** Many faces complexity

* "$\frac{1}{r}$-Cuttings on the average"

$\boxed{1}$

P - set of points $|P| = m$

$\mathcal{L}$ - set of lines $|\mathcal{L}| = n$

$4 < r < n$, $0 < \alpha \leq 1$

$s \leq 3r^2$

$\mathcal{L} = \{l_1, l_2, \dots, l_n\}$ $P = \{p_1, p_2, \dots, p_m\}$

$\Delta_1, \Delta_2, \dots, \Delta_s$

m_i - points in $P \cap \Delta_i$

n_i - lines hitting interior of Δ_i

$\mathcal{L}_i \subseteq \mathcal{L}$, lines intersecting Δ_i interior

$P_i = P \cap \Delta_i$, $n_i = |\mathcal{L}_i|$, $m_i = |P_i|$.

$\Rightarrow$ * (i) $\sum_{i=1}^{s} n_i \leq c_1 n r$

** (ii) $\sum_{i=1}^{s} m_i n_i^{\alpha} \leq c_2 m \left(\frac{n}{r}\right)^{\alpha}$

[Clarkson et al. 1990]

$\rightarrow$ Consider $p_j \in P$ lying inside (unique) Δ_R of $\mathcal{A}^*(R)$, $R \subseteq \mathcal{L}$, $|R| = r$

$\rightarrow$ Let q_j be number of lines of $\mathcal{L}$ intersecting Δ_R.

Consider $E\left(\sum_{i=1}^{s} m_i n_i^{\alpha}\right)$ over all $\binom{n}{r}$ possibilities for R.

Write this expectation by rewriting the summation $\sum_{i=1}^{s} m_i n_i^{\alpha}$ as $\sum_{j=1}^{m} q_j^{\alpha}$. (Simple change of variables.)

$\boxed{2}$

- Putting $\alpha = 1$ gives $\sum_{i=1}^{s} m_i n_i^{\alpha} = \sum_{i=1}^{s} m_i n_i \le c_2 m \left(\frac{n}{r}\right)$ for at least one sample $R \subseteq \mathcal{L}$, where $|R| = r$. — (I)
- Observe that the average "conflict" size of the $s (\le 3r^2)$ trapezoids in $A^*(R)$ with respect to lines in $\mathcal{L} \setminus R$ is at most $c_2\left(\frac{n}{r}\right)$ for the m faces (even assuming the m faces are all distinct
- $\sum_{i=1}^{s} n_i \le c_1 n r \Rightarrow$ Average "conflict" size with $\mathcal{L} \setminus R$ on all s faces is $O\left(\frac{n}{r}\right)$.

  Therefore, we call such a (random) sample, an "$\frac{1}{r}$-cutting on the average".

Now
$$E\left[\sum_{j=1}^{m} (q_j)^{\alpha}\right] = \sum_{j=1}^{m} E\left[q_j^{\alpha}\right] \le \sum_{j=1}^{m} \left(E[q_j]\right)^{\alpha} \quad \text{— (II)}$$

(the inequality above is due to Jensen's inequality). [Ex. 11.5 Pach et al.]

Now we bound $E[q_j]$ for a fixed j, for $p_j \in P$, over all $\binom{n}{r}$ samples R, where $|R| = r$, $R \subseteq \mathcal{L}$, $|\mathcal{L}| = n$ and show that

$$E[q_j] \le \frac{4n}{r-4} = O\left(\frac{n}{r}\right)$$

This would imply $E\left[\sum_{i=1}^{s} m_i n_i^{\alpha}\right] \le m \cdot \max_{1 \le j \le m} \left(E(q_j)\right)^{\alpha} \le m \left(\frac{4n}{r-4}\right)^{\alpha}$

establishing (I) above (using (II) above)

CS60086 S. P. Pal 29/01/14

- For $E[q_j]$ we consider $\mathcal{D}_{\mathcal{L}}$ – the family of all trapezoids Δ, containing p_j, where the trapezoid is defined by some $R_\Delta \subset \mathcal{L}$ with $|R_\Delta| = r_\Delta \leq 4$. Let n_Δ be the number of lines of $\mathcal{L}$ in "conflict" with $\Delta \in \mathcal{D}_{\mathcal{L}}$.
- For any such $\Delta \in \mathcal{D}_{\mathcal{L}}$, define Pr_Δ to be the probability that $\Delta_R = \Delta$, where Δ_R is the trapezoid in $\mathcal{A}^*(R)$ containing p_j. [Here, the space of experiments is the $\binom{n}{r}$ possibilities for $R \subseteq L$, each with equal probability.

The "average" of q_j over all $\binom{n}{r}$ possibilities for R is clearly n_Δ times the probability that this Δ appears "unscathed" in $\mathcal{A}^*(R)$. So, $E[q_j] = \sum_{\Delta \in \mathcal{D}_{\mathcal{L}}} n_\Delta \, Pr_\Delta$.

For Δ to be "unscathed" in $\mathcal{A}^*(R)$, no line in R must "conflict" with Δ, and defining r_Δ lines of R_Δ must be in R. So, we commit r_Δ lines and freely choose only $r - r_\Delta$ lines for R, avoiding $n_\Delta + r_\Delta$ lines from n.

So, $$Pr_\Delta = \binom{n - n_\Delta - r_\Delta}{r - r_\Delta} \Big/ \binom{n}{r} \leq \frac{\frac{n}{r-4}\binom{n - n_\Delta - r_\Delta}{r - r_\Delta - 1}}{\binom{n}{r}}$$

S. P. Pal CS60086 29/01/14

$\boxed{4}$

So, $E(q_i) \leq \dfrac{n}{r-4} \displaystyle\sum_{\Delta \in \mathcal{D}_\mathcal{L}} n_\Delta \binom{n-n_\Delta-r_\Delta}{r-r_\Delta-1} \Big/ \binom{n}{r}$

We show that the sum above is at most 4, whence

$E[q_i] \leq 4n/r-4$

$$\sum_{\Delta \in \mathcal{D}_\mathcal{L}} n_\Delta \binom{n-n_\Delta-r_\Delta}{r-r_\Delta-1} \Big/ \binom{n}{r} = \sum_{\Delta \in \mathcal{D}_\mathcal{L}} \sum_{j=1}^{n_\Delta} \binom{n-n_\Delta-r_\Delta}{r-r_\Delta-1} \Big/ \binom{n}{r}$$

$$= \sum_{j=1}^{n_\Delta} \sum_{\Delta \in \mathcal{D}_\mathcal{L}} \binom{n-n_\Delta-r_\Delta}{r-r_\Delta-1} \Big/ \binom{n}{r}$$

$$= \sum_{j=1}^{n_\Delta} \sum_{\Delta \in \mathcal{D}_\mathcal{L}}$$ Probability that $\Delta = \Delta_{R-\{l\}}$ for any one $l \in \mathcal{L}_\Delta \subseteq \mathcal{L}$, where Δ meets exactly one line $l \in \mathcal{L}_\Delta \subseteq \mathcal{L}$ & $l \in R \subseteq \mathcal{L}$

= the expected number of bounding edges of $\Delta = \Delta R$ for $p_j \leq r_\Delta = 4$.

S. P. Pal CS60086
29/01/14

So, $(\delta-1)^{1/r} m\, n^{1-1/r} + (r-1)n \geq |E|$

follows from $(\delta-1)\, m^r \geq n \left(\frac{|E|}{n} - r + 1\right)^r$

since $(\delta-1)\binom{m}{r} \geq n \binom{|E|/n}{r}$

[numerator in LHS is overestimated].

[r's in both denominators cancel out.]

[numerator on right is underestimated.]

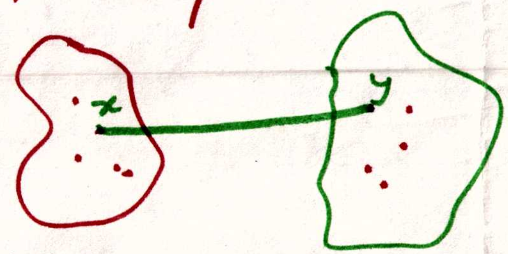
Theorem.. For fixed r, s s.t. $r \leq s$, a $K_{r,s}$-free graph $G(V,E)$, where $|V| = n$ has at most $|E| \leq c_s\, n^{2-1/r}$ edges.

[C_s depends only on s.]

Proof: $\exists\, G' \subseteq G$, where G' is bipartite with $\lfloor \frac{n}{2} \rfloor$ & $\lceil \frac{n}{2} \rceil$ partites, s.t. $|E(G)| \leq 2 |E(G')|$. ⊛ Apply Kovari, Sos, Turan to G', giving $E(G') \leq c_{r,s} \left(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil^{1-1/r} + \lfloor \frac{n}{2} \rfloor \right) \leq c' \left(n^{2-1/r}\right)$ ■

Theorem The vertex set V of $G(V,E)$ can be split in to disjoint parts V_1 & V_2, s.t. $\big||V_1| - |V_2|\big| \leq 1$, and the number of edges $xy \in E(G)$ with $x \in V_1, y \in V_2$ is at least $|E(G)|/2$.

Proof: Use of the probabilistic method helps.



$|V_1| = \lfloor n/2 \rfloor$ $\quad$ $|V_2| = \lceil \frac{n}{2} \rceil$ $\Longrightarrow$ Such are the only possibilities.

$\binom{n}{\lfloor \frac{n}{2} \rfloor}$ ways in all. How many are favourable? xy & yx for $xy \in E$.

$$\text{Probability} = \frac{2\binom{n-2}{\lfloor n/2 \rfloor - 1}}{\binom{n}{\lfloor n/2 \rfloor}}$$

Expected number across the cut is $\displaystyle\sum_{xy \in E(G)} 2\binom{n-2}{\lfloor \frac{n}{2} \rfloor - 1} \Big/ \binom{n}{\lfloor n/2 \rfloor}$.

With $|E(G)|$ summands this expectation is at least $|E(G)|/2$ ■

$$\binom{n-2}{\lfloor \frac{n}{2} \rfloor - 1} \Big/ \binom{n}{\lfloor n/2 \rfloor}$$

$$= \left(\frac{(n-2)!}{(\lfloor \frac{n}{2} \rfloor - 1)!}\right) \times \frac{1}{(n-1-\lfloor \frac{n}{2} \rfloor)!} \times \frac{\lfloor \frac{n}{2} \rfloor! \, (n - \lfloor \frac{n}{2} \rfloor)!}{n!}$$

$$= \frac{1}{n(n-1)} \times \frac{\lfloor \frac{n}{2} \rfloor}{1} \times \frac{(n - \lfloor \frac{n}{2} \rfloor)}{1} = \frac{\lfloor \frac{n}{2} \rfloor (n - \lfloor \frac{n}{2} \rfloor) \frac{1}{n^2}}{1 \cdot [1 - \frac{1}{n}]}$$

$$= \frac{\geq \frac{1}{4}}{\leq 1} \geq \frac{1}{4}$$

So, the expectation with $|E(G)|$ summands is at least $\frac{|E(G)|}{2}$.

Therefore, there is a partition as claimed with cut size at least $\frac{|E(G)|}{2}$.