Section 1.2

Problems: 3ab, 8, 10ac

Problem 3

a). False. We define $A_n = \{n, n+1, n+2, \dots\}$ for all $n \in \mathbb{N}$. We see that $A_1 \supseteq A_2 \supseteq A_3 \dots$, however $\bigcap_{n=1}^{\infty} A_n = \emptyset$, thus is not infinite. The proof below would establish that $\bigcap_{n=1}^{\infty} A_n$ is indeed empty.

Proof. We will use a proof by contradiction. Assume $a \in \bigcap_{n=1}^{\infty} A_n$, we know that $a \in A_n$ for all $n \in \mathbb{N}$. But wee see that $A_{a+1} = \{a+1, a+2, a+3, \dots\}$, and that $a \notin A_{a+1}$. Thus we have arrived at our contradiction. $\square$

b). True.

Problem 8

a). Example: $f : \mathbb{N} \to \mathbb{N}$, $f(x) = x + 1$. It is not surjective since no element of the domain maps to 1 in the co-domain.

b). Example: $f : \mathbb{N} \to \mathbb{N}$, $f(x) = \lceil \log x \rceil$

c). Example: $f : \mathbb{N} \to \mathbb{Z}$, $f(x) = (-1)^x \lceil \frac{x}{2} \rceil$

Problem 10

a). False. If $a = b$, we see that $a = b < b + \epsilon$ for some $\epsilon > 0$. However, we see that $a \not< b$, and thus the statement is false.

c). True.

Proof. We will first establish the forward direction of the proof. We know that $b < b + \epsilon$ for all $\epsilon > 0$, and so $a \le b < b + \epsilon$. Consequently, $a < b + \epsilon$. Now we will show the converse of this statement using contradiction. That is, we assume $a < b + \epsilon$ for all $\epsilon > 0$ and $a > b$. Because $a > b$, we know that there exists some $r \in \mathbb{R}$ such that $r = a - b$. We see that $a > b + \frac{r}{2}$ as $\frac{r}{2} < r$. We now arrived at our contraction because we know that $a < b + \epsilon$ for all $\epsilon > 0$. $\square$

Section 1.3

Problems: 2, 6, 8

Problem 2

a). A = {12}, inf B = 12 = sup B

b). It is not possible, as in a finite set, the largest element will be the supremum.

c). $A = \{x \mid \pi < x < 5\}$

Problem 6

a). *Proof.* If $c \in A + B$, we know that $c = a + b$ for some $a \in A$ and $b \in B$. Because $s = supA \geq a$ for all $a \in A$, and $t = supB \geq b$ for all $b \in B$, therefore $s + t \geq c$ for all $c \in A + B$. And by definition, $s + t$ is an uppper bound for $A + B$. □

b). *Proof.* We will use a proof by contradiction. We assume that for all u that is an upper bound for $A + B$, $a \in A$, and $t > u - a$. Thus we see $t + a > u$ for all upper bound u. We know that $t + a \leq t + s$ since $s = supA \geq a$, and from (a). we know that $s + t$ is an upper bound. We let $u = s + t + 1$, we know that $u > s + t \geq t + a$. However, we assumed that $u < a + t$, hence we have arrived at our contradiction. □

c). *Proof.* We let $u = sup(A + B)$. From (b). we know that $t \leq u - a$ and $s \leq u - b$. Therefore $t+s \leq 2u-a-b$, and so $t+s+(a+b) \leq 2u$. We see $t+s+(a+b) \leq t+s+u \leq 2u$. Consequently, $s + t \leq u$, and establishing the fact that $Sup(A) + Sup(B) = Sup(A + B)$. □

d). *Proof.* We know that $s - \epsilon < a$ for some $a \in A$ and $\epsilon > 0$, and $t - \epsilon < b$ for some $b \in B$ and $\epsilon > 0$. Thus, we see $s + t - 2\epsilon < a + b$, and further more that $(a + b) \in A + B$. We let $\epsilon_1 \equiv 2\epsilon$, and $c \equiv a + b$. And we see that $s + t - \epsilon_1 < c$ for all $c \in A + B$ and $\epsilon_1 > 0$. Consequently, by Lemma 1.3.8, this is equivalent to $s + t = sup(A + B)$. □

Problem 8

a). suprema: 1
infima: 0

b). suprema: 1
infima: -1

c). suprema: $\frac{1}{3}$
infima: $\frac{1}{4}$

d). suprema: 1
infima: 0