The gregarious model structure for double categories

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Plan

1. Motivation: What is a double ∞-category?
2. The gregarious model structure for double categories
3. Double quasi-categories

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1. Motivation: What is a double ∞-category?

2. The gregarious model structure for double categories

3. Double quasi-categories

Double categories

A **double category** consists of:

- objects a,
- horizontal morphisms $f \colon a \longrightarrow b$,
- vertical morphisms $u \colon a \longrightarrow\!\!\!\!\!\!\bullet\,\,\, c$,
- double cells

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\alpha"] & b \arrow[d, "v"] \\ c \arrow[r, "g"'] & d \end{tikzcd}$$

- horizontal composition operations for horizontal morphisms and double cells,
- vertical composition operations for vertical morphisms and double cells,
- identity horizontal and vertical morphisms and identity double cells,

all subject to associativity, unit, and interchange axioms.

Double categories and **double functors** form the category **DblCat**.

Double categories as bisimplicial sets

Let **ssSet** $= [\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathbf{Set}]$ denote the category of **bisimplicial sets**.

Let X be a bisimplicial set. For each $n \geq 0$, we call the simplicial sets $X_{n,\bullet}$ and $X_{\bullet,n}$ the n**th column** and the n**th row** of X respectively.

There is a fully faithful **nerve functor** $N\colon \mathbf{DblCat} \longrightarrow \mathbf{ssSet}$, induced by the functor $\Delta \times \Delta \longrightarrow \mathbf{ssSet}$ that sends $([m],[n])$ to the exterior product $[m] \boxtimes [n]$ (the "free-living m-by-n grid of double cells"). So, for each double category A, $(NA)_{0,0}$, $(NA)_{1,0}$, $(NA)_{0,1}$, and $(NA)_{1,1}$ are the sets of objects, vertical morphisms, horizontal morphisms, and double cells of A respectively.

A bisimplicial set X is isomorphic to the nerve of a double category if and only if each column $X_{m,\bullet}$ and each row $X_{\bullet,n}$ of X is isomorphic to the nerve of a category, i.e. if and only if the horizontal and vertical Segal maps

$$X_{m,n} \longrightarrow X_{m,1} \times_{X_{m,0}} \cdots \times_{X_{m,0}} X_{m,1}$$

$$X_{m,n} \longrightarrow X_{1,n} \times_{X_{0,n}} \cdots \times_{X_{0,n}} X_{1,n}$$

are bijections for all $m, n \geq 0$.

Double Segal spaces I

Question

What is the "correct" ∞-categorical generalisation of the notion of double category?

Let $\mathscr{S}$ denote the ∞-category of ∞-groupoids.

Candidate 1

A **double Segal space** is a functor $X \colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathscr{S}$ such that each column $X_{m,\bullet}$ and each row $X_{\bullet,n}$ is a Segal space.

Drawback: This definition is too general. It lacks a "completeness" condition, à la Rezk's complete Segal spaces, to tie the homotopical structure of a double Segal space to its categorical structure.

But what is the missing "completeness" condition?

Double Segal spaces II

Question

What should be a "complete" double Segal space?

Candidate 2

A **double ∞-category** is a Segal object in the ∞-category of ∞-categories, i.e. a double Segal space X such that each column $X_{m,\bullet}$ is a complete Segal space.

Drawback: This definition is not symmetric in the horizontal and vertical directions. So the transpose of a double ∞-category is not always a double ∞-category.

Candidate 3

A **double complete Segal space** is a double Segal space X such that each column $X_{m,\bullet}$ and each row $X_{\bullet,n}$ is a complete Segal space.

Drawback: This definition is too restrictive. For example, two objects in such an X are equivalent in the complete Segal space $X_{0,\bullet}$ if and only if they are equivalent in the complete Segal space $X_{\bullet,0}$. This excludes important examples, such as the pseudo double category of rings, ring homomorphisms, and bimodules. (Ring isomorphism $\neq$ Morita equivalence.)

Plan

Model structures

A model structure on a category $\mathcal{E}$ enables one to "do homotopy theory" in $\mathcal{E}$.

Model structures

A **model structure** on a category $\mathcal{E}$ consists of three classes of morphisms $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ in $\mathcal{E}$ – called **cofibrations**, **weak equivalences**, and **fibrations** – such that:

- $\mathcal{W}$ satisfies the 2-out-of-3 property, and
- $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorisation systems in $\mathcal{E}$.

An object A is **cofibrant** if $0 \longrightarrow A$ is a cofibration.
An object X is **fibrant** if $X \longrightarrow 1$ is a fibration.

The morphisms in the classes $\mathcal{C} \cap \mathcal{W}$ and $\mathcal{W} \cap \mathcal{F}$ are called **trivial cofibrations** and **trivial fibrations** respectively.

Lemma

A model structure on a category is determined by its fibrant objects and trivial fibrations.

The gregarious model structure for double categories

Lemma

A model structure on a category is determined by its fibrant objects and trivial fibrations.

Lack's model structure for 2-categories

There exists a unique model structure on **2-Cat** in which:

- every 2-category is fibrant, and
- a 2-functor is a trivial fibration iff it is surjective on objects, full on 1-morphisms, and fully faithful on 2-cells.

Theorem (The gregarious model structure for double categories)

There exists a unique model structure on **DblCat** *in which:*

- *every double category is fibrant, and*
- *a double functor is a trivial fibration iff it is surjective on objects, full on horizontal morphisms, full on vertical morphisms, and fully faithful on double cells.*

Some functors from **2-Cat** to **DblCat**

Let A be a 2-category.

The double category $\mathbb{S}\mathrm{q}(A)$ – called the **double category of squares** of A – has the same objects as A, its horizontal and vertical morphisms are the morphisms of A, and its double cells are the squares in A:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\alpha"] & b \arrow[d, "v"] \\ c \arrow[r, "g"'] & d \end{tikzcd}$$

Let $\mathbb{H}(A)$ denote the locally full sub double category of $\mathbb{S}\mathrm{q}(A)$ whose horizontal morphisms are the morphisms of A, and whose vertical morphisms are the identity morphisms of A. Dually, one has the double category $\mathbb{V}(A)$ whose vertical morphisms are the morphisms of A, but whose horizontal morphisms are the identity morphisms of A.

These three constructions define functors $\mathbb{S}\mathrm{q}, \mathbb{H}, \mathbb{V} \colon$ **2-Cat** $\longrightarrow$ **DblCat**.

Some functors from **DblCat** to **2-Cat**

Let A be a double category. Let $\mathcal{H}(A)$ – the **horizontal** 2-**category** of A – denote the 2-category with the same objects as A, whose morphisms are the horizontal morphisms of A, and whose 2-cells are the double squares of A of the form displayed on the left below.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\alpha"] & b \arrow[d, equal] \\ a \arrow[r, "g"'] & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, equal] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\beta"] & a \arrow[d, "v"] \\ b \arrow[r, equal] & b \end{tikzcd}$$

Dually, let $\mathcal{V}(A)$ – the **vertical** 2-**category** – denote the 2-category with the same objects as A, whose morphisms are the vertical morphisms of A, and whose 2-cells are the double cells of A of the form displayed on the right above.

These two constructions define functors $\mathcal{H}, \mathcal{V} \colon$ **DblCat** $\longrightarrow$ **2-Cat**.

We shall sometimes refer to the invertible 2-cells in the 2-categories $\mathcal{H}(A)$ and $\mathcal{V}(A)$ as **globular isomorphisms** in A.

Companion pairs

Let A be a double category.

Companion pairs

A **companion pair** of morphisms in A consists of:

- a horizontal morphism $f\colon a \longrightarrow b$ in A,
- a vertical morphism $u\colon a \longrightarrow\!\!\!\!\!\!\bullet\;\;\; b$ in A, and
- double cells in A

$$\begin{tikzcd} a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & a \arrow[d, "u"] \\ a \arrow[r, "f"'] & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow\varepsilon"] & b \arrow[d, equal] \\ b \arrow[r, equal] & b \end{tikzcd}$$

such that $\eta \boxbar \varepsilon = 1_f$ and $\eta \boxminus \varepsilon = 1_u$.

A companion pair in A amounts precisely to a double functor $\mathbb{S}\mathrm{q}(\mathbf{2}) \longrightarrow A$.

We say that a horizontal morphism $f\colon a \longrightarrow b$ in A has a **vertical companion** if it extends to a companion pair in A. Dually for vertical morphisms.

Gregarious equivalences I

Let A be a double category.

Definition (gregarious equivalence)

A companion pair of morphisms (f, u) in A is said to be a **gregarious equivalence** in A if f is an equivalence in the horizontal 2-category $\mathcal{H}(A)$ and u is an equivalence in the vertical 2-category $\mathcal{V}(A)$.

Let $\mathbb{E}$ denote the "free-living adjoint equivalence" 2-category.

Lemma

A companion pair of morphisms (f, u) in A is a gregarious equivalence iff there exists an extension

$$\begin{tikzcd} \mathbb{S}\mathrm{q}(\mathbf{2}) \arrow[r, "{(f,u)}"] \arrow[d] & A \\ \mathbb{S}\mathrm{q}(\mathbb{E}) \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

Gregarious equivalences II

Lemma

Let $f\colon a \longrightarrow b$ be a horizontal morphism in A. The following are equivalent.

1. *f extends to a gregarious equivalence in A.*
2. *f has a vertical companion u in A, f is an equivalence in $\mathcal{H}(A)$, and u is an equivalence in $\mathcal{V}(A)$.*
3. *f has a pseudo-inverse g in $\mathcal{H}(A)$, and both f and g have vertical companions in A.*

By an abuse of language, we shall say that a horizontal morphism in A is a **gregarious equivalence** if it satisfies the equivalent properties of the lemma.

Dually for vertical morphisms.

Weak equivalences in the gregarious model structure

Proposition

*A double functor $F\colon A \longrightarrow B$ is a **weak equivalence** in the gregarious model structure for double categories iff it is :*

1. *surjective on objects up to gregarious equivalence (i.e. for every object $b \in B$, there exists an object $a \in A$ and a gregarious equivalence $Fa \to b$ in B),*
2. *full on horizontal morphisms up to globular isomorphism,*
3. *full on vertical morphisms up to globular isomorphism, and*
4. *fully faithful on double cells.*

By (2) we mean that for each pair of objects a, b in A, and each horizontal morphism $g\colon Fa \longrightarrow Fb$ in B, there exists a horizontal morphism $f\colon a \longrightarrow b$ in A and a vertically invertible double cell in B as below.

$$\begin{tikzcd} Fa \arrow[r, "Ff"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\cong"] & Fb \arrow[d, equal] \\ Fa \arrow[r, "g"'] & Fb \end{tikzcd}$$

Dually for (3).

Fibrations in the gregarious model structure

Proposition

*A double functor $F \colon A \longrightarrow B$ is a **fibration** in the gregarious model structure for double categories iff:*

1. *for every object $a \in A$, and every gregarious equivalence $g \colon Fa \to b$ in B, there exists a gregarious equivalence $f \colon a \to a'$ in A such that $F(f) = g$,*
2. *the 2-functor $\mathcal{H}(F)$ is an isofibration on hom-categories, and*
3. *the 2-functor $\mathcal{V}(F)$ is an isofibration on hom-categories.*

Let $\mathbb{I}_2$ denote the "free-living invertible 2-cell" 2-category.

Proposition

A double functor is a fibration in the gregarious model structure for double categories iff it has the right lifting property with respect to the following three double functors:

1. $\{0\} \longrightarrow \mathbb{S}\mathrm{q}(\mathbb{E})$,
2. $\mathbb{H}(\mathbf{2}) \longrightarrow \mathbb{H}(\mathbb{I}_2)$, *and*
3. $\mathbb{V}(\mathbf{2}) \longrightarrow \mathbb{V}(\mathbb{I}_2)$.

The proof of existence of the gregarious model structure

The proof of the existence of the gregarious model structure follows by standard arguments from the following propositions. This follows the argument of Lack's proof of the existence of his model structure for bicategories.

Proposition

The class of "gregarious" weak equivalences in **DblCat** *satsfies the 2-out-of-6 property.*

Proposition

A "gregarious" fibration in **DblCat** *is a "gregarious" weak equivalence iff it is a trivial fibration.*

Proposition

The fibrations and trivial fibrations are the right classes of cofibrantly generated weak factorisation systems in **DblCat**.

Proposition

Every double category A admits a "path object", i.e. a factorisation of the diagonal $A \longrightarrow A \times A$ into a weak equivalence followed by a fibration.

Cofibrations in the gregarious model structure

Recall that a 2-functor is a cofibration in Lack's model structure on **2-Cat** iff its underlying functor has the LLP in **Cat** wrt all surjective-on-objects-and-full functors.

Proposition

A double functor $F\colon A \longrightarrow B$ is a ***cofibration*** *in the gregarious model structure iff the 2-functors $\mathcal{H}(F)$ and $\mathcal{V}(F)$ are cofibrations in Lack's model structure on* **2-Cat**.

Corollary

A double category A is ***cofibrant*** *in the gregarious model structure iff its underlying categories $\mathcal{H}(A)_0$ and $\mathcal{V}(A)_0$ of horizontal morphisms and vertical morphisms are free.*

A **double pseudofunctor** is a kind of morphism of double categories that is weak (i.e. pseudofunctorial) in both directions.

Proposition

Let A be a double category. Then the ***double pseudofunctor classifier*** *QA of A is a cofibrant double category, and the counit double functor $QA \longrightarrow A$ is a bijective-on-objects weak equivalence. That is, QA is a* ***cofibrant replacement*** *of A.*

The double category of squares functor

The "double category of squares" functor $\mathbb{S}\mathrm{q}\colon \mathbf{2\text{-}Cat} \longrightarrow \mathbf{DblCat}$ has both a left adjoint (called *String* by Ehresmann & Ehresmann), and a right adjoint $\mathcal{C}\colon \mathbf{DblCat} \longrightarrow \mathbf{2\text{-}Cat}$.

Note that, for each double category A, the morphisms (resp. equivalences) in $\mathcal{C}(A)$ are the companion pairs (resp. gregarious equivalences) in A.

Proposition

The functor $\mathbb{S}\mathrm{q}\colon \mathbf{2\text{-}Cat} \longrightarrow \mathbf{DblCat}$:

- *is both left and right Quillen wrt Lack's model structure on* $\mathbf{2\text{-}Cat}$ *and the gregarious model structure on* $\mathbf{DblCat}$*,*
- *creates Lack's model structure on* $\mathbf{2\text{-}Cat}$ *from the gregarious model structure on* $\mathbf{DblCat}$*,*
- *is homotopically fully faithful.*

Work in progress

Here are a couple of properties of the gregarious model structure which I expect to be true, but which I haven't yet proved.

- The gregarious model structure on **DblCat** is proper.
- The gregarious model structure on **DblCat** is monoidal with respect to Böhm's Gray tensor product for double categories.

Theorem

There exists a left Bousfield localisation of the gregarious model structure on **DblCat** *in which:*

1. *a double functor is a "local"* ***weak equivalence*** *iff it is surjective on objects up to horizontal equivalence, full on horizontal morphisms up to globular isomorphism, surjective on vertical morphisms up to horizontal equivalence, and fully faithful on double cells (i.e. a weak equivalence in the* ***Moser–Sarazola–Verdugo*** *model structure, see arXiv:2004.14233);*
2. *a double functor $F\colon A \longrightarrow B$ is a "local"* ***fibration*** *iff $\mathcal{H}(F)$ is an equifibration, $\mathcal{V}(F)$ is an isofibration on homs, and any horizontal equivalence in A sent by F to a gregarious equivalence in B is a gregarious equivalence in A.*

In particular, a double category A is fibrant in this model structure if and only if every horizontal equivalence in A has a vertical companion.

[After this talk, this localised model structure was independently constructed by Moser–Sarazola–Verdugo in arXiv:2007.00588.]

Plan

Cisinski model structures

Inspired by Ara's definition of 2-quasi-categories, we shall define double quasi-categories as the fibrant objects of a Cisinski model structure.

Localisers

Let $\mathbf{A}$ be a small category. A morphism in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ is a **trivial fibration** if it has the right lifting property with respect to all monomorphisms in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$.
An **A-localiser** is a class W of morphisms in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ such that:

1. W satisfies the two-out-of-three property,
2. every trivial fibration belongs to W,
3. the class of monomorphisms belonging to W is stable under pushout and transfinite composition.

Cisinski model structures

Let S be a small set of morphisms in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$. There exists a unique model structure on $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ whose cofibrations are the monomorphisms, and whose class of weak equivalences is the smallest $\mathbf{A}$-localiser containing S. We will call this model structure the **Cisinski model structure generated by** S.

Double quasi-categories

Let $\boxtimes\colon \mathbf{sSet} \times \mathbf{sSet} \longrightarrow \mathbf{ssSet}$ denote the **exterior product** functor, defined by $(X \boxtimes Y)_{m,n} = X_m \times Y_n$.

For each $n \geq 2$, let $I[n] \longrightarrow \Delta[n]$ denote the **spine inclusion** of the free-living n-simplex.

The model structure for double quasi-categories

We define the **model structure for double quasi-categories** on **ssSet** to be the Cisinski model structure generated by the morphisms:

- $I[m] \boxtimes \Delta[n] \longrightarrow \Delta[m] \boxtimes \Delta[n]$, for all $m \geq 2$, $n \geq 0$,
- $\Delta[m] \boxtimes I[n] \longrightarrow \Delta[m] \boxtimes \Delta[n]$, for all $m \geq 0$, $n \geq 2$,
- $N\mathbb{H}(\mathbb{I}_2 \longrightarrow \mathbf{2})$,
- $N\mathbb{V}(\mathbb{I}_2 \longrightarrow \mathbf{2})$.

Definition

A **double quasi-category** is a bisimplicial set that is fibrant in the model structure for double quasi-categories.

A homotopy coherent nerve for double categories

Watson's nerve functor

Define the **coherent nerve** functor $N_W \colon \mathbf{DblCat} \longrightarrow \mathbf{ssSet}$ to be the nerve (or singular) functor induced by the functor $\Delta \times \Delta \longrightarrow \mathbf{DblCat}$ that sends $([m],[n])$ to the normal double pseudofunctor classifier of $[m] \boxtimes [n]$.

So, for each double category A, the elements of the set $(N_W A)_{m,n}$ are the **normal double pseudofunctors** $[m] \boxtimes [n] \longrightarrow A$.

Theorem

The coherent nerve functor $N_W \colon \mathbf{DblCat} \longrightarrow \mathbf{ssSet}$*:*

- *is a right Quillen functor from the gregarious model structure for double categories to the model structure for double quasi-categories;*
- *right-induces the gregarious model structure for double categories from the model structure for double quasi-categories;*
- *is homotopically fully faithful.*

Complete double Segal spaces

Say that a double Segal space $X\colon \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathscr{S}$ is **local** with respect to a morphism of bisimplicial sets $f\colon A \longrightarrow B$ when the induced map between weighted limits $f \pitchfork X\colon B \pitchfork X \longrightarrow A \pitchfork X$ is an equivalence of ∞-groupoids.

Definition (complete double Segal space)

A double Segal space X is **complete** if it is local with respect to the following morphisms of bisimplicial sets:

1. $N\mathbb{H}(\mathbb{I}_2 \longrightarrow \mathbf{2})$,
2. $N\mathbb{V}(\mathbb{I}_2 \longrightarrow \mathbf{2})$,
3. the projection $N\mathbb{S}\mathrm{q}(\mathbb{I}) \times \Delta[m,n] \longrightarrow \Delta[m,n]$, for all $m, n \geq 0$.

I expect (but have not proved) that is suffices to take $m = n = 0$ in (3).

Theorem

The ∞-category of complete double Segal spaces is the ∞-category presented by the model category of bisimplicial sets equipped with the model structure for double quasi-categories.

Thank you!