Moment Generating Functions

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1 What to know beforehand

- Taylor Series Approximations of Functions
- Probability Mass Function vs. Probability Distribution Function
- Finding Expected Values of Distributions

2 Introduction

Moment Generating Function is defined as

$$M_x[s] = E[e^{(sx)}]$$

Since we know the Taylor Series for e^x

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Then we know the Taylor Series for $e^{(sx)}$

$$e^{(sx)} = \sum_{k=0}^{\infty} \frac{(sx)^k}{k!}$$

Taking Taylor Series Sum of the MGF gives us

$$E[e^{(sx)}] = E[\sum_{k=0}^{\infty} \frac{(sx)^k}{k!}]$$

$$E[e^{(sx)}] = E[\sum_{k=0}^{\infty} \frac{s^k x^k}{k!}]$$

Since expected value is only affected by values of x

$$E[e^{(sx)}] = \sum_{k=0}^{\infty} \frac{s^k}{k!}(E[x^k])$$

Taking the Derivative of both sides with respect to s gives us

$$\frac{d^k}{ds^k}[M_x[s]] = \frac{d^k}{ds^k}[\sum_{k=0}^{\infty} \frac{s^k}{k!}(E[x^k])]$$

$$\frac{d^k}{ds^k}[M_x[s]] = \frac{d^k}{ds^k}[0 + \frac{s}{1}(E[x]) + \frac{s^2}{2}(E[x^2])...]$$

Since x is a constant when deriving with respect to s

$$\frac{d^k}{ds^k}[M_x[s]] = [0 + 0 + 0... + \frac{1 * k!}{k!}(E[x^k]) + \frac{s}{1}(E[x^{k+1}]) + \frac{s^2}{2}(E[x^{k+2}])]$$

If we set s = 0, since s is a placeholder variable $s \in R$

$$\frac{d^k}{ds^k}[M_x[s]] = [0 + 0 + 0... + (E[x^k]) + 0 + 0...]$$

This gives us 2 insights into the MGF

1. kth derivative of MGF gives us the kth "raw" moment
2. THE MGF is the sum total of all "raw" moments multiplied by a constant, specifically the taylor series coefficients

   Sidenote: If Y = X1 + X2 + X3 ... (Y, X1, X2, X3... are all random independent variables)

   $$M_Y[s] = E[e^{(sY)}] = E[e^{(s(X1+X1+X3...))}]$$
   $$M_Y[s] = E[e^{(sX1)}e^{(sX2)}e^{(sX3)}...]$$

   Since X1, X2, X3 are all independent variables

   $$M_Y[s] = E[e^{(sX1)}]E[e^{(sX2)}]E[e^{(sX3)}]...$$

   $$M_Y[s] = M_{X1}[s]M_{X2}[s]M_{X3}[s]...$$