

## Section 1.5

Problems: 4abc, 5, 6ab, 10a

### Problem 4

- a). We see that  $f(x) = \frac{x}{x^2-1}$  has derivative  $f'(x) = -\frac{1+2x^2}{(x^2-1)^2}$ . And for all  $x \in (-1, 1)$  we see that  $f'(x) < 0$ . Consequently,  $f(x)$  doesn't "loop back" (for  $a, b \in (-1, 1)$ , if  $a < b$ , then  $f(a) > f(b)$ ), and hence it is injective. We will now show that it is surjective. We see  $\lim_{x \rightarrow -1^+} f(x) = +\infty$  and  $\lim_{x \rightarrow 1^-} f(x) = -\infty$ , and invoking the intermediate value theorem, we can see that it has a range of  $(-\infty, \infty)$ , consequently surjective. We see the function  $g(x) = \frac{x - \frac{a-b}{2}}{(\frac{a-b}{2})}$  describes merely a shift along with a scaling from  $(a, b)$  to  $(-1, 1)$ , thus it is bijective. And  $h(x) = f(g(x))$  is a bijective function (composition of bijective functions is bijective) which shows  $(a, b) \sim (-\infty, \infty)$ .
- b). We let  $g(x) = \frac{2}{x-a+1} - 1$  for  $x \in (a, \infty)$ . We will first show that the function is injective using contradiction. For  $x_1 \neq x_2$ , we assume  $f(x_1) = f(x_2)$ , we see

$$\begin{aligned} \frac{2}{x_1 - a + 1} - 1 &= \frac{2}{x_2 - a + 1} - 1 \\ \implies \frac{2}{x_1 - a + 1} &= \frac{2}{x_2 - a + 1} \\ \implies x_1 &= x_2 \end{aligned}$$

Consequently,  $g(x)$  is injective. We see that  $g(x)$  is continuous for  $x \in (a, \infty)$ , and  $\lim_{x \rightarrow a^+} g(x) = 1$  and  $\lim_{x \rightarrow \infty} g(x) = -1$ . Thus, using the intermediate value, we know that  $g$  is surjective to the co-domain  $(-1, 1)$ . We know that the composition of two bijective functions is bijective, thus, we define  $p(x) = g(h(x))$  using  $h(x)$  from part a). Finally this shows that  $(a, \infty) \sim \mathbb{R}$ .

$$c). f(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{x}{2} & x = \frac{1}{2^n} \text{ for } n \in \mathbb{N} \\ x & \text{else} \end{cases}$$

We see that in the case when  $f(x) = \frac{1}{2}$ , the only solution is  $x = 0$ . For  $f(x) = \frac{1}{2^n}$ , where  $n \in \mathbb{N}$ ,  $n \geq 2$ , the only solution is  $x = \frac{1}{2^{n-1}}$ . And for all  $f(x) \neq \frac{1}{2^n}$  for some  $n \in \mathbb{N}$ , the only solution is  $x = x$ . Consequently, we have shown that  $f(x)$  has exactly one solution for all  $x \in (0, 1)$ , thus it is bijective, and so  $[0, 1) \sim (0, 1)$ .

### Problem 5

- a). For every  $x \in A$ ,  $f(x) = x$  is a bijection from  $A$  to  $A$ . Consequently,  $A \sim A$ .
- b). Because  $A \sim B$ , we know that there exists a bijective function  $f(x)$  such that  $f: A \rightarrow B$ . And thus  $f^{-1}(x)$  is also bijective and  $f^{-1}: B \rightarrow A$ . Consequently,  $B \sim A$ .

- c). Because  $A \sim B$ , we know that there exists a bijective function  $f(x)$  such that  $f : A \rightarrow B$ . From  $B \sim C$ , we know that there exists a bijective function  $g(x)$  such that  $g : B \rightarrow C$ . The composition of two bijective functions with the co-domain of the first equivalent to the domain of the second is also bijective. Thus the bijective function  $h(x) = f(g(x))$  has  $h : A \rightarrow C$ . Consequently, we have  $A \sim C$ .

### Problem 6

- a).  $I_1 = (1, 2)$ ,  $I_2 = (2, 3)$ , and  $I_n = (n, n + 1)$ , for  $n \in \mathbb{N}$   
b).

### Problem 10

## Section 2.2

*Problems: 1, 2b*

### Problem 1

### Problem 2