# The Continuum Hypothesis

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The Continuum Hypothesis has been riddling the minds of the greatest mathematicians since it was first proposed in 1878 by Greg Cantor, a genius Russian mathematician born in 1845. Cantor was a religious man who was curious about infinity. He completed his doctorate by 1867 and even studied with Weierstrass, another largely influential mathematician. The Continuum Hypothesis deals with the cardinality of infinite sets. Cantor claimed that there exists no set $\lambda$ such that $\aleph_0 \leq |\lambda| \leq |\mathbb{R}|$. In this paper, the history, theorems, and philosophy behind this famous Continuum Hypothesis will be discussed.

Cantor created a subject in math called Set Theory. He was making bold claims that were not agreed with by the mathematicians of his time. He proposed the cardinality of infinite sets. Cardinality is a number representation of how many elements are in a specific set. For example, if there exists a set of the members of my family, denoted by $F = \{Anna, Levi, Lydia, Kris, Rachael\}$, the cardinality of this set would be the number 5, denoted by $|F| = 5$. This, however, is clearly a finite set, and Cantor's questions were on the transfinite sets. An example of this would be the set of natural numbers, integers, real numbers etc. These are their notations:

$$\mathbb{N} = \{1, 2, 3, \ldots, \infty\}$$
$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$
$$\mathbb{R} = \text{the set of Real numbers}$$
$$\mathbb{Q} = \{\frac{p}{q} \text{ such that p,q are integers and q } \neq 0\}$$

The set of irrational numbers has no official notation, but its definition is numbers that cannot be written as a fraction like $\pi$ or $\sqrt{2}$. We will use these notations for proofs later. Cantor proposed that two sets M and N are equivalent if they can be shown to have a one to one correspondence. In the book *Journey through Genius*, William Dunham describes finding a one to one correspondence like this:

> "Imagine an audience filtering into a large auditorium to answer the question of whether there are as many spectators as seats. We could go through the tedious process of counting both audience and chairs, and then comparing our final count. But instead, we simply ask all in attendance to sit down. If each person had a seat and if each seat had a person then the answer is "yes", since the very process of sitting has exhibited a perfect one to one correspondence."

This example would technically show the sets are bijective, which is both one to one and onto, but nonetheless this is a phenomenal example of how mathematicians can prove two sets to be one to one. If we can show that the set of N is one to one with the set Z, then we know that these 2 sets of infinite elements have the same cardinality. Cantor did just this. We begin with listing the two sets:

$$\mathbb{N} = \{1, 2, 3, \ldots, \infty\}$$
$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Then we re-arrange them such that there is some pattern for every $\mathbb{N} \to \mathbb{Z}$.

$$\mathbb{N} = 1, 2, 3, 4, 5, ...$$
$$\mathbb{Z} = 0, -1, 1, -2, 2, ...$$

If we align each element in $\mathbb{N}$ with the corresponding one below it in $\mathbb{Z}$ we find the function's formula for $f : \mathbb{N} \to \mathbb{Z}$

$$f(n) = \begin{cases} \frac{-n}{2} \text{ if n is even} \\ \frac{n-1}{2} \text{ if n is odd} \end{cases}$$

$$\therefore \text{there exists a one to one correspondence for } \mathbb{N} \to \mathbb{Z}$$

Cantor then expanded this by showing that $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$ in a similar way. So now we know the infinite sets of natural numbers, integers, and rationals numbers have the same number of elements. He defined this unison of transfinite cardinality as $\aleph_0$, the first letter in the Hebrew Alphabet. Here it is representing the smallest transfinite cardinal. According to its notation, $\aleph_1$ would be the second largest transfinite cardinal, $\aleph_2$ would be the third and so on. We know what $\aleph_0$ is, but what about the next smallest transfinite cardinal? Cantor had one last question before the great Continuum Hypothesis, what about the set of irrational numbers? This was his break point, he needed to show that there is no existing one to one correspondence between the set of irrational numbers and $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. Cantor proved this by showing that the unit interval

$$[0,1] := \{\, x \in \mathbb{R} : 0 \leq x \leq 1 \} \text{ is not one to one to one with } \mathbb{N}.$$

This following proof is from source [5].
**Proof:** This is proven by contradiction. We will use the fact that every real number x $\in$ [0,1] has a decimal representation x $= 0.b_1b_2b_3...$, where $b_i =$ 0,1,...,9. Suppose that there is an enumeration $x_1, x_2, x_3...$ of all numbers in [0,1] which can be displayed as:

$$\begin{aligned}
&x_1 = 0.\mathbf{b_{11}}b_{12}b_{13}...b_{1n}...,\\
&x_2 = 0.b_{21}\mathbf{b_{22}}b_{23}...b_{2n}...,\\
&x_3 = 0.b_{31}b_{32}\mathbf{b_{33}}...b_{3n}...,\\
&...\\
&x_n = 0.b_{n1}b_{n2}b_{n3}...\mathbf{b_{nn}}...,
\end{aligned}$$

We now define a real number $y := 0.y_1y_2y_3...y_n...$ by setting

$$y_n = \begin{cases} 2 \text{ if } b_{nn} \geq 5, \\ 7 \text{ if } b_{nn} \leq 4 \end{cases}$$

Then $y \in [0,1]$. Note that the number $y$ is not equal to any of the numbers with two decimal representations, since $y_n \neq 0.9$, for all $n \in \mathbb{N}$. Further, since $y$ and $x_n$ differ in $n$th decimal place then $y \neq x_n$, for any $n \in \mathbb{N}$. Therefore, $y$ is not included in the enumeration of [0,1], contradicting the hypothesis.

$$\therefore \text{the unit interval [0,1] is not countable}$$

This is to say that there is no one to one correspondence to $\mathbb{N}$. Because we know that if $A \subset B$ and A is not countable, then B is not countable, and

[0,1] $\subset \mathbb{R}$. So then we can conclude that $\mathbb{R}$ is also not countable. So now we have concluded that a one to one correspondence is not possible for $\mathbb{R}$. Now, we know that the cardinality of $\mathbb{N}$ is transfinite, and the cardinality of $\mathbb{R}$ is transfinite, but they are not equal infinities. This indicates that one infinity would be larger than another. Cantor follows this with a proof that $|\mathbb{Z}| < |\mathbb{R}|$ meaning that $\aleph_0 < |\mathbb{R}|$. Thus, in 1878 Cantor proposed the Continuum Hypothesis: There exists no set $\lambda$ such that $\aleph_0 \leq |\lambda| \leq |\mathbb{R}|$. He essentially claimed that the set $\mathbb{R}$ is the next largest transfinite set to $\aleph_0$, and there is no set with a cardinality in between them. Stanford's article written by Peter Koellner describes the hypothesis like so:

> "Cantor immediately tried to determine whether there were any infinite sets of real numbers that were of intermediate size, that is, whether there was an infinite set of real numbers that could not be put into one-to-one correspondence with the natural numbers and could not be put into one-to-one correspondence with the real numbers. The continuum hypothesis (under one formulation) is simply the statement that there is no such set of real numbers."

Now that we have covered sets, cardinality, $\aleph_0$, and The Continuum Hypothesis, we can move forward with the more recent discoveries towards this hypothesis.

There are three mathematicians; Kurt Gödel, Paul Cohen, and Hugh Woodin that have made huge steps in the progress of this hypothesis, though it remains unsolved. In 1937 Kurt Gödel proved that this hypothesis cannot be proven false using Zermelo-Fraenkel Set Theory, which contains axioms that all mathematicians use for proofs in set theory. Robert Passmann, a logician working on his PhD at the University of Amsterdam describes Gödel's work as a creation of some "set-theoretic world in which the continuum hypothesis is true: the so-called constructible universe." Gödel assumed there existed a proof that the Continuum Hypothesis was false using the axioms in set theory. Then "As the axioms of set theory hold in Gödel's constructible universe, it must follow that the continuum hypothesis is false in this universe. But Gödel showed that it's true — a contradiction!" Then in 1963 Paul Cohen proved that no contradiction would arise if the negation of the hypothesis were added to set theory. He received the Fields medal for this proof. To do this, Paul Cohen invented something called the forcing technique, in which he creates set-theoretic universes. This resulted in a proof that using our axioms of set theory there can be no constructed proof that the continuum hypothesis is true. So, now we are aware that we cannot prove this hypothesis true or false using what we currently know of set theory. Mathematician Hugh Woodin has dedicated his life to this hypothesis and found a solution, but one that does not fit into the universe of sets. Koellner describes it as an effective failure. In Woodin's speech at the World Science Festival, he claimed that "We can't find the Continuum Hypothesis with some clever proof, we know that. Somehow there has to be a solution that forms to our intuitions, there has to be clues. The continuum hypothesis is just one instance of an infinite sequence of questions. It makes no sense to have an answer to the continuum hypothesis without having some global solution to the universe of sets. So that becomes the problem." Thus, we see how this hypothesis suddenly turns into a mathematical hypothesis with philosophical implications.

According to Koellner, there are two main philosophical approaches to this hypothesis, the pluralists and non-pluralists. The pluralist view is that there are many set-theory universes in which there exist different axioms. According to Joel Hamkins, professor of logic at Oxford University, all set-theory universes together are true. "I would argue that the continuum hypothesis is settled on the multiverse view by our extensive knowledge about how it behaves in the multiverse, and as a result it can no longer be settled in the manner formerly hoped for [by the universists]" Hamkins says. Koellner points out that Woodin also used the multiverse approach along with Paul Cohen and Kurt Gödel. Essentially the multiverse view is that there is no objective mathematical realm. Then we find the non-pluralists or universists. The universist's view is that there is one true mathematical universe, in which everything created within this universe is either true or false. There is nothing outside of this, and so universalists believe there can be new axioms in alignment with our current axioms, created to prove the continuum hypothesis true or false, we just have not discovered them yet. This is essentially the belief that there is an objective mathematical realm which we have access to and can comprehend but have yet to do so. What I find interesting is that any progress made on this hypothesis came from the pluralists, people that dared to believe in and explore something outside of what we hold to be objectively true like Paul Cohen did in set-forcing.

I doubt that Greg Cantor knew the gravity of his proposition when he originally made the claim in 1878, much like most profound mathematicians have done. The continuum hypothesis has been proven by Paul Cohen and Kurt Gödel to be a problem beyond the scope of what we know right now. Even as mathematicians like Hugh Woodin dedicate their lives to this project, there may be no solution found within the present axioms of set-theory. As Woodin said himself, to solve the continuum hypothesis would be to have a global solution to the universe of sets. Pluralists think that this solution requires multiple set-theoretical universes, while universists believe that it can be found within an objective mathematical universe of this realm. Thus, while it remains unsolved, it does not remain unsolvable so long as mathematicians continue to stretch the known into the unknown.

# References


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