair of simplicial sets A and B, the category $\mathbf{Cyl}(A,B)$ of (A,B)-cylinders is equivalent to the category of presheaves over the product $\Delta/A \times \Delta/B$ of the categories of simplices of A and B.

$$\mathbf{Cyl}(A,B) \simeq [(\Delta/A \times \Delta/B)^{\mathrm{op}}, \mathbf{Set}] \tag{3.2}$$

Under this equivalence, an (A,B)-cylinder X with structure map $p\colon X \longrightarrow A \star B$ corresponds to the presheaf on $\Delta/A \times \Delta/B$ whose value at the object $(([m],\alpha),([n],\beta)) \in \Delta/A \times \Delta/B$ is the set of $(m+1+n)$-simplices σ of X such that $p(\sigma) = \alpha \star \beta$, i.e. such that the diagram below commutes.

$$\begin{tikzcd} & X \arrow[d, "p"] \\ \Delta[m] \star \Delta[n] \arrow[ur, "\sigma"] \arrow[r, "\alpha\star\beta"'] & A \star B \end{tikzcd}$$

We refer the reader to [Joy08b, Chapter 7] for further details of this equivalence.

Combining the equivalence (3.2) with the standard equivalences

$$\mathbf{sSet}/C \simeq [(\Delta/C)^{\mathrm{op}}, \mathbf{Set}], \tag{3.3}$$

we obtain further equivalences between $\mathbf{Cyl}(A,B)$ and the functor categories displayed below.

$$[(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B] \simeq \mathbf{Cyl}(A,B) \simeq [(\Delta/B)^{\mathrm{op}}, \mathbf{sSet}/A] \tag{3.4}$$

In the sequel, we shall sometimes use the word *vertical* (resp. *horizontal*) to indicate that we are thinking of (A,B)-cylinders in terms of their corresponding functor $(\Delta/A)^{\mathrm{op}} \longrightarrow \mathbf{sSet}/B$ (resp. $(\Delta/B)^{\mathrm{op}} \longrightarrow \mathbf{sSet}/A$).

These equivalences of categories enable us to employ various standard constructions (drawn from [JT07, §7], [Ara14, §3], and [RV14, §4], to which we refer the reader for further details) in our analysis of the ambivariant model structure on $\mathbf{Cyl}(A,B)$. Following a brief survey of these constructions, the main argument of this section will begin in earnest with Proposition 3.14, in which we use these constructions, together with a result of Stevenson (Lemma 3.13), to give an alternative characterisation of the ambifibrant objects of $\mathbf{Cyl}(A,B)$.

3.5. **Exterior (Leibniz) products.** Recall that we may define the *exterior product* bifunctor

$$[(\Delta/A)^{\mathrm{op}}, \mathbf{Set}] \times [(\Delta/B)^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\boxtimes} [(\Delta/A \times \Delta/B)^{\mathrm{op}}, \mathbf{Set}]$$

in the way made manifest by the formula $(X \boxtimes Y)_{\alpha,\beta} = X_\alpha \times Y_\beta$. Furthermore, we may define the corresponding *exterior Leibniz product* (or *exterior pushout-product*) bifunctor between arrow categories

$$[(\Delta/A)^{\mathrm{op}}, \mathbf{Set}]^{\mathbf{2}} \times [(\Delta/B)^{\mathrm{op}}, \mathbf{Set}]^{\mathbf{2}} \xrightarrow{\widehat{\boxtimes}} [(\Delta/A \times \Delta/B)^{\mathrm{op}}, \mathbf{Set}]^{\mathbf{2}},$$

which sends a pair of morphisms $(f\colon M \longrightarrow N, g\colon S \longrightarrow T)$ to the pushout-corner map

$$f \mathbin{\widehat{\boxtimes}} g\colon (M \boxtimes T) \cup_{M \boxtimes S} (N \boxtimes S) \longrightarrow N \boxtimes T$$

of the commutative square displayed below.

$$\begin{tikzcd} M \boxtimes S \arrow[r, "1\boxtimes g"] \arrow[d, "f\boxtimes 1"'] & M \boxtimes T \arrow[d, "f\boxtimes 1"] \\ N \boxtimes S \arrow[r, "1\boxtimes g"'] & N \boxtimes T \end{tikzcd}$$

It is straightforward to show that, under the equivalences (3.2) and (3.3), the exterior product bifunctor corresponds to the composite bifunctor

$$\mathbf{sSet}/A \times \mathbf{sSet}/B \xrightarrow{\star} \mathbf{sSet}/(A \star B) \xrightarrow{L} \mathbf{Cyl}(A, B),$$

where L denotes the reflection described in §2.6. Furthermore, the exterior Leibniz product bifunctor corresponds under these equivalences to the composite bifunctor

$$(\mathbf{sSet}/A)^{\mathbf{2}} \times (\mathbf{sSet}/B)^{\mathbf{2}} \xrightarrow{\widehat{\star}} (\mathbf{sSet}/(A \star B))^{\mathbf{2}} \xrightarrow{L^{\mathbf{2}}} \mathbf{Cyl}(A, B)^{\mathbf{2}},$$

whose first factor is the *Leibniz join* (or *pushout-join*) bifunctor. We shall henceforth denote these two composite bifunctors by $\boxtimes$ and $\widehat{\boxtimes}$ respectively, so that $(X, p) \boxtimes (Y, q) = L(X \star Y, p \star q)$ and $f \mathbin{\widehat{\boxtimes}} g = L(f \mathbin{\widehat{\star}} g)$.

3.6. **Left and right division.** The exterior product bifunctor

$$\mathbf{sSet}/A \times \mathbf{sSet}/B \xrightarrow{\boxtimes} \mathbf{Cyl}(A, B)$$

is cocontinuous in each variable, and therefore forms part of a two-variable adjunction

$$(\mathbf{sSet}/B)(S, M\backslash X) \cong (\mathbf{Cyl}(A, B))(M \boxtimes S, X) \cong (\mathbf{sSet}/A)(M, X/S),$$

where, following [JT07, §7], we denote[4] the two right adjoint bifunctors

$$(\mathbf{sSet}/A)^{\mathrm{op}} \times \mathbf{Cyl}(A, B) \xrightarrow{\backslash} \mathbf{sSet}/B \qquad \mathbf{Cyl}(A, B) \times (\mathbf{sSet}/B)^{\mathrm{op}} \xrightarrow{/} \mathbf{sSet}/A$$

by the symbols of left division and right division.

The Yoneda lemma implies the important observation that the functor $(\Delta/A)^{\mathrm{op}} \longrightarrow \mathbf{sSet}/B$ to which an object $X \in \mathbf{Cyl}(A, B)$ corresponds under the equivalence (3.4) is naturally isomorphic to the composite

$$(\Delta/A)^{\mathrm{op}} \longrightarrow (\mathbf{sSet}/A)^{\mathrm{op}} \xrightarrow{-\backslash X} \mathbf{sSet}/B$$

of the Yoneda embedding and the "left division of X" functor.

Furthermore, under the equivalences of §3.1, these *division* bifunctors correspond to the *weighted limit* bifunctors indicated below.

$$[(\Delta/A)^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{op}} \times [(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B] \xrightarrow{\backslash} \mathbf{sSet}/B$$

$$[(\Delta/B)^{\mathrm{op}}, \mathbf{sSet}/A] \times [(\Delta/B)^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{op}} \xrightarrow{/} \mathbf{sSet}/A$$

Thus, for $M \in \mathbf{sSet}/A$ and $X \in \mathbf{Cyl}(A, B)$, the object $M\backslash X \in \mathbf{sSet}/B$ is the limit of the functor $X \colon (\Delta/A)^{\mathrm{op}} \longrightarrow \mathbf{sSet}/B$ weighted by the functor $M \colon (\Delta/A)^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

3.7. *Observation* (Leibniz joins and lifting problems). Let $X \in \mathbf{Cyl}(A, B)$ with structure map $p \colon X \longrightarrow A \star B$, and consider a pair of morphisms in $\mathbf{sSet}/A$ and $\mathbf{sSet}/B$ as displayed below.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "nf"'] & & N \arrow[dl, "n"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "g"] \arrow[dr, "tg"'] & & T \arrow[dl, "t"] \\ & B & \end{tikzcd}$$

By adjointness (see e.g. [JT07, Proposition 7.6], [Ara14, Proposition 3.3], or [RV14, Observation 4.11]) and the formula $f \mathbin{\widehat{\boxtimes}} g = L(f \mathbin{\widehat{\star}} g)$ (see §3.5 above), the following are equivalent.

[4]As in [JT07], so too in this paper is there no risk of confusing these constructions with the similarly denoted slice constructions defined in [Joy02, §3], since the latter do not here appear.

(i) Any lifting problem in **sSet** of the form

$$\begin{tikzcd}
(M \star T) \cup_{M\star S} (N \star S) \arrow[r] \arrow[d, "\widehat{f\star g}"'] & X \arrow[d, "p"] \\
N \star T \arrow[r, "n\star t"'] \arrow[ur, dotted] & A \star B
\end{tikzcd}$$

has a solution.

(ii) The cylinder X has the right lifting property in $\mathbf{Cyl}(A, B)$ with respect to the exterior Leibniz product $f \mathbin{\widehat{\boxtimes}} g \colon (M \boxtimes T) \cup_{M\boxtimes S} (N \boxtimes S) \longrightarrow N \boxtimes T$.

(iii) The morphism $f\backslash X \colon N\backslash X \longrightarrow M\backslash X$ has the right lifting property in $\mathbf{sSet}/B$ with respect to the morphism $g\colon (S, tg) \longrightarrow (T, t)$.

(iv) The morphism $X/g\colon X/T \longrightarrow X/S$ has the right lifting property in $\mathbf{sSet}/A$ with respect to the morphism $f\colon (M, nf) \longrightarrow (N, n)$.

3.8. *Observation* (cellular presentations of monomorphisms). Every monomorphism in $\mathbf{sSet}/A$ can be expressed as a countable composite of pushouts of coproducts of the boundary inclusions $b_m\colon (\partial\Delta[m], \partial\alpha) \longrightarrow (\Delta[m], \alpha)$ for $([m], \alpha) \in \Delta/A$. Furthermore, every monomorphism in $\mathbf{Cyl}(A, B)$ can be expressed as a countable composite of pushouts of coproducts of the exterior Leibniz products

$$b_m \mathbin{\widehat{\boxtimes}} b_n \colon (\Delta[m], \alpha) \boxtimes (\partial\Delta[n], \partial\beta) \cup (\partial\Delta[m], \partial\alpha) \boxtimes (\Delta[n], \beta) \longrightarrow (\Delta[m], \alpha) \boxtimes (\Delta[n], \beta)$$

for $([m], \alpha) \in \Delta/A$ and $([n], \beta) \in \Delta/B$. These two statements follow from the Eilenberg–Zilber lemma applied to the categories Δ/A and $\Delta/A \times \Delta/B$ equipped with their EZ-Reedy structures inherited from the standard EZ-Reedy structure on Δ; see for instance [Cis19, §1.3].

Before we can state Proposition 3.14, we must first make a couple of definitions.

3.9. *Recall* (the covariant model structure). There exists a model structure on the slice category $\mathbf{sSet}/B$ (due to Joyal, see [Joy08b, Chapter 8]) whose cofibrations are the monomorphisms, and whose fibrant objects are the left fibrations with codomain B, which we will sometimes call the *left fibrant* objects of $\mathbf{sSet}/B$. A morphism between left fibrant objects is a fibration if and only if it is a left fibration. This model structure is called the *covariant model structure* on $\mathbf{sSet}/B$. A morphism in $\mathbf{sSet}/B$ is said to be a *covariant equivalence* if it is a weak equivalence in this model structure.

3.10. **Definition** (Reedy fibrant cylinders)**.** A cylinder $X \in \mathbf{Cyl}(A, B)$ is said to be *vertically Reedy left fibrant* if the morphism on the left below is a left fibration in $\mathbf{sSet}/B$ for all $([m], \alpha) \in \Delta/A$.

$$b_m\backslash X \colon (\Delta[m], \alpha)\backslash X \longrightarrow (\partial\Delta[m], \partial\alpha)\backslash X \qquad\qquad X/b_n\colon X/(\Delta[n], \beta) \longrightarrow X/(\partial\Delta[n], \partial\beta)$$

Dually, an object $X \in \mathbf{Cyl}(A, B)$ is said to be *horizontally Reedy right fibrant* if the morphism on the right above is a right fibration in $\mathbf{sSet}/A$ for all $([n], \beta) \in \Delta/B$.

3.11. *Observation* (Reedy fibrant cylinders recast). It follows from Observation 3.8 that a cylinder $X \in \mathbf{Cyl}(A, B)$ is vertically Reedy left fibrant if and only if the morphism $f\backslash X\colon N\backslash X \longrightarrow M\backslash X$ is a left fibration in $\mathbf{sSet}/B$ for every monomorphism $f\colon M \longrightarrow N$ in $\mathbf{sSet}/A$. In particular, if $X \in \mathbf{Cyl}(A, B)$ is vertically Reedy left fibrant, then the object $M\backslash X \in \mathbf{sSet}/B$ is left fibrant for every object $M \in \mathbf{sSet}/A$.

Furthermore, Observation 3.7 implies that a cylinder $X \in \mathbf{Cyl}(A, B)$ is horizontally Reedy right fibrant if and only if the morphism

$$h^k_m\backslash X \colon (\Delta[m], \alpha)\backslash X \longrightarrow (\Lambda^k[m], \Lambda^k[\alpha])\backslash X,$$

induced by the horn inclusion $h^k_m\colon \Lambda^k[m] \longrightarrow \Delta[m]$, is a trivial fibration in $\mathbf{sSet}/A$ for all $m \geq 1$, $0 < k \leq m$, and $([m], \alpha) \in \Delta/A$.

For each $m \geq 1$, we let $i_m\colon \Delta[0] \longrightarrow \Delta[m]$ denote the *final vertex map*, which picks out the final vertex m of $\Delta[m]$.

3.12. **Definition** (vertically right local cylinders)**.** A cylinder $X \in \mathbf{Cyl}(A,B)$ is said to be *vertically right local* if the morphism

$$i_m \backslash X \colon (\Delta[m],\alpha)\backslash X \longrightarrow (\Delta[0],\alpha_m)\backslash X$$

is a covariant equivalence in $\mathbf{sSet}/B$ for all $m \geq 1$ and $([m],\alpha) \in \Delta/A$.

Recall that a class C of monomorphisms in a category is said to have the *right cancellation property* if $(u \in \mathsf{C}$ and $vu \in \mathsf{C}) \implies v \in \mathsf{C}$ for any composable pair of monomorphisms u and v.

3.13. **Lemma** (Stevenson)**.** *Let C be a class of monomorphisms of simplicial sets which is closed under composition, stable under pushout, and has the right cancellation property. Then the following are equivalent.*

(i) *C contains the horn inclusion $\Lambda^k[m] \longrightarrow \Delta[m]$ for every $m \geq 1$ and $0 < k \leq m$.*
(ii) *C contains the final vertex map $i_m \colon \Delta[0] \longrightarrow \Delta[m]$ for every $m \geq 1$.*

Proof. See [Ste17, Proposition 2.6]. □

3.14. **Proposition.** *Let $X \in \mathbf{Cyl}(A,B)$. The following are equivalent.*

(i) *The canonical morphism $X \longrightarrow A \star B$ is an inner fibration.*
(ii) *X is vertically Reedy left fibrant and horizontally Reedy right fibrant.*
(iii) *X is vertically Reedy left fibrant and vertically right local.*

Proof. First, we prove the equivalence (i) $\iff$ (ii). By definition, the canonical morphism $X \longrightarrow A \star B$ is an inner fibration if and only if every lifting problem of the form displayed below,

$$\begin{tikzcd} \Lambda^k[l] \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[l] \arrow[r, "\sigma"'] \arrow[ur, dotted] & A \star B \end{tikzcd}$$

where $l \geq 2$ and $0 < k < l$, has a solution. These lifting problems fall into two cases. If $\sigma \colon \Delta[l] \longrightarrow A \star B$ factors through the inclusion $A + B \longrightarrow A \star B$, then the lifting problem is solved trivially, since the pullback of $X \longrightarrow A \star B$ along $A + B \longrightarrow A \star B$ is an isomorphism (see §2.6). Otherwise, σ is of the form $\alpha \star \beta \colon \Delta[m] \star \Delta[n] \longrightarrow A \star B$ for some $m, n \geq 0$, in which case the displayed inner horn inclusion is either the Leibniz join (see [Joy02, Lemma 3.3])

$$(\Lambda^k[m] \to \Delta[m]) \mathbin{\widehat{\star}} (\partial\Delta[n] \to \Delta[n])$$

if $0 < k \leq m$, or the Leibniz join

$$(\partial\Delta[m] \to \Delta[m]) \mathbin{\widehat{\star}} (\Lambda^j[n] \to \Delta[n])$$

if $0 \leq j < n$, where $j = k - m - 1$.

It thus follows by Observation 3.7 that the canonical morphism $X \longrightarrow A \star B$ is an inner fibration if and only if $(\Delta[m],\alpha)\backslash X \longrightarrow (\partial\Delta[m],\partial\alpha)\backslash X$ is a left fibration for all $([m],\alpha) \in \Delta/A$, and $X/(\Delta[n],\beta) \longrightarrow X/(\partial\Delta[n],\partial\beta)$ is a right fibration for all $([n],\beta) \in \Delta/B$, that is, if and only if X is vertically Reedy left fibrant and horizontally Reedy right fibrant.

Next, we prove the equivalence (ii) $\iff$ (iii). Suppose that X is vertically Reedy left fibrant. Let C denote the class of monomorphisms $f \colon M \longrightarrow N$ in $\mathbf{sSet}$ with the property that, for any morphism $n \colon N \longrightarrow A$, the induced morphism $f\backslash X \colon (N,n)\backslash X \longrightarrow (M,nf)\backslash X$ is a covariant equivalence, or equivalently a trivial fibration (since X is vertically Reedy left fibrant), in $\mathbf{sSet}/B$. This class of monomorphisms C is closed under composition, stable under pushout, and has the right cancellation property. Hence Lemma 3.13 implies (via Observation 3.11) that X is horizontally Reedy right fibrant if and only if X is vertically right local. □

We now use the equivalence $\mathbf{Cyl}(A,B) \simeq [(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B]$ (see §3.1) to construct a "Reedy" model structure on $\mathbf{Cyl}(A,B)$, whose weak equivalences are described in the following definition.

3.15. **Definition** (vertical covariant equivalences)**.** A morphism $f\colon X \longrightarrow Y$ in $\mathbf{Cyl}(A,B)$ is said to be a *vertical covariant equivalence* if the morphism

$$(\Delta[m],\alpha)\backslash f\colon (\Delta[m],\alpha)\backslash X \longrightarrow (\Delta[m],\alpha)\backslash Y$$

is a covariant equivalence in $\mathbf{sSet}/B$ for all $([m],\alpha) \in \Delta/A$.

3.16. **Proposition.** *There exists a model structure on the category* $\mathbf{Cyl}(A,B)$ *whose cofibrations are the monomorphisms, whose weak equivalences are the vertical covariant equivalences, and whose fibrant objects are the vertically Reedy left fibrant objects.*

Proof. By for instance [RV14, Theorem 4.18], the Reedy structure on Δ/A (inherited from the standard Reedy structure on Δ) and the covariant model structure on $\mathbf{sSet}/B$ (see Recollection 3.9) cooperate to endow the functor category $[(\Delta/A)^{\mathrm{op}}, \mathbf{sSet}/B]$ with a Reedy model structure, which transports along the equivalence (3.4) to a model structure on $\mathbf{Cyl}(A,B)$. By construction, the weak equivalences of this model structure are the vertical covariant equivalences; by [RV14, Corollary 6.7] and Observation 3.8, the cofibrations are the monomorphisms.

By construction, an object $X \in \mathbf{Cyl}(A,B)$ is fibrant in this model structure if and only if the morphism

$$b_m\backslash X\colon (\Delta[m],\alpha)\backslash X \longrightarrow (\partial\Delta[m],\partial\alpha)\backslash X$$

is a fibration in the covariant model structure on $\mathbf{sSet}/B$ for every $([m],\alpha) \in \Delta/A$. But it follows (as in Observation 3.11) that X is fibrant if and only if the morphism $b_m\backslash X$ is a fibration between fibrant objects in the covariant model structure on $\mathbf{sSet}/B$ for every $([m],\alpha) \in \Delta/A$. Hence, by Observation 3.11 and Recollection 3.9, an object of $\mathbf{Cyl}(A,B)$ is fibrant in this model structure if and only if it is vertically Reedy left fibrant. $\square$

We call the model structure of Proposition 3.16 the *vertical Reedy covariant model structure* on $\mathbf{Cyl}(A,B)$.

The following theorem is phrased in terms of the theory of Bousfield localisations of model categories (see e.g. [Joy08b, Appendix E]), from which we recall a few basic definitions and results.

3.17. **Definition** (Bousfield localisation)**.** Let M and N be model structures on a category $\mathcal{C}$. The model structure N is said to be a *Bousfield localisation* of the model structure M if N has the same class of cofibrations as M, and if every fibrant object of N is fibrant in M.

3.18. **Definition** (local objects)**.** Let M and N be model structures on a category $\mathcal{C}$, and suppose that N is a Bousfield localisation of M. An object of $\mathcal{C}$ is said to be *local* (with respect to this Bousfield localisation) if it is weakly equivalent in M to a fibrant object of N.

3.19. **Lemma.** *Let* M *and* N *be model structures on a category* $\mathcal{C}$*, and suppose that* N *is a Bousfield localisation of* M*. Every weak equivalence in* M *is a weak equivalence in* N*. Conversely, any weak equivalence in* N *between local objects is a weak equivalence in* M*.*

Proof. See e.g. [Joy08b, Proposition E.1.10] and [Joy08b, Proposition E.2.21]. $\square$

We now use Proposition 3.14 to prove the main result of this section.

3.20. **Theorem.** *On the category* $\mathbf{Cyl}(A,B)$*, the ambivariant model structure is a Bousfield localisation of the vertical Reedy covariant model structure. An object of* $\mathbf{Cyl}(A,B)$ *is local with respect to this Bousfield localisation if and only if it is vertically right local.*

Proof. In both model structures, the cofibrations are precisely the monomorphisms. By Proposition 3.14, every ambifibrant object of $\mathbf{Cyl}(A,B)$ is vertically Reedy left fibrant. This proves the first statement.

As indicated by the diagram below, it is immediate from the definitions and the two-out-of-three property that an object of $\mathbf{Cyl}(A,B)$ is vertically right local if it is weakly equivalent in the vertical Reedy covariant model structure to a vertically right local object.

$$\begin{tikzcd}
(\Delta[m],\alpha)\backslash X \arrow[r, "i_m\backslash X"] \arrow[d, "\sim"'] & (\Delta[0],\alpha_m)\backslash X \arrow[d, "\sim"] \\
(\Delta[m],\alpha)\backslash Y \arrow[r, "i_m\backslash Y"'] & (\Delta[0],\alpha_m)\backslash Y
\end{tikzcd}$$

Hence, to prove the second statement, it suffices to prove that a vertically Reedy left fibrant object of $\mathbf{Cyl}(A, B)$ is ambifibrant if and only if it is vertically right local. But this was shown in Proposition 3.14. ☐

3.21. **Corollary.** *In the category* $\mathbf{Cyl}(A, B)$*, every vertical covariant equivalence is an ambivariant equivalence. Conversely, any ambivariant equivalence between vertically right local objects of* $\mathbf{Cyl}(A, B)$ *is a vertical covariant equivalence.*

Proof. This follows from Theorem 3.20 by Lemma 3.19. ☐

3.22. *Remark* (duality).

3.23. *Remark* (alternative constructions of the ambivariant model structure).

4. Change of base

The goal of this section is to carry out step (4) of the proof of Joyal's conjecture outlined in §1.3. We prove (Proposition 4.2) that, for each pair of morphisms of simplicial sets u and v, the *pushforward–pullback adjunction* $(u, v)_! \dashv (u, v)^*$ (see §4.1) is a Quillen adjunction with respect to the Joyal and ambivariant model structures (constructed in §2). Furthermore, we use the results of §3 to prove (Theorem 4.7) that, if u and v are weak categorical equivalences, then the pushforward–pullback adjunction $(u, v)_! \dashv (u, v)^*$ is a Quillen equivalence with respect to the ambivariant model structures.

4.1. **The pushforward–pullback adjunction.** Let $u\colon A \longrightarrow A'$ and $v\colon B \longrightarrow B'$ be a pair of morphisms of simplicial sets. Recall from [Joy08a, §14.6] that there is an adjunction

$$\begin{tikzcd}
\mathbf{Cyl}(A,B) \arrow[r, shift left=2, "(u,v)_!"] \arrow[r, phantom, "\perp"] & \mathbf{Cyl}(A',B') \arrow[l, shift left=2, "(u,v)^*"]
\end{tikzcd}$$

whose left adjoint sends an (A, B)-cylinder X to the *pushforward* (A', B')-cylinder $(u, v)_!(X)$ defined by the pushout square on the left below,

$$\begin{tikzcd}
A+B \arrow[r, "u+v"] \arrow[d] & A'+B' \arrow[d] \\
X \arrow[r] & (u,v)_!(X) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad
\begin{tikzcd}
(u,v)^*(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
A\star B \arrow[r, "u\star v"'] & A'\star B'
\end{tikzcd}$$

and whose right adjoint sends an (A', B')-cylinder Y to the *pullback* (A, B)-cylinder $(u, v)^*(Y)$ defined by the pullback square on the right above.

4.2. **Proposition.** *The pushforward–pullback adjunction*

$$\begin{tikzcd}
\mathbf{Cyl}(A,B) \arrow[r, shift left=2, "(u,v)_!"] \arrow[r, phantom, "\perp"] & \mathbf{Cyl}(A',B') \arrow[l, shift left=2, "(u,v)^*"]
\end{tikzcd}$$

is a Quillen adjunction between the Joyal (resp. ambivariant) model structures on $\mathbf{Cyl}(A, B)$ *and* $\mathbf{Cyl}(A', B')$*. In particular, the pushforward functor* $(u, v)_!\colon \mathbf{Cyl}(A, B) \longrightarrow \mathbf{Cyl}(A', B')$ *preserves weak categorical equivalences and ambivariant equivalences.*

Proof. It suffices to show that the left adjoint $(u,v)_!$ preserves monomorphisms, weak categorical equivalences, and (by Proposition 2.12) inner anodyne extensions. The proof is a reiteration of the proof of Proposition 2.9. The second statement follows from the first because every object of $\mathbf{Cyl}(A,B)$ is cofibrant in both the Joyal model structure and the ambivariant model structure. ☐

4.3. *Observation* (pushforward as left Kan extension). Under the equivalences (3.2), the pushforward–pullback adjunction $(u,v)_! \dashv (u,v)^*$ corresponds to the adjunction

$$[(\Delta/A\times\Delta/B)^{\mathrm{op}},\mathbf{Set}] \underset{(\Delta/u\times\Delta/v)^*}{\overset{(\Delta/u\times\Delta/v)_!}{\rightleftarrows}} [(\Delta/A'\times\Delta/B')^{\mathrm{op}},\mathbf{Set}]$$

defined by left Kan extension and restriction along (the opposite of) the functor

$$\Delta/A\times\Delta/B \xrightarrow{\Delta/u\times\Delta/v} \Delta/A'\times\Delta/B'.$$

In the special case in which u is the identity morphism 1_A, the pushforward–pullback adjunction corresponds furthermore to the adjunction

$$[(\Delta/A)^{\mathrm{op}},\mathbf{sSet}/B] \underset{[1,v^*]}{\overset{[1,v_!]}{\rightleftarrows}} [(\Delta/A)^{\mathrm{op}},\mathbf{sSet}/B']$$

induced from the adjunction

$$\mathbf{sSet}/B \underset{v^*}{\overset{v_!}{\rightleftarrows}} \mathbf{sSet}/B'$$

whose left and right adjoints are given by composition with, and pullback along, the morphism $v\colon B\longrightarrow B'$ respectively. Hence, by §3.6, there exists an isomorphism

$$(\Delta[m],\alpha)\backslash(1_A,v)_!(X)\cong v_!((\Delta[m],\alpha)\backslash X) \tag{4.4}$$

natural in $([m],\alpha)\in\Delta/A$ and $X\in\mathbf{Cyl}(A,B)$.

We now use the results and constructions of §3 to deduce the main theorem of this section from the following theorem of Joyal.

4.5. **Theorem** (Joyal). *If $v\colon B\longrightarrow B'$ is a weak categorical equivalence, then the adjunction*

$$\mathbf{sSet}/B \underset{v^*}{\overset{v_!}{\rightleftarrows}} \mathbf{sSet}/B'$$

is a Quillen equivalence between the covariant model structures on $\mathbf{sSet}/B$ and $\mathbf{sSet}/B'$.

Proof. See [Joy08b, Theorem 10.2] and [Cis19, Theorem 5.2.14]. ☐

4.6. **Lemma.** *For any simplicial set A and any morphism of simplicial sets $v\colon B\longrightarrow B'$, the pushforward functor $(1_A,v)_!\colon \mathbf{Cyl}(A,B)\longrightarrow\mathbf{Cyl}(A,B')$ preserves vertically right local objects.*

Proof. Let X be a vertically right local object of $\mathbf{Cyl}(A,B)$. By the natural isomorphism (4.4), for each object $([m],\alpha)\in\Delta/A$, the morphism

$$i_m\backslash(1_A,v)_!(X)\colon (\Delta[m],\alpha)\backslash(1_A,v)_!(X)\longrightarrow(\Delta[0],\alpha_m)\backslash(1_A,v)_!(X)$$

is isomorphic in $\mathbf{sSet}/B'$ to the morphism

$$v_!(i_m\backslash X)\colon v_!((\Delta[m],\alpha)\backslash X)\longrightarrow v_!((\Delta[0],\alpha_m)\backslash X).$$

But this latter morphism is a covariant equivalence in $\mathbf{sSet}/B'$, since X is vertically right local by assumption and since the functor $v_!\colon\mathbf{sSet}/B\longrightarrow\mathbf{sSet}/B'$ preserves covariant equivalences by [Joy08b, Theorem 10.2]. Hence $(1_A,v)_!(X)$ is a vertically right local object of $\mathbf{Cyl}(A,B')$. ☐

4.7. **Theorem.** *Suppose $u\colon A \longrightarrow A'$ and $v\colon B \longrightarrow B'$ are weak categorical equivalences. Then the pushforward–pullback adjunction*

$$\begin{tikzcd} \mathbf{Cyl}(A,B) \arrow[r, shift left=1.5ex, "(u,v)_!"] \arrow[r, phantom, "\perp"] & \mathbf{Cyl}(A',B') \arrow[l, shift left=1.5ex, "(u,v)^*"] \end{tikzcd}$$

is a Quillen equivalence between the ambivariant model structures on $\mathbf{Cyl}(A,B)$ and $\mathbf{Cyl}(A',B')$. In particular, the pushforward functor $(u,v)_!\colon \mathbf{Cyl}(A,B) \longrightarrow \mathbf{Cyl}(A',B')$ reflects ambivariant equivalences.

Proof. By Proposition 4.2, the pushforward–pullback adjunction is a Quillen adjunction between the ambivariant model structures.

Observe that the pushforward–pullback adjunction is isomorphic to the following composite adjunction.

$$\begin{tikzcd} \mathbf{Cyl}(A,B) \arrow[r, shift left=1.5ex, "(1_A,v)_!"] \arrow[r, phantom, "\perp"] & \mathbf{Cyl}(A,B') \arrow[l, shift left=1.5ex, "(1_A,v)^*"] \arrow[r, shift left=1.5ex, "(u,1_{B'})_!"] \arrow[r, phantom, "\perp"] & \mathbf{Cyl}(A',B') \arrow[l, shift left=1.5ex, "(u,1_{B'})^*"] \end{tikzcd}$$

Hence it suffices to consider the case in which u is an identity and the case in which v is an identity. We will prove the Quillen equivalence in the first case, from which the second case follows by Observation 2.13.

We first prove that the pushforward functor $(1_A,v)_!\colon \mathbf{Cyl}(A,B) \longrightarrow \mathbf{Cyl}(A,B')$ reflects ambivariant equivalences. Let $f\colon X \longrightarrow Y$ be a morphism in $\mathbf{Cyl}(A,B)$ and suppose that the morphism

$$(1_A,v)_!(f)\colon (1_A,v)_!(X) \longrightarrow (1_A,v)_!(Y)$$

is an ambivariant equivalence in $\mathbf{Cyl}(A,B')$. Since the pushforward functor $(1_A,v)_!$ preserves ambivariant equivalences by Proposition 4.2, we may suppose without loss of generality that X and Y are ambifibrant, and hence vertically right local by Proposition 3.14. Hence it follows from Lemma 4.6 that $(1_A,v)_!(X)$ and $(1_A,v)_!(Y)$ are vertically right local objects of $\mathbf{Cyl}(A,B')$. It then follows from Corollary 3.21 that the morphism $(1_A,v)_!(f)$ is a vertical covariant equivalence in $\mathbf{Cyl}(A,B')$, i.e. that the morphism

$$(\Delta[m],\alpha)\backslash(1_A,v)_!(f)\colon (\Delta[m],\alpha)\backslash(1_A,v)_!(X) \longrightarrow (\Delta[m],\alpha)\backslash(1_A,v)_!(Y)$$

is a covariant equivalence in $\mathbf{sSet}/B'$ for each $([m],\alpha) \in \Delta/A$. By the natural isomorphism (4.4), this latter morphism is isomorphic in $\mathbf{sSet}/B'$ to the morphism

$$v_!((\Delta[m],\alpha)\backslash f)\colon v_!((\Delta[m],\alpha)\backslash X) \longrightarrow v_!((\Delta[m],\alpha)\backslash Y),$$

which is therefore also a covariant equivalence in $\mathbf{sSet}/B'$. Therefore, since the functor $v_!\colon \mathbf{sSet}/B \longrightarrow \mathbf{sSet}/B'$ reflects covariant equivalences by Theorem 4.5, we may deduce that the morphism $f\colon X \longrightarrow Y$ is a vertical covariant equivalence, and hence an ambivariant equivalence in $\mathbf{Cyl}(A,B)$ by Corollary 3.21.

It remains to prove that, for each ambifibrant (and therefore vertically Reedy left fibrant by Proposition 3.14) object $Y \in \mathbf{Cyl}(A,B')$, the counit morphism

$$\varepsilon_Y\colon (1_A,v)_!(1_A,v)^*(Y) \longrightarrow Y$$

is an ambivariant equivalence in $\mathbf{Cyl}(A,B')$. Indeed, this morphism is a vertical covariant equivalence (and hence an ambivariant equivalence by Corollary 3.21), i.e., for each object $([m],\alpha) \in \Delta/A$, the morphism

$$(\Delta[m],\alpha)\backslash\varepsilon_Y\colon (\Delta[m],\alpha)\backslash(1_A,v)_!(1_A,v)^*(Y) \longrightarrow (\Delta[m],\alpha)\backslash Y$$

is a covariant equivalence in $\mathbf{sSet}/B'$. For, by Observation 4.3, this latter morphism is isomorphic in $\mathbf{sSet}/B'$ to the counit morphism

$$v_!v^*((\Delta[m],\alpha)\backslash Y) \longrightarrow (\Delta[m],\alpha)\backslash Y$$

of the adjunction $v_! \dashv v^*$, and is therefore a covariant equivalence by Theorem 4.5, since, by Observation 3.11 and the assumption that Y is vertically Reedy left fibrant, $(\Delta[m], \alpha)\backslash Y$ is a left fibrant object of $\mathbf{sSet}/B$.

Hence the pushforward–pullback adjunction is a Quillen equivalence between the ambivariant model structures. This implies the second statement because every object of $\mathbf{Cyl}(A, B)$ is cofibrant in the ambivariant model structure. □

5. Joyal's cylinder conjecture

In this section, we use the results of the preceding sections to complete the proof of Joyal's conjecture outlined in §1.3.

We begin with a recollection (see [Ste18a, Lemmas 3.7 and 3.8]) of the proof of the easy case of Joyal's conjecture. Recall from Theorem 2.11 that, for each pair of simplicial sets A and B, an object $X \in \mathbf{Cyl}(A, B)$ is fibrant in the ambivariant model structure on $\mathbf{Cyl}(A, B)$ if and only if the canonical morphism $X \longrightarrow A \star B$ is an inner fibration.

5.1. **Lemma.** *Suppose A and B are quasi-categories. Then, in the Joyal model structure on $\mathbf{Cyl}(A, B)$, an object X is fibrant if and only if the canonical morphism $X \longrightarrow A \star B$ is an inner fibration, and a morphism between fibrant objects is a fibration if and only if it is an inner fibration.*

Proof. Recall from Theorem 2.10 that a morphism in $\mathbf{Cyl}(A, B)$ is a fibration in the Joyal model structure on $\mathbf{Cyl}(A, B)$ if and only if it (more precisely, its underlying morphism of simplicial sets) is a fibration in the Joyal model structure for quasi-categories on $\mathbf{sSet}$. Since every fibration in the Joyal model structure for quasi-categories is in particular an inner fibration, it remains to prove the sufficiency of each condition.

Let $X \in \mathbf{Cyl}(A, B)$, and suppose that the canonical morphism $p\colon X \longrightarrow A \star B$ from X to the terminal object of $\mathbf{Cyl}(A, B)$ is an inner fibration. Since p is an inner fibration between quasi-categories (by [Joy08b, Corollary 3.23]), to prove that p is a fibration, it suffices to prove that the induced functor between homotopy categories $\mathrm{ho}(p)\colon \mathrm{ho}(X) \longrightarrow \mathrm{ho}(A \star B)$ is an isofibration. In fact, this functor is a *discrete* isofibration (i.e. has the *unique* isomorphism lifting property): for any isomorphism in the category $\mathrm{ho}(A \star B) \cong \mathrm{ho}(A) \star \mathrm{ho}(B)$ must belong to one of its full subcategories $\mathrm{ho}(A)$ or $\mathrm{ho}(B)$, and the pullback of $\mathrm{ho}(p)$ along either full subcategory inclusion $\mathrm{ho}(A) \longrightarrow \mathrm{ho}(A \star B)$ or $\mathrm{ho}(B) \longrightarrow \mathrm{ho}(A \star B)$ is an isomorphism (since $X \in \mathbf{Cyl}(A, B)$). Hence X is a fibrant object in the Joyal model structure on $\mathbf{Cyl}(A, B)$.

Now, let $f\colon X \longrightarrow Y$ be a morphism between fibrant objects in the Joyal model structure on $\mathbf{Cyl}(A, B)$, and suppose that f is an inner fibration. Once again, it suffices to prove that the functor $\mathrm{ho}(f)$ is an isofibration. Let $q\colon Y \longrightarrow A \star B$ denote the canonical morphism. By the previous paragraph, the functors $\mathrm{ho}(q)$ and $\mathrm{ho}(qf) = \mathrm{ho}(q) \circ \mathrm{ho}(f)$ are discrete isofibrations, which implies that the functor $\mathrm{ho}(f)$ is also a discrete isofibration. Hence f is a fibration in the Joyal model structure on $\mathbf{Cyl}(A, B)$. □

5.2. **Corollary.** *Suppose A and B are quasi-categories. On the category $\mathbf{Cyl}(A, B)$, the Joyal model structure and the ambivariant model structure coincide. In particular, a morphism in $\mathbf{Cyl}(A, B)$ is a weak categorical equivalence if and only if it is an ambivariant equivalence.*

Proof. The two model structures have the same cofibrations (namely, the monomorphisms) and, by Lemma 5.1, the same fibrant objects. Hence the two model structures coincide by [Joy08b, Proposition E.1.10]. □

We now use the results of the preceding sections to deduce the general case of Joyal's conjecture from this special case.

5.3. **Theorem.** *Let A and B be a pair of simplicial sets. On the category $\mathbf{Cyl}(A, B)$, the Joyal model structure and the ambivariant model structure coincide.*

Proof. Since the two model structures have the same cofibrations (namely, the monomorphisms), it suffices to show that they have the same weak equivalences, i.e., that a morphism in $\mathbf{Cyl}(A, B)$

is a weak categorical equivalence if and only if it is an ambivariant equivalence. Since every fibration in the Joyal model structure is in particular an inner fibration, it follows by [Joy08b, Proposition E.1.10] that every ambivariant equivalence in $\mathbf{Cyl}(A, B)$ is a weak categorical equivalence. It remains to prove the converse.

Let f be a weak categorical equivalence in $\mathbf{Cyl}(A, B)$. Let $u \colon A \longrightarrow A'$ and $v \colon B \longrightarrow B'$ be weak categorical equivalences such that A' and B' are quasi-categories (as may be constructed by the small object argument). By Proposition 4.2, the pushforward morphism $(u, v)_!(f)$ is a weak categorical equivalence in $\mathbf{Cyl}(A', B')$. Then, since A' and B' are quasi-categories, Corollary 5.2 implies that $(u, v)_!(f)$ is an ambivariant equivalence in $\mathbf{Cyl}(A', B')$. Finally, since u and v are weak categorical equivalences, Theorem 4.7 implies that f is an ambivariant equivalence in $\mathbf{Cyl}(A, B)$. □

Therefore, for each pair of simplicial sets A and B, everything we have proved about the ambivariant model structure on $\mathbf{Cyl}(A, B)$ is true also of the Joyal model structure on $\mathbf{Cyl}(A, B)$, for the two are one. In particular, we may deduce the following corollary.

5.4. **Theorem** (Joyal's cylinder conjecture)**.** *Let A and B be a pair of simplicial sets. In the Joyal model structure on $\mathbf{Cyl}(A, B)$, an object X is fibrant if and only if the canonical morphism $X \longrightarrow A \star B$ is an inner fibration, and a morphism between fibrant objects is a fibration if and only if it is an inner fibration.*

Proof. Since the Joyal and ambivariant model structures on $\mathbf{Cyl}(A, B)$ coincide by Theorem 5.3, they must have the same fibrant objects and the same fibrations between fibrant objects. By the description of the ambivariant model structure given in Theorem 2.11, this proves the theorem. □

6. Covariant equivalences

In this final section, we use our proof of Joyal's cylinder conjecture to give a new, direct proof (see Theorem 6.5) of a characterisation of covariant equivalences due to Lurie [Lur09, Chapter 2], which avoids the use of the straightening theorem [Lur09, Theorem 2.2.1.2].

6.1. **The left cone functor.** Let B be a simplicial set. We recall from [Lur09, Definition 2.4.1.2] the *left cone functor* $C^{\triangleleft} \colon \mathbf{sSet}/B \longrightarrow \mathbf{sSet}$, which sends an object $p \colon X \longrightarrow B$ of $\mathbf{sSet}/B$ to its *left cone*, that is, the simplicial set $C^{\triangleleft}(X, p)$ defined by the pushout square below.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "p"'] & \Delta[0] \star X \arrow[d] \\ B \arrow[r] & C^{\triangleleft}(X, p) \end{tikzcd}$$

Observe that $C^{\triangleleft}(X, p)$ is the underlying simplicial set of a $(\Delta[0], B)$-cylinder; indeed, the left cone functor is none other than the composite

$$\mathbf{sSet}/B \xrightarrow{\Delta[0] \boxtimes (-)} \mathbf{Cyl}(\Delta[0], B) \longrightarrow \mathbf{sSet}$$

of the exterior product functor (see §3.5; here $A = \Delta[0]$) and the forgetful functor.

6.2. **Lurie's characterisation of covariant equivalences.** In [Lur09, Chapter 2], Lurie proves that, for each simplicial set B, a morphism in $\mathbf{sSet}/B$ is a covariant equivalence if and only if it is sent by the left cone functor $C^{\triangleleft} \colon \mathbf{sSet}/B \longrightarrow \mathbf{sSet}$ to a weak categorical equivalence.

6.3. *Remark.* While this characterisation of covariant equivalences does not appear to be stated explicitly in [Lur09, Chapter 2], it is nonetheless, by the following argument, a consequence of the results of that chapter.

By [Lur09, Proposition 2.1.4.7], there exists a model structure on $\mathbf{sSet}/B$ whose cofibrations are the monomorphisms, and whose weak equivalences are the morphisms in $\mathbf{sSet}/B$ that are sent by the composite

$$\mathbf{sSet}/B \xrightarrow{C^{\triangleleft}} \mathbf{sSet} \xrightarrow{\mathfrak{C}} \mathbf{sSet}\text{-}\mathbf{Cat}$$

of the left cone functor and the homotopy coherent realisation functor to Dwyer–Kan equivalences of simplicially enriched categories.

By [Lur09, Corollary 2.2.3.12] (proved as a corollary of the straightening theorem [Lur09, Theorem 2.2.1.2]), the fibrant objects of this model structure are the left fibrations with codomain B. Hence, by [Joy08b, Proposition E.1.10], this model structure on $\mathbf{sSet}/B$ coincides with the covariant model structure (as defined and constructed in [Joy08b, Chapter 8]). In particular, a morphism in $\mathbf{sSet}/B$ is a covariant equivalence if and only if it is sent by the above composite functor to a Dwyer–Kan equivalence.

So the characterisation follows at last from [Lur09, Proposition 2.2.5.8] (whose proof depends on [Lur09, Theorem 2.4.6.1]), which states that a morphism of simplicial sets is a weak categorical equivalence if and only if it is sent by the homotopy coherent realisation functor to a Dwyer–Kan equivalence.

We now use the results of the preceding sections to give a direct proof of Lurie's characterisation of covariant equivalences. We note that this proof does not depend on the straightening theorem.

6.4. **Theorem.** *Let B be a simplicial set. The adjunction*

$$\begin{tikzcd} \mathbf{Cyl}(\Delta[0],B) \arrow[r, shift right, "\Delta[0]\backslash(-)"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet}/B \arrow[l, shift right, "\Delta[0]\boxtimes(-)"'] \end{tikzcd}$$

is a Quillen equivalence between the Joyal model structure on $\mathbf{Cyl}(\Delta[0],B)$ and the covariant model structure on $\mathbf{sSet}/B$.

Proof. Under the equivalence (3.4), this adjunction corresponds to the adjunction

$$\begin{tikzcd} {[\Delta^{\mathrm{op}},\mathbf{sSet}/B]} \arrow[r, shift right, "\mathrm{ev}_0"'] \arrow[r, phantom, "\perp"] & \mathbf{sSet}/B \arrow[l, shift right, "\mathrm{cst}"'] \end{tikzcd}$$

whose left adjoint sends an object X of $\mathbf{sSet}/B$ to the constant simplicial object in $\mathbf{sSet}/B$ with value X, and whose right adjoint sends a simplicial object in $\mathbf{sSet}/B$ to its value at $[0]\in\Delta$. Moreover, by Theorems 3.20 and 5.3, the model structure on $[\Delta^{\mathrm{op}},\mathbf{sSet}/B]$ which corresponds under this equivalence to the Joyal model structure on $\mathbf{Cyl}(\Delta[0],B)$ is the Bousfield localisation of the Reedy model structure (with respect to the covariant model structure on $\mathbf{sSet}/B$) whose local objects are the weakly constant ones. But this is precisely the "canonical model structure" (in the sense of [RSS01, Theorem 3.1]) on $[\Delta^{\mathrm{op}},\mathbf{sSet}/B]$ with respect to the covariant model structure on $\mathbf{sSet}/B$, so [RSS01, Theorem 3.9] implies that this adjunction is a Quillen equivalence. □

6.5. **Theorem** (Lurie)**.** *Let B be a simplicial set. A morphism in $\mathbf{sSet}/B$ is a covariant equivalence if and only if it is sent by the left cone functor $C^{\triangleleft}\colon \mathbf{sSet}/B \longrightarrow \mathbf{sSet}$ to a weak categorical equivalence.*

Proof. As observed in §6.1, the left cone functor is the composite

$$\mathbf{sSet}/B \xrightarrow{\Delta[0]\boxtimes(-)} \mathbf{Cyl}(\Delta[0],B) \longrightarrow \mathbf{sSet}$$

of the exterior product functor and the forgetful functor. Thus the result follows from Theorem 6.4 (since every object of $\mathbf{sSet}/B$ is cofibrant in the covariant model structure). □

6.6. *Remark* (two definitions of covariant equivalences).

Appendix A. The parametrised Joyal model structure

[*To be polished.*]

In this appendix, we introduce and study, for each simplicial set B, a model structure on the slice category $\mathbf{sSet}/B$ whose fibrant objects are the inner fibrations with codomain B, which we call the *parametrised Joyal model structure* on $\mathbf{sSet}/B$ (see Theorem A.6). We use this family of model structures to define a new class of morphisms of simplicial sets, which we call the *absolute weak categorical equivalences* (see Definition A.17), using which we prove some new results about inner anodyne extensions and inner fibrations (see §A.16), building on [Cam20], and give a new proof of a theorem of Stevenson (see Proposition A.23).

A.1. *Remark.* The constructions and results of this section are inspired by, but not an instance of, the general theory presented in [Cis19, §2.5]. The existence of the parametrised Joyal model structures studied in this section is prefigured in [Cis19, Remark 5.1.22], where it is observed that the inner fibrations with codomain B are (among the) fibrant objects in the minimal Cisinski model structure on $\mathbf{sSet}/B$.

A.2. **Cisinski model structures.** We will construct the aforementioned parametrised Joyal model structures by Cisinski's method for constructing model structures on presheaf categories presented in [Cis06, §1.3] and [Cis19, §2.4].

Recall that one constructs a model structure on a given presheaf category by this method by specifying an *exact cylinder* (see [Cis06, Définition 1.3.6] or [Cis19, Definition 2.4.8]) and a class of *anodyne extensions* relative to this exact cylinder (see [Cis06, Définition 1.3.10] or [Cis19, Definition 2.4.11]). The resulting model structure is cofibrantly generated, its cofibrations are the monomorphisms, and an object is fibrant (resp. a morphism between fibrant objects is a fibration) if and only if it has the right lifting property with respect to the specified class of anodyne extensions (see [Cis06, Théorème 1.3.22, Proposition 1.3.36] or [Cis19, Theorem 2.4.19]). Any cofibrantly generated model structure on a presheaf category whose cofibrations are the monomorphisms arises by this method.

A.3. **The Joyal model structure for quasi-categories.** Recall the *Joyal model structure* on the category $\mathbf{sSet}$, whose cofibrations are the monomorphisms and whose fibrant objects are the quasi-categories, and in which a morphism between quasi-categories is a fibration if and only if it is an isofibration. This model structure may be constructed by Cisinski's method as follows. The input data is the exact cylinder $X \mapsto J \times X$ (where, following Joyal, J denotes the nerve of the contractible groupoid with two objects, 0 and 1), and the class of J-anodyne extensions defined to be the saturation of the set of monomorphisms displayed below.

$$\Big\{\{0\} \longrightarrow J\Big\} \cup \Big\{\Lambda^k[n] \longrightarrow \Delta[n] :\ n \geq 2,\, 0 < k < n\Big\}$$

The verification that this saturated class is indeed a class of J-anodyne extensions essentially boils down to the following fundamental lemma of quasi-category theory.

A.4. **Lemma.** *For each $n > 0$, the Leibniz product*

$$(J \times \partial\Delta[n]) \cup_{\{0\}\times\partial\Delta[n]} (\{0\} \times \Delta[n]) \longrightarrow J \times \Delta[n]$$

is an inner anodyne extension.

Proof. Since the codomain of this morphism is a quasi-category, it suffices to show that this morphism has the left lifting property with respect to every inner fibration $p\colon X \longrightarrow Y$ between quasi-categories. By adjointness, it thus suffices to show that the morphism $X^{\Delta[n]} \longrightarrow X^{\partial\Delta[n]} \times_{Y^{\partial\Delta[n]}} Y^{\Delta[n]}$, which is an inner fibration between quasi-categories, is an isofibration. This follows from the Leibniz product version of Joyal's lifiting theorem, see e.g. [Joy08b, Lemma 5.8] or (for a simpler proof) [Cis19, Theorem 3.5.9]. □

We will see that the construction of the *parametrised* Joyal model structures depends on the exact same fundamental lemma. First, we describe the fibrations between fibrant objects in this model structure.

A.5. **Definition** (fibrewise isofibrations)**.** Let $f\colon (X,p) \longrightarrow (Y,q)$ be a morphism in $\mathbf{sSet}/B$, and suppose that all three morphisms f, p, and q are inner fibrations. We say that f is a *fibrewise isofibration over B* if, for each 0-simplex $b \in B$, the morphism $f_b\colon X_b \longrightarrow Y_b$, which is an inner fibration between quasi-categories, is moreover an isofibration.

A.6. **Theorem** (the parametrised Joyal model structure)**.** *For each simplicial set B, there exists a model structure on the slice category* $\mathbf{sSet}/B$ *whose cofibrations are the monomorphisms and whose fibrant objects are the inner fibrations with codomain B. A morphism between fibrant objects is a fibration if and only if it is an inner fibration and a fibrewise isofibration over B.*

Proof. We define an exact cylinder (see [Cis06, Définition 1.3.6] or [Cis19, Definition 2.4.8]) on $\mathbf{sSet}/B$ by $(X, p) \mapsto (J \times X, p \circ \mathrm{pr}_2)$. Let An denote the (weak) saturation of the set of morphisms in $\mathbf{sSet}/B$ consisting of the inclusions

$$\begin{tikzcd} \Delta[0] \arrow[rr, "\partial_0"] \arrow[dr, "b"'] & & J \arrow[dl, "b!"] \\ & B & \end{tikzcd}$$

for each 0-simplex $b \in B_0$, and the inner horn inclusions

$$\begin{tikzcd} \Lambda^k[n] \arrow[rr] \arrow[dr] & & \Delta[n] \arrow[dl, "\beta"] \\ & B & \end{tikzcd}$$

for each $n \geq 2$, $0 < k < n$, and $\beta \in B_n$.

It is easy to show that every inner anodyne extension over B belongs to the class An. The verification that this class An defines a class of "anodyne extensions" relative to the above exact cylinder (see [Cis06, Définition 1.3.10] or [Cis19, Definition 2.4.11]) ultimately boils down to the following facts:

(i) the class of monomorphisms in $\mathbf{sSet}/B$ is the saturation of the set of boundary inclusions $i \colon (\partial\Delta[n], \beta i) \longrightarrow (\Delta[n], \beta)$ for $n \geq 0$ and $\beta \in B_n$,

(ii) the Leibniz product of a monomorphism with an inner anodyne extension is inner anodyne,

(iii) Lemma A.4.

It then follows from [Cis06, Théorème 1.3.22, Proposition 1.3.36] or [Cis19, Theorem 2.4.19] that there exists a (necessarily unique) model structure on $\mathbf{sSet}/B$ whose cofibrations are the monomorphisms, and in which a morphism with fibrant codomain is a fibration if and only if it has the right lifting property with respect to the class of morphisms An. It is straightforward to show that these morphisms are precisely those described in the statement of the theorem. □

We call the model structure of Theorem A.6 on $\mathbf{sSet}/B$ the *parametrised Joyal model structure* over B. In the case $B = \Delta[0]$, this model structure coincides with Joyal's model structure for quasi-categories on $\mathbf{sSet}$. More generally, if B is a quasi-category whose homotopy category has no non-identity isomorphisms, then this model structure on $\mathbf{sSet}/B$ coincides with the one induced from Joyal's model structure for quasi-categories. This coincidence does not occur in general, e.g. when $B = J$.

A.7. *Perspective* (parametrised categories). [*In which the parametrised Joyal model structures are related to the parametrised categories of* [SS88].]

We will use the following lemma to recognise left Quillen functors out of the parametrised Joyal model structure on $\mathbf{sSet}/B$.

A.8. **Lemma.** *Let B be a simplicial set, let $\mathcal{M}$ be a model category, and let $F \colon \mathbf{sSet}/B \longrightarrow \mathcal{M}$ be a cocontinuous functor that sends monomorphisms to cofibrations. Then F sends the weak equivalences in the parametrised Joyal model structure on* $\mathbf{sSet}/B$ *to weak equivalences in $\mathcal{M}$ if and only if it sends the following morphisms to weak equivalences in $\mathcal{M}$:*

(i) *for each* 0*-simplex* $b \in B_0$*, the morphism*

$$\begin{tikzcd} J \arrow[rr, "!"] \arrow[dr] & & \Delta[0] \arrow[dl, "b"] \\ & B & \end{tikzcd}$$

(ii) *for each* $n \geq 2$*,* $0 < k < n$*, and* n*-simplex* $\beta \in B_n$*, the morphism*

$$\begin{tikzcd} \Lambda^k[n] \arrow[rr] \arrow[dr] & & \Delta[n] \arrow[dl, "\beta"] \\ & B & \end{tikzcd}$$

Proof. This is immediate from the construction of the model structure given in the proof of Theorem A.6. Alternatively, one can argue directly from the statement of that theorem by using [JT07, Proposition 7.15]. ☐

This lemma has some immediate consequences.

A.9. **Proposition.** *For each morphism of simplicial sets* $u\colon A \longrightarrow B$*, the adjunction*

$$\begin{tikzcd} \mathbf{sSet}/A \arrow[r, shift left=2, "u_!"] \arrow[r, phantom, "\perp"] & \mathbf{sSet}/B \arrow[l, shift left=2, "u^*"] \end{tikzcd}$$

is a Quillen adjunction between the parametrised Joyal model structures over A *and* B*. In particular, for any simplicial set* A*, every weak equivalence in the parametrised Joyal model structure on* $\mathbf{sSet}/A$ *is a weak categorical equivalence.*

Proof. Apply Lemma A.8. ☐

A.10. **Example.** The converse to the second statement of Proposition A.9 is not true in general. For example, the morphism $\partial_0\colon \Delta[0] \longrightarrow J$ is a weak categorical equivalence, but the morphism $\partial_0\colon (\Delta[0], \partial_0) \longrightarrow (J, \mathrm{id}_J)$ in $\mathbf{sSet}/J$ is not a weak equivalence in the parametrised Joyal model structure (since it is a morphism between cofibrant-fibrant objects, but there exists no morphism in the opposite direction).

A.11. **Proposition.** *For each simplicial set* B*, the parametrised Joyal model structure on* $\mathbf{sSet}/B$ *is enriched over the Joyal model structure on* $\mathbf{sSet}$ *via the standard simplicial enrichment of the slice category* $\mathbf{sSet}/B$*.*

Proof. This follows by a couple of applications of Lemma A.8, together with the facts that the Leibniz product in $\mathbf{sSet}$ of a monomorphism with a monomorphism (resp. inner anodyne extension) is again a monomorphism (resp. inner anodyne extension), and that J is an injective object of $\mathbf{sSet}$. ☐

We will denote the simplicial hom for the standard enrichment of $\mathbf{sSet}/B$ by $\mathbf{Fun}_B(-,-)$.

A.12. *Observation* (the ∞-cosmos of B-parametrised quasi-categories)*.* We thus obtain, for each simplicial set B, an ∞-cosmos (in the sense of Riehl–Verity [RV17]) whose objects are the inner fibrations with codomain B.

The following lemma makes precise the sense in which the model category $\mathbf{sSet}/B$ (with the parametrised Joyal model structure) is generated by the 1-simplices of B under homotopy colimits.

A.13. **Lemma.** *Let* B *be a simplicial set and let* D *be a class of objects of* $\mathbf{sSet}/B$ *with the following properties:*

(i) D *is saturated by monomorphisms in* $\mathbf{sSet}/B$ *(see* [Cis06, Définition 1.1.12] *or* [Cis19, Definition 1.3.9]*),*

(ii) D *is replete with respect to weak equivalences in the parametrised model structure on* $\mathbf{sSet}/B$*, and*
(iii) *for every* 1*-simplex* f *of* B*, the object* $f\colon \Delta[1] \longrightarrow B$ *belongs to* D*.*

Then every object of $\mathbf{sSet}/B$ *belongs to* D*.*

Proof. It suffices to show that every representable $\beta\colon \Delta[n] \longrightarrow B$ belongs to D. Every 1-simplex of B belongs to D by assumption (iii). Since any 0-simplex of B is a retract of its degenerate 1-simplex, every 0-simplex of B belongs to D by (i). By (i), any object $I[n] \longrightarrow B$ (where $I[n]$ denotes the n-spine) belongs to D for each $n \geq 2$. For each $n \geq 2$, the spine inclusion $I[n] \longrightarrow \Delta[n]$ is inner anodyne; hence by (ii), every simplex of B belongs to D. ☐

A.14. **Proposition** (equivalences of parametrised quasi-categories)**.** *Let* B *be a simplicial set,* $p\colon X \longrightarrow B$ *and* $q\colon Y \longrightarrow B$ *inner fibrations, and* $u\colon X \longrightarrow Y$ *a morphism such that* $qu = p$*. Then the following are equivalent.*

(i) $u\colon (X,p) \longrightarrow (Y,q)$ *is a weak equivalence in the parametrised Joyal model structure on* $\mathbf{sSet}/B$*.*
(ii) *For every* 1*-simplex* f *of* B*, the induced morphism on fibres* $u\colon X_f \longrightarrow Y_f$ *is an equivalence of quasi-categories.*
(iii) *For every* 1*-simplex* f *of* B*, the morphism*
$$\mathbf{Fun}_B(1,u)\colon \mathbf{Fun}_B((\Delta[1],f),X) \longrightarrow \mathbf{Fun}_B((\Delta[1],f),Y)$$
is an equivalence of quasi-categories.
(iv) *For every* 0*-simplex* $b \in B_0$*, the induced morphism on fibres* $u\colon X_b \longrightarrow Y_b$ *is essentially surjective on objects, and for each pair of* 0*-simplices* $x,y \in X_0$ *and each* 1*-simplex* $f\colon p(x) \to p(y)$ *in* B*, the morphism* $u\colon \mathrm{Hom}_{X_f}(x,y) \longrightarrow \mathrm{Hom}_{Y_f}(u(x),u(y))$ *is an equivalence of Kan complexes.*

Proof. (i) $\Longrightarrow$ (ii) since pullback along $f\colon \Delta[1] \longrightarrow B$ is right Quillen by Proposition A.9, and so preserve weak equivalences between fibrant objects, together with the observation that the parametrised Joyal model structure on $\mathbf{sSet}/\Delta[1]$ coincides with the usual slice Joyal model structure.

(ii) $\Longrightarrow$ (iii) since $\mathbf{Fun}_B((\Delta[1],f),X) \cong \mathbf{Fun}_{\Delta[1]}(\Delta[1],X_f)$ and since $\mathbf{Fun}_{\Delta[1]}(\Delta[1],-)$ is right Quillen by Proposition A.11.

(iii) $\Longrightarrow$ (i) from Lemma A.13 by considering the class of objects A of $\mathbf{sSet}/B$ for which $\mathbf{Fun}_B(A,u)$ is an equivalence of quasi-categories.

(iv) is an evident rephrasing of (ii), using the standard theorem that a morphism of quasi-categories is an equivalence if and only if it is essentially surjective on objects and fully faithful. ☐

From the point of view described in Perspective A.7, the properties given in the condition (iv) above might be called "parametrised essentially surjective on objects" and "parametrised fully faithful".

The following proposition is a corollary of the preceding proposition; note however that it can also be easily proved directly.

A.15. **Proposition.** *Let* $p\colon X \longrightarrow B$ *be an inner fibration. The following are equivalent.*

(i) $p\colon X \longrightarrow B$ *is a trivial fibration.*
(ii) *For every* 1*-simplex* f *of* B*, the fibre* X_f *is categorically equivalent to* $\Delta[1]$*.*
(iii) *For every* 1*-simplex* f *of* B*, the pullback* $X_f \longrightarrow \Delta[1]$ *of* p *along* $f\colon \Delta[1] \longrightarrow B$ *is a trivial fibration.*
(iv) p *is surjective on objects and, for each pair of* 0*-simplices* $x,y \in X_0$ *and each* 1*-simplex* $f\colon p(x) \longrightarrow p(y)$ *in* B*, the hom-space* $\mathrm{Hom}_{X_f}(x,y)$ *is a contractible Kan complex.*

Proof. This follows from the previous proposition applied to the fibration between fibrant objects $p\colon (X,p) \longrightarrow (B,1_B)$ in $\mathbf{sSet}/B$. (ii) is intended to be a snappy rephrasing of (iii). ☐

A.16. **Absolute weak categorical equivalences.** For many years, the following questions (here suggestively posed) were open:

(1) Is a monomorphism inner anodyne if and only if it is a surjective-on-0-simplices weak categorical equivalence?
(2) Is an inner fibration a trivial fibration if and only if it is a surjective-on-0-simplices weak categorical equivalence?

In [Cam20], I showed that the answer to both of these questions is no (amusingly, a single morphism was a counterexample to both questions); note that both statements are true if the codomains of the morphisms in question are assumed to be quasi-categories. But that paper still left unresolved the general question of how to "intrinsically" characterise the inner anodyne extensions among the monomorphisms, and the trivial fibrations among the inner fibrations.

In this section I will show that these statements may be corrected by replacing the property "surjective-on-0-simplices weak categorical equivalence" by the new property *absolute weak categorical equivalence* (which we introduce in Definition A.17); that is, we prove (see Proposition A.20):

(1) A monomorphism is inner anodyne if and only if it is an absolute weak categorical equivalence.
(2) An inner fibration is a trivial fibration if and only if it is an absolute weak categorical equivalence.

Note that we have in fact already proved another characterisation of the trivial fibrations among the inner fibrations (see Proposition A.15):

(3) An inner fibration $X \longrightarrow B$ is a trivial fibration if and only if the fibre X_f over each 1-simplex f of B is categorically equivalent to $\Delta[1]$.

This triple of statements should be thought of as analogous to the following triple:

(1) A monomorphism is right anodyne if and only if it is final.
(2) A right fibration is a trivial fibration if and only if it is final.
(3) A right fibration $X \longrightarrow B$ is a trivial fibration if and only if the fibre X_b over each 0-simplex b of B is homotopy equivalent to $\Delta[0]$.

A.17. **Definition** (absolute weak categorical equivalence)**.** A morphism of simplicial sets $u\colon A \longrightarrow B$ is said to be an *absolute weak categorical equivalence* if, for every simplicial set C and every morphism $g\colon B \longrightarrow C$, the morphism $u\colon (A, gu) \longrightarrow (B, g)$ is a weak equivalence in the parametrised Joyal model structure on $\mathbf{sSet}/C$.

A.18. **Example.** Any inner anodyne extension and any trivial fibration is an absolute weak categorical equivalence.

A.19. **Proposition.** *The class of absolute weak categorical equivalences is closed under composition and satisfies the right cancellation property.*

Proof. This is immediate from the definition. □

A.20. **Proposition.**

(1) *A morphism of simplicial sets $u\colon A \longrightarrow B$ is an absolute weak categorical equivalence if and only if the morphism $u\colon (A, u) \longrightarrow (B, 1_B)$ is a weak equivalence in the parametrised Joyal model structure on* $\mathbf{sSet}/B$.
(2) *An inner fibration is a trivial fibration if and only if it an absolute weak categorical equivalence.*
(3) *A morphism of simplicial sets is an absolute weak categorical equivalence if and only if it factors as an inner anodyne extension followed by a trivial fibration.*
(4) *A morphism of simplicial sets is inner anodyne if and only if it is a monic absolute weak categorical equivalence.*

Proof. (1) Necessity is immediate from the definition, and sufficiency follows from Proposition A.9.

(2) Let $p\colon X \longrightarrow B$ be an inner fibration. Then $p\colon (X, p) \longrightarrow (B, 1_B)$ is a fibration between fibrant objects in the parametrised Joyal model structure on $\mathbf{sSet}/B$, and is therefore a weak equivalence therein if and only if it is a trivial fibration.

(3) Any inner anodyne extension and any trivial fibration is an absolute weak categorical equivalence, and the class of absolute weak categorical equivalences is closed under composition. Conversely, let u be an absolute weak categorical equivalence, and et $u = pi$ be a factorisation of u as an inner anodyne extension i followed by an inner fibration p. It follows from the right cancellation property of the class of absolute weak categorical equivalences that p is an absolute weak categorical equivalence, and hence a trivial fibration by (2).

(4) Every inner anodyne extension is both monic and an absolute weak categorical equivalence. The converse follows from part (3). $\square$

A.21. **Proposition.** *Every absolute weak categorical equivalence is a surjective-on-*0*-simplices weak categorical equivalence. Any surjective-on-*0*-simplices weak categorical equivalence whose codomain is a quasi-category is an absolute weak categorical equivalence.*

Proof. The first statement follows from part (3) of Proposition A.20. For the partial converse, let $u\colon A \longrightarrow B$ be a surjective-on-0-simplices weak categorical equivalence whose codomain is a quasi-category. Let $u = pi$ be a factorisation of u into an inner anodyne extension i followed by an inner fibration p. Then p is an inner fibration between quasi-categories which is also a surjective-on-0-simplices weak categorical equivalence. Hence $\mathrm{ho}(p)$ is a surjective equivalence, and in particular an isofibration, and so p is an isofibration, and hence a trivial fibration. Hence u is an absolute weak categorical equivalence. $\square$

A.22. **Example.** Not every surjective-on-0-simplices weak categorical equivalence is an absolute weak categorical equivalence. The morphism $f\colon \Delta[1] \longrightarrow S$ defined in [Cam20] is a counterexample.

Finally, we recover Stevenson's result [Ste18b, Theorem 1.5] that the class of inner anodyne extensions satisfies the right cancellation property among monos.

A.23. **Proposition** (Stevenson)**.** *Let $u\colon A \longrightarrow B$ and $v\colon B \longrightarrow C$ be a composable pair of monomorphisms of simplicial sets. If u and vu are inner anodyne, then so is v.*

Proof. This follows from part (4) of Proposition A.20 and from Proposition A.19. $\square$

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