Section 1.5

Problems: 4abc, 5, 6ab, 10a

Problem 4

a). We see that $f(x) = \frac{x}{x^2-1}$ has derivative $f'(x) = -\frac{1+2x^2}{(x^2-1)^2}$. And for all $x \in (-1, 1)$ we see that $f'(x) < 0$. Conseqeuntly, $f(x)$ doesn't "loop back" (for $a, b \in (-1, 1)$, if $a < b$, then $f(a) > f(b)$), and hence it is injective. We will now show that it is surjective. We see $\lim_{x\to-1^+} f(x) = +\infty$ and $\lim_{x\to 1^-} f(x) = -\infty$, and invoking the intermediate value theorem, we can see that it has a range of $(-\infty, \infty)$, consequently surjective. We see the function $g(x) = \frac{x-\frac{a-b}{2}}{(\frac{a-b}{2})}$ describes merely a shift along with a scaling from (a, b) to $(-1, 1)$, thus it is bijective. And $h(x) = f(g(x))$ is a bejective function(composition of bijective functions is bejiective) which shows $(a, b) \sim (-\infty, \infty)$.

b). We let $g(x) = \frac{2}{x-a+1} - 1$ for $x \in (a, \infty)$. We will first show that the function is injective using contradiction. For $x_1 \neq x_2$, we assume $f(x_1) = f(x_2)$, we see

$$\begin{aligned}
\frac{2}{x_1 - a + 1} - 1 &= \frac{2}{x_2 - a + 1} - 1 \\
\implies \frac{2}{x_1 - a + 1} &= \frac{2}{x_2 - a + 1} \\
\implies x_1 &= x_2
\end{aligned}$$

Conseqeuntly, $g(x)$ is injective. We see that $g(x)$ is continuous for $x \in (a, \infty)$, and $\lim_{x\to a^+} g(x) = 1$ and $\lim_{x\to\infty} g(x) = -1$. Thus, using the intermediate value, we know that x is surjective to the co-domain $(-1, 1)$. We know that the composition of two bijective functions is bijective, thus, we we define $p(x) = g(h(x))$ using $h(x)$ from part a). Finally this shows that $(a, \infty) \sim \mathbb{R}$.

c). $f(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{x}{2} & x = \frac{1}{2^n} \text{ for } n \in \mathbb{N} \\ x & \text{else} \end{cases}$

We see that in the case when $f(x) = \frac{1}{2}$, the only solution is $x = 0$. For $f(x) = \frac{1}{2^n}$, where $n \in \mathbb{N}$, $n \geq 2$, the only solution is $x = \frac{1}{2^{n-1}}$. And for all $f(x) \neq \frac{1}{2^n}$ for some $n \in \mathbb{N}$, the only solution is $x = x$. Conseqeuntly, we have show that f(x) have exactly one solution for all $x \in (0, 1)$, thus it is bijective, and so $[0, 1) \sim (0, 1)$.

Problem 5

a). For every $x \in A$, $f(x) = x$ is a bijiection from A to A. Conseqeuntly, $A \sim A$.

b). Because $A \sim B$, we know that there exists a bijective function $f(x)$ such that $f : A \to B$. And thus $f^{-1}(x)$ is also bijective and $f^{-1} : B \to A$. Conseqeuntly, $B \sim A$

c). Because $A \sim B$, we know that there exists a bijective function $f(x)$ such that $f : A \to B$. From $B \sim C$, we know that there exists a bijective function $g(x)$ such that $g : B \to C$. The composition of two bijective functions with the co-domain of the first equivalent to the domain of the second is also bijective. Thus the bijective function $h(x) = f(g(x))$ has $h : A \to C$. Conseqeuntly, we have $A \sim C$.

Problem 6

a). $I_1 = (1, 2)$, $I_2 = (2, 3)$, and $I_n = (n, n + 1)$, for $n \in \mathbb{N}$

b). We let M be an uncountable set, and let I_m for $m \in M$ to be the collection of uncountable disjoint intervals. Namely, $I_m = (a_m, b_m)$. From the theorem of the density of the rational number which was proven in chapter 1, we know that between any two $a, b \in \mathbb{R}$, there exists a $c \in \mathbb{Q}$ such that $a < c < b$, therefore it is surjective. The sole thing left to be established is the injectivity between $c \in \mathbb{Q}$ and (a_m, b_m). We know that the interval is disjoint, thus the $c \in \mathbb{Q}$ cannot be an element of the other intervals, thus injectivity. We can thus rename this set as I_c for some $c \in \mathbb{Q}$.

Problem 10

a). *Proof.* We will use a proof by contradiction. Assume not, that is $C \subseteq [0, 1]$ is uncountable and that for all $a \in (0, 1)$, $C \cap [a, 1]$ is countable. We see that $\frac{1}{n} \in (0, 1)$ for all $n \geq 2$, $n \in \mathbb{N}$. Let $I_n = [\frac{1}{n}, 1]$, and let A_n be the set consisted of the elements of $C \cap I_n$. We know that

$$\cup_{n=2}^{\infty} C \cap I_n = C \ \cup_{n=2}^{\infty} I_n$$

is countable. We see $\lim_{n\to\infty} \frac{1}{n} = 0$, thus $\cup_{n=2}^{\infty} I_n = (0, 1]$ And so

$$A_n = \{C \cup (0, 1]\}$$

is countable. And we know that the union of two countable set is countable, thus

$$A_n \cup \{0\} = \{C \cup (0, 1] \cup 0\} = C \cup [0, 1] = C$$

is countable. Conseqeuntly, we have arrived at our contradiction because we know that C is uncountable, and thus our original statement is true. That is, if $C \subseteq [0, 1]$ is uncountable, there exists $a \in (0, 1)$, such that $C \cap [a, 1]$ is countable. □

Section 2.2

Problems: 1, 2b

Problem 1

$a_n = (1, 1, 1, 1, 1, 1, \dots)$, we see that $|a_n - 0| < 2$. Because there exists $\epsilon > 0$ ($\epsilon = 2$ in this case) such that for all $N \in \mathbb{N}$, $n \geq N$, a_n verconges to 0, thus it is vercongent. We know that the sequence $a_n = (1, -1, 1, -1, \dots, (-1)^n)$ diverges, but we see that $|a_n - 0| < 2$, thus it also verconges to 0, conseqeuntly, this is an example of a divergent sequence that is vercongent.

Claim. *A sequence a_n that verconges to a verconges to b for any $b \in \mathbb{R}$.*

Proof. We know that $b = a + c$ for some $c \in \mathbb{R}$. we see

$$|a_n - b| = |a_n - (a + c)| \leq |a_n - a| + |-c| = |a_n - a| + |c|$$

We know that $|a_n - a| < \epsilon_1$ for some $\epsilon_1 > 0$. Conseqeuntly, $|a_n - b| < \epsilon_1 + |c|$, for some $\epsilon_1 + |c| > 0$, a_n verconges to b. □

We see that a sequence can verconge to two different values. This strange definition seems to describe the idea of bounded of a sequence that is shifted.

Problem 2

Proof. For all $\epsilon > 0$, we choose N such that $N > \frac{2}{\epsilon}$. For $n \geq N$, we see

$$n > \frac{2}{\epsilon}$$
$$\frac{2}{n} < \epsilon$$
$$\frac{2n^2}{n^3} < \epsilon$$
$$\frac{2n^2}{n^3 + 3} < \frac{2n^2}{n^3} < \epsilon$$
$$\left| \frac{2n^2}{n^3 + 3} - 0 \right| < \epsilon$$

Consequently, we see that $\lim_{n \to \infty} = 0$. □