Section 5.3

Problems: 3, 6a, 7

Problem 3

a). *Proof.* We set $g(x) := h(x) - x$, and observe that it is continous on $[0,3]$. We see that $g(0) = 1$ and $g(3) = -1$. Thus based on the Intermediate Value Theorem, there exists $d \in [0,3]$, such that $g(d) = 0$, which means $h(d) - d = 0$, thus $h(d) = d$. □

b). *Proof.* Because h is a differentiable function on $[0,3]$, thus we can invoke the mean value theorem. That is, there exists $c \in (0,3)$ such that $h'(c) = \frac{h(3)-h(0)}{3-0} = \frac{2-1}{3-0} = \frac{1}{3}$. □

c). *Proof.* Similar to part b), using the mean value theorem, we see there exists $c \in (0,1)$ such that $f'(c) = \frac{f(1)-f(0)}{1-0} = 1$. We see h is differentiable on $[1,3]$ and we see that $h(1) = 2 = h(3)$. Thus being fancy, we can utilise Rolle's theorem to show that there exists $d \in (1,3)$ such that $f'(d) = 0$. Finally, we see that $0 < \frac{1}{4} < 1$, and using Darboux's theorem, we know that there exists $L \in (c,d)$ such that $h'(L) = \frac{1}{4}$ (reminder: $(c,d) \subseteq [0,3]$). □

Problem 6

a). *Proof.* We know that for all $x \in [0,a]$, there exists $c \in (0,x)$ such that $g'(c) = \frac{g(x)-g(0)}{x-0} = \frac{g(x)}{x}$. And we know that $|g'(c)| \leq M$ for all $c \in [0,a]$, so $|\frac{g(x)}{x}| \leq M$. Because $x \in [0,a]$, we know that

$$\begin{aligned} \left|\frac{g'(x)}{x}\right| &\leq M \\ \implies |g'(x)| &\leq Mx \end{aligned}$$

since $x > 0$. □

Problem 7

Proof. We will use a indirect proof. Assume for contradiction that there exists more than one fixed points. That is, there exists a, b, which are elements of the interval such that $a \neq b$ and $f(a) = a$ and $f(b) = b$. We know that f is differentiable on the interval $[a,b]$(Notice: this also implies continuity). Thus we can invoke the Mean Value Theorem. That is there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1$$

However, we know that $f'(x) \neq 1$. Consequently, we have arrived at our contradiction. □

Section 6.2

Problems: 2a, 8

Problem 2

a).

Claim. f_n *is continous at* 0.

Proof. Given $\epsilon > 0$, we let $\delta = \frac{1}{n}$. We see that $|x - 0| = |x| < \frac{1}{n} \implies |f_n(0) - f_n(x)| = |0 - 0| < \epsilon$. Notice that we have chosen x in such a way such that $|x| < \frac{1}{n} \implies -\frac{1}{n} < x < \frac{1}{n}$, thus $f_n(x)$ always equals 0. □

Claim. f *is not continuous at 0.*

Proof. $f(x) = \lim_{n\to\infty} f_n(x)$. We let $(a_n) = \frac{1}{n}$ for $n \in \mathbb{N}$. We see that $(a_n) \to 0$, but $|f(0) - f(a_n)| = 1 > 0$. Thus we se $f(x_n)$ does not converge to $f(0)$, and by the Criterion for Discontinuity, we may conclude that f is not continuous at c. □

Claim. f_n *does not converge uniformly on* $\mathbb{R}$.

Proof. We will use a indirect proof. Assume for contradiction that $f_n \to f$ uniformly. Let (f_n) be a sequence of functions that converges to f. Using the contrapostive statement of the Continuous Limit Theorem, if f is not continuous at 0, then there exists f_n is not continous at 0. However, we know that f_n is continous at 0 for all $n \in \mathbb{N}$. Thus we have arrived at our contradiction. □

Problem 8

Proof. We see

$$\left|\frac{1}{g(x)} - \frac{1}{g_n(x)}\right| = |g_n(x) - g(x)| \left|\frac{1}{g(x)g_n(x)}\right|$$

We know that K is a compact set and $g(x) \neq 0$, thus we know $|g(x)|$ has a minimum $M > 0$, that is $M \leq |g(x)|$, so $\frac{1}{|g(x)|} \leq M$. Now we will show $\frac{1}{|g_n(x)|}$ is bounded. We know for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|g_n(x) - g(x)| < \epsilon$ whenever $n \geq N$ and $x \in K$. Thus, we choose some $\epsilon_1 < M$, and we know that

$$\begin{aligned}
&|g(x) - g(x)_n| < \epsilon_1 \\
\implies &|g(x)| < \epsilon_1 + |g_n(x)| \\
\implies &|g_n(x)| > |g(x)| - \epsilon_1 \\
\implies &|g_n(x)| > M - \epsilon_1 \\
\implies &\frac{1}{|g_n(x)|} < \frac{1}{M - \epsilon_1}
\end{aligned}$$

let $P \coloneqq \frac{1}{M(M-\epsilon_1)}$, we see that

$$\frac{1}{|g(x)g_(x)|} < \frac{1}{M(M-\epsilon_1)} = P$$

Consequently, $\left|\frac{1}{g(x)} - \frac{1}{g_n(x)}\right| = |g(x) - g_n(x)| \left|\frac{1}{g(x)g_n(x)}\right|$. And so for any $\epsilon > 0$, we can choose an $N \in \mathbb{N}$ such that $n \geq N \implies |g(x) - g_n(x)| < \frac{\epsilon}{P} \implies \left|\frac{1}{g(x)} - \frac{1}{g_n(x)}\right| < \epsilon$, and we got our desired result.

$\square$