Robust Reinforcement Learning with Dynamic Distortion Risk Measures

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Joint work with
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Statistical Sciences
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Reinforcement Learning (RL)

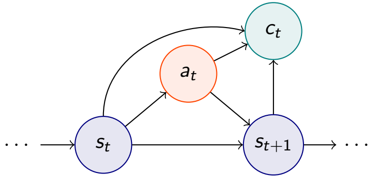
Principled model-agnostic framework for learning-based control

During a training phase, the agent:

↳ interacts with a virtual environment

↳ observes feedback in the form of costs

↳ updates its behaviour; finds best course of action



Applications of interest:

- Portfolio allocation
- Pricing and hedging
- Robot control
- Route optimisation
- Resource allocation
- Healthcare treatments
- Self-driving vehicles
- Control in agriculture
- etc.

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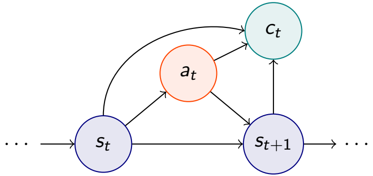

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Robust Risk-Aware RL

Standard RL: aim at optimising problems of the form $\min_\theta \mathbb{E}[Y^\theta]$, where $Y^\theta = \sum_t \gamma^t c_t^\theta$

- ✗ Ignores the risk of the costs!

Risk-aware RL: e.g. expected utility [Nass et al., 2019], risk-constrained $\mathbb{E}$ [Di Castro et al., 2019], coherent risk [Tamar et al., 2016], etc.

- ✗ Optimising static risk measures leads to optimal precommitment policies!

Robust risk-aware RL: e.g. distributional RL and KL divergence [Smirnova et al., 2019], risk-neutral RL and Wasserstein ball [Abdullah et al., 2019], distributional RL and ϕ-divergence [Clavier et al., 2022], RDEU and Wasserstein ball [Jaimungal et al., 2022], etc.

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- ✗ Gives time-inconsistent optimal policies!

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Robust Risk-Aware RL (cont'd)

Time-consistent approaches: e.g. recursive risk measures [Chu and Zhang, 2014; Bäuerle and Glauner, 2022], dynamic risk measures [Tamar et al., 2016; Ahmadi et al., 2021; Cheng and Jaimungal, 2022], etc.

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- ✓ Accounts for model uncertainty via Bayesian perspective
- × Requires finite state and action spaces

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- × Does not allow robustification

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Contributions

Goal: develop deep RL algorithms to solve robust risk-aware problems with dynamic risk

- ✓ Actor-critic algorithm optimising dynamic robust risk measures
- ✓ Accounts for model uncertainty and risk in a time-consistent manner
- ✓ Analysis with uncertainty sets induced by the conditional Wasserstein distance
- ✓ Derivation of deterministic policy gradient formulas
- ✓ Universal approximation theorem of the value function
- ✓ Performance evaluation on a portfolio allocation example

Agenda

Dynamic Risk Measures

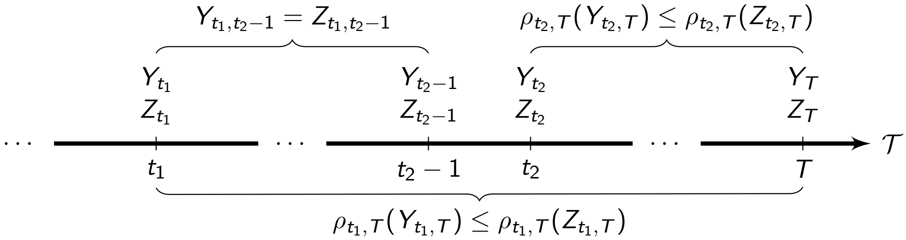
- Let $\mathcal{T} := \{0, 1, \dots, T\}$
- We work on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{T}}, \mathbb{P})$
- $\mathcal{F}_t$-measurable bounded random costs: $\mathcal{Y}_t := \mathcal{L}^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$
- $\mathcal{Y}_{t_1,t_2} := \mathcal{Y}_{t_1} \times \cdots \times \mathcal{Y}_{t_2}$

Dynamic risk measure: A sequence of maps $\{\rho_{t,T}\}_{t \in \mathcal{T}}$ such that $\rho_{t,T} : \mathcal{Y}_{t,T} \to \mathcal{Y}_t$

Time-Consistency

Strong time-consistency: For any $Y_{t_1,T}, Z_{t_1,T} \in \mathcal{Y}_{t_1,T}$ and $0 \le t_1 < t_2 \le T$, we have

$$\begin{matrix} Y_{t_1,t_2-1} = Z_{t_1,t_2-1} \\ \rho_{t_2,T}(Y_{t_2,T}) \le \rho_{t_2,T}(Z_{t_2,T}) \end{matrix} \implies \rho_{t_1,T}(Y_{t_1,T}) \le \rho_{t_1,T}(Z_{t_1,T})$$



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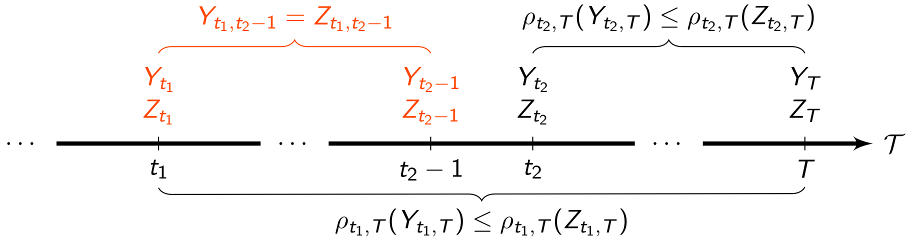

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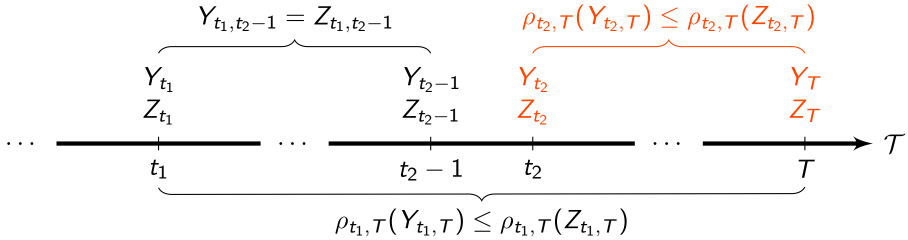

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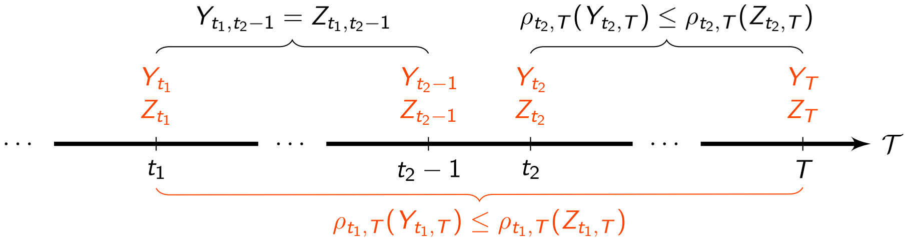


Time-Consistent Dynamic Risk

Theorem 1 of Ruszczyński [2010]

Let $\{\rho_{t,T}\}_{t\in\mathcal{T}}$ be a time-consistent, dynamic risk measure. Suppose that it satisfies

- $\rho_{t,T}(Y_t, Y_{t+1}, \dots, Y_T) = Y_t + \rho_{t,T}(0, Y_{t+1}, \dots, Y_T)$
- $\rho_{t,T}(0, \dots, 0) = 0$
- $Y \leq Z$ a.s. $\implies \rho_{t,T}(Y) \leq \rho_{t,T}(Z)$

Then $\{\rho_{t,T}\}_{t\in\mathcal{T}}$ may be expressed as

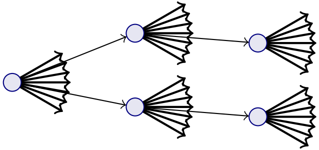
$$\rho_{t,T}(Y_{t,T}) = Y_t + \rho_t\Bigg(Y_{t+1} + \rho_{t+1}\bigg(Y_{t+2} + \cdots + \rho_{T-2}\Big(Y_{T-1} + \rho_{T-1}\big(Y_T\big)\Big)\cdots\bigg)\Bigg),$$

where each one-step conditional risk measure $\rho_t : \mathcal{Y}_{t+1} \to \mathcal{Y}_t$ satisfies $\rho_t(Y) = \rho_{t,t+1}(0, Y)$ for any $Y \in \mathcal{Y}_{t+1}$.

Elicitability

Nested simulations are computationally expensive...

- Simulation of N episodes with T periods
- Additional M inner transitions for each state



ρ_t is k-elicitable [Gneiting, 2011] iff there exists a scoring function $S : \mathbb{R}^k \times \mathbb{Y} \to \mathbb{R}$ s.t.

$$\rho_t(Y) = \arg\min_{\mathfrak{a} \in \mathbb{R}^k} \mathbb{E}_{Y \sim F_{Y|\mathcal{F}_t}} \Big[S(\mathfrak{a}, Y) \Big].$$

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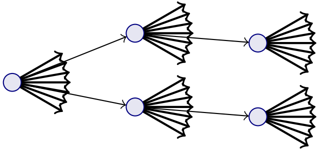

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Elicitable Mappings

Expectation: $\mathbb{E}[Y] = \underset{\mathfrak{a}\in\mathbb{R}}{\arg\min}\, \mathbb{E}_{Y\sim F_Y}\left[(\mathfrak{a} - Y)^2\right]$

Pair VaR_α-CVaR_α: $\Big(\mathrm{VaR}_\alpha(Y), \mathrm{CVaR}_\alpha(Y)\Big) = \underset{(\mathfrak{a}_1,\mathfrak{a}_2)\in\mathbb{R}^2}{\arg\min}\, \mathbb{E}_{Y\sim F_Y}\Big[S(\mathfrak{a}_1, \mathfrak{a}_2, Y)\Big]$, with

$$
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S(\mathfrak{a}_1, \mathfrak{a}_2, y) = \Big(\mathbb{1}_{\{y\leq\mathfrak{a}_1\}} - \alpha\Big)\Big(G_1(\mathfrak{a}_1) - G_1(y)\Big) - G_2(\mathfrak{a}_2) + G_2(y) \\
+ G_2'(\mathfrak{a}_2)\left[\mathfrak{a}_2 + \frac{1}{1-\alpha}\left(\mathfrak{a}_1\Big(\mathbb{1}_{\{y>\mathfrak{a}_1\}} - (1-\alpha)\Big) - y\,\mathbb{1}_{\{y>\mathfrak{a}_1\}}\right)\right]
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Conditional elicitable maps:

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\rho_t(Y \mid s_t = s) = \underset{h\,:\,\mathcal{S}\to\mathbb{R}}{\arg\min}\, \mathbb{E}_{Y\sim F_Y}\Big[S(h(s), Y)\Big]
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Problem Setup

Problems of the form

$$\min_{\pi} \rho_{0,T}\Big(\{c_t^\pi\}_t\Big) = \min_{\pi} \rho_0\Bigg(c_0^\pi + \rho_1\Big(c_1^\pi + \cdots + \rho_{T-1}\Big(c_{T-1}^\pi + \rho_T\big(c_T^\pi\big)\Big)\cdots\Big)\Bigg)$$

where $c_t^\pi = c(s_t, \pi(s_t), s_{t+1}^\pi)$ are $\mathcal{F}_{t+1}$-measurable random costs.

Running risk-to-go satisfies dynamic programming equations:

$$V_t(s;\pi) = \rho_t\Big(c_t^\pi + V_{t+1}(s_{t+1}^\pi;\pi) \;\Big|\; s_t = s\Big)$$

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Account for Model Uncertainty

Training experience should reflect events similar to those likely to occur during testing

↳ What if there is model uncertainty?

We include uncertainty sets within dynamic risk measures [Moresco et al., 2024]

↳ Leads to time-consistent optimal policies

↳ Provides a general equivalence between time-consistency and robust dynamic risk

↳ Shows equivalence between uncertainty on the entire stochastic process and one-step uncertainty sets

Robust one-step conditional risk: For an uncertainty set $\varphi^\epsilon : \mathcal{Y}_{t+1} \to 2^{\mathcal{Y}_{t+1}}$, define

$$\varrho_t^\epsilon(Y) = \operatorname{ess\,sup} \left\{ \rho_t(Y^\phi) \ : \ Y^\phi \in \varphi_Y^\epsilon \right\}$$

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Dynamic Robust Distortion Risk Measures

We aim to optimise a class of dynamic robust distortion risk measures with piecewise linear γ_s and uncertainty sets induced by the conditional 2-Wasserstein distance

$$\varrho_t^{\epsilon_s,\gamma_s}(Y_t^\pi) = \operatorname*{ess\,sup}_{Y^\phi \in \varphi^{\epsilon_s}_{Y_t^\pi}} \left\langle \gamma_s, \breve{F}_\phi(\cdot|s,a) \right\rangle \quad \text{with} \quad Y_t^\pi := c_t(s,a,s') + V_{t+1}(s';\pi).$$

- **w/o moment constraints**:

$$\vartheta_Y^\epsilon = \left\{ Y^\phi \in \mathcal{Y}_{t+1} : \ \left\| \breve{F}_{Y|\mathcal{F}_t} - \breve{F}_{Y^\phi|\mathcal{F}_t} \right\| \le \epsilon \right\}$$

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Optimal Quantile Function w/o Moments

[Thm. 3.9, Pesenti and Jaimungal, 2023]

Consider dynamic robust distortion risk measures, where $\varphi^{\epsilon_s}_{Y^\theta_t} = \vartheta^{\epsilon_s}_{Y^\theta_t}$. The optimal quantile function is given by

$$\breve{F}^*_\phi(\cdot|s,a) = \left(\breve{F}_{Y^\theta_t}(\cdot|s,a) + \frac{\gamma_s(\cdot)}{2\lambda^*}\right)^{\uparrow},$$

where $\lambda^* > 0$ is such that $\left\|\breve{F}^*_\phi(\cdot|s,a) - \breve{F}_{Y^\theta_t}(\cdot|s,a)\right\|^2 = \epsilon_s^2$ and $F^\uparrow := \arg\min_{G\in\mathbb{F}}\{\|G-F\|^2\}$ denotes the isotonic projection of a function F, where

$$\mathbb{F} = \{F \in \mathbb{L}^2([0,1]) \,:\, F \text{ is nondecreasing and left-continuous}\}.$$

Optimal Quantile Function w/o Moments (cont'd)

If γ_s is nondecreasing, then $\breve{F}^*_\phi(\cdot|s,a) = \breve{F}_{Y_t^\theta}(\cdot|s,a) + \frac{\epsilon_s \gamma_s}{\|\gamma_s\|}$. In addition, we obtain

$$
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Q_t(s,a;\theta) &= \operatorname*{ess\,sup}_{Y_t^\phi \in \vartheta^{\epsilon_s}_{Y_t^\theta}} \left\langle \gamma_s, \breve{F}_{Y_t^\phi}(\cdot|s,a) \right\rangle \\
&= \left\langle \gamma_s, \breve{F}_{Y_t^\theta}(\cdot|s,a) \right\rangle + \epsilon_s \|\gamma_s\| \\
&= \left\langle \gamma_s, \breve{F}_{\underbrace{(c_t + \epsilon_s \|\gamma_s\|)}_{c'_t} + Q_{t+1}(s_{t+1}, \pi^\theta(s_{t+1}); \theta)}(\cdot|s,a) \right\rangle
\end{aligned}
$$

↳ The $\mathcal{F}_t$-measurable shift $\epsilon_s \|\gamma_s\|$ may be included as part of the cost function

↳ State-independent ϵ, γ lead to identical robust and non-robust optimal policies

Optimal Quantile Function w/o Moments (cont'd)

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Q_t(s,a;\theta) &= \operatorname*{ess\,sup}_{Y_t^\phi \in \vartheta^{\epsilon_s}_{Y_t^\theta}} \left\langle \gamma_s, \breve{F}_{Y_t^\phi}(\cdot|s,a) \right\rangle \\
&= \left\langle \gamma_s, \breve{F}_{Y_t^\theta}(\cdot|s,a) \right\rangle + \epsilon_s \|\gamma_s\| \\
&= \left\langle \gamma_s, \breve{F}_{\underbrace{(c_t + \epsilon_s \|\gamma_s\|)}_{c'_t} + Q_{t+1}(s_{t+1}, \pi^\theta(s_{t+1}); \theta)}(\cdot|s,a) \right\rangle
\end{aligned}
$$

↳ The $\mathcal{F}_t$-measurable shift $\epsilon_s \|\gamma_s\|$ may be included as part of the cost function

↳ State-independent ϵ, γ lead to identical robust and non-robust optimal policies

Optimal Quantile Function w/ Moments

We cast in a dynamic setting [Thm. 3.1, Bernard et al., 2023]:

Theorem [C., Jaimungal, 2024]

Consider dynamic robust distortion risk measures, where γ_s is nondecreasing and

$$\varsigma_{Y_t^\theta}^{\epsilon_s} = \left\{ Y^\phi \in \mathcal{Y}_{t+1} : \|\breve{F}_{Y_t^\theta|_{\mathcal{F}_t}} - \breve{F}_{Y^\phi|_{\mathcal{F}_t}}\| \le \epsilon_s, \quad \mu = \langle \breve{F}_{Y^\phi|_{\mathcal{F}_t}}, 1 \rangle, \quad \mu^2 + \sigma^2 = \|\breve{F}_{Y^\phi|_{\mathcal{F}_t}}\|^2 \right\}.$$

The optimal quantile function is then given by

$$\breve{F}_\phi^*(u|s,a) = \mu + \frac{\lambda^* \left(\breve{F}_{Y_t^\theta}(u|s,a) - \mu \right) + \gamma_s(u) - 1}{b_{\lambda^*}},$$

where λ^* and b_{λ^*} depend non-trivially on ϵ_s, γ_s, and $\breve{F}_{Y_t^\theta}$.

Additionally, the optimal solution remains valid with $\lambda^* = 0$ if the tolerance ϵ_s is sufficiently large.

Deterministic Gradient

Theorem [C., Jaimungal, 2024]

Consider dynamic robust distortion risk measures, where γ_s is non-decreasing and

$$\varsigma_{Y_t^\theta}^{\epsilon_s} = \left\{ Y^\phi \in \mathcal{Y}_{t+1} \,:\, \|\breve{F}_{Y_t^\theta|\mathcal{F}_t} - \breve{F}_{Y^\phi|\mathcal{F}_t}\| \leq \epsilon_s, \quad \mu = \langle \breve{F}_{Y^\phi|\mathcal{F}_t}, 1 \rangle, \quad \mu^2 + \sigma^2 = \|\breve{F}_{Y^\phi|\mathcal{F}_t}\|^2 \right\}.$$

The gradient of the value function is given by

$$\begin{aligned}\nabla_\theta V_t(s;\theta) = \nabla_a Q_t(s,a;\theta)\Big|_{a=\pi^\theta(s)} \nabla_\theta \pi^\theta(s) \\ - \frac{b_{\lambda^*} - \lambda^*}{b_{\lambda^*}} \mathbb{E}_{t,s}\left[\Big((b_{\lambda^*} - \lambda^*)(Y_t^\theta - \mu) + 1 \Big) \frac{\nabla_a F_{Y_t^\theta}(x|s,a)}{\nabla_x F_{Y_t^\theta}(x|s,a)} \Bigg|_{(x,a)=(Y_t^\theta, \pi^\theta(s))} \right] \nabla_\theta \pi^\theta(s).\end{aligned}$$

↳ Reduces to deterministic policy gradient [Silver et al., 2014] when $\epsilon_s \downarrow 0$

Algorithm

We parameterise the functionals by neural networks, and wish to optimise the value function $V_t(s, \theta) = Q_t(s, \pi^\theta(s); \theta)$ over policies θ via policy gradient approach:

$$\theta \leftarrow \theta - \eta \nabla_\theta V(\cdot; \theta)$$

Actor-critic style algorithm composed of interleaved procedures:

- ✓ estimate the distribution of costs-to-go
- ✓ approximate the running risk-to-go
- ✓ update the policy via deterministic policy gradient

Algorithm (cont'd)

Step 1: Estimate the distribution $F_{Y_t^\theta|_{(s,a)}}$ where $Y_t^\theta := c_t(s,a,s') + Q_{t+1}^\theta(s', \pi^\theta(s'))$

↳ Continuous ranked probability score as strictly proper scoring rule

↳ Requires an estimation of the Q-function...

Step 2: Approximate the running risk-to-go $Q_t^\theta(s,a) = \operatorname*{ess\,sup}_{\breve{F}_\phi \in \varphi^{\epsilon_s}_{\breve{F}_{Y_t^\theta|_{(s,a)}}}} \left\langle \gamma_s, \breve{F}_\phi(\cdot|s,a) \right\rangle$

↳ Known optimal quantile function $\breve{F}_\phi^*$, and class of elicitable one-step risk measures

↳ Changes the distribution of Y_t^θ...

Step 3: Update π^θ with the analytical deterministic gradient formula

↳ Convex optimisation over the space of quantile functions

Algorithm (cont'd)

Step 1: Estimate the distribution $F_{Y_t^\theta|_{(s,a)}}$ where $Y_t^\theta := c_t(s,a,s') + Q_{t+1}^\theta(s', \pi^\theta(s'))$

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Algorithm (cont'd)

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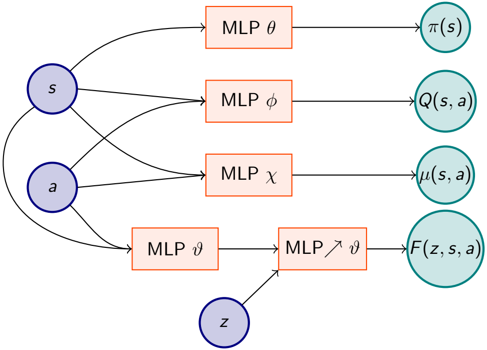
Neural Network Structure



- For layers that are descendant of z, we constrain the weights to non-negative values and use monotonic activation function to ensure a nondecreasing mapping [Sill, 1997]
- There exists a sufficiently large ANN approximating Q to any arbitrary accuracy

Agenda

Experimental Setup

Consider a market with multiple assets, where an agent

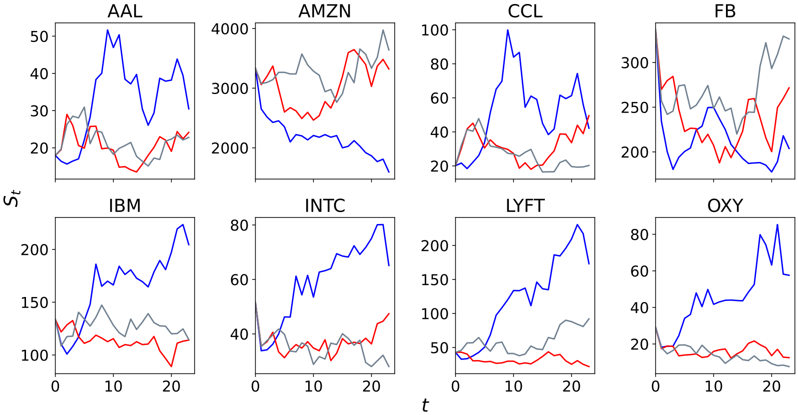
- ↳ observes the time and asset prices
- ↳ decides on the proportion of wealth to invest in each asset
- ↳ receives feedback from P&L differences
- ↳ assumes a null interest rate, no leveraging nor short-selling

We estimate a co-integration model with daily data from different stocks and use the resulting estimated model as a simulation engine to generate price paths

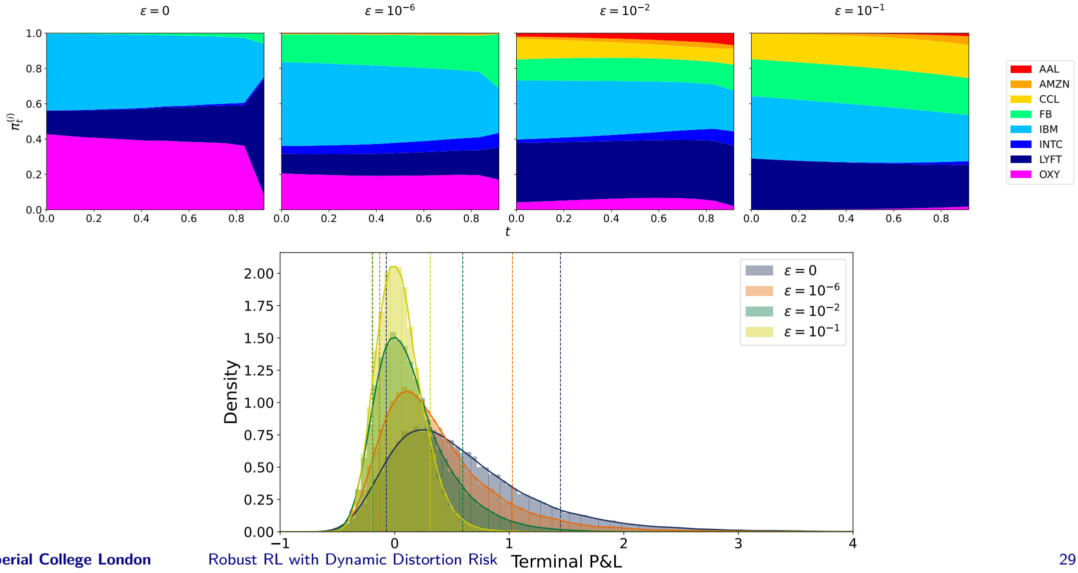
$$\Delta S_\tau = \alpha \beta^\intercal S_{\tau-1} + \Gamma_1 \Delta S_{\tau-1} + \cdots + \Gamma_{k_{ar}-1} \Delta S_{\tau-k_{ar}+1} + C D_\tau + u_\tau$$

Simulation Engine

Co-integration model using daily data from eight different stocks listed on the NASDAQ exchange between September 31, 2020 and December 31, 2021.



Robust Portfolio Allocation



Agenda

Motivations

Risk Assessment

Problem Setup

Algorithm

Experiments

Discussion

Future Directions

Practical algorithm for risk-sensitive RL with dynamic robust risk measures

↳ Accounts simultaneously for **risk** and **model uncertainty**

↳ Utilises **elicitable mappings** to avoid nested simulations

↳ Proves that classical deterministic policy gradient is a limiting case

Future directions:

- Other classes of dynamic robust risk measures
- Multi-agent RL with dynamic risk measures
- Identification of risk-aversion using inverse RL
- Model-based methods for partially observable MDPs

Thank you!

More info and slides:

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