Section 2.5

Problems: 5

Problem 5

Proof. We will use a proof by contradiction. Assume that it is not the case that lim $= a$. That is there exists $\epsilon > 0$, such that for all $N \in \mathbb{N}$, $n \geq N$ and $|a_n - a| \geq \epsilon$. And thus we can construct a subsequence of a_n such that $|a_{n_k} - a| \geq \epsilon$. We know that (a_n) is bounded, so (a_{n_k}) is bounded, since it is a subsequence of (a_n). And from the Bolzano-Weierstrass theorem, we know that (a_{n_k}) contains a converging subsequence $(a_{n_{k_j}})$. And we know that there exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$, $|a_{n_{k_k}} - a| \geq \epsilon$, therefore the subsequence does not converge to a. However, we see that $(a_{n_{k_j}})$ is all so a subsequence of a_n, so by assumption, it converges to a. And consequently, we have arrived at our contradiction, and thus establishing the validity of our original statement. $\square$

Section 2.6

Problems: 2

Problem 2

a).

b).

c).

d).

Section 2.7

Problems: 4, 7a

Problem 4

a). $\sum x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ diverges
$\sum y_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ diverges
We see that

$$\sum x_n y_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

which converges(in the form $\frac{1}{n^p}$, where $p > 1$).

b). We know that $\sum x_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1}(\frac{1}{n})$ converges. And we take a bounded sequence $(y_n) = (1, -1, 1, \ldots, (-1)^n + 1)$. We see that $\sum x_n y_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, which diverges.

c). The request is impossible. Since $\sum(x_n + y_n)$ and $\sum x_n$ converges, using the algebraic limit theorem, we know that $\sum(x_n + y_n) - \sum x_n = \sum y_n$ converges. But we know that $\sum y_n$ diverges, hence we have arrived at our contradiction.

d). We can let $x_n = (0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, \ldots)$. We see that $\sum(-1)^n x_n = \sum \frac{1}{2n} = 2 \sum \frac{1}{n}$. We konw that $\sum \frac{1}{n} = (2)(\frac{1}{2n})$ diverges, thus $2 \sum \frac{1}{n}$ diverges.

Problem 7

a). *Proof.* Because $\lim(na_n) = l$, we know that $l > 0$ (The proof is trivial employing conradiction and the definition is limit, assume $l < 0$ and set $\epsilon = \frac{|l|}{2}$, show a is both positive and negative). $\forall \epsilon > 0$, $N \in \mathbb{N}$, $n \geq N \implies |n(a_n) - l| < \epsilon$. We can set $\epsilon = \frac{l}{2}$, we see that $\frac{l}{2} < n(a_n) < \frac{3l}{2}$. And we see that $\frac{l}{2n} < a_n$. Because $\sum \frac{l}{2n} = \frac{l}{2} \sum \frac{1}{n}$, and we know the harmonic series diverges, so it also diverges. And we know $\frac{l}{2n} < a_n$ and $0 < \frac{l}{2n} < a_n$ so by the comparison test, we know that $\sum a_n$ diverges. $\square$

Section 3.2

Problems: 2, 4a, 8ab, 11a

Problem 2

Problem 4

a).

Problem 8

a).

b).

Problem 11

a).