Section 6.5

Problems: 1a, 6

Problem 1

a). We see that $g(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, and thus $|g(x)| = \sum_{n=1}^{\infty}\frac{x^n}{n}$. We know that the geometric series $\sum_{n=1}^{\infty}x^n$ converges for all $x \in (-1,1)$. Because $0 \leq |g(x)| \leq \sum_{n=1}^{\infty}x^n$, we know that $g(x)$ converges absolutely. And due to the Absolute Convergence Test, we know that $\sum_{n=1}^{\infty}g(x)$ converges for all $x \in (-1,1)$ Consequently, we see that $g(x)$ is defined for all $x \in (-1,1)$. For when $x = 1$, we see that $g(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is the alternating harmonic series, and we know that converges (simply check that it fits the criteria for the Alternating Series Test). And so from Abel's Test, we know that it thus converges uniformly on $[0,1]$. and also converges absolutely for any x such that $|x| < 1$, which implies it converges uniformly for any compact subset of $(-1,1)$. And using the Term-by-Term Continuity Theorem (we know that each partial sum is continous because it is a polynomial), we can say that f is continous on $(-1,1]$ as uniform convergence is present. Notice, this also implies that g is continous on $(-1,1)$. On $[-1,1]$, $g(x)$ is not even defined, because we see that $g(-1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ is the harmonic series, which we know diverges. And we also know that it is not possible for $g(x)$ to converge for any other points $|x| > 1$. As if we assume such point exists, then we know $|-1| < |x|$, and so $g(-1)$ converges, and we know that to be a contradiction.

Problem 6

Because it is a power series, we know the differentiated series converges uniformly on compact sets in $(-1,1)$. And utilising the Term-by-term Differentiability Theorem, we know that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(x^n)' = \sum_{n=1}^{\infty}nx^{n-1}$$

Take $x = \frac{1}{2}$, we have

$$\begin{aligned}\frac{1}{(1-\frac{1}{2})^2} &= \sum_{n=1}^{\infty}n\left(\frac{1}{2}\right)^{n-1}\\ \implies \frac{1}{2(1-\frac{1}{2})^2} &= \sum_{n=1}^{\infty}\frac{n}{2^n}\\ \implies \sum_{n=1}^{\infty}\frac{n}{2^n} &= 2\end{aligned}$$

For $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$, we will leverage a similar process. Because it is yet another power series, we could differentiate both side and obtain

$$\frac{2}{(1-x)^3} = \sum_{n=1}^{\infty}n(n-1)x^{n-2} = \sum_{n=1}^{\infty}\left(n^2x^{n-2} - nx^{n-2}\right)$$

We may set $x = \frac{1}{2}$, and get

$$
\begin{aligned}
\frac{2}{\left(1-\frac{1}{2}\right)^3} &= \sum_{n=1}^{\infty} \frac{n^2}{2^{n-2}} - 4\sum_{n=1}^{\infty} \frac{n}{2^n} \\
\implies \sum_{n=1}^{\infty} \frac{n^2}{2^{n-2}} &= 4\frac{n=1}{\infty}\frac{n}{2^n} + \frac{2}{\left(\frac{1}{2}\right)^3} = 4(2) + 16 = 24 \\
\implies \sum_{n=1}^{\infty} \frac{n^2}{2^n} &= 6
\end{aligned}
$$

Section 6.6

Problems: 2a

Problem 2

a). From the section, we know that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

And it converges for all intervals of the form $[-R, R]$. Thus, it is valid for all $x \in \mathbb{R}$. We can differentiate both sides and get

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$

With simply manipulations, we see

$$
\begin{aligned}
\cos(x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{2n!} \\
\implies x\cos(x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2n!}
\end{aligned}
$$

Section 7.2

Problems: 4

Problem 4

We know that there exists a partition P with $L(g, P) = U(g, P)$. P can be expressed as $x_0, x_1, \ldots, x_n$, and we see that

$$\sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$
$$\implies \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) = 0$$

We know that $x_k - x_{k-1} \neq 0$, thus $M_k = m_k$. This indicates that $\sup\{g(x) : x \in [x_{k-1}, x_k]\}$ $= \inf\{g(x) : x \in [x_{k-1}, x_k]\}$. We can easily see that $f(x)$ is a constant over the interval $[x_{k-1}, x_k]$ (A rather simple proof using contradiction), and $g(x_{k-1}) = g(x_k)$. And thus we see $g(x_0) = g(x_1) = \cdots = g(x_n)$ (via induction), and so we know that $g(x)$ is constant over the entire interval [a,b]. This means it is integrable (constant implies continuity implies integrability). And the value of $\int_a^b g = (b-a)g(c)$ for any $c \in [a, b]$.