Section 1.3

Problems: 9

Problem 9

a). *Proof.* We will use a direct proof. Because $\sup A < \sup B$, we see $\sup B - \sup A > 0$. We know that $\sup A$ and $\sup B \in \mathbb{R}$, thus there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \sup B - \sup A$. And we also know that for all $\epsilon > 0$, there exists $b \in B$ such that $b > \sup B - \epsilon$. Let $\epsilon = \frac{1}{n_0}$, we see $b > \sup B - \frac{1}{n_0} > \sup A$ for some $b \in B$. Consequently, we have shown that there exists an element $b \in B$ that is an upper bound for A if $\sup B > \sup A$. □

b). $A = (0, 1)$
$B = (0, 1)$
$\sup A = \sup B$

Section 1.4

Problems: 1, 4, 5, 8

Problem 1

a). *Proof.* If $a, b \in \mathbb{Q}$, by definition, $a = \frac{p}{q}$ and $b = \frac{s}{t}$ for $p, q, s, t \in \mathbb{Z}$ and $q, t \neq 0$. We will first show that $ab \in \mathbb{Q}$. We see

$$\begin{aligned} &ab \\ =&(\frac{p}{q})(\frac{s}{t}) \\ =&\frac{ps}{qt} \end{aligned}$$

It is clear that $ps, qt \in \mathbb{Z}$ and $qt \neq 0$. Consequently, we see that $ab \in \mathbb{Q}$ We will now show that $a + b \in \mathbb{Q}$. We see

$$\begin{aligned} &a + b \\ =&(\frac{p}{q}) + (\frac{s}{t}) \\ =&\frac{pt + qs}{qt} \end{aligned}$$

One can easily see that $pt + qs, qt \in \mathbb{Z}$ and $qt \neq 0$. Consequently, we see that $a + b \in \mathbb{Q}$

□

b). *Proof.* We will use contradiction to show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$. Assume not, that is, $a \in \mathbb{Q}$, $t \in \mathbb{I}$, and $a + t \in \mathbb{Q}$. Because $a + t \in \mathbb{Q}$, $a + t = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. In addition, because $a \in \mathbb{Q}$, $a = \frac{c}{d}$ for $c, d \in \mathbb{Z}$ and $d \neq 0$. We see

$$\begin{aligned} a + t &= a + t \\ \frac{c}{d} + t &= \frac{p}{q} \\ t &= \frac{p}{q} + \frac{c}{d} \\ t &= \frac{pd + qc}{qd} \end{aligned}$$

Because $pd + cq, pd \in \mathbb{Z}$ adn $pd \neq 0$, we know that $t \in \mathbb{Q}$. Thus we have arrived at our contradiction because we know that $t \in \mathbb{R}$. $\square$

Proof. We will use contradiction to show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $at \in \mathbb{I}$. Assume not, that is, $a \in \mathbb{Q}$, $t \in \mathbb{I}$, and $at \in \mathbb{Q}$. Because $at \in \mathbb{Q}$, $at = \frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $q \neq 0$. In addition, because $a \in \mathbb{Q}$, $a = \frac{c}{d}$ for $c, d \in \mathbb{Z}$ and $d \neq 0$. We see

$$\begin{aligned} at &= at \\ \frac{c}{d}t &= \frac{p}{q} \\ t &= \frac{pc}{qd} \end{aligned}$$

Because $pc, pd \in \mathbb{Z}$ and $pd \neq 0$, we know that $t \in \mathbb{Q}$. Thus we have arrived at our contradiction because we know that $t \in \mathbb{R}$. $\square$

c). $\mathbb{I}$ is not closed under addition and multiplication e.g. we see $\sqrt{2} \times \sqrt{2} = 2$, and $2 \in \mathbb{Q}$. We do not know whether $s + t$ or $s + t$ will be a rational number or irrational number.

Problem 4

Proof. We will first show that b is an uppper bound. We know that for all $x \in \mathbb{Q} \cap [a, b]$, $a \leq x \leq b$, consequently, b is an upper bound of $\mathbb{Q} \cap [a, b]$. Now we will use a proof by contradiction to show that for any upper bound c, $b \leq c$. Assume there exists upper bound c such that $c < b$. We know that there exists $r \in \mathbb{Q}$ such that $c < r < b$ because of the rational density theorem. We have

$$a < c < x < b$$

However, we can see $x \in [a, b] \cap \mathbb{Q}$, and so c is not a upper bound. Consequently, we have arrived at our contradiction, and know that for any upper bound d, $b \leq d$. Thus we have shown that b is the least upper bound.

$\square$

Problem 5

Proof. We know any $c \in \mathbb{R}$ can be expressed as $c = a - \sqrt{2}$, where $a \in \mathbb{R}$. And for any $d \in \mathbb{R}$, $d = b - \sqrt{2}$ for some $b \in \mathbb{R}$. In addtion, we know that for some $s \in \mathbb{Q}$, if $t = s - \sqrt{2}$, then $t \in \mathbb{I}$. We let $c < d$, and we want to show that

$$\begin{aligned} &c < t < d \\ \implies &a - \sqrt{2} < s - \sqrt{2} < b - \sqrt{2} \\ \implies &a < s < b \end{aligned}$$

That is, for some $a, b \in \mathbb{R}$, $a < b$, there exists $s \in \mathbb{Q}$ such that $a < s < b$ (this is theorem 1.4.3 of the text book). Because of the Archimedean property, we know that there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. And because $s \in \mathbb{Q}$, we let $s = \frac{m}{n}$ for some $m \in \mathbb{Z}$. We pick the smallest m so that $m - 1 \leq an < m$. We can easily see that $a < \frac{m}{n}$, thus completing half of our proof. Because $\frac{1}{n} < b - a$, we see $b > \frac{1}{n} + a$. We also see that $m - 1 \leq an$, so $\frac{m}{n} \leq a + \frac{1}{n}$. Therefore

$$\begin{aligned} &\frac{m}{n} \leq a + \frac{1}{n} < b \\ \implies &s < b \end{aligned}$$

Consequently, we see that for some $a, b \in \mathbb{R}$, $a < b$, there exists $s \in \mathbb{Q}$ such that $a < s < b$. And thus $a - \sqrt{2} < s - \sqrt{2} < b - \sqrt{2}$ is also true, meaning there exists an irrational number between any two real number.

□

Problem 8

a). $A = \{\frac{1}{2n}, n \in \mathbb{N}\}$
$B = \{\frac{1}{2n+1}, n \in \mathbb{N}\}$