Section 4.2

Problems: 5, 6

Problem 5

a). $\lim_{x\to 2}(3x+4) = 10$

Proof. For any $\epsilon > 0$, we let $\delta = \frac{\epsilon}{3}$, we see that

$$\begin{aligned} |x-2| &< \frac{\epsilon}{3} \\ \implies |3x-6| &< \epsilon \\ \implies |(3x+4)-10| &< \epsilon \end{aligned}$$

Consequently, we are done. □

b). $\lim_{x\to 0} x^3 = 0$

Proof. For any $\epsilon > 0$, we let $\delta = \sqrt[3]{\epsilon}$, we see that

$$\begin{aligned} |x-0| &< \sqrt[3]{\epsilon} \\ \implies |x|^3 &< \epsilon \\ \implies |x^3 - 0| &< \epsilon \end{aligned}$$

Consequently, we are done. □

Problem 6

a). *Proof.* True. Let the function be $\lim_{x\to c} f(x) = L$. We see that for δ_s such that $0 < \delta_s < \delta$, we have $|x-c| < \delta_s \implies |x-c| < \delta$. And we already know that the orginal δ is a suitable response to the particular ϵ challenge, thus δ_s also works (since $|x-c| < \delta \implies |f(x)-L| < \epsilon$). □

b). False.
Consider the piece-wise function as counter-example:

$$f(x) = \begin{cases} x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

We see that $\lim_{x\to 0} f(x) = 0$, but $0 \neq f(a) = 1$.

Section 4.3

Problems: 1, 4, 6

Problem 1

a). Let $\epsilon > 0$ be arbitrary, we let $\delta = \epsilon^3$, we see that

$$\begin{aligned} |x - 0| &< \epsilon^3 \\ \implies \sqrt[3]{|x|} &< \epsilon \\ \implies |\sqrt[3]{x} - 0| &< \epsilon \end{aligned}$$

Consequently, we are done.

b).

Claim. *If $|x-c| < |c|$, then $c > 0 \implies x > 0$, and $c < 0 \implies x < 0$.*

Proof. If $c > 0$, we see

$$\begin{aligned} -|c| + c < x &< |c| + c \\ 0 < x &< 2c \end{aligned}$$

Consequently, $x > 0$. And if $c < 0$,

$$\begin{aligned} -|c| + c < x &< |c| + c \\ 2c < x &< 0 \end{aligned}$$

So $x < 0$. □

Proof. Given $\epsilon > 0$, we let $\delta = \min\{|c|, \epsilon|\sqrt[3]{c^2}|\}$. We see that

$$\begin{aligned} |x - c| &< \epsilon|\sqrt[3]{c^2}| \\ \implies \frac{|x-\epsilon|}{|\sqrt[3]{c^2}|} &< \epsilon \\ \implies \frac{x-c}{\sqrt[3]{x^2} + \sqrt[3]{xc} + \sqrt[3]{x^2}} < \frac{|x-\epsilon|}{|\sqrt[3]{c^2}|} &< \epsilon \\ \implies |\sqrt[3]{x} - \sqrt[3]{c}| &< \epsilon \end{aligned}$$

Consequently, we are done. □

Problem 4

a). Example:

$$f(x) = 0$$

$$g(x) = \begin{cases} x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

We see that $\lim_{x\to 0} g(x) = 0$, but $\lim_{x\to 0} g(f(x)) = 1$.

Problem 6

a). We see that if

$$f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

$f(x)g(x) = 0$, a constant function, hence continous at 0. And $f(x) + g(x) = 1$, a constant function, hence continous at 0.

b). The request is impossible. We see that $f(x)$ and $f(x)+g(x)$ are continuous at 0. Thus we see that $g(x) = f(x) + g(x) - f(x)$ is continuous, hence a contraditiction ($g(x)$ is assumed to be not continous).

Section 4.4

Problems: 1

Problem 1

a). *Proof.* We know that $f(x) = x$ is continuous, and we know that the product of continous funtion is continous, so $g(x) = x^3 = f(x)f(x)f(x)$ is continuous.

□

We can produce an alternative proof employing the definition of convergence.

Proof. Given $\epsilon > 0$, we let $\delta = \min\{1, \frac{\epsilon}{|(|c|+1)^2|+|(|c|+1)c|+c^2}\}$, thus we are able to bound x, $c - 1 < x < c + 1 \leq |c| + 1$. We see that

$$\begin{aligned} |x-c| &< \frac{\epsilon}{|(|c|+1)^2| + |(|c|+1)c| + c^2} \\ \implies (|(|c|+1)^2| + |(|c|+1)c| + c^2)|x-c| &< \epsilon \\ \implies |(x^2+xc+c^2)(x-c)| = |x^3-c^3| &< \epsilon \end{aligned}$$

□

b). We let $(x_n) = n$ for $n \in \mathbb{N}$ and $(y_n) = n + \frac{1}{n}$ for all $n \in \mathbb{N}$, we see that $|y_n - x_n| = |\frac{1}{n}|$, and thus $|y_n - x_n| \to 0$. We let $\epsilon_0 = 0.5$, we see that $|f(y_n) - f(x_n)| = |n^3 + 3n^2(\frac{1}{n}) + 3n(\frac{1}{n})^2 + (\frac{1}{n})^3 - n^3| > |3n| > 0.5$. Consequently, it fits the sequential criterion for the absence of uniform convergence.

c). *Proof.* We left A be a bounded subset of the $\mathbb{R}$, we see that for all $|a| \in A$, $a < M$ for some $M > 0$, $M \in \mathbb{R}$. Notice, for any $x, y \in A$, $x < M$ and $y < M$. Given any $\epsilon > 0$, we let $\delta = \frac{\epsilon}{3M^2}$, we see

$$|x - y| < \frac{\epsilon}{3M^2}$$
$$3M^2|x - y| < \epsilon$$
$$3M^2|x - y| < \epsilon$$
$$|x^3 - y^3| = |(x - y)(x^3 + xy + y^2)| < |(x - y)(M^2 + (M)(M) + M^2)| < 3M^2|x - y| < \epsilon$$

Consequently, we are done. □