Section 5.3

Problems: 3, 6a, 7

Problem 3

a). *Proof.* We set $g(x) \coloneqq h(x) - x$, and observe that it is continous on $[0, 3]$. We see that $g(0) = 1$ and $g(3) = -1$. Thus based on the Intermediate Value Theorem, there exists $d \in [0, 3]$, such that $g(d) = 0$, which means $h(d) - d = 0$, thus $h(d) = d$. $\square$

b). *Proof.* Because h is a differentiable function on $[0, 3]$, thus we can invoke the mean value theorem. That is, there exists $c \in (0, 3)$ such that $h'(c) = \frac{h(3)-h(0)}{3-0} = \frac{2-1}{3-0} = \frac{1}{3}$. $\square$

c). *Proof.* Similar to part b), using the mean value theorem, we see there exists $c \in (0, 1)$ such that $f'(c) = \frac{f(1)-f(0)}{1-0} = 1$. We see h is differentiable on $[1, 3]$ and we see that $h(1) = 2 = h(3)$. thus being fancy, we can utilise Rolle's theorem to show that there exists $d \in (1, 3)$ such that $f'(d) = 0$. Finally, we see that $0, \frac{1}{4} < 1$, and using Darboux's theorem, we know that there exists $L \in (c, d)$ such that $h'(c) = \frac{1}{4}$ (reminder: $(c, d) \subseteq [0, 3]$). $\square$

Problem 6

a). *Proof.* We know that for all $x \in [0, a]$, there exists $c \in (0, x)$ such that $g'(c) = \frac{g(x)-g(0)}{x-0} = \frac{g(x)}{x}$. And we know that $|g'(c)| \leq M$ for all $c \in [0, a]$, so $|\frac{g(x)}{x}| \leq M$. Because $x \in [0, a]$, we know that

$$\left|\frac{g'(x)}{x}\right| \leq M$$
$$\implies |g'(x)| \leq Mx$$

since $x > 0$. $\square$

Problem 7

Proof. We will use a indirect proof. Assume for contradiction that there exists more than one fixed points. That is, there exists a, b, which are elements of the interval such that $a \neq b$ and $f(a) = a$ and $f(b) = b$. We know that f is differentiable on the interval $[a, b]$(Notice: this also implies continuity). Thus we can invoke the Mean Value Theorem. That is there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1$$

However, we know that $f'(x) \neq 0$. Consequently, we have arrived at our contradiction. $\square$

Section 6.2

Problems: 2a, 8

Problem 2

a).

Claim. *f_n is continous at 0.*

Proof. Given $\epsilon > 0$, we let $\delta = \frac{1}{n}$. We see that $|x - 0| = |x| < \frac{1}{n} \implies |fn(0) - f_n(x)| = |0 - 0| < \epsilon$. Notice that we have chosen x in such a way such that $|x| < \frac{1}{n} \implies -\frac{1}{n} < x < \frac{1}{n}$, thus $f_n(x)$ always equals 0. □

Claim. *f is not continuous at 0.*

Proof. $f(x) = \lim_{n\to\infty} f_n(x)$. We let $(a_n) = \frac{1}{n}$ for $n \in \mathbb{N}$. We see that $(a_n) \to 0$, but $|f(0) - f(a_n)| = 1 > 0$. Thus we se $f(x_n)$ does not converge to $f(0)$, and by the Criterion for Discontinuity, we may conclude that f is not continuous at c. □

Claim. *f_n does not converge uniformly on $\mathbb{R}$.*

Proof. We will use a indirect proof. Assume for contradiction that $f_n \to f$ uniformly. Let (f_n) be a sequence of functions that converges to f. Using the contrapostive statement of the Continuous Limit Theorem, if f is not conitnous at 0, then there exists f_n is not continous at 0. However, we know that f_n is continous at 0 for all $n \in \mathbb{N}$. Thus we have arrived at our contradiction. □

Problem 8

Proof. □