Optimal Trading Across Coexisting Exchanges: Limit-Order Books & Automated Market Makers

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IMPERIAL

ETH - Hong Kong - Imperial Mathematical Finance Workshop
April 22-25, 2025

Trading Venues

Consider two assets X (e.g., USDC), Y (e.g., ETH) and

- **centralised LOB** with immediate linear impact $\lambda > 0$
- **decentralised AMM** with bonding curve $f(x, y) = xy = L^2$, $L > 0$

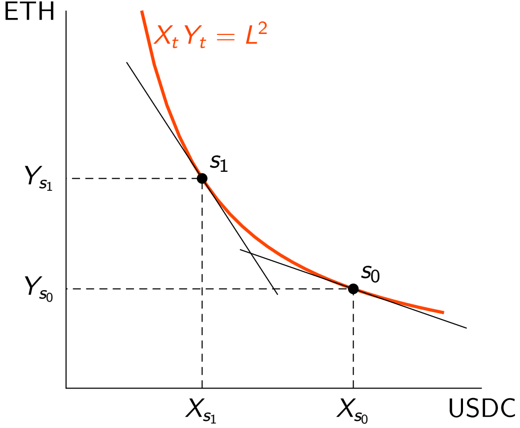
For any swap $\Delta X \longleftrightarrow \Delta Y$, we must have

$$(X_t - \Delta X)(Y_t + \Delta Y) = X_t Y_t = L^2$$

Price on the AMM:

$$P_t = \frac{f_x(X_t, Y_t)}{f_y(X_t, Y_t)} = \frac{Y_t}{X_t} = \left(\frac{L}{X_t}\right)^2$$

Bonding Curve for Constant Product AMM



Optimal Trading Across Exchanges

To the best of our knowledge, there is no paper which has jointly modeled LOB and AMM in a dynamic model, the closest being static game theoretical models [Aoyagi and Ito, 2021]

- **Liquidity trading on CEXs**: Alfonsi et al. [2010]; Almgren [2012]; Obizhaeva and Wang [2013]; Cheridito and Sepin [2014]; Neuman and Voß [2022]; Hey et al. [2025]; etc....
- **Liquidity trading on DEXs**: Zhang et al. [2022]; Angeris et al. [2022]; Wang et al. [2022]; Hasbrouck et al. [2022]; Cartea et al. [2023]; He et al. [2024]; etc....

We develop a dynamic model, internally consistent with minimal reduced-form cuts and nonlinear price dynamics, to study optimal trading across exchanges

Aggregate Market

Without fees, arbitrageurs can realise a true arbitrage if they observe a discrepancy between quotes in the LOB and AMM. Define $\ell := 2/L$ and the map $\kappa(P) := \frac{\lambda \ell P^{3/2}}{\lambda + \ell P^{3/2}}$.

Optimal routing of ΔX_t under the inf-convolution:

$$\begin{aligned} &\frac{\kappa(P_t)}{\lambda}\Delta X_t + \frac{3\kappa(P_t)^3}{4\lambda\ell P_t^{5/2}}(\Delta X_t)^2 + O((\Delta X_t)^3) \quad \text{in LOB} \\ &\frac{\kappa(P_t)}{\ell P_t^{3/2}}\Delta X_t - \frac{3\kappa(P_t)^3}{4\lambda\ell P_t^{5/2}}(\Delta X_t)^2 + O((\Delta X_t)^3) \quad \text{in AMM} \end{aligned}$$

After the arbitrageurs' intervention, prices in both venues are the same, the aggregate market

↳ Aggregate market's depth $1/\kappa(P_t)$ behaves like the sum of the local depths

↳ Best to split the order flow proportionally to each market's depth

Continuous-Time Scaling Limits

Order flow of a large trader $(Q_t)_t$ and of other market participants $(\overline{Q}_t)_t$

Unaffected price, fully driven by $(\overline{Q}_t)_t$

$$d\overline{P}_t = \overline{P}_t\Big(\nu_t dt + \varsigma_t dW_t\Big)$$

Aggregate exchange rate, with arbitrageurs exploiting mispricings relative to $\overline{P}_t$

$$dP_t = \kappa(P_t)\Big(d\overline{Q}_t + dQ_t - \beta\left(P_t - \overline{P}_t\right)dt\Big) + \frac{3\kappa(P_t)^3}{4\ell P_t^{5/2}}\Big(d\langle\overline{Q}\rangle_t + d\langle Q\rangle_t + 2d\langle\overline{Q}, Q\rangle_t\Big)$$

- ✓ LOB limit ($\ell \to \infty$): Obizhaeva-Wang model with $\kappa(P) \to \lambda$
- ✓ AMM limit ($\lambda \to \infty$): Price-dependent $\kappa(P) \to \ell P^{3/2}$

Optimal Trading with Alpha Signals

Consider a large risk-neutral trader who

- has a forecast for $\mathbb{E}_t[\overline{P}_\tau]$ at a later time; i.e., $\alpha_t := \mathbb{E}_t[\overline{P}_\tau] - \overline{P}_t$, $\tau > T$
- cannot frontrun other market participants, with a cash account $(C_t)_t$

$$dC_t = -P_t dQ_t - \frac{\kappa(P_t)}{2}\Big(d\langle Q\rangle_t + 2d\langle Q, \overline{Q}\rangle_t\Big)$$

Large trader's goal functional: maximise the expected PnL

$$\sup_{(Q_t)_{t\in[0,T]}} \mathbb{E}[\mathrm{PnL}_T] = \sup_{(Q_t)_{t\in[0,T]}} \mathbb{E}\left[Q_T \mathbb{E}_T[\overline{P}_\tau] + \int_{0+}^{T-} dC_t + \sum_{s\in\{0,T\}} \Delta Q_s \mathbb{E}_s[\overline{P}_\tau] + \Delta C_s\right]$$

Passage to Impact Space

We redefine the control variable from order flow $(Q_t)_t$ to aggregate price $(P_t)_t$ – this "passage to impact space" allows **closed-form expressions** and **pointwise optimisation**

↳ One-to-one correspondence between order flow and exchange rate

$$dQ_t = \frac{1}{\kappa(P_t)} dP_t - \frac{3}{4\ell P_t^{5/2}} d\langle P \rangle_t + \beta \left(P_t - \overline{P}_t \right) dt - d\overline{Q}_t$$

↳ Integration by parts to obtain the alpha signal

$$Q_T \mathbb{E}_T[\overline{P}_\tau] + \int_0^T dC_t = \int_0^T (\alpha_t + \overline{P}_t - P_t) dQ_t + [\ldots \text{quadratic variation terms} \ldots]$$

↳ Replace $\int dQ_t$ in expected PnL, and $\int dP_t$ using Itô's formula

↳ Cancellations between variations and expression of order flows in terms of prices

Equivalence

Theorem. The large trader's optimisation problem in **"order flow space"**

$$\sup_{(Q_t)_{t\in[0,T]}} \mathbb{E}\left[Q_T \mathbb{E}_T[\overline{P}_\tau] + \int_{0+}^{T-} dC_t + \sum_{s\in\{0,T\}} \Delta Q_s \mathbb{E}_s[\overline{P}_\tau] + \Delta C_s \right]$$

is equivalent to the following problem in **"impact space"**

$$\sup_{(P_t)_{t\in(0,T]}} \mathbb{E}\left[\int_{0+}^{T-} \left((\alpha_t + \overline{P}_t - P_t)\left(\beta(P_t - \overline{P}_t) - \frac{\nu_t \overline{P}_t}{\kappa(\overline{P}_t)} + \frac{3\varsigma_t^2}{4\ell \overline{P}_t^{1/2}} \right) + \frac{\varsigma_t^2 \overline{P}_t^2 \kappa(P_t)}{2\kappa(\overline{P}_t)^2} \right) dt - d\langle \overline{Q}_t, \mathbb{E}_t[\overline{P}_\tau] \rangle_t \right]$$
$$- \frac{P_T^2}{2\lambda} - \frac{2(\alpha_T + \overline{P}_T + P_T)}{\ell P_T^{1/2}} + \frac{P_T(\alpha_T + \overline{P}_T)}{\lambda}.$$

Solutions

Theorem. The optimal price is $P_T^* = \alpha_T + \overline{P}_T$ and $P_t^*, t \in (0, T)$ such that

$$\underbrace{\beta(\alpha_t - 2(P_t - \overline{P}_t)) + \frac{\nu_t \overline{P}_t}{\kappa(\overline{P}_t)}}_{\text{strength of alpha signal}} + \underbrace{\frac{3\varsigma_t^2}{4\ell \overline{P}_t^{1/2}} \left(\frac{\overline{P}_t^{5/2} \kappa(P_t)^2}{P_t^{5/2} \kappa(\overline{P}_t)^2} - 1 \right)}_{\text{"LVR" profit from mispricings}} = 0$$

We recover the corresponding optimal order flow via the mapping

$$Q_t = \int_0^t \frac{1}{\kappa(P_s)} dP_s - \int_0^t \frac{1}{\kappa(\overline{P}_s)} d\overline{P}_s - \int_0^t \frac{3}{4\ell P_s^{5/2}} d\langle P \rangle_s + \int_0^t \frac{3\varsigma_s^2}{4\ell \overline{P}_s^{1/2}} ds + \int_0^t \beta \left(P_s - \overline{P}_s \right) ds$$

↳ Optimal price of $P_t^* = \overline{P}_t$ without alpha signal, no price manipulation

Approximation and Limiting Cases

Optimal price when the volatility of $\overline{P}_t$ is zero:

$$P_t^\circ = \overline{P}_t + \frac{1}{2}\left(\alpha_t + \frac{\nu_t \overline{P}_t}{\beta\kappa(\overline{P}_t)}\right)$$

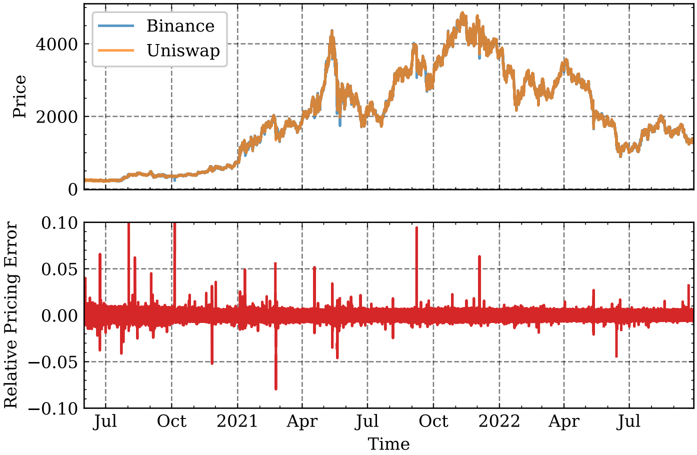
Approximation for small ς_t^2 via the implicit function theorem around P_t°:

$$P_t^* = P_t^\circ + \frac{3\varsigma_t^2}{8\ell\beta\overline{P}_t^{1/2}}\left(\frac{\overline{P}_t^{5/2}\kappa(P_t^\circ)^2}{(P_t^\circ)^{5/2}\kappa(\overline{P}_t)^2} - 1\right) + O(\varsigma_t^4)$$

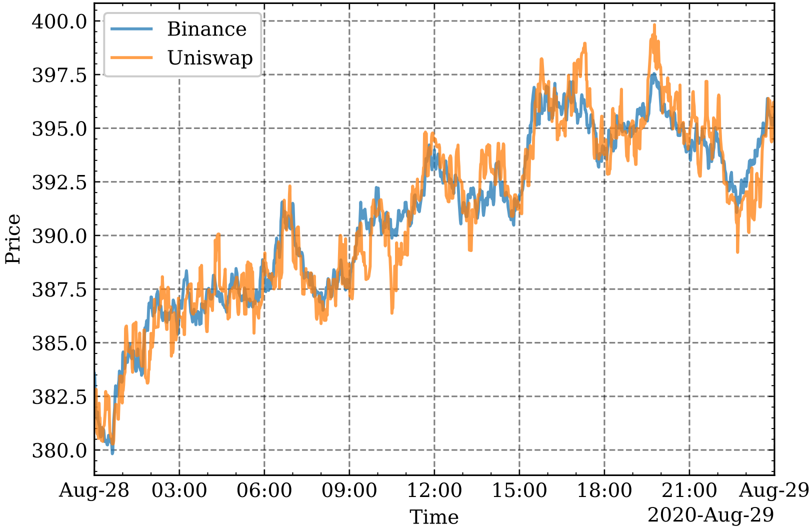
- ✓ LOB limit ($\ell \to \infty$): $P_t^* = \overline{P}_t + \frac{1}{2}\left(\alpha_t + \frac{\nu_t \overline{P}_t}{\beta\lambda}\right)$
- ✓ AMM limit ($\lambda \to \infty$): FOC is quadratic in $\sqrt{P_t}$, analytic optimal price

Empirical Study

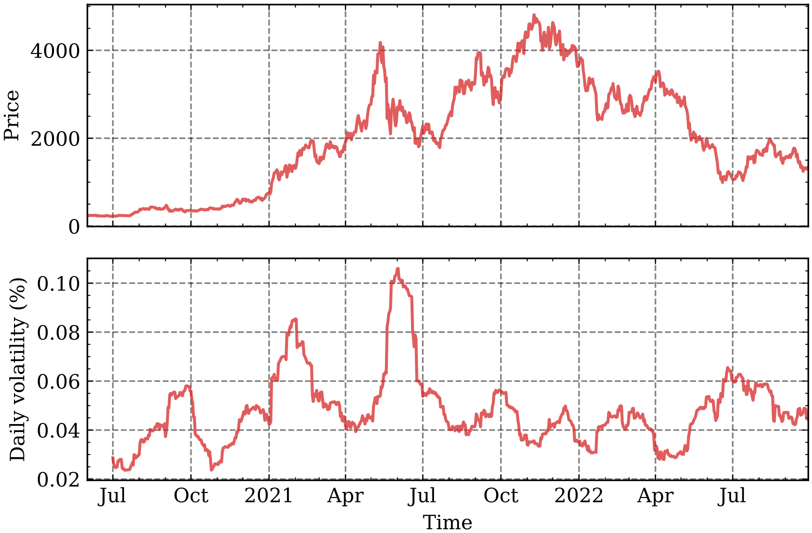
We have 10-second bins for price and trade data from **Binance** (LOB) and **Uniswap** (AMM) between Aug. 2020 and Sept. 2022 for the pair ETH-USDC



Prices on August 28, 2020



Price Tracking



Empirical Study (cont'd)

We fit the Obizhaeva-Wang model to the aggregate trades [see e.g. Muhle-Karbe et al., 2024]

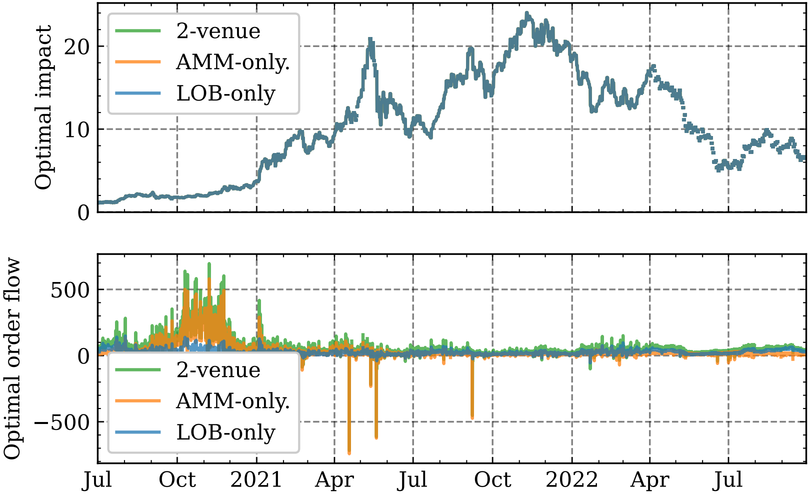
$$\overline{P}_{t+\Delta h} - \overline{P}_t = \underbrace{-\overbrace{\beta\kappa(\overline{P}_t)}^{\beta'} I_t \Delta h + \overbrace{\kappa(\overline{P}_t)}^{\lambda'} \left(\Delta\overline{Q}_t^{\mathrm{LOB}} + \Delta\overline{Q}_t^{\mathrm{AMM}}\right)}_{I_{t+\Delta h} - I_t} + \varepsilon$$

Analysis repeated daily, with a rolling window of 30 days and optimal flow routing:

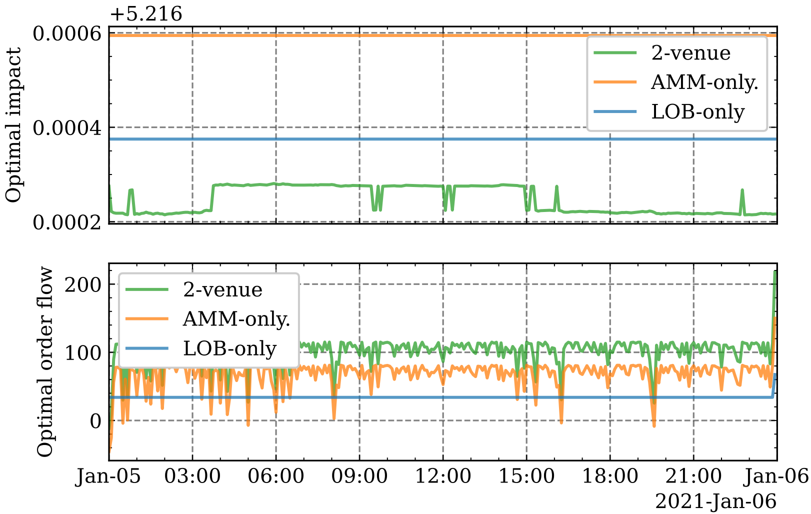
- Estimation of market activity by an exponentially weighted moving average
- Estimation of price impact parameters by a linear regression
- Grid search for the decay parameter

Trading Volumes

Optimal Impact & Order Flow



Optimal Impact & Order Flow



Contributions

We study optimal trading across a LOB and AMM:

↳ **Nonlinear dynamics** of the aggregate exchange with **diffusive trading strategies**

↳ **Tractable solution** in "impact space"; results can be generalised to G3Ms

↳ Empirical case study on Binance & Uniswap data

Future directions:

↳ Inclusion of fees, multi-dimensional model for all prices

↳ Interactions between liquidity takers and liquidity providers

↳ Competition between many LOBs and AMMs

↳ Etc.

Thank you!

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