1 Deriving the gradient functions for stress

1.1 Definitions

- D, the table distance
- d, the map distance
- x, the x coordinate of the antigen.
- a, the x coordinate of the serum.
- y, the y coordinate of the antigen.
- b, the y coordinate of the serum.

1.2 Detectable titers

The stress (S) for a detectable titer is:

$$S = (D - \sqrt{((x-a)^2 + (y-b)^2)})^2 \tag{1}$$

This can be differentiated as follows:

$$\frac{\mathrm{d}S}{\mathrm{d}x}\left(D - \sqrt{((x-a)^2 + (y-b)^2)}\right)^2 \tag{2}$$

Using chain rule:

$$= 2(D - \sqrt{((x-a)^2 + (y-b)^2)}).(\frac{\mathrm{d}S}{\mathrm{d}x}\left(D - \sqrt{((x-a)^2 + (y-b)^2)}\right)) \tag{3}$$

Differentiating $D - \sqrt{((x-a)^2 + (y-b)^2)}$:

Lose the D (it's a constant).

$$-\sqrt{((x-a)^2 + (y-b)^2)} \tag{4}$$

Using the chain rule again:

$$\left(-0.5((x-a)^2 + (y-b)^2)^{-0.5}\right).(\frac{\mathrm{d}S}{\mathrm{d}x}\left((x-a)^2 + (y-b)^2\right)) \tag{5}$$

Differentiating $(x-a)^2 + (y-b)^2$:

Lose $(y-b)^2$ (it does not depend on x) leaves:

$$2(x-a) \tag{6}$$

Substituting back into equation 5 gives:

$$(-0.5((x-a)^2 + (y-b)^2)^{-0.5}).2(x-a) \tag{7}$$

This simplifies to:

$$-\frac{(x-a)}{\sqrt{((x-a)^2 + (y-b)^2)}} \tag{8}$$

Substituting back into equation 3 gives:

$$2(D - \sqrt{((x-a)^2 + (y-b)^2)}).\left(-\frac{(x-a)}{\sqrt{((x-a)^2 + (y-b)^2)}}\right) \tag{9}$$

Which simplifies to:

$$-\frac{2(x-a)\left(D-\sqrt{((x-a)^2+(y-b)^2)}\right)}{\sqrt{((x-a)^2+(y-b)^2)}} \tag{10}$$

Substituting in d and taking out $(x-a)$ we get:

$$-(x-a)\left(\frac{2(D-d)}{d}\right) \tag{11}$$

The second part being the `inc_base`.

1.3 Non-detectable titers

The stress (S) for a non-detectable titer is:

$$S = \delta^2 . f_\sigma(\delta) \tag{12}$$

Where δ is defined as:

$$\delta = D - \sqrt{((x-a)^2 + (y-b)^2)} + 1 \tag{13}$$

This can be differentiated as follows:

$$\frac{\mathrm{d}S}{\mathrm{d}x}(\delta)^2 . f_\sigma(\delta) \tag{14}$$

Using the product rule:

$$\delta^2 \left(f'_\sigma(\delta)\right) + f_\sigma(\delta)\left(\frac{\mathrm{d}S}{\mathrm{d}x}\delta^2\right) \tag{15}$$

Differentiating δ^2 :

$$\frac{\mathrm{d}S}{\mathrm{d}x}\left(D - \sqrt{((x-a)^2 + (y-b)^2)} + 1\right)^2 \tag{16}$$

Using the chain rule:

$$2(D - \sqrt{((x-a)^2 + (y-b)^2)} + 1).(\frac{\mathrm{d}S}{\mathrm{d}x}\left(D - \sqrt{((x-a)^2 + (y-b)^2)} + 1\right)) \tag{17}$$

Differentiating $D - \sqrt{((x-a)^2 + (y-b)^2)} + 1$:

Lose the D and the 1 (they are constants).

$$-\sqrt{((x-a)^2 + (y-b)^2)} \tag{18}$$

This has already been differentiated in equation 8 :

$$-\frac{(x-a)}{\sqrt{((x-a)^2 + (y-b)^2)}} \tag{19}$$

and substitute back into equation 17 :

$$2\left(D-\sqrt{((x-a)^2+(y-b)^2)}+1\right).\left(-\frac{(x-a)}{\sqrt{((x-a)^2+(y-b)^2)}}\right) \tag{20}$$

and substitute back into equation 15 :

$$\delta^2\left(f'_\sigma(\delta)\right)+f_\sigma(\delta)\left(2\left(D-\sqrt{((x-a)^2+(y-b)^2)}+1\right).\left(-\frac{(x-a)}{\sqrt{((x-a)^2+(y-b)^2)}}\right)\right) \tag{21}$$

And we can replace some with some δ and d:

$$\delta^2\left(f'_\sigma(\delta)\right)+f_\sigma(\delta)\left(2(\delta)\right).\left(-\frac{(x-a)}{d}\right) \tag{22}$$

Now going back to equation 15 we need to differentiate the sigma function:

$$f_\sigma(\delta)=\frac{1}{1+e^{-10\delta}} \tag{23}$$

Using the chain rule:

$$\left(-(1+e^{-10\sigma})^{-2}\right).(\left(\frac{\mathrm{d}S}{\mathrm{d}x}(-10\delta)\right)e^{-10\delta}) \tag{24}$$

This simplifies to:

$$\left(\frac{\mathrm{d}S}{\mathrm{d}x}(-10\delta)\right).\left(-(1+e^{-10\delta})^{-2}\right).\left(-e^{-10\delta}\right) \tag{25}$$

And further to:

$$\left(\frac{\mathrm{d}S}{\mathrm{d}x}(-10\delta)\right)\left(\frac{e^{-10\delta}}{(1+e^{-10\delta})^2}\right) \tag{26}$$

You can now do some rearranging to reach:

$$\left(\frac{\mathrm{d}S}{\mathrm{d}x}(-10\delta)\right)\left(\frac{1}{1+e^{-10\delta}}\right)\left(\frac{e^{-10\delta}}{1+e^{-10\delta}}\right) \tag{27}$$

And some more rearranging to reach:

$$\left(\frac{\mathrm{d}S}{\mathrm{d}x}(-10\delta)\right)\left(\frac{1}{1+e^{-10\delta}}\right)\left(\frac{(1+e^{-10\delta})-1}{1+e^{-10\delta}}\right) \tag{28}$$

and:

$$\left(\frac{\mathrm{d}S}{\mathrm{d}x}(-10\delta)\right)\left(\frac{1}{1+e^{-10\delta}}\right)\left(\frac{1+e^{-10\delta}}{1+e^{-10\delta}}-\frac{1}{1+e^{-10\delta}}\right) \tag{29}$$

and:

$$\left(\frac{\mathrm{d}S}{\mathrm{d}x}(-10\delta)\right)\left(\frac{1}{1+e^{-10\delta}}\right)\left(1-\frac{1}{1+e^{-10\delta}}\right) \tag{30}$$

substituting back in $f_\sigma(\delta)$ you get:

$$\left(\frac{\mathrm{d}S}{\mathrm{d}x}(-10\delta)\right)(f_\sigma(\delta))\,(1-f_\sigma(\delta)) \tag{31}$$

Now differentiating -10δ :

$$\frac{\mathrm{d}S}{\mathrm{d}x}(-10\delta)=\frac{\mathrm{d}S}{\mathrm{d}x}-10(D-\sqrt{((x-a)^2+(y-b)^2)}+1) \tag{32}$$

Again we can lose the D and the 1 and we get a similar result to equation 19:

$$10\left(\frac{(x-a)}{\sqrt{((x-a)^2+(y-b)^2)}}\right) \tag{33}$$

Substituting back into equation 31 :

$$\left(10\left(\frac{(x-a)}{\sqrt{((x-a)^2+(y-b)^2)}}\right)\right)(f_\sigma(\delta))\,(1-f_\sigma(\delta)) \tag{34}$$

Substitute d and $f'_\sigma(\delta)$:

$$\left(10\left(\frac{(x-a)}{d}\right)\right)(f'_\sigma(\delta)) \tag{35}$$

Finally substituting everything back into equation 22 :

$$\delta^2\left(\left(10\left(\frac{(x-a)}{d}\right)\right)(f'_\sigma(\delta))\right)+f_\sigma(\delta)\,(2(\delta))\,.\left(-\frac{(x-a)}{d}\right) \tag{36}$$

And simplify :

$$-\frac{(x-a)}{d}\left(\delta^2\,(10(f'_\sigma(\delta)))+f_\sigma(\delta)\,(2(\delta))\right) \tag{37}$$

And simplify :

$$-\frac{(x-a)}{d}\left(10\delta^2(f'_\sigma(\delta))+f_\sigma(\delta)\,(2(\delta))\right) \tag{38}$$

And finally we end up with :

$$-(x-a)\left(\frac{10\delta^2.f'_\sigma(\delta)+2\delta.f_\sigma(\delta)}{d}\right) \tag{39}$$

Again the second part being the `inc_base`.

1.4 Introducing weights

1.4.1 For detectable titers

When introducing weights for detectable titers, the stress function simply changes from:

$$S = \left(D - \sqrt{((x-a)^2 + (y-b)^2)}\right)^2 \tag{40}$$

to :

$$S = \boldsymbol{w}\left(D - \sqrt{((x-a)^2 + (y-b)^2)}\right)^2 \tag{41}$$

When we differentiate it we simple find

$$\frac{\mathrm{d}S}{\mathrm{d}x}\boldsymbol{w}\left(D - \sqrt{((x-a)^2 + (y-b)^2)}\right)^2 \tag{42}$$

When we then use the chain rule we find:

$$= 2\boldsymbol{w}(D - \sqrt{((x-a)^2 + (y-b)^2)}).(\frac{\mathrm{d}S}{\mathrm{d}x}\left(D - \sqrt{((x-a)^2 + (y-b)^2)}\right)) \tag{43}$$

Instead of the original:

$$= 2(D - \sqrt{((x-a)^2 + (y-b)^2)}).(\frac{\mathrm{d}S}{\mathrm{d}x}\left(D - \sqrt{((x-a)^2 + (y-b)^2)}\right)) \tag{44}$$

Everything else happens as before and we end up with:

$$-(x-a)\boldsymbol{w}\left(\frac{2(D-d)}{d}\right) \tag{45}$$

So to implement weights you would simply need to multiply `inc_base` by w.

1.4.2 For non detectable titers

It is the same story, the stress function changes from:

$$S = \delta^2.f_\sigma(\delta) \tag{46}$$

to :

$$S = \boldsymbol{w}\delta^2.f_\sigma(\delta) \tag{47}$$

When we use the product rule instead of finding:

$$\delta^2\left(f'_\sigma(\delta)\right) + f_\sigma(\delta)\left(\frac{\mathrm{d}S}{\mathrm{d}x}\delta^2\right) \tag{48}$$

We can find:

$$\boldsymbol{w}\delta^2\left(f'_\sigma(\delta)\right) + f_\sigma(\delta)\left(\frac{\mathrm{d}S}{\mathrm{d}x}\boldsymbol{w}\delta^2\right) \tag{49}$$

As before, when we find $\frac{\mathrm{d}S}{\mathrm{d}x}\boldsymbol{w}\delta^2$, the $\boldsymbol{w}$ remains outside, in detail:

Differentiating $\boldsymbol{w}\delta^2$:

$$\frac{\mathrm{d}S}{\mathrm{d}x}\boldsymbol{w}\left(D - \sqrt{((x-a)^2 + (y-b)^2)} + 1\right)^2 \tag{50}$$

Using the chain rule:

$$2\boldsymbol{w}(D - \sqrt{((x-a)^2 + (y-b)^2)} + 1).(\frac{\mathrm{d}S}{\mathrm{d}x}\left(D - \sqrt{((x-a)^2 + (y-b)^2)} + 1\right)) \tag{51}$$

and again everything comes out the same otherwise.

When we combine everything again, the $\boldsymbol{w}$ in both sides factors out and we end up with:

$$-(x-a)\boldsymbol{w}\left(\frac{10\delta^2.f'_\sigma(\delta) + 2\delta.f_\sigma(\delta)}{d}\right) \tag{52}$$

1.5 Generalising

Here we've shown finding the gradient with regard to the x coordinate of the antigens, I hope it is pretty evident that the same working applies for y coordinates. In the case of sera, and differentiating with respect to a or b, you end up with the same but gradients but without the minus. since here we are doing antigen position minus serum position.