European Field Experiments Summer School
Potential Outcomes

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- Defining treatment: The variable d_i indicates whether the ith subject is treated.
- $d_i = 1$ means the ith subject receives the treatment.
- $d_i = 0$ means the ith subject does not receive the treatment.
- It is assumed that d_i is observed for every subject.

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- In general, potential outcomes may be written $Y_i(d)$, where the argument d indexes the treatment.

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- $Y_i(1)$ is the potential outcome if the ith subject was treated.
- $Y_i(0)$ is the potential outcome if the ith subject was not treated.
- Potential outcomes are fixed attributes of each subject and represent the outcome that would be observed hypothetically if that subject were treated or untreated.

Conditional potential outcomes

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- $Y_i(d) \mid X = x$ denotes potential outcomes when the condition X = x holds.

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Treatments as random variables

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- We use D_i when talking about the statistical properties of treatments.

Example

- What's the effect of private tutoring on exam scores (ranging from 1 to 6)?

Full schedule of potential outcomes

	$Y_i(0)$	$Y_i(1)$
	Test score if	
subject i	not tutored	tutored
1	3	4.5
2	5	5
3	5	4.5
4	4.5	5
5	4	5.5
6	6	6

Definition of a subject-level treatment effect

- The individual level treatment effect τ_i for a given subject i is defined as:

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Full schedule of potential outcomes

subject i	$Y_i(0)$ Test score if not tutored	$Y_i(1)$ Test score if tutored	τ_i Treatment effect
1	3	4.5	1.5
2	5	5	0
3	5	4.5	-0.5
4	4.5	5	0.5
5	4	5.5	1.5
6	6	6	0

Definition of Average Treatment Effect

$\mu_{(Y(1))} - \mu_{(Y(0))}$,

where

$\mu_{(Y(1))}$ is the average value of $Y_i(1)$ for all subjects and
$\mu_{(Y(0))}$ is the average value of $Y_i(0)$ for all subjects.

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5	4	5.5	1.5
6	6	6	0
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3	5	4.5	-0.5
4	4.5	5	0.5
5	4	5.5	1.5
6	6	6	0
Average	4.58	5.08	0.5

Potential and observed outcomes

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- The $Y_i(1)s$ are observed for subjects who are treated, and the $Y_i(0)s$ are observed for subjects who are not treated. For any given subject, we observe either $Y_i(1)$ or $Y_i(0)$, never both at the same time.

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- The fact that we observe only one potential outcome is called the "fundamental problem of causal inference" (Holland 1986).
- A subject's treatment effect is unobserved.

Potential and observed outcomes

- The connection between the observed outcome Y_i and the underlying potential outcomes is given by the "switching equation":

$$Y_i = d_i Y_i(1) + (1 - d_i) Y_i(0)$$

Independence assumption

Treatment status is statistically independent of potential outcomes and background attributes (X):

$$D_i \perp\!\!\!\perp Y_i(0), Y_i(1), X$$

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If a subject is randomly assigned to treatment, knowing whether a subject is treated provides no information about the subject's potential outcomes, or background attributes.

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- The exclusion restriction breaks down if treatment assignment z_i sets in motion causes of Y_i other than the treatment d_i.

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- Each subject is unaffected by the treatments and assignments of other units.

Expectations

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where $Pr[X = x]$ denotes the probability that X takes on the value x, and where the summation is taken over all possible values of x.

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$$E[Y_i(1) \mid D_i = 1] = E[Y_i(1) \mid D_i = 0]$$

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$$= E[Y_i(1)] - E[Y_i(0)] = E[\tau_i] = ATE.$$

Estimator and estimand

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- Our data set is just one of many possible data sets that could have been created via random assignment. If we would redo the exact same random assignment procedure, different units would be allocated to treatment and control groups!
- The average estimated ATE across all possible random assignments is equal to the true ATE.
- On average we recover the true ATE. Our *estimator* is unbiased.

Observed Outcomes

subject i	$Y_i(0)$ Test score if not tutored	$Y_i(1)$ tutored	τ_i Treatment effect
1	?	4.5	?
2	5	?	?
3	?	4.5	?
4	4.5	?	?
5	4	?	?
6	?	6	?

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4	4.5	?	?
5	4	?	?
6	?	6	?
Average	4.5	5	

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4	4.5	?	?
5	4	?	?
6	?	6	?
Average	4.5	5	0.5

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3	?	4.5	?
4	4.5	?	?
5	?	5.5	?
6	6	?	?

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5	?	5.5	?
6	6	?	?
Average	5.17	4.83	

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3	?	4.5	?
4	4.5	?	?
5	?	5.5	?
6	6	?	?
Average	5.17	4.83	-0.32

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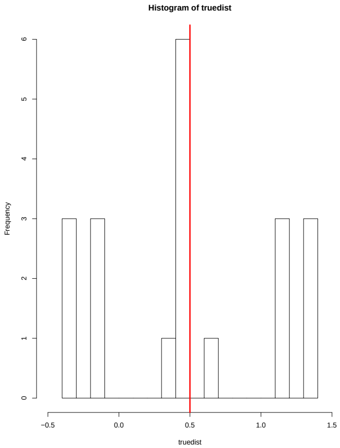
$$\frac{N!}{m!(N-m)!} = \frac{6!}{3!3!} = \frac{720}{36} = 20$$

ATEs

	Estimated ATEs	Frequency
1	-0.33	3
2	-0.17	3
3	0.33	1
4	0.50	6
5	0.67	1
6	1.17	3
7	1.33	3
	0.5	20

Sampling Distribution of estimated ATEs



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- How to:

Standard error

- Sampling distribution: The collection of estimates that could have been generated by every possible random assignment.
- Sampling distribution of the ATE: The collection of ATE estimates that could have been generated by every possible random assignment.
- The standard error is the standard deviation of the sampling distribution.
- How to: Calculate the squared deviation of each estimate from the average estimate, divide by the number of possible randomizations, and take the square root of the result.

Example

	Estimated ATEs	Frequency
1	-0.33	3
2	-0.17	3
3	0.33	1
4	0.50	6
5	0.67	1
6	1.17	3
7	1.33	3
	0.5	20

True standard error

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Sum of squared deviations =

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$(-0.33-0.5)^2+(-0.33-0.5)^2+(-0.33-0.5)^2+(-0.17-0.5)^2+(-0.17-0.5)^2+(-0.17-0.5)^2+(0.33-0.5)^2+(0.5-0.5)^2+(0.5-0.5)^2+(0.5-0.5)^2+(0.5-0.5)^2+(0.5-0.5)^2+(0.5-0.5)^2+(0.67-0.5)^2+(1.17-0.5)^2+(1.17-0.5)^2+(1.17-0.5)^2+(1.33-0.5)^2+(1.33-0.5)^2+(1.33-0.5)^2$

=

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$= 6.8846$

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Square root of average squared deviation =

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Square root of average squared deviation $= \sqrt{\frac{1}{20} * 6.8846} =$

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$= 6.8846$

Square root of average squared deviation $= \sqrt{\frac{1}{20} * 6.8846} = 0.587$

The true standard error using potential outcomes notation

$$SE(\widehat{ATE}) =$$

$$\sqrt{\frac{1}{N-1}\{\frac{mVar(Y_i(0))}{N-m} + \frac{(N-m)Var(Y_i(1))}{m} + 2Cov(Y_i(0), Y_i(1))\}}$$

where

$$Var(Y_i(1)) = \frac{1}{N}\sum_1^N (Y_i(1) - \frac{\sum_1^N Y_i(1)}{N})^2$$

$$Cov(Y_i(0), Y_i(1)) =$$

$$\frac{1}{N}\sum_1^N (Y_i(0) - \frac{\sum_1^N Y_i(0)}{N})\ (Y_i(1) - \frac{\sum_1^N Y_i(1)}{N})$$

Estimating the standard error

$$\widehat{SE} = \sqrt{\frac{\widehat{Var}(Y_i(0))}{N-m} + \frac{\widehat{Var}(Y_i(1))}{m}}$$

We don't know the covariance between $Y_i(0)$ and $Y_i(1)$, so we (conservatively) assume that the treatment effect is the same for all subjects (correlation between $Y_i(0)$ and $Y_i(1) = 1$).

Time for questions.