Game comonads & generalised quantifiers

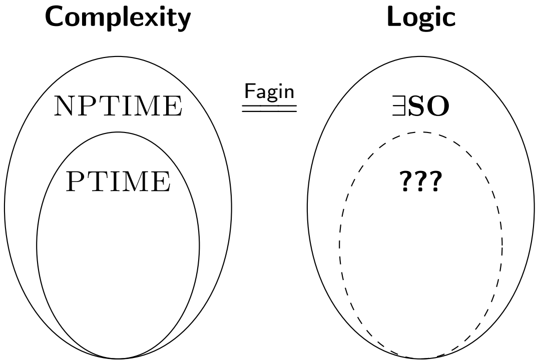
Adam Ó Conghaile
joint with *Anuj Dawar*

March 2020

Content

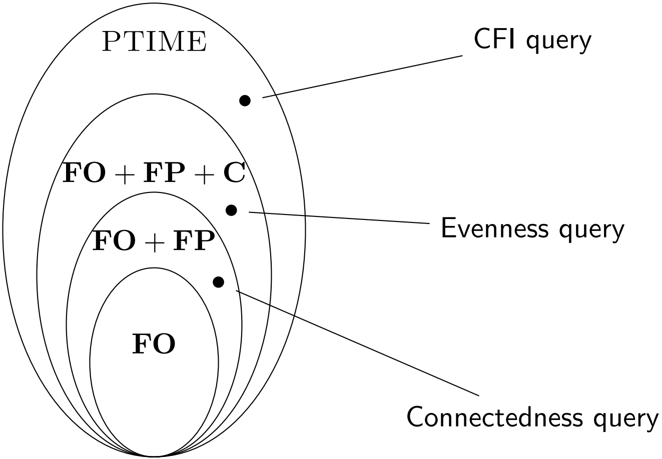
- Quantifiers & descriptive complexity
- Generalised quantifiers & Hella's game
- Game comonads & (co)resources
- Quantifier arity as a resource: a comonad for Hella's game
- Coalgebras and generalised tree-decomposition
- Future work

The search for a logic for PTIME



- Since Fagin showed that $\exists\mathbf{SO}$ captures NPTIME, finite model theorists have tried to find a logic that *captures* PTIME.
- It was noticed early on that $\mathbf{FO}$ is not enough (e.g. graph connectedness is not in $\mathbf{FO}$)
- To gain more power we need to add new types of computation to the logic. This can be done through quantifiers

Computation and quantifiers



Inflationary fix-point quantifiers

For $\mathcal{A}$ a τ-structure, $\mathbf{a}, \mathbf{b}$ tuples of elements of $\mathcal{A}$ of lengths n and m and some logic $\mathcal{L}$, let X be a second-order variable standing for an m-ary and $\mathbf{x}$, $\mathbf{y}$ be a tuples of m and n first-order variables, respectively then any formula $\phi(X, \mathbf{x}, \mathbf{y})$ in $\mathcal{L}[\tau \cup \{X\}]$ defines a sequence

$$X_0^{\mathbf{a}} = \emptyset; \quad \dots \quad X_{i+1}^{\mathbf{a}} = \{\mathbf{b} \mid \mathcal{A}, \mathbf{a}, \mathbf{b} \models \phi(X_i^{\mathbf{a}}, \mathbf{x}, \mathbf{y})\}$$

and if $\mathbf{b} \in \bigcup_i X_i^{\mathbf{a}}$ then we write

$$\mathcal{A}, \mathbf{a}, \mathbf{b} \models \mathbf{ifp}_{\mathbf{x},\mathbf{X}}(\phi(X, \mathbf{x}, \mathbf{y}))[\mathbf{z}]$$

Counting quantifiers

For $\mathcal{A}$ a τ-structure, $\mathbf{a}$ a tuple of elements of $\mathcal{A}$ of lengths n and some logic $\mathcal{L}$, any formula $\phi(\mathbf{x}, \mathbf{y})$ in $\mathcal{L}[\tau]$ with $n+m$ free variables defines a set

$$S_\phi^{\mathbf{a}} = \{\mathbf{b} \mid \mathcal{A}, \mathbf{a}, \mathbf{b} \models \phi(\mathbf{x}, \mathbf{y})\}$$

and for any $m \in \mathbb{N}$, if $|S_\phi^{\mathbf{a}}| \geq m$ we write

$$\mathcal{A}, \mathbf{a} \models \exists^{\geq m}\mathbf{y}.\ \phi(\mathbf{x}, \mathbf{y})$$

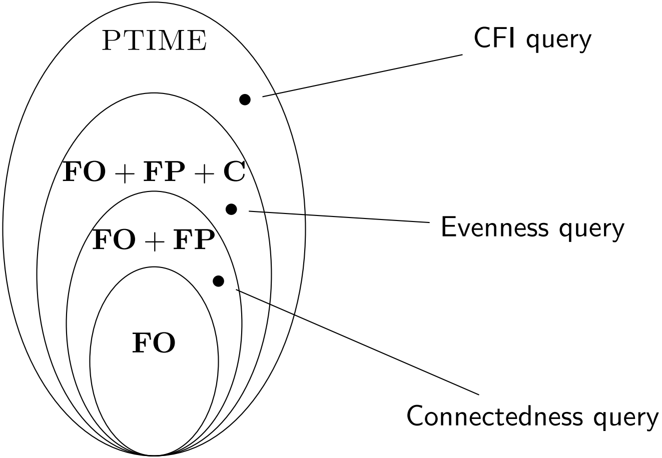
Games for judging the strength of quantifiers

Spoiler-duplicator games are used to test the strength of new quantifiers we may add to $\mathbf{FO}$. This is usually done by containing the logic $\mathbf{FO} + \mathbf{Q}$ in some logic which is graded by some syntactic resource such as variables or quantifier depth.

Table below shows logics and each of the gradings and games use. Each of these can witness an example showing the $\mathbf{FO} + \mathbf{Q}$ in question does not capture PTIME

Logic		Hierarchy	Corresponding Game
FO	$=$	$\bigcup_n \mathcal{L}_n$	n-round E-F game
FO $+$ **FP**	$\subset$	$\bigcup_k \mathcal{L}^k_{\infty\omega}$	k-pebble game
FO $+$ **FP** $+ \mathbf{C}$	$\subset$	$\bigcup_k \mathcal{C}^k_{\infty\omega}$	k-pebble bijection game

Computation and quantifiers



Generalised quantifiers

So quantifiers can add expressive power to a logic in a similar way to how oracles add computational power to some model of computation.

Given a query q (i.e. an isomorphism-closed class of structures) which can't be expressed in $\mathbf{FO}$ how do we add q to $\mathbf{FO}$

This is the idea behind generalised (or Lindström) quantifiers.

Generalised quantifiers

Lindström generalised quantifiers (Syntax)

Given a logic $\mathcal{L}[\sigma]$ and an isomorphism-closed class q of $\{R_1, \dots R_m\}$-structures the logic $\mathcal{L}(\mathcal{Q}_q)[\sigma]$ is the smallest extension $\mathcal{L}'$ of $\mathcal{L}[\sigma]$ closed under the normal syntactic rules for $\mathcal{L}$ and the following construction: given $\{\psi_i\}_{1 \leq i \leq m} \in \mathcal{L}'$ with $\{\mathbf{x}_i\}_{1 \leq i \leq m}$ tuples of free variables of length $n_i = \text{arity}(R_i)$ among the free variables of ψ_i then

$$\mathcal{Q}_q \mathbf{x}_1 \dots \mathbf{x}_m (\psi_1, \dots, \psi_m) \ \in \ \mathcal{L}'$$

Where the variables $\mathbf{x}_1 \dots \mathbf{x}_m$ are bound in this formula

Generalised quantifiers

Lindström generalised quantifiers (Semantics)

For a σ-structure $\mathcal{A}$ and formulas $\psi_i(\mathbf{x}_i, \mathbf{y}_i)$, define the sets

$$\psi_i^{\mathcal{A}}(\cdot, \mathbf{b}_i) := \{\mathbf{a} \mid \mathcal{A}, \mathbf{a}, \mathbf{b}_i \models \psi_i(\cdot, \mathbf{y}_i)\} \subset A^{n_i}$$

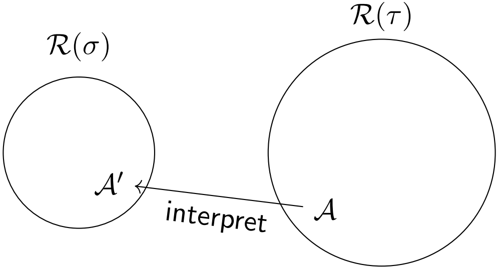
then for q, τ and ψ_i as above we say

$$\mathcal{A}\mathbf{b}_1, \dots \mathbf{b}_m \models \mathcal{Q}_q \mathbf{x}_1 \dots \mathbf{x}_m(\psi_1, \dots, \psi_m)[\mathbf{y}_1, \dots \mathbf{y}_m]$$

if and only if

$$\langle A, \psi_1^{\mathcal{A}}(\cdot, \mathbf{b}_1), \dots \psi_m^{\mathcal{A}}(\cdot, \mathbf{b}_m)\rangle \in q$$

Generalised quantifiers



Generalised quantifiers

These can be weird

Henkin quantifiers

$$\mathcal{A} \models \left\{ \begin{array}{cc} \forall x & \exists y \\ \forall u & \exists v \end{array} \right\} R(x,y,u,v) \iff \begin{array}{l} \exists f, g : A \to A \text{ s.t } \forall x, u \in A \\ (x, f(x), u, g(u)) \in R^{\mathcal{A}} \end{array}$$

And very strong

3-COL quantifier

$q = 3\mathrm{COL}$ or worse q could be undecidable!

but they are everything when it comes to the search for a logic for PTIME

Dawar 1995

If there is a logic (in the sense of Gurevich 1988) which captures PTIME then there is such a logic of the form $\mathbf{FO} + \mathbf{FP} + \mathbf{Q}$ where $\mathbf{Q}$ is a collection of generalised quantifiers.

Arity of quantifier, a graded power

Definition

For a Boolean query q on σ-structures, the arity of the quantifier $\mathcal{Q}_q$ is the maximum arity of the relations of σ
Let $\mathcal{Q}_n$ denote the collection of all Lindström quantifiers of arity $\leq n$ and $\mathcal{L}^k_{\infty\omega}(\mathcal{Q}_n)$ denote k variable infinitary first order logic extended by all quantifiers in $\mathcal{Q}_n$

Extending $\mathcal{L}^k_{\infty\omega}$ by all arity n quantifiers can be seen as adding oracles for deciding all queries over signatures with arity $\leq n$.

The arity of generalised quantifiers gives us a new way of grading the strength of some extension of $\mathbf{FO}$. In particular, $\bigcup_n \bigcup_k \mathcal{L}^k_{\infty\omega}(\mathcal{Q}_n)$ is a grading of $\mathbf{FO}$ extended with **all** Lindström quantifiers.

A game for generalised quantifiers

Hella introduced a game to to test the expressive power given by this new resource.

k-pebble n-bijective game between $\mathcal{A}$ and $\mathcal{B}$

With game configurations as in the k pebble game, at the start of each round

- Duplicator provides a bijection $f : A \to B$
- Spoiler moves pebbles $i_1, \dots i_n$ to $a_1, \dots a_n$
- Duplicator's pebbles $i_1, \dots i_n$ move to $f(a_1) \dots f(a_n)$
- If the subset of $A \times B$ given by the span $A \leftarrow [k] \rightarrow B$ is not a partial homomorphism Spoiler wins.

A game for generalised quantifiers

Theorem (Hella 1996)

The following are equivalent:

- Duplicator has a winning strategy for the k-pebble n-bijective game between $\mathcal{A}$ and $\mathcal{B}$
- $\mathcal{A} \equiv \mathcal{B}$ (over $\mathcal{L}^k_{\infty\omega}(\mathcal{Q}_n)$)
- $\mathcal{A} \equiv \mathcal{B}$ (over $\mathbf{FO}^k(\mathcal{Q}_n)$)

Games for quantifiers

Logic		Hierarchy	Corresponding Game
FO	$=$	$\bigcup_n \mathcal{L}_n$	n-round E-F game
FO + **FP**	$\subset$	$\bigcup_k \mathcal{L}^k_{\infty\omega}$	k-pebble game
FO + **FP** + **C**	$\subset$	$\begin{cases} \bigcup_k \mathcal{C}^k_{\infty\omega} \\ \bigcup_k \mathcal{L}^k_{\infty\omega}(\mathcal{Q}_1) \end{cases}$	k-pebble bijection game
FO + **ALL**	$\subset$	$\bigcup_n \bigcup_k \mathcal{L}^k_{\infty\omega}(\mathcal{Q}_n)$	n-bijective k-pebble game

Game comonads, recap

Game comonads give us a categorical semantics relating the gradings of syntactic structure studied by finite model theorists to the power of the query languages they represent.

Grading	Game	Comonad
$\textbf{FO} = \bigcup_n \mathcal{L}_n$	n round EF game	$\mathbb{E}_n$
$\textbf{FO} + \mathbf{FP} \subset \bigcup_n \mathcal{L}^k_{\infty\omega}$	k pebble game	$\mathbb{P}_k$
$\textbf{ML} = \bigcup_n \mathbf{ML}_n$	n round bisimulation game	$\mathbb{M}_n$

Towards a game comonad for generalised quantifiers

In our work we wanted to relate the two tiered hierarchy used by Hella to this language of game semantics

Aim: Arity as a resource

Find a graded comonad for which the Kleisli category captures entailment and equivalence of structures in (fragments of) $\mathcal{L}^k_{\infty\omega}(\mathcal{Q}_n)$ in the same way that previous examples do for other resources.

Towards a game comonad for generalised quantifiers

But this had some initial challenges

Challenges

- Multiple pebble moves per round
- Duplicator has new restriction in giving a bijection
- Hella's game generalises the bijection game not the pebble game. What's the approximation of homomorphism?

Relaxing Hella's game

We chose a game that resembled Hella's game, except in every round Duplicator gives a *function* $f : A \to B$ instead of a bijection.

k pebble n-function game

With game configurations as in the k pebble game, at the start of each round

- Duplicator provides a bijection $f : A \to B$
- Spoiler moves pebbles $i_1, \dots i_n$ to $a_1, \dots a_n$
- Duplicator's pebbles $i_1, \dots i_n$ move to $f(a_1) \dots f(a_n)$
- If the subset of $A \times B$ given by the span $A \leftarrow [k] \rightarrow B$ is not a partial homomorphism Spoiler wins.

Relaxing Hella's game

We prove that this game corresponds to an important fragment of $\mathcal{L}^k_{\infty\omega}(\mathcal{Q}_n)$ namely k variable infinitary logic extended with the arity n generalised quantifiers for **homomorphism-closed** queries, $\mathcal{Q}^H_n$

Theorem 1

TFAE:

- Duplicator wins the k-pebble n-function game from $\mathcal{A}$ to $\mathcal{B}$
- $\mathcal{A} \Rrightarrow \mathcal{B}$ (over $\exists + \mathcal{L}^k_{\infty\omega}(\mathcal{Q}^H_n)$) i.e.

$$\forall \phi \in \mathcal{L}^k_{\infty\omega}(\mathcal{Q}^H_n) \mathcal{A} \models \phi \implies \mathcal{B} \models \phi$$

Strategies for
k pebble n function game

$\updownarrow$

Strategies for
the k pebble game
with restrictions on Duplicator

$\updownarrow$

Homomorphisms
$\mathbb{P}_k A / \approx_n \to B$

$\longrightarrow$

The new comonad is
$\mathbb{P}_k(-) / \approx_n \to B$

Step 1: Moving to the k pebble game

Observation

Duplicator strategies for k-pebble n-function game can be seen as a restricted set of strategies for the k pebble game

Indeed, a Duplicator with a strategy for the k-pebble n-function game would play the k-pebble game as follows:

- At the end of turn mn, (for $m \geq 0$), Duplicator writes looks at the game configuration (history of Spoiler moves) and writes dow the function f_m they would have played on round m of the n-function game
- For rounds $nm+1, \ldots, n(m+1)$, if Spoiler places a pebble on $a \in A$, Duplicator responds with $f_m(a) \in B$
- Repeat.

It is not hard to see the translated strategy is winning of the k-pebble game if and only if the original strategy was winning for the n-function k-pebble game.

Step 2: Quotienting $\mathbb{P}_k$

However, it is clear that the translation cannot go the other way. So to identify the homomorphisms which do indeed come from a winning strategy to the n-function k pebble game we need to identify Spoiler histories $s \in \mathbb{P}_k\mathcal{A}$ which will be treated the same by any such strategy. Noting that a choice of f_m depending on the Spoiler moves up to (and including) Round nm, determines Duplicators responses in these strategies for rounds $nm+1, \ldots n(m+1)$ we chose the equivalence relation

$\approx_n$

For any $n \geq 1$ we define $\approx_n$ to be the equivalence relation on $\mathbb{P}_k A$ where if $s = [(a_1, p_1), \ldots (a_{nm+l}, p_{nm+l})]$ and $t = [(b_1, q_1), \ldots (b_{nm+r}, q_{nm+r})]$ for some $m \in \mathbb{N}$ and $0 < l, r \leq n$ then

$$s \approx_n t \iff a_{nm+l} = b_{nm+r} \text{ and } \forall 1 \leq j \leq nm,\ a_j = b_j \text{ and } p_j = q_j$$

Step 2: Quotienting $\mathbb{P}_k$

Lemma 2

Given a "natural" equivalence relation $\approx_A$ on $\mathbb{P}_k A$ then choosing relations on $\mathbb{P}_k A/\approx$ such that the quotient map reflects relations, we have that if the counit and comultplication are well-defined w.r.t. $\approx$ then $\mathbb{P}_k(-)/\approx$ is a comonad

The Kleisli category

Theorem 2 (Power Theorem)

(i)	$\mathbb{P}_{n,k}\mathcal{A} \to B$	$\iff$	$\mathcal{A} \Rrightarrow \mathcal{B}$	over $\exists + \mathcal{L}^k(\mathbf{Q}_n^H)$
(ii)	$\mathcal{A} \leftrightarrow_{\mathcal{K}(\mathbb{P}_{n,k})} B$	$\iff$	$\mathcal{A} \equiv \mathcal{B}$	over $\mathcal{L}^k(\mathbf{Q}_n^H)$
(iii)	$\mathcal{A} \cong_{\mathcal{K}(\mathbb{P}_{n,k})} B$	$\iff$	$\mathcal{A} \equiv \mathcal{B}$	over $\mathcal{L}^k_{\infty\omega}(\mathcal{Q}_n)$

Coalgebras & generalised treewidth

Theorem 3 (Structure Theorem)

Coalgebras of $\mathbb{P}_{n,k}$, $\alpha : \mathcal{A} \to \mathcal{A} \to \mathbb{P}_{n,k}\mathcal{A}$ correspond to n-ary generalised tree-decompositions of $\mathcal{A}$ with width $< k$

Generalised tree-decomposition

Definition (Generalised Treewidth)

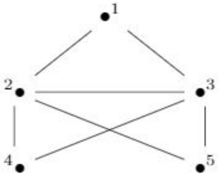
An n-ary tree-decomposition of a σ-structure A is defined by (T, β_1, β_2) where:

- T is a tree
- $\beta_1, \beta_2 : T \to 2^A$ identify the *active* and *floating* bags at each node
- $\beta(t) = \beta_1(t) \cup \beta_2(t)$

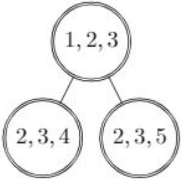
such that:

1. For every $x \in A$, there is at most one $t \in T$ s.t. $x \in \beta_2(t)$.
2. For every $x \in A$, the set $\{t \in T \mid x \in \beta(t)\}$ is a subtree
3. For any $t_1 \prec t_2$ in T it must be the case that
    1. $|\beta_1(t_2) \cap \beta_2(t_1)| \leq r$ and
    2. $\beta_2(t_2) \cap \beta(t_1) = \emptyset$
4. For any $(a_1, \ldots a_m) \in R^A$ there is a $t \in T$ s.t. $\{a_1, \ldots a_n\} \subset \beta(t)$ and $|\{a_1, \ldots a_n\} \cap \beta_2(t)| \leq n$

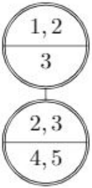
Generalised tree-decomposition



Example of a graph $\mathcal{G}$ of treewidth 2



A standard tree decomposition of $\mathcal{G}$



1-ary tree decomposition of $\mathcal{G}$

Figure 1: Two types of tree decompositions of a graph

Conclusions & future work

- We've demonstrated that $\mathbb{P}_k$ can be generalised to give categorical semantics to games for generalised quantifiers.
- We've come up with new methods of building new game comonads from old ones.
- Next we'd like to do the same for games with more restricted forms of generalised quantifiers e.g. Dawar, Grädel and Pakusa's $\mathbf{LA}^{\omega}(\mathcal{Q})$ ($\mathcal{L}^{\omega}_{\infty\omega}$ extended with all **linear algebraic** quantifiers over $\mathbb{F}_p$ for each $p \in \mathcal{Q}$)