classy

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1 Grammar

Definition 1 (Terms)**.**

$$t, u ::= \mathbb{R} \mid \blacksquare \mid x \mid \langle t, u \rangle \mid [t, u] \mid \{t, u\}$$

$$\frac{}{x\colon A, x\colon A^{\perp}}$$

$$\frac{\Gamma}{\Gamma, \blacksquare\colon A}\ \text{weakening}$$

$$\frac{\Gamma, t\colon A, u\colon A}{\Gamma, \{t, u\}\colon A}\ \text{contraction}$$

$$\frac{\Gamma, t\colon A, u\colon B}{\Gamma, [t, u]\colon A \vee B}$$

$$\frac{\Gamma, t\colon A \qquad \Delta, u\colon B}{\Gamma, \Delta, \langle t, u \rangle\colon A \wedge B}$$

$$\frac{\Gamma, t\colon A \qquad \Delta, u\colon A^{\perp}}{\Gamma, \Delta}\ \text{cut}$$

Definition 2 (Programs)**.** *A* cut expression *has the form $t * u$ where t and u are terms. The cut operator $*$ is commutative.*

Programs are denoted by Π and are unordered sequences of cut expressions.

$$\Pi ::= \epsilon \mid \Pi, t * u$$

Definition 3 (Reduction rules)**.** *.*

$$\begin{array}{llll}
\textit{Destruct} & \Pi, <t_1, t_2> * [u_1, u_2] & \to_1 & \Pi, t_1 * u_1, t_2 * u_2 \\
\textit{Send} & \Pi, x * t & \to_2 & \Pi\,\{x \Leftarrow t\} \\
\textit{Copy} & \Pi, \{t_1, t_2\} * u & \to_3 & \Pi^\alpha, t_1 * u, t_2 * u^\alpha \\
 & \Pi, <t_1, t_2> * \blacksquare & \to & \Pi, t_1 * \blacksquare, t_2 * \blacksquare \\
 & \Pi, [t_1, t_2] * \blacksquare & \to & \Pi, t_1 * \blacksquare, t_2 * \blacksquare
\end{array}$$

The Copy rule makes a copy of the term u by also duplicating the free variables that occur in it. We denote by u^α the term obtained from u by replacing each free variable x with a new fresh variable x'; the program Π^α is defined as:

$$\Pi^\alpha := \Pi\,\{x \Leftarrow \{x, x'\} \mid x \textit{ free in } u\}$$

Example 1 (Divergent program)**.**

$$\omega := [\{<x, y>, x\}, y]$$

$$\begin{array}{rl}
\Omega := & \omega * <\omega', \mathbb{R}> \\
\to_1 & \{<x, y>, x\} * \omega', y * \mathbb{R} \\
= & \{<x, y>, x\} * [\{<x', y'>, x'\}, y'], y * \mathbb{R} \\
\to_3 & <x, y> * [\{<x', y'>, x'\}, y'], x * \omega'', y * \mathbb{R} \\
\to_1 & x * \{<x', y'>, x'\}, y * y', x * \omega'', y * \mathbb{R} \\
\to_2 & x * \{<x', y'>, x'\}, x * \omega'', y' * \mathbb{R} \\
\to_2 & \{<x', y'>, x'\} * \omega'', y' * \mathbb{R}
\end{array}$$

2 λ-terms

We denote by σ a sequence of assignment of variables to terms of the form $(x \mapsto t)$.

We denote by σ_x the restriction of an assignment σ to the variable x:

$$\sigma_x := \{t \mid (x \mapsto t) \in \sigma\}.$$

The translation $\tau(t)$ of a λ-term into a program relies on an auxiliary function $\tau_{\text{aux}}(t)$, which takes in input a λ-term and returns a thriple $(\Pi; \sigma; t')$ composed of a program, a substitution, and a term.

The auxiliary function $\tau_{\text{aux}}(t)$ is defined as follows:

$\tau(\lambda x.t) = (\Pi; \sigma \setminus \sigma_x; \mathtt{[}\{\sigma_x\}, t'\mathtt{]})$ where $(\Pi; \sigma; t') \coloneqq \tau(t)$

$\tau(x t_1 \dots t_n) = (\Pi_1 \dots \Pi_n; \sigma_1 \dots \sigma_n (x \mapsto \mathtt{<} t'_1, \dots \mathtt{<} t'_n, z \mathtt{>>}); z)$ where $(\Pi_i; \sigma_i; t'_i) \coloneqq \tau(t_i)$ and z is a fresh var

$\tau(t_0 t_1 \dots t_n) = (\Pi_0 \Pi_1 \dots \Pi_n, t'_0 * \mathtt{<} t'_1, \dots \mathtt{<} t'_n, z \mathtt{>>}; \sigma_0 \dots \sigma_n; z)$ where $(\Pi_i; \sigma_i; t'_i) \coloneqq \tau(t_i)$ and z is a fresh v

The translation $\tau(t)$ is then defined as follows:

$$\tau(t) \coloneqq \Pi\{x \Leftarrow \mathbb{R}\} \text{ where } (\Pi; \sigma; x) \coloneqq \tau_{\text{aux}}(t).$$

3 Correctness

Definition 4 (Π-path)**.** *Let Π be a program. A Π-path is given by:*

- $n > 0$
- *a sequence of variables $x_0, \dots, x_n$ in Π*
- *terms $t_0 \circledast u_0, \dots, t_n \circledast u_n$ in Π (where $t \circledast u$ is either $\mathtt{<} t, u \mathtt{>}$ or $t * u$)*
- *such that $x_i \in t_i$ and $x_{i+1} \in u_i$ for $i = 0 \dots n-1$.*

Definition 5. *A program Π is valid if*

- *All subterms are unique, except variables: each variable occurs exactly twice.*
- *There are no cyclic Π-paths (cyclic meaning a Π-path $x_0, \dots, x_n$ such that $x_0 = x_n$).*

4 Simulation

Theorem 1. *For every λ-terms t and u, if $t \to_\beta u$, then $\tau(t) \to \tau(u)$.*

Proof. Let C be an evaluation context such that $t = C\langle(\lambda x.t')u'\rangle$ and $u = C\langle t'\{x \Leftarrow u'\}\rangle$. □

5 Stub correctness proofs

$$\frac{}{\parallel x, x \quad \mathrm{wf}}$$

$$\frac{\Pi \parallel \Gamma \quad \mathrm{wf}}{\Pi \parallel \Gamma, \blacksquare \quad \mathrm{wf}}$$

$$\frac{\Pi \parallel \Gamma, t, u \quad \mathrm{wf}}{\Pi \parallel \Gamma, \{t, u\} \quad \mathrm{wf}}$$

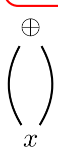
$$\frac{\Pi \parallel \Gamma, t, u \quad \mathrm{wf}}{\Pi \parallel \Gamma, \texttt{[}t, u\texttt{]} \quad \mathrm{wf}}$$

$$\frac{\Pi \parallel \Gamma, t \quad \mathrm{wf} \qquad \Pi' \parallel \Delta, u \quad \mathrm{wf}}{\Pi, \Pi' \parallel \Gamma, \Delta, \texttt{<}t, u\texttt{>} \quad \mathrm{wf}}$$

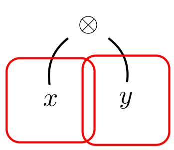
$$\frac{\Pi \parallel \Gamma, t \quad \mathrm{wf} \qquad \Pi' \parallel \Delta, u \quad \mathrm{wf}}{\Pi, \Pi', t * u \parallel \Gamma, \Delta \quad \mathrm{wf}}$$

- $x * a$
- $x * b$
- $< a, b >$

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$



$$T := \texttt{[}\{\texttt{<} - x, -y\texttt{>}, +x\}, +y\texttt{]}$$



$$\frac{}{\parallel x, x \quad \text{wf}}$$

$$\frac{\Pi \parallel \Gamma \quad \text{wf}}{\Pi \parallel \Gamma, \blacksquare \quad \text{wf}}$$

$$\frac{\Pi \parallel \Gamma, t, u \quad \text{wf}}{\Pi, \alpha = \{t, u\} \parallel \Gamma, \alpha \quad \text{wf}}$$

$$\frac{\Pi \parallel \Gamma, t, u \quad \text{wf}}{\Pi, \alpha = [t, u] \parallel \Gamma, \alpha \quad \text{wf}}$$

$$\frac{\Pi \parallel \Gamma, t \quad \text{wf} \qquad \Pi' \parallel \Delta, u \quad \text{wf}}{\alpha = \langle \Pi t, \Pi' u \rangle \parallel \Gamma, \Delta, \alpha \quad \text{wf}}$$

$$\frac{\Pi \parallel \Gamma, t \quad \text{wf} \qquad \Pi' \parallel \Delta, u \quad \text{wf}}{\alpha = \Pi t * \Pi' u \parallel \alpha \quad \text{wf}}$$