Graphs

Isomorphism

Two graphs, A and B are said to be isomorphic if $\exists\ f : A_V \to B_V$, such that the same vertices stay connected. Denoted as $A \cong B$.

Two graphs that are isomorphic must have:

- The same number of:
  - Vertices (of any given degree)
  - Edges
  - Cycles/Circuits (of any given length)
- Same connectedness
- Same degree sequence (list of degrees in decreasing order)
- Cycle types

A graph is *connected* if there is a path between every pair of vertices in the graph. (this might be slightly wrong, you should probably read the question closely)

(Spanning) Trees

A graph, B is said to be a *spanning* graph of A if $\forall v \in A_V, v \in B_V$. There may be variation in edges.

A *tree* is a graph where each vertex only has one incoming edge, except for one (the root).

Tree Facts:

- Any tree with n vertices has $n - 1$ edges.
- Any *connected* graph with n vertices and $n - 1$ edges is necessarily a tree.

Minimal Spanning Trees:

- A spanning tree with the least total weight.
- If a graph has unique edge weights, it has a unique MST; otherwise, it may have many correct MSTs

Prim's Algorithm:

1. Start at any vertex, highlight it.
2. Identify all edges coming out of that vertex and choose the edge with the least weight.
3. While we still have unvisited vertices:
   a. Identify all edges coming out of all visited vertices to unvisited vertices
   b. Choose the edge with the minimum cost

Kruskal's Algorithm:

1. Start with an empty graph T
2. While we still have less than $n - 1$ edges:
   a. Identify an edge of minimum weight in G
   b. If adding this edge to T doesn't form a cycle, then add it to T and delete it from G

Adjacency Matrices

- Raising an adjacency matrix to the n-th power tells how many walks of length n there are between two vertices.
- Adding $M^1 + M^2 + M^3 + \cdots + M^k$ gives the number of nontrivial walks of length k or less.

Relations can be expressed as graphs.

- A relation R is transitive $\Leftrightarrow M^2 \leq M$, where M is the adjacency matrix for R.

Trails, Walks, Cycles

- *Walk*: a list of alternating vertices and edges.
  - In a simple graph, there is no need to indicate the edges.
  - The length of a walk is its number of edges.
  - A walk is trivial if its length is 0.
  - A walk is *closed* if it starts and ends on the same vertex.
- *Trail*: a walk with no repeated edges.
- *Circuit*: a closed trail.
- *Path*: a walk with no repeated vertices.
- *Cycle*: a nontrivial circuit in which the only repeated vertex is the first/last one.

Eulerianness

- A trail or circuit is *Eulerian* if it uses every edge in the graph exactly once (allowing for revisiting vertices).
- A graph is Eulerian if it has an Eulerian trail/circuit containing every edge (technically, semi-Eulerian if only a trail).
- A graph is only Eulerian if every vertex has an even degree.
- A graph is semi-Eulerian if only 0 or 2 vertices have an odd degree.
- A Hamiltonian cycle is one that visits each vertex exactly once (allowing for revisiting edges).

Counting

	Order matters	Order does not matter
Repetitions are allowed	Ordered list (permutations with repetitions)	Unordered list or bag (combination with repetitions)
Repetitions are not allowed	Permutations (ordered list without repetition)	Combination or set (unordered list without repetition)

- *Rule of Products*: If there are n ways to do task A and m ways to do task B regardless of how A was performed, then there are $m \cdot n$ ways to do AB.
- *Rule of Sums*: If there are n ways to do task A and m ways to do task B, then there are $m + n$ ways to do task A or B but not both.
- *Choose*: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ *Permute*: $P(n, r) = \frac{n!}{(n-r)!}$

Example Problems

How many ways to rearrange the letters in the word "MISSISSIPPI":

- 11 letters means 11! permutations, but this is an overcount
- Each set of identical letters is overcounted, so divide by those permutations:
- $\frac{11!}{4! \cdot 4! \cdot 2! \cdot 1!}$

How many sequences (x, y, z) of non-negative integers satisfy $x + y + z = 10$?

- The solution is a sequence of three numbers in $\mathbb{Z}^{\geq 0}$ that add up to 10. Note:
  - Order matters, because it is a sequence
  - Repeats are allowed
  - This is not an unordered list structure
- How many binary sequences of length 12 have exactly two 1's and ten 0's?
- Precisely equal to the desired answer.
- The number of binary sequences of length $r + n - 1$ containing exactly r 0's is $\binom{r+n-1}{r}$.

How many outcomes frrom four throws of a six-sided die sum to 14?

1. Count solutions where all $x_i \geq 1$.
2. Count solutions where some $x_j \geq 7$.
3. Required answer is (1) - (2).

$x_1 + x_2 + x_3 + x_4 = 14$.

1:

- The minimum number that appears on a die is 1, so 4 out of 14 is already guaranteed.
- The same as counting solutions where $y_i \geq 0$ for the equation $y_1 + y_2 + y_3 + y_4 = 10$.
- We know that the number of solutions with $y_i \geq 0$ is $\binom{10+4-1}{10} = 286$.

2: Count solutions where at least one $x_j \geq 7$ for $x_1 + x_2 + x_3 + x_4 = 14$.

- Equivalent to "How many bags of 14 pieces of fruit can be bought from a store that sells apples, bananas, oranges, and pears, if we get at least 7 of one kind and one of each other kind?"
- One way to pick 7 apples, 1 banana, 1 orange, and 1 pear.
  - How many ways to pick the remaining 4 pieces of fruit to fill the bag?
  - Equivalent to $z_1 + z_2 + z_3 + z_4 = 4$.
  - $\binom{4+4-1}{4}$.
- One way to pick 1 apple, 7 bananas, 1 orange, and 1 pear.
- So on, and so forth...
- Resulting quantity is $\binom{4+4-1}{4} \cdot 4$.

3: Required answer is (1) - (2).