Enriched algebraic weak factorisation systems

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Enriched (co)fibrant replacement

Theorem (Folklore: Garner, Riehl, Shulman, ...)

Let $\mathcal{V}$ be a monoidal model category in which every object is cofibrant. Then any cofibrantly generated model $\mathcal{V}$-category has:

- *a cofibrant replacement $\mathcal{V}$-comonad, and*
- *a fibrant replacement $\mathcal{V}$-monad.*

Examples

$\mathcal{V} = \mathbf{sSet}, \mathbf{Cat}$.

Theorem (Lack–Rosický)

Let $\mathcal{V}$ be a monoidal model category with cofibrant unit object. If $\mathcal{V}$ has a cofibrant replacement $\mathcal{V}$-comonad, then every object of $\mathcal{V}$ is cofibrant.

The problem of enriched (co)fibrant replacement

Question

If $\mathcal{V}$ is a monoidal model category in which not every object is cofibrant, then what extra structure, if not an enrichment in the ordinary sense, is naturally possessed by the (co)fibrant replacement (co)monad of a model $\mathcal{V}$-category?

An analysis of the monoidal model category $\mathcal{V} = \mathbf{2\text{-}Cat}$ suggests the decisive concept:

$$\boxed{\textbf{locally weak } \mathcal{V}\textbf{-functor}}$$

Outline

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The monoidal category of 2-categories

Recall (Gray)

The category **2-Cat** of (small) 2-categories and 2-functors is a symmetric monoidal closed category with:

- unit object 1,
- tensor product $A \otimes B$ the (pseudo) Gray tensor product of 2-categories,
- internal hom $\mathbf{Gray}(A, B)$ the 2-category of 2-functors $A \longrightarrow B$, pseudonatural transformations, and modifications.

Categories enriched over this monoidal category are called **Gray**-categories. The self-enrichment of **2-Cat** is called **Gray**.

The monoidal model category of 2-categories

Recall (Lack)

There is a model structure on **2-Cat**, whose weak equivalences are the biequivalences, and which is monoidal with respect to the Gray monoidal structure.
A 2-category is cofibrant if and only if its underlying category is free on a graph. In particular the unit 2-category 1 is cofibrant.

Since not every 2-category is cofibrant, it follows from the argument of Lack and Rosický that there does not exist a **Gray**-enriched cofibrant replacement comonad on **2-Cat**.

The model category **2-Cat** has a canonical cofibrant replacement comonad, which is induced by the adjunction

$$\begin{tikzcd} \mathbf{2\text{-}Cat} \arrow[r, phantom, "\perp"] & \mathbf{Bicat} \arrow[l, bend right, "\mathbf{st}"'] \end{tikzcd}$$

where **Bicat** is the category of bicategories and pseudofunctors, the right adjoint is the inclusion, and the left adjoint **st** sends a bicategory to its "strictification".

Hence this canonical cofibrant replacement $\mathbf{st}A$ of a 2-category A is its "pseudofunctor classifier"; i.e. it has the universal property:

$$\frac{\mathbf{st}A \longrightarrow B \qquad \text{2-functors}}{A \rightsquigarrow B \qquad \text{pseudofunctors}}$$

The strictification multiadjunction

Theorem (C.)

The strictification adjunction extends to an adjunction of multicategories, i.e. an adjunction in the 2-category of multicategories.

$$\begin{tikzcd} \mathbf{2\text{-}Cat} \arrow[r, bend right, ""{name=D}] & \mathbf{Bicat} \arrow[l, bend right, "\mathbf{st}"', ""{name=U}] \arrow[phantom, from=U, to=D, "\perp"] \end{tikzcd}$$

The multicategory structure on **2-Cat** is represented by the Gray monoidal structure. Its n-ary morphisms are the "cubical functors of n variables".
The multicategory structure on **Bicat**, introduced in Verity's PhD thesis, is closed but not representable. Its n-ary morphisms are the "cubical pseudofunctors of n variables".
Hence the comonad **st** on **2-Cat** extends to a comonad in the 2-category of multicategories. But the multicategory structure on **2-Cat** is representable, so **st** in fact extends to a monoidal comonad on **2-Cat**.

How strict is strictification?

Corollary

The strictification comonad **st** *is a monoidal comonad on* **2-Cat**.

By adjointness, a monoidal comonad on a monoidal closed category is equally a closed comonad, so **st** comes equipped with 2-functors

$$\mathbf{st}\left(\mathbf{Gray}(A, B)\right) \longrightarrow \mathbf{Gray}(\mathbf{st}A, \mathbf{st}B)$$

which, by the universal property of the pseudofunctor classifier, are equally pseudofunctors

$$\mathbf{Gray}(A, B) \rightsquigarrow \mathbf{Gray}(\mathbf{st}A, \mathbf{st}B)$$

making $\mathbf{st} \colon \mathbf{Gray} \to \mathbf{Gray}$ into a "locally weak **Gray**-functor".

Corollary (C.)

The strictification comonad **st** *is a locally weak* **Gray***-comonad on* **Gray**.

Outline

Locally weak $\mathcal{V}$-functors

Let $(Q, \varphi, \varphi_0, \ldots)$ be a monoidal comonad on a monoidal category $\mathcal{V}$. We think of morphisms $QX \longrightarrow Y$ in $\mathcal{V}$ as "weak morphisms" $X \rightsquigarrow Y$ in $\mathcal{V}$.

Definition (C.)

Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories. A *locally Q-weak $\mathcal{V}$-functor* $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ consists of:

(i) a function $F\colon \mathrm{ob}\mathcal{A} \longrightarrow \mathrm{ob}\mathcal{B}$,

(ii) for each $A, B \in \mathcal{A}$, a morphism $\psi_{A,B}\colon Q\mathcal{A}(A,B) \longrightarrow \mathcal{B}(FA,FB)$ in $\mathcal{V}$, i.e. a "weak morphism" $\psi_{A,B}\colon\ \mathcal{A}(A,B) \rightsquigarrow \mathcal{B}(FA,FB)$ in $\mathcal{V}$,

subject to the following two axioms.

$$\begin{tikzcd}
Q\mathcal{A}(B,C) \otimes Q\mathcal{A}(A,B) \arrow[r, "\varphi"] \arrow[d, "\psi\otimes\psi"'] & Q\left(\mathcal{A}(B,C) \otimes \mathcal{A}(A,B)\right) \arrow[r, "QK"] & Q\mathcal{A}(A,C) \arrow[d, "\psi"] \\
\mathcal{B}(FB,FC) \otimes \mathcal{B}(FA,FB) \arrow[rr, "K"'] & & \mathcal{B}(FA,FC)
\end{tikzcd}
\qquad
\begin{tikzcd}
QI \arrow[r, "Qj"] & Q\mathcal{A}(A,A) \arrow[d, "\psi"] \\
I \arrow[u, "\varphi_0"] \arrow[r, "j"'] & \mathcal{B}(FA,FA)
\end{tikzcd}$$

The Kleisli 2-category of locally weak $\mathcal{V}$-functors

Let Q be a monoidal comonad on a monoidal category $\mathcal{V}$. Change of base along Q defines a 2-comonad on the 2-category $\mathcal{V}$-**Cat**.

The Kleisli 2-category of this 2-comonad has:

- objects: $\mathcal{V}$-categories,
- morphisms: locally Q-weak $\mathcal{V}$-functors,
- 2-cells: locally Q-weak $\mathcal{V}$-natural transformations.

A (co)monad in this 2-category is called a *locally Q-weak $\mathcal{V}$-(co)monad*.

Outline

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category with finite colimits and finite limits, such that $\otimes$ preserves finite colimits in each variable.

By Day convolution, the arrow category $\mathcal{V}^{\mathbf{2}}$ is a monoidal category with:

- unit: $0 \longrightarrow I$,
- tensor product $(\, A \xrightarrow{f} B \,)\widehat{\otimes}(\, C \xrightarrow{g} D \,)$ given by:

$$
\begin{tikzcd}
A \otimes C \arrow[r, "1 \otimes g"] \arrow[d, "f \otimes 1"'] & A \otimes D \arrow[d] \arrow[ddr, bend left, "f \otimes 1"] & \\
B \otimes C \arrow[r] \arrow[drr, bend right, "1 \otimes g"'] & \cdot \arrow[dr, "f \widehat{\otimes} g"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& & B \otimes D
\end{tikzcd}
$$

- definition of the associativity and unit constraints requires the above assumption on colimits.

Let $\mathcal{A}$ be a $\mathcal{V}$-category. Then $\mathcal{A}^{\mathbf{2}}$ is a $\mathcal{V}^{\mathbf{2}}$-category, with homs $\widehat{\mathcal{A}(f,g)}$ given by:

$$\begin{tikzcd}
\mathcal{A}(B,C) \arrow[dr, "\widehat{\mathcal{A}(f,g)}"] \arrow[drr, bend left, "\mathcal{A}(1,g)"] \arrow[ddr, bend right, "\mathcal{A}(f,1)"'] & & \\
& \mathbf{Sq}(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \mathcal{A}(B,D) \arrow[d, "\mathcal{A}(f,1)"] \\
& \mathcal{A}(A,C) \arrow[r, "\mathcal{A}(1,g)"'] & \mathcal{A}(A,D)
\end{tikzcd}$$

$\mathbf{Sq}(f,g)$ is the $\mathcal{V}$-object of squares $f \to g$.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & \mathcal{A}(B,C) \arrow[d, "\widehat{\mathcal{A}(f,g)}"] \\
I \arrow[r, "(u{,}v)"'] & \mathbf{Sq}(f,g)
\end{tikzcd}
\leftrightarrow
\begin{tikzcd}
I \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "="] & \mathcal{A}(B,D) \arrow[d, "\mathcal{A}(f,1)"] \\
\mathcal{A}(A,C) \arrow[r, "\mathcal{A}(1,g)"'] & \mathcal{A}(A,D)
\end{tikzcd}
\leftrightarrow
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & C \arrow[d, "g"] \\
B \arrow[r, "v"'] & D
\end{tikzcd}$$

Weak factorisation systems

A *weak factorisation system* (WFS) on a category $\mathcal{C}$ consists of two classes of morphisms $(\mathcal{L}, \mathcal{R})$ in $\mathcal{C}$ subject to closure axioms, such that:

(i) every morphism f in $\mathcal{C}$ has a factorisation

$$\begin{tikzcd} \cdot \arrow[rr, "f"] \arrow[dr, "\mathcal{L}\ni l"'] & {} \arrow[d, phantom, "="] & \cdot \\ & \cdot \arrow[ur, "r\in\mathcal{R}"'] & \end{tikzcd}$$

(ii) every square $l \to r$ has a diagonal filler:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathcal{L}\ni l"'] & C \arrow[d, "r\in\mathcal{R}"] \\ B \arrow[r] \arrow[ur, dotted, "\exists"] & D \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} \mathcal{C}(B, C) \arrow[d, "\widehat{\mathcal{C}(l,r)}"] \\ \mathbf{Sq}(l, r) \end{tikzcd} \quad \text{is surjective } \forall l \in \mathcal{L}, r \in \mathcal{R}.$$

Enriched weak factorisation systems

Let $(\mathcal{L}, \mathcal{R})$ be a WFS on a monoidal category $\mathcal{V}$. A WFS $(\mathcal{H}, \mathcal{M})$ on a $\mathcal{V}$-category $\mathcal{A}$ is said to be *enriched over* $(\mathcal{L}, \mathcal{R})$ if for each $A \xrightarrow{f} B$ in $\mathcal{H}$ and each $C \xrightarrow{g} D$ in $\mathcal{M}$, the morphism $\mathcal{A}(B, C) \xrightarrow{\widehat{\mathcal{A}(f,g)}} \mathbf{Sq}(f, g)$ in $\mathcal{V}$ belongs to $\mathcal{R}$.

Examples

(a) Every WFS is enriched over the (injective, surjective) WFS on **Set**.

(b) A WFS enriched over the (all, iso) factorisation system on **Set** is precisely an orthogonal factorisation system.

(c) Let $\mathcal{V}$ be a monoidal model category. The two defining WFS of a model $\mathcal{V}$-category are enriched over the (cofibration, trivial fibration) WFS on $\mathcal{V}$.

Algebraic weak factorisation systems

An *algebraic weak factorisation system* (AWFS) on a category $\mathcal{C}$ consists of a comonad L and a monad R on the arrow category $\mathcal{C}^{\mathbf{2}}$, subject to various axioms, including that every morphism f has the canonical factorisation:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "Lf"'] & \arrow[d, phantom, "="] & B \\ & Ef \arrow[ur, "Rf"'] & \end{tikzcd}$$

Note that $E\colon \mathcal{C}^{\mathbf{2}} \longrightarrow \mathcal{C}$ is a functor.

"L-map" $\equiv$ L-coalgebra

$$\begin{tikzcd} A \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\ B \arrow[r, "1"'] \arrow[ur, "s"'] & B \end{tikzcd}$$

"R-map" $\equiv$ R-algebra

$$\begin{tikzcd} C \arrow[r, "1"] \arrow[d, "Lg"'] & C \arrow[d, "g"] \\ Eg \arrow[r, "Rg"'] \arrow[ur, "p"'] & D \end{tikzcd}$$

Each square in $\mathcal{C}$ from an L-coalgebra (f, s) to an R-algebra (g, p)

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

has the canonical diagonal filler $p \circ E(u, v) \circ s$.

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "1"'] & A \arrow[r, "u"] \arrow[d, "Lf"] & C \arrow[r, "1"] \arrow[d, "Lg"'] & C \arrow[d, "g"] \\ A \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[r, "E(u,v)"'] \arrow[d, "Rf"] & Eg \arrow[r, "Rg"'] \arrow[d, "Rg"'] \arrow[ur, "p"'] & D \arrow[d, "1"] \\ B \arrow[r, "1"'] \arrow[ur, "s"'] & B \arrow[r, "v"'] & D \arrow[r, "1"'] & D \end{tikzcd}$$

If (L, R) is an AWFS on a category $\mathcal{C}$ with an initial object 0, then factorisation of morphisms of the form

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "="] & & A \\ & QA \arrow[ur, "\varepsilon_A"'] & \end{tikzcd}$$

defines a comonad Q on $\mathcal{C}$, called the *cofibrant replacement comonad* for (L, R). Q-coalgebras are called *algebraically cofibrant objects*. The Kleisli category $\mathcal{C}_Q$ for this comonad is called the *category of weak maps* for (L, R).

Dually, if $\mathcal{C}$ has a terminal object 1, then factorisation of morphisms of the form $A \longrightarrow 1$ defines a monad on $\mathcal{C}$, called the fibrant replacement monad.

Monoidal AWFS

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category with finite colimits and finite limits, and such that $\otimes$ preserves finite colimits in each variable.
Recall that a WFS $(\mathcal{L}, \mathcal{R})$ on $\mathcal{V}$ is said to be a *monoidal* WFS if $f, g \in \mathcal{L}$ implies $f \widehat{\otimes} g \in \mathcal{L}$.

Definition (Riehl, C.)

An AWFS (L, R) on $\mathcal{V}$ is said to be a *monoidal* AWFS when it is equipped with:

(i) a natural transformation $\varphi \colon Ef \otimes Eg \longrightarrow E(f \widehat{\otimes} g)$,

(ii) a morphism $\varphi_0 \colon I \longrightarrow QI$,

making:

(iii) $\otimes \colon \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ a two-variable oplax morphism of AWFS,

(iv) $E \colon \mathcal{V}^{\mathbf{2}} \longrightarrow \mathcal{V}$ a monoidal functor,

(v) I an algebraically cofibrant object.

Axiom (iii) ($\otimes$ is a two-variable oplax morphism of AWFS) implies, inter alia, the following result.

Proposition (Riehl)

The tensor product $\widehat{\otimes}$ on $\mathcal{V}^{\mathbf{2}}$ lifts to a functor

$$\widehat{\otimes} \colon L\text{-}\mathrm{Coalg} \times L\text{-}\mathrm{Coalg} \longrightarrow L\text{-}\mathrm{Coalg}.$$

Moreover, by the definition of two-variable oplax morphisms of AWFS, φ defines natural transformations

$$Lf \widehat{\otimes} Lg \xrightarrow{\Phi} L(f \widehat{\otimes} g) \qquad Lf \widehat{\otimes} Rg \xrightarrow{\Sigma} R(f \widehat{\otimes} g) \qquad Rf \widehat{\otimes} Lg \xrightarrow{\Pi} R(f \widehat{\otimes} g)$$

which, together with the remaining axioms, prove the following theorem.

Cofibrant replacement is a monoidal comonad

Let (L, R) be a monoidal AWFS on $\mathcal{V}$.

Theorem (C.)

(i) *L is a monoidal comonad on $\mathcal{V}^{\mathbf{2}}$.*

(ii) *R is a L-bistrong monad on $\mathcal{V}^{\mathbf{2}}$.*

(iii) *The cofibrant replacement comonad Q is a monoidal comonad on $\mathcal{V}$.*

(iv) *The fibrant replacement monad P is a Q-bistrong monad on $\mathcal{V}$.*

Corollary

(i) *The monoidal structure on $\mathcal{V}^{\mathbf{2}}$ lifts to a monoidal structure on $L\text{-}\mathrm{Coalg}$.*

(ii) *$R\text{-}\mathrm{Kl}$ is a two-sided $(L\text{-}\mathrm{Coalg})$-actegory.*

(iii) *The monoidal structure on $\mathcal{V}$ lifts to a monoidal structure on $Q\text{-}\mathrm{Coalg}$.*

(iv) *$P\text{-}\mathrm{Kl}$ is a two-sided $(Q\text{-}\mathrm{Coalg})$-actegory.*

Let (L, R) be a monoidal AWFS on $\mathcal{V}$ with cofibrant replacement comonad Q. Recall that the Kleisli category $\mathcal{V}_Q$ for Q is called the category of weak maps for (L, R).

Corollary

The Kleisli adjunction for Q extends to an adjunction of multicategories.

$$\begin{tikzcd} \mathcal{V} \arrow[r, phantom, "\perp"] \arrow[r, bend right] & \mathcal{V}_Q \arrow[l, bend right, "Q"'] \end{tikzcd}$$

n-ary morphisms $(X_1, \ldots, X_n) \longrightarrow Y$ in the multicategory structure on $\mathcal{V}_Q$ are morphisms $QX_1 \otimes \cdots \otimes QX_n \longrightarrow Y$ in $\mathcal{V}$.

Enriched AWFS

Let (L, E, R) be a monoidal AWFS on $\mathcal{V}$.

Definition (Riehl, C.)

An AWFS (H, N, M) on a $\mathcal{V}$-category $\mathcal{A}$ is said to be *enriched over* (L, R) when it is equipped with:

(i) a natural transformation $\psi \colon E\widehat{\mathcal{A}(f,g)} \longrightarrow \mathcal{A}(Nf, Ng)$,

making:

(ii) $\mathcal{A}(-,-) \colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathcal{V}$ a two-variable lax morphism of AWFS,

(iii) $(N, E) \colon (\mathcal{A}^2, \mathcal{V}^2) \longrightarrow (\mathcal{A}, \mathcal{V})$ a morphism of enriched categories.

$$\begin{tikzcd} E\widehat{\mathcal{A}(g,h)} \otimes E\widehat{\mathcal{A}(f,g)} \arrow[r, "\varphi"] \arrow[d, "\psi\otimes\psi"'] & E\left(\widehat{\mathcal{A}(g,h)}\widehat{\otimes}\widehat{\mathcal{A}(f,g)}\right) \arrow[r, "E\widehat{K}"] & E\widehat{\mathcal{A}(f,h)} \arrow[d, "\psi"] \\ \mathcal{A}(Ng,Nh) \otimes \mathcal{A}(Nf,Ng) \arrow[rr, "K"'] & & \mathcal{A}(Nf,Nh) \end{tikzcd} \qquad \begin{tikzcd} QI \arrow[r, "E\widehat{j}"] & E\widehat{\mathcal{A}(f,f)} \arrow[d, "\psi"] \\ I \arrow[u, "\varphi_0"] \arrow[r, "j"'] & \mathcal{A}(Nf,Nf) \end{tikzcd}$$

Axiom (ii) ($\mathcal{A}(-,-)$ is a two-variable lax morphism of AWFS) implies, inter alia, the following result.

Proposition (Riehl)

The $\mathcal{V}^2$-valued hom $\mathcal{A}\widehat{(-,-)}$ on $\mathcal{A}^2$ lifts to a functor

$$\mathcal{A}\widehat{(-,-)}\colon H\text{-}\mathrm{Coalg} \times M\text{-}\mathrm{Alg} \longrightarrow R\text{-}\mathrm{Alg}.$$

Moreover, by the definition of two-variable lax morphisms of AWFS, ψ defines natural transformations

$$R\mathcal{A}\widehat{(f,g)} \xrightarrow{\Theta} \mathcal{A}\widehat{(Hf,Mg)}$$

$$L\mathcal{A}\widehat{(f,g)} \xrightarrow{\Psi} \mathcal{A}\widehat{(Hf,Hg)} \qquad L\mathcal{A}\widehat{(f,g)} \xrightarrow{\Omega} \mathcal{A}\widehat{(Mf,Mg)}$$

which, together with the remaining axioms, prove the following theorem.

Theorem (C.)

Let (H, M) be an (L, R)-enriched AWFS *on $\mathcal{A}$. Then the following are true.*

(i) *H is a locally L-weak $\mathcal{V}^{\mathbf{2}}$-comonad on $\mathcal{A}^{\mathbf{2}}$.*

(ii) *M is a locally L-weak $\mathcal{V}^{\mathbf{2}}$-monad on $\mathcal{A}^{\mathbf{2}}$.*

(iii) *The cofibrant replacement comonad for (H, M) is a locally Q-weak $\mathcal{V}$-comonad on $\mathcal{A}$.*

(iv) *The fibrant replacement monad for (H, M) is a locally Q-weak $\mathcal{V}$-monad on $\mathcal{A}$.*

Let (L, R) be a monoidal AWFS on $\mathcal{V}$ with cofibrant replacement comonad Q. Let (H, M) be a (L, R)-enriched AWFS on a $\mathcal{V}$-category $\mathcal{A}$ with cofibrant replacement comonad S.

Corollary (C.)

*The Kleisli adjunction for S extends to a $\mathcal{V}_Q$-enriched adjunction, i.e. an adjunction in the 2-category $\mathcal{V}_Q$-**Cat** of categories enriched over the multicategory of weak maps for (L, R).*

$$\begin{tikzcd} \mathcal{A} \arrow[r, phantom, "\perp"] \arrow[r, bend right, shift right] & \mathcal{A}_S \arrow[l, bend right, shift right, "S"'] \end{tikzcd}$$

The hom-objects in the $\mathcal{V}_Q$-category $\mathcal{A}_S$ are $\mathcal{A}_S(A, B) = \mathcal{A}(SA, B)$.

Examples

(a) Every monoidal AWFS on a monoidal closed category is enriched over itself.

(b) The (all,iso) factorisation system on a monoidal category $\mathcal{V}$ is a monoidal AWFS (with canonical factorisation $f = 1 \circ f$) . An AWFS on a $\mathcal{V}$-category $\mathcal{A}$ enriched over this monoidal AWFS is precisely a $\mathcal{V}$-enriched orthogonal factorisation system on $\mathcal{A}$.

(c) The "split epi" AWFS on **Set** (in which $f\colon X \to Y$ factors through $X + Y$) is monoidal with respect to cartesian product. Every AWFS is canonically enriched over this monoidal AWFS.

(d) Let $\mathcal{V}$ be a monoidally cocomplete category, so that $U = \mathcal{V}(I, -)\colon \mathcal{V} \to \mathbf{Set}$ has a left adjoint F. The "U-split epi" AWFS on $\mathcal{V}$ (in which $f\colon X \to Y$ factors through $X + FUY$) is monoidal. Every AWFS on a $\mathcal{V}$-category is canonically enriched over this monoidal AWFS.