Section 2.5

Problems: 5

Problem 5

Proof. We will use a proof by contradiction. Assume that it is not the case that lim = a. That is there exists $\epsilon > 0$, such that for all $N \in \mathbb{N}$, $n \geq N$ and $|a_n - a| \geq \epsilon$. And thus we can construct a subsequence of a_n such that $|a_{n_k} - a| \geq \epsilon$. We know that (a_n) is bounded, so (a_{n_k}) is bounded, since it is a subsequence of (a_n). And from the Bolzano-Weierstrass theorem, we know that (a_{n_k}) contains a converging subsequence $(a_{n_{k_j}})$. And we know that there exists $\epsilon > 0$ such that for all $n \in \mathbb{N}$, $|a_{n_{k_k}} - a| \geq \epsilon$, therefore the subsequence does not converge to a. However, we see that $(a_{n_{k_j}})$ is all so a subsequence of a_n, so by assumption, it converges to a. And consequently, we have arrived at our contradiction, and thus establishing the validity of our original statement. $\square$

Section 2.6

Problems: 2

Problem 2

a). We see that $(a_n) = (1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1}(\frac{1}{n}))$ is Cauchy, but it is not monotone ($a_1 > a_2$ and $a_2 < a_3$). Hence, we have found our example.

b). Impossible, since we know that all Cauchy sequence is convergent, and all the subsequences of a convergent sequence is also convergent, and thus all convergent sequences are bounded, and so all the subsequences are bounded and an unbounded subsequence does not exists.

c). Impossible. We name the squence (a_n), and we know that it is a divergent monotone sequence. Hence it is not bounded, in addition, we know that $|a_n| \leq |a_{n+1}|$ (It would be bounded otherwise). Because a_n it is not bounded, $\forall M \in \mathbb{R}$, $\exists N \in \mathbb{N}$, such that $|a_N| \geq M$. And we see that $\forall n \geq N$, $|a_n| \geq M$. consequently, we see that all the subsequences of (a_n) is unbounded (when $k \geq N$, $n_k \geq k \geq N$). However, we know that there exists a Cauchy subsequence, hence it is convergent, hence it is bounded. But we know that all subsequenes are unbounded, therefore we have arrived at our contradiction.

d). $(a_n) = (1, 0, 2, 0, 4, 0, 5, \dots)$ has subsequence $(0, 0, 0, \dots)$ which is Cauchy.

Section 2.7

Problems: 4, 7a

Problem 4

a). $\sum x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ diverges

$\sum y_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ diverges

We see that
$$\sum x_n y_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$
which converges(in the form $\frac{1}{n^p}$, where $p > 1$).

b). We know that $\sum x_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1}(\frac{1}{n})$ converges. And we take a bounded sequence $(y_n) = (1, -1, 1, \ldots, (-1)^n + 1)$. We see that $\sum x_n y_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, which diverges.

c). The request is impossible. Since $\sum(x_n + y_n)$ and $\sum x_n$ converges, using the algebraic limit theorem, we know that $\sum(x_n + y_n) - \sum x_n = \sum y_n$ converges. But we know that $\sum y_n$ diverges, hence we have arrived at our contradiction.

d). We can let $x_n = (0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, \ldots)$. We see that $\sum(-1)^n x_n = \sum \frac{1}{2n} = 2 \sum \frac{1}{n}$. We konw that $\sum \frac{1}{n} = (2)(\frac{1}{2n})$ diverges, thus $2 \sum \frac{1}{n}$ diverges.

Problem 7

a). *Proof.* Because $\lim(na_n) = l$, we know that $l > 0$ (The proof is trivial employing conradiction and the definition is limit, assume $l < 0$ and set $\epsilon = \frac{|l|}{2}$, show a is negative, hence contradiction). $\forall \epsilon > 0$, $N \in \mathbb{N}$, $n \geq N \implies |n(a_n) - l| < \epsilon$. We can set $\epsilon = \frac{l}{2}$, we see that $\frac{l}{2} < n(a_n) < \frac{3l}{2}$. And we see that $\frac{l}{2n} < a_n$. Because $\sum \frac{l}{2n} = \frac{l}{2} \sum \frac{1}{n}$, and we know the harmonic series diverges, so it also diverges. And we know $\frac{l}{2n} < a_n$ and $0 < \frac{l}{2n} < a_n$ so by the comparison test, we know that $\sum a_n$ diverges. $\square$

Section 3.2

Problems: 2, 4a, 8ab, 11a

Problem 2

a). limit points:
For set A: -1 and 1.
For set B: $[0, 1]$.

b). Both sets are not closed, as they both don't contain all of their limit points. We see that set A doesn't contain -1, which is a limit point. And we see that for all $\epsilon > 0$, $V_e(\frac{\sqrt{2}}{2}) \cap B \neq \emptyset$ or $\frac{\sqrt{2}}{2}$ (rationals are dense), thus $\frac{\sqrt{2}}{2}$ is a limit point not contained. And we see that both A and B are not open, because we know that irrationals are dense, and so irrationals are contained in any $V_\epsilon(x)$ for A. consequently, causing $V_\epsilon(x)$ to not be a subset of A. The reasoning for B is the same.

c). For set A, we see that all points except for 1 is an isolated point. Because we know that for any other arbirary point, a_n, $\exists \epsilon > 0$ such that $V_\epsilon(a) \cap A = \emptyset$, namely, let $\epsilon = \frac{1}{n} - \frac{1}{n+2}$. We see that B does not contain any limit point, as the rationals are also dense, hence in any ϵ interval, there exists rationals.

d). We see that $\bar{A} = A \cup \{-1\}$, and we see that $\bar{B} = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$

Problem 4

a). *Proof.* We see that if $s = a$ for some $a \in A$, then we see that $s \in A \subseteq \bar{A}$, thus we are done. And for $s \neq a$ for some $a \in A$, we know that for all $\epsilon > 0$, $s - \epsilon < a_1 < s < s + \epsilon$ for some $a_1 \in A$. We see that for all $\epsilon > 0$, $V_\epsilon(s) \cap A \neq \emptyset$ or equal to s (since $s \notin A$). Consequently, s is a limit point of A, $s \in L$, and thus $s \in \bar{A}$. $\square$

Problem 8

We know that the only sets that are both open and close are $\mathbb{R}$ and $\emptyset$.

a). We know that the closure of any set, in this case $A \cup B$, is closed. Thus, definitly closed

b). We know that $A - B$ is equivalent to $A \cap B^{\complement}$. And because B is closed, so we know that $B^{\complement}$ is open. The interseciton of an finite number of open sets is open. Thus, we see that $A - B$ is open.

Problem 11

a). *Proof.* We will show that $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ and $\overline{A \cup B} \supseteq \bar{A} \cup \bar{B}$, thus they are equivalent. We know that $\overline{A} = A \cup L_A$, and same for B. So we see that $A \cup B \subseteq \overline{A} \cup \overline{B}$ (unions of sets are commutative). And it is easy to see from definition that $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$ ($\forall \epsilon > 0$, $V_\epsilon(x) \cap (A \cup B) \neq \{x\}$ and not empty $\implies$ $V_\epsilon(x) \cap (\overline{A} \cup \overline{B}) \neq \{x\}$ and not empty). We know that the union of two close set is a closed set, thus $\overline{A} \cup \overline{B}$ is a close set, and so $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Thus, $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. To show that the other way is true, we see that $\overline{A} \subseteq \overline{A \cup B}$, and $\overline{B} \subseteq \overline{A \cup B}$ (can be easily seen through definition of limit point, like above). Thus, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. And finally we have shown that they are equivalent.

$\square$