

## Section 2.3

Problems: 1a, 5, 7ab

### Problem 1

- a). *Proof.* Because for all  $\epsilon > 0$ ,  $|x_n| < \epsilon$  and  $\epsilon^2 > 0$ , we let  $|x_n| < \epsilon^2$ . We see that  $x_n < \epsilon^2$  since  $x_n \geq 0$ . And consequently  $\sqrt{x_n} < \epsilon$  for all  $\epsilon > 0$ .  $\square$

### Problem 5

*Proof.* (forwards direction) We want to show  $(z_n)$  is convergent if  $(y_n)$  and  $(x_n)$  are both convergent with  $\lim x_n = \lim y_n$ . Notice that  $x_n = (z_1, z_3, z_5, \dots, z_{2n-1})$ , which is the odd numbered terms of  $z_n$ . And we see that  $y_n = (z_2, z_3, z_4, \dots, z_{2n})$ , which is the even numbered terms of  $z_n$ . For all  $\epsilon > 0$ , there exists  $N_1 \in \mathbb{N}$ , such that  $n \geq N_1 \implies |x_n - c| = |z_{2n-1} - c| < \epsilon$  for some  $c$ . And for all  $\epsilon > 0$ , there exists  $N_2 \in \mathbb{N}$ , such that  $n \geq N_2 \implies |y_n - c| = |z_{2n} - c| < \epsilon$ . We let  $N = \max(N_1, N_2)$ , and we know that both the even and odd terms of  $z_n$  converges, consequently,  $z_n$  is convergent.

(backwards direction) We want to show that  $(x_n)$  and  $(y_n)$  are both convergent with  $\lim x_n = \lim y_n$ . We see that  $(x_n) = (z_1, z_3, z_5, \dots, z_{2n-1})$  and  $(y_n) = (z_2, z_4, z_6, \dots, z_{2n})$ . We know for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $n \geq N \implies |z_n - c| < \epsilon$ . We know that  $2n - 1 \geq n$  and  $x = z_{2n-1}$ , consequently,  $|x_n - c| < \epsilon$ . Same reasoning,  $2n > n$ , and thus  $|y_n - c| < \epsilon$ .  $\square$

### Problem 7

- a). Let  $(x_n) = (1, -1, 1, \dots, (-1)^{n-1})$  for  $n \in \mathbb{N}$  and  $(y_n) = (-1, 1, -1, \dots, (-1)^n)$ . We see that  $x_n$  and  $y_n$  both diverges, and  $(x_n + y_n) = (0, 0, 0, \dots)$  converges.
- b). Because  $(x_n)$  and  $(x_n + y_n)$  converges. We see that  $(-x_n)$  converges, and thus  $(x_n + y_n) + (-x_n) = (y_n)$  converges. However, it was given that  $(y_n)$  diverges, consequently, the request is impossible by referencing the proper theorems.

## Section 2.4

Problems: 1abc, 2a

### Problem 1

- a). *Proof.* We can show that the sequence is Monotone convergent by showing that it is bounded and monotone. We will use induction to show that the sequence is monotone.

We see that  $x_1 = 3$  and  $x_2 = \frac{1}{4-3} = 1$ , and  $3 > 1$ . We assume that  $x_k < x_{k-1}$  and we need to show that  $x_{k+1} < x_k$ . We see

$$\begin{aligned}
 & x_{k-1} > x_k \\
 \implies & -x_{k-1} < -x_k \\
 \implies & 4 - x_{k-1} < 4 - x_k \\
 \implies & \frac{1}{4 - x_{k-1}} > \frac{1}{4 - x_k} \\
 \implies & x_k > x_{k+1}
 \end{aligned}$$

We have thus showed that  $(x_n)$  is decreasing for all  $n \in \mathbb{N}$ . Now we have to show that  $(x_n)$  is bounded. Because all terms are strictly decreasing, therefore no term can be greater than  $x_1 = 3$ . Notice that  $4 - x_n > 0$  if  $x_n < 4$ , and we know that  $x_1 = 3 < 4$ , so consequently,  $\frac{1}{4-x_n} > 0$  for all  $n \in \mathbb{N}$  (can be easily proven using induction). consequently,  $4 > x_n > 0$  and we see that  $|x_n| < 4$  for all  $n \in \mathbb{N}$ . Then, by the Montone convergent theorem, we know that the sequence defined by  $x_1 = 3$  and  $x_{n+1} = \frac{1}{4-x_n}$  is convergent.  $\square$

b). *Proof.* We know that  $\lim x_n$  exists, let it be  $x$ . Therefore, we know that for all  $\epsilon > 0$ , there exists a  $N \in \mathbb{N}$ , such that  $n \geq N \implies |x_n - x| < \epsilon$ . We know that  $n+1 > n \geq N$ , so consequently,  $|x_{n+1} - x| < \epsilon$ . And thus,  $\lim x_{n+1}$  exists and equals to the same value.  $\square$

c). We see

$$\begin{aligned}
 x_{n+1} &= \frac{1}{4 - x_n} \\
 \lim x_{n+1} &= \lim \left( \frac{1}{4 - x_n} \right) \\
 \lim x_{n+1} &= \frac{\lim(1)}{\lim(4 - x_n)} \\
 \lim x_{n+1} &= \frac{\lim(1)}{\lim(4 - x_n)}
 \end{aligned}$$

Because  $\lim x_n = \lim x_{n+1}$ , we let it be  $x$ . We have

$$\begin{aligned}x &= \frac{1}{4-x} \\x(4-x) &= 1 \\4x - x^2 &= 1 \\x^2 - 4x + 1 &= 0 \\x^2 - 4x + 4 &= 3 \\(x-2)^2 &= 3 \\x &= 2 \pm \sqrt{3}\end{aligned}$$

We know that  $x_n < 3$  and is decreasing, therefore  $\lim x_n = 2 - \sqrt{3}$ .

## Problem 2

- a). We see that we can express  $(y_n)$  explicitly as  $(x_n) = (1, 2, 1, 2, 1, \dots)$ , and it does not converge. Therefore this argument is wrong because both  $y_{n+1}$  and  $y_n$  does not converge to a value.

## Section 2.5

*Problems: 1ab*

### Problem 1

- a). The request is impossible, as we know that the subsequence of the subsequence is also a subsequence of the original sequence. And because the subsequence is bounded, therefore via the Bolzano-Weierstrass Theorem, we know that the subsequence contains a convergent subsequence, therefore the original sequence also contains that convergent subsequence.
- b). By definition, we know that that a sequence is a function whose domain is the natural numbers. Thus we let

$$f(x) = \begin{cases} \frac{1}{n} & n \text{ is odd}, n \in \mathbb{N} \\ 1 - \frac{1}{n} & n \text{ is even}, n \in \mathbb{N} \end{cases}$$

We see  $a_n = (1, \frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \dots)$ . the Subsequence, namely  $a_{n_k}$  where  $n_k = 2k - 1$  for  $k \in \mathbb{N}$  converges to 0. The subsequence of  $a_{n_k}$  where  $n_k = 2k$  for  $k \in \mathbb{N}$  converges to 1. And addition, we know that the original sequence diverges due to the Divergence criteria.