Section 5.3

Problems: 3, 6a, 7

Problem 3

a). *Proof.* We set $g(x) := h(x) - x$, and observe that it is continous on $[0, 3]$. We see that $g(0) = 1$ and $g(3) = -1$. Thus based on the Intermediate Value Theorem, there exists $d \in [0, 3]$, such that $g(d) = 0$, which means $h(d) - d = 0$, thus $h(d) = d$. □

b). *Proof.* Because h is a differentiable function on $[0, 3]$, thus we can invoke the mean value theorem. That is, there exists $c \in (0, 3)$ such that $h'(c) = \frac{h(3)-h(0)}{3-0} = \frac{2-1}{3-0} = \frac{1}{3}$. □

c). *Proof.* Similar to part b), using the mean value theorem, we see there exists $c \in (0, 1)$ such that $f'(c) = \frac{f(1)-f(0)}{1-0} = 1$. We see h is differentiable on $[1, 3]$ and we see that $h(1) = 2 = h(3)$. thus being fancy, we can utilise Rolle's theorem to show that there exists $d \in (1, 3)$ such that $f'(d) = 0$. Finally, we see that $0, \frac{1}{4} < 1$, and using Darboux's theorem, we know that there exists $L \in (c, d)$ such that $h'(c) = \frac{1}{4}$ (reminder: $(c, d) \subseteq [0, 3]$). □

Problem 6

a). *Proof.* We know that for all $x \in [0, a]$, there exists $c \in (0, x)$ such that $g'(c) = \frac{g(x)-g(0)}{x-0} = \frac{g(x)}{x}$. And we know that $|g'(c)| \leq M$ for all $c \in [0, a]$, so $|\frac{g(x)}{x}| \leq M$. Because $x \in [0, a]$, we know that

$$\begin{aligned} \left|\frac{g'(x)}{x}\right| &\leq M \\ \implies |g'(x)| &\leq Mx \end{aligned}$$

since $x > 0$. □

Problem 7

Proof. We will use a indirect proof. Assume for contradiction that there exists more than one fixed points. That is, there exists a, b, which are elements of the interval such that $a \neq b$ and $f(a) = a$ and $f(b) = b$. We know that f is differentiable on the interval $[a, b]$(Notice: this also implies continuity). Thus we can invoke the Mean Value Theorem. That is there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1$$

However, we know that $f'(x) \neq 0$. Consequently, we have arrived at our contradiction. □

Section 6.2

Problems: 2a, 8

Problem 2

a).

Problem 8