Section 2.3

Problems: 1a, 5, 7ab

Problem 1

a). *Proof.* Because for all $\epsilon > 0$, $|x_n| < \epsilon$ and $\epsilon^2 > 0$, we let $|x_n| < \epsilon^2$. We see that $x_n < \epsilon^2$ since $x_n \geq 0$. And consequently $\sqrt{x_n} < \epsilon$. □

Problem 5

Proof. (forwards direction) We want to show (z_n) is convergent if (y_n) and (x_n) are both convergent with $\lim x_n = \lim y_n$. Notice that $x_n = (z_1, z_3, z_5, \ldots, z_{2n-1})$, which is the odd numbered terms of z_n. And we see that $y_n = (z_2, z_3, z_4, \ldots, z_{2n})$, which is the even numbered terms of z_n. For all $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that $n \geq N_1 \implies |x_n - c| = |z_{2n-1} - c| < \epsilon$ for some c. And for all $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$, such that $n \geq N_2 \implies |y_n - c| = |z_{2n} - c| < \epsilon$. We let $N = \max(N_1, N_2)$, and we know that both the even and odd terms of z_n converges, conseqeuntly, z_n is convergent.

(backwards direction) We want to show that (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$. We see that $(x_n) = (z_1, z_3, z_5, \ldots, z_{2n-1})$ and $(y_n) = (z_2, z_4, z_6, \ldots, z_{2n})$. We know for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |z_n - c| < \epsilon$. We know that $x_n = z_{2n-1} \geq z_n$, conseqeuntly, $|x_n - c| < \epsilon$. Same reasoning, $y_n = z_{2n} > z_n$, and thus $|y_n - c| < \epsilon$. □

Problem 7

a). Let $(x_n) = (1, -1, 1, \ldots, (-1)^{n-1})$ for $n \in \mathbb{N}$ and $(y_n) = (-1, 1, -1, \ldots .(-1)^n)$. We see that x_n and y_n both converges, and $(x_n + y_n) = (0, 0, 0, \ldots)$ diverges.

b). Because (x_n) and $(x_n + y_n)$ converges. We know that $(-x_n)$ converges, and thus $(x_n + y_n) + (-x_n) = (y_n)$ converges. However, it was given that (y_n) diverges, consequently, the request is impossible by referencing the proper theorems.

Section 2.4

Problems: 1abc, 2a

Problem 1

a).

b).

c).

Problem 2

a).

Section 2.5

Problems: 1ab

Problem 1

a).

b).