Section 2.3

Problems: 1a, 5, 7ab

Problem 1

a). *Proof.* Because for all $\epsilon > 0$, $|x_n| < \epsilon$ and $\epsilon^2 > 0$, we let $|x_n| < \epsilon^2$. We see that $x_n < \epsilon^2$ since $x_n \geq 0$. And consequently $\sqrt{x_n} < \epsilon$. □

Problem 5

Proof. (forwards direction) We want to show (z_n) is convergent if (y_n) and (x_n) are both convergent with $\lim x_n = \lim y_n$. Notice that $x_n = (z_1, z_3, z_5, \ldots, z_{2n-1})$, which is the odd numbered terms of z_n. And we see that $y_n = (z_2, z_3, z_4, \ldots, z_{2n})$, which is the even numbered terms of z_n. For all $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that $n \geq N_1 \implies |x_n - c| = |z_{2n-1} - c| < \epsilon$ for some c. And for all $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$, such that $n \geq N_2 \implies |y_n - c| = |z_{2n} - c| < \epsilon$. We let $N = \max(N_1, N_2)$, and we know that both the even and odd terms of z_n converges, conseqeuntly, z_n is convergent.

(backwards direction) We want to show that (x_n) and (y_n) are both convergent with $\lim x_n = \lim y_n$. We see that $(x_n) = (z_1, z_3, z_5, \ldots, z_{2n-1})$ and $(y_n) = (z_2, z_4, z_6, \ldots, z_{2n})$. We know for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N \implies |z_n - c| < \epsilon$. We know that $x_n = z_{2n-1} \geq z_n$, conseqeuntly, $|x_n - c| < \epsilon$. Same reasoning, $y_n = z_{2n} > z_n$, and thus $|y_n - c| < \epsilon$. □

Problem 7

a). Let $(x_n) = (1, -1, 1, \ldots, (-1)^{n-1})$ for $n \in \mathbb{N}$ and $(y_n) = (-1, 1, -1, \ldots. (-1)^n)$. We see that x_n and y_n both converges, and $(x_n + y_n) = (0, 0, 0, \ldots)$ diverges.

b). Because (x_n) and $(x_n + y_n)$ converges. We know that $(-x_n)$ converges, and thus $(x_n + y_n) + (-x_n) = (y_n)$ converges. However, it was given that (y_n) diverges, consequently, the request is impossible by referencing the proper theorems.

Section 2.4

Problems: 1abc, 2a

Problem 1

a). *Proof.* We can show that the sequence is Monotone convergent by showing that it is bounded and monotone. We will use induction to show that the sequence is monotone.

We see that $x_1 = 3$ and $x_2 = \frac{1}{4-3} = 1$, and $3 > 1$. We assume that $x_k > x_{k-1}$ and we need to show that $x_{k+1} > x_k$. We see

$$\begin{aligned} x_{n-1} &< x_n \\ \implies -x_{n-1} &> -x_n \\ \implies 4 - x_{n-1} &> 4 - x_n \\ \implies \frac{1}{4-x_{n-1}} &< \frac{1}{4-x_n} \\ \implies x_n &< x_{x+1} \end{aligned}$$

We have thus showed that (x_n) is decreasing for all $n \in \mathbb{N}$. Now we have to show that (x_n) is bounded. Because all terms are strictly decreasing, therefore no term can be greater than $x_1 = 3$. Notice that $4 - x_n > 0$ if $x_n < 4$, and we know that $x_1 = 3 < 4$, so conseqeuntly, $\frac{1}{4-x_n} > 0$ for all $n \in \mathbb{N}$ (can be easily proven using induction). conseqeuntly, $4 > x_n > 0$ and we see that $|x_n| < 4$ for all $n \in \mathbb{N}$. Then, by the Montone convergent theorem, we know that the sequence defined by $x_1 = 3$ and $x_{n+1} = \frac{1}{4-x_n}$ is convergent. □

b). *Proof.* We know that $\lim x_n$ exists, let it be x. Therefore, we know that for all $\epsilon > 0$, there exists a $N \in \mathbb{N}$, such that $n \geq N \implies |x_n - x| < \epsilon$. We know that $n+1 > n \geq N$, so conseqeuntly, $|x_{n+1} - x| < \epsilon$. And thus, $\lim x_{n+1}$ exists and equals to the same value. □

c). We see

$$\begin{aligned} x_{n+1} &= \frac{1}{4-x_n} \\ \lim x_{n+1} &= \lim\left(\frac{1}{4-x_n}\right) \\ \lim x_{n+1} &= \frac{\lim(1)}{\lim(4-x_n)} \\ \lim x_{n+1} &= \frac{\lim(1)}{\lim(4-x_n)} \end{aligned}$$

Because $\lim x_n = \lim x_{n+1}$, we let it be x. We have

$$\begin{aligned} x &= \frac{1}{4-x} \\ x(4-x) &= 1 \\ 4x - x^2 &= 1 \\ x^2 - 4x + 1 &= 0 \\ x^2 - 4x + 4 &= 3 \\ (x-2)^2 &= 3 \\ x &= 2 \pm \sqrt{3} \end{aligned}$$

We know that $x_n < 3$ and is decreasing, therefore $\lim x_n = 2 - \sqrt{3}$.

Problem 2

a). We see that we can express (y_n) explicitly as $(x_n) = (1, 2, 1, 2, 1, \dots)$, and it does not converge. Therefore this argument is wrong because both y_{n+1} and y_n does not converge to a value.

Section 2.5

Problems: 1ab

Problem 1

a). The request is impossible, as we know that the subsequence of the subsequence is also a subsequence of the original sequence. And because the subsequence is bounded, therefore via the Bolzano-Weierstrass Theorem, we know that the subsequence contains a convergent subsequence, therefore the orignal sequence also contains that convergent subsequence.

b). By definition, we know that that a sequence is a funciton whose domain is the natural numbers. Thus we let

$$f(x) = \begin{cases} \frac{1}{n} & \text{n is odd}, n \in \mathbb{N} \\ 1 - \frac{1}{n} & \text{n is even} n \in \mathbb{N} \end{cases}$$

We see $a_n = (1, 0, \frac{1}{3}, \frac{3}{4}, \dots)$. The subsequence of a_{n_k} where $n_k = 2k - 1$ for $k \in \mathbb{N}$ converges to 0, and The subsequence of a_{n_k} where $n_k = 2k$ for $k \in \mathbb{N}$ converges to 1, and