1 Deep Inelastic Scattering Kinematics

1.1 Conventions and Basic Relations

We choose the direction of the hadron beam as the $+z$ direction, with the lepton beam traveling in the opposite direction. The initial hadron and electron energies are E_{h} and E_ℓ, respectively. The scattering angle of the lepton is denoted by θ ($\theta = \pi$ for the initial lepton direction), and the angle of the final state hadronic system is denoted by γ ($\gamma = 0$ for the initial hadron direction). We denote the momentum four-vectors of the initial lepton and the inital hadron as k and P, respectively, and the scattered four-momenta as k' and P'. The final state energies of the lepton and hadronic system are denote E'_ℓ and E'_{h}, respectively.

The squared center-of-mass energy, s, is given by:

$$s = 4E_\ell E_{\mathrm{h}} \tag{1}$$

The negative squared four-momentum transferred in deep inelastic scattering (DIS) between the lepton and hadron is

$$Q^2 \equiv -q^2 = -(k - k')^2 = sxy \tag{2}$$

where x is the Bjorken scaling variable and y is the fractional energy transfer from the lepton, defined as:

$$x = \frac{Q^2}{2P \cdot q} \tag{3}$$

$$y = \frac{P \cdot q}{P \cdot k} \tag{4}$$

1.2 Lepton Variables

The DIS variables using the scattered lepton variables are given by:

$$Q^2 = 2E_\ell E'_\ell (1 + \cos\theta) \tag{5}$$

$$y = 1 - \frac{E'_\ell}{2E_\ell} (1 - \cos\theta) \tag{6}$$

The variable x can be found from Eq.(2).

An expression for Q^2 as a function of x based on the scattered lepton energy E'_ℓ can be found by solving for $\cos\theta$ in Eq.(5) and plugging into Eq.(6), then using Eq.(2):

$$Q^2 = \frac{1 - E'_\ell/E_\ell}{\frac{1}{sx} - \frac{1}{4E_\ell^2}} \tag{7}$$

Likewise, an expression for Q^2 as a function of x based on the scattered lepton angle θ can be found by solving for E'_ℓ in Eq.(5) and plugging into plugging into Eq.(6), then using Eq.(2):

$$Q^2 = \frac{s}{\frac{1}{x} + \frac{E_\mathrm{h}}{E_\ell}(\frac{1-\cos\theta}{1+\cos\theta})} = \frac{sx}{1 + x\,\frac{E_\mathrm{h}}{E_\ell}\,\tan^2(\frac{\theta}{2})} \tag{8}$$

Graphs showing the dependence of Q^2 vs. x for various choices of the scattered lepton energy and pseudorapidity are shown in the left plots of Figs. 1, 2, and 3 for different machine configurations.

We can invert Eq. (7) to find E'_ℓ in terms of Q^2 and x:

$$E'_\ell = E_\ell\left[1 - \frac{Q^2}{4E_\ell}\left(\frac{1}{xE_\mathrm{h}} - \frac{1}{E_\ell}\right)\right] \tag{9}$$

and also invert Eq. (8) to obtain θ (or η) in terms of Q^2 and x:

$$e^{-\eta} = \tan\frac{\theta}{2} = \sqrt{\frac{4E_\ell^2}{Q^2} - \frac{E_\ell}{xE_\mathrm{h}}} \tag{10}$$

When the second term in the radical is negligible (for all but the smallest x values), the scattered lepton angle depends only on Q^2:

$$\eta \approx -\ln\left(\frac{2E_\ell}{\sqrt{Q^2}}\right) \tag{11}$$

1.3 Hadron variables

The DIS variables using the scattered hadronic variables are given by:

$$Q^2 = \frac{E'^2_\mathrm{h}\sin^2\gamma}{1-y} \tag{12}$$

$$y = \frac{E'_{\rm h}}{2E_\ell}\,(1-\cos\gamma) \tag{13}$$

These variables can be found from the measured hadronic energy flow via the Jacquet-Blondel method:

$$Q^2 = \frac{(\sum_i p_{{\rm T},i})^2}{1-y} \tag{14}$$

$$y = \frac{\sum_i (E_{{\rm h},i} - p_{z,i})}{2E_\ell} \tag{15}$$

In other words,

$$\sin\gamma = \frac{\sum_i p_{{\rm T},i}}{E'_{\rm h}} \tag{16}$$

The variable x can be found from Eq.(2).

An expression for Q^2 as a function of x based on the scattered hadron energy $E'_{\rm h}$ can be found by solving for $\cos\gamma$ in Eq.(13) and plugging into Eq.(12), expanding out $\sin^2\gamma = (1+\cos\gamma)(1-\cos\gamma)$, then using Eq.(2):

$$Q^2 = \frac{sx\,(1-\frac{E'_{\rm h}}{x\,E_{\rm h}})}{(1-\frac{E_\ell}{x\,E_{\rm h}})} \tag{17}$$

Likewise, an expression for Q^2 as a function of x based on the scattered hadronic angle γ can be found by solving for $E'_{\rm h}$ in Eq.(13) and plugging into plugging into Eq.(12), then using Eq.(2):

$$Q^2 = \frac{sx}{1+\frac{4E_\ell^2}{sx}\frac{(1+\cos\gamma)}{(1-\cos\gamma)}} = \frac{sx}{1+\frac{E_\ell}{x\,E_{\rm h}}\cot^2(\frac{\gamma}{2})} \tag{18}$$

Graphs showing the dependence of Q^2 vs. x for various choices of the scattered hadron energy and pseudorapidity are shown in the right plots of Figs. 1, 2, and 3 for different machine configurations.

We can invert Eq. (17) to find $E'_{\rm h}$ in terms of Q^2 and x:

$$E'_{\rm h} = xE_{\rm h} + \frac{Q^2}{4E_\ell}\left(\frac{E_\ell}{xE_{\rm h}} - 1\right) \tag{19}$$

which for $x \ll E_\ell/E_{\rm h}$ is approximated by:

$$E'_{\rm h} = yE_\ell \tag{20}$$

We can also invert Eq. (18) to obtain γ (or η) in terms of Q^2 and x:

$$e^{-\eta} = \tan\frac{\gamma}{2} = \frac{1}{\sqrt{\frac{4E_\mathrm{h}^2 x^2}{Q^2} - x\frac{E_\mathrm{h}}{E_\ell}}} \tag{21}$$

which again for $x \ll E_\ell/E_\mathrm{h}$ can approximated by:

$$\eta \approx -\ln\left(\frac{\sqrt{Q^2}}{2E_\mathrm{h}\, x}\right) \tag{22}$$

1.3.1 Σ and T variables

A variation of the hadron method is to define the Σ and T variables:

$$\Sigma = \sum_i (E_{\mathrm{h},i} - p_{z,i}) = 2E_\ell\, y \tag{23}$$

$$T = \sqrt{(\sum_i p_{x,i})^2 + (\sum_i p_{y,i})^2} = E'_\mathrm{h}\sin\gamma \tag{24}$$

which again are sums over the hadron energies and momenta. The variable T is essentialy the transverse momentum (p_T) of the hadron system (i.e. jet from the struck parton). From these definitions we can find Q^2 vs. x in terms of Σ and T:

$$Q^2 = 2E_\mathrm{h}\, x\, \Sigma \tag{25}$$

and using the quadratic equation:

$$Q^2 = \frac{sx}{2}\left(1 \pm \sqrt{1 - \frac{4T^2}{sx}}\right) \tag{26}$$

Graphs showing the dependence of Q^2 vs. x for various choices of Σ and T are shown in Fig. 4. Since it can be shown that:

$$T^2 = (1-y)\, Q^2 \tag{27}$$

$T \approx Q$ for small y.

1.4 Kinematic Resolutions

1.4.1 Q^2, Lepton Variables

In the far backward region, $\theta \approx \pi$, we can express the relation for Q^2 in terms of the lepton angle from the $-z$ axis, where $\alpha = \pi - \theta$:

$$Q^2 = 2E_\ell E'_\ell \,(1 - \cos\alpha) \tag{28}$$

From this relation, the relative resolution on Q^2 in terms of the relative resolution on the measured scattered lepton energy E', assuming negligible uncertainty on the scattered angle, is given by:

$$\frac{dQ^2}{Q^2} = \frac{dE'_\ell}{E'_\ell} \tag{29}$$

The dependence of the relative uncertainty on Q^2 on the uncertainty of the measured scattering angle for small α (θ near π) can be found from $\cos\alpha \approx 1 - \alpha^2/2$, such that:

$$Q^2 \approx E\, E'_\ell\, \alpha^2 \tag{30}$$

This implies:

$$\frac{dQ^2}{Q^2} = 2\frac{d\alpha}{\alpha} \tag{31}$$

1.4.2 y, Lepton Variables

We can express the relation for y in terms of the lepton angle from the $-z$ axis, where $\alpha = \pi - \theta$:

$$y = 1 - \frac{E'_\ell}{2E_\ell}\,(1 + \cos\alpha) \tag{32}$$

We can then express the relative uncertainty on $(1-y)$ in terms of the relative resolution on the measured scattered lepton energy E' as:

$$\frac{d(1-y)}{1-y} = \frac{dE'_\ell}{E'_\ell} \tag{33}$$

Or put another way:

$$dy = \frac{dE'_\ell}{E'_\ell}(1-y) \tag{34}$$

This implies that the reconstruction of y from the scattered lepton energy is good for large y, but degrades and reaches 100% uncertainty for $y \approx dE'_\ell/E'_\ell$.

For the dependence of the uncertainty on $1-y$ from the uncertainty on the measured scattering angle α we get:

$$\frac{d(1-y)}{1-y} \approx \frac{\alpha\, d\alpha}{(2-\alpha^2/2)} \approx \frac{1}{2}\alpha\, d\alpha \tag{35}$$

1.4.3 y, Hadron Variables

From Eq.(13) we can then express the relative uncertainty on y in terms of the relative resolution on the measured scattered hadron energy E'_h as:

$$\frac{dy}{y} = \frac{dE'_\mathrm{h}}{E'_\mathrm{h}} \tag{36}$$

For small angles from the far backward region, $\gamma \approx \pi$, we can express the relation for y in terms of the hadron angle from the $-z$ axis, where $\beta = \pi - \gamma$:

$$y = \frac{E'_\mathrm{h}}{2E_\ell}\,(1+\cos\beta) \approx \frac{E'_\mathrm{h}}{2E_\ell}\,(2-\beta^2/2) \tag{37}$$

The dependence of the relative uncertainty on y on the uncertainty of the measured scattering angle for small β is therefore:

$$\frac{dy}{y} = \frac{\beta\, d\beta}{(2-\beta^2/2)} \tag{38}$$

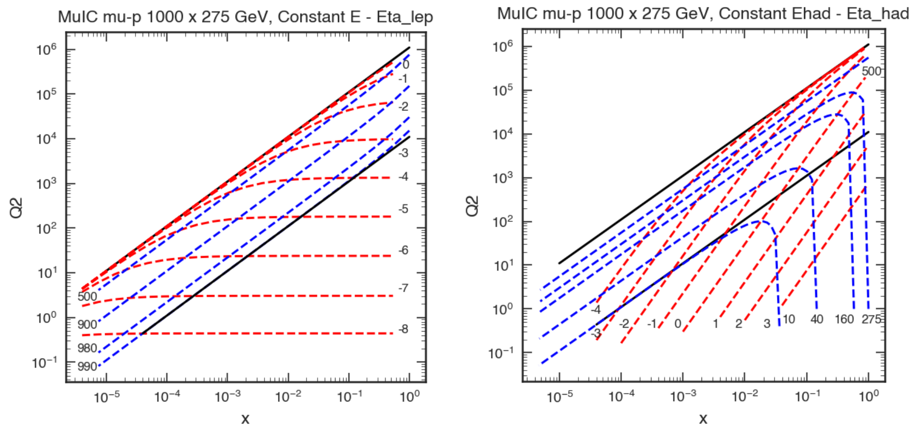

Figure 1: Kinematics of the scattered muon (left) and final-state hadrons (right) in the Q^2-x plane for muon-proton deep inelastic scattering with a 1000 GeV muon beam colliding with a 275 GeV proton beam. The dashed blue lines correspond to constant energy in GeV and the dashed red lines to constant pseudorapidity, respectively. The lower diagonal solid line corresponds to the inelasticity $y = 0.01$.

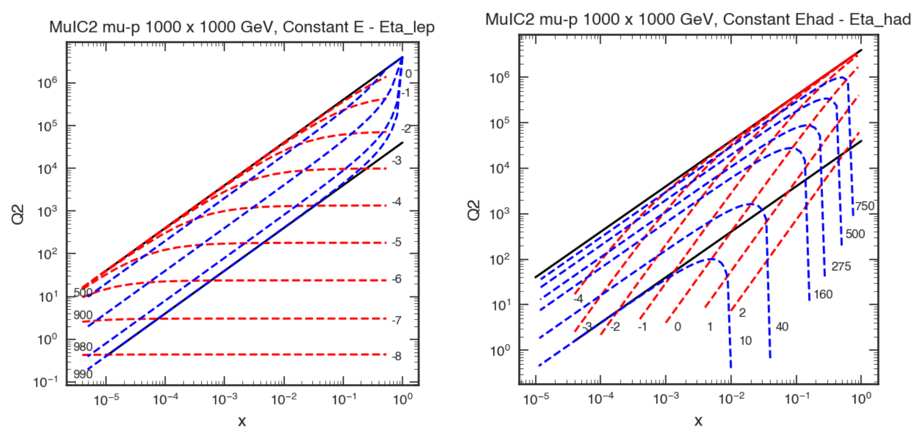

Figure 2: Kinematics of the scattered muon (left) and final-state hadrons (right) in the Q^2-x plane for muon-proton deep inelastic scattering with a 1000 GeV muon beam colliding with a 1000 GeV proton beam. The dashed blue lines correspond to constant energy in GeV and the dashed red lines to constant pseudorapidity, respectively. The lower diagonal solid line corresponds to the inelasticity $y = 0.01$.

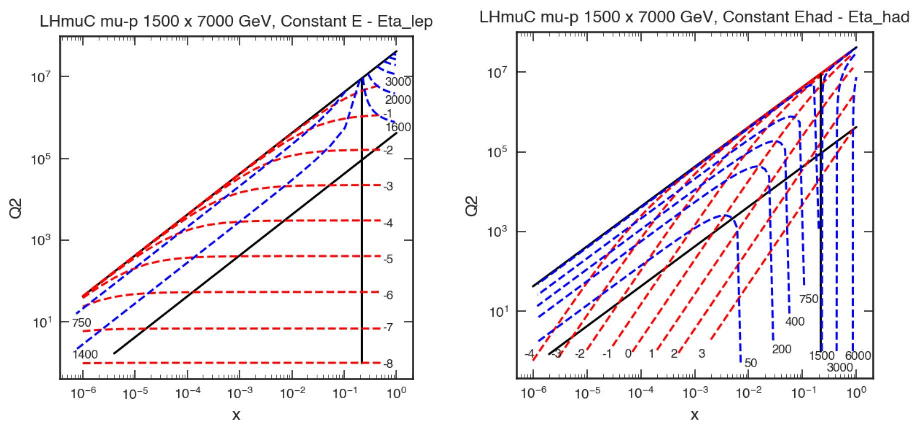

Figure 3: Kinematics of the scattered muon (left) and final-state hadrons (right) in the Q^2-x plane for muon-proton deep inelastic scattering with a 1500 GeV muon beam colliding with a 7000 GeV proton beam. The dashed blue lines correspond to constant energy in GeV and the dashed red lines to constant pseudorapidity, respectively. The lower diagonal solid line corresponds to the inelasticity $y = 0.01$.

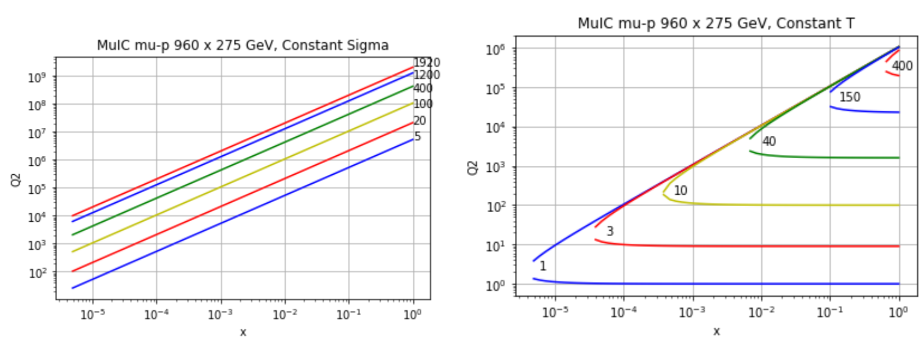

Figure 4: Σ (left) and T (right) in the Q^2-x plane for muon-proton deep inelastic scattering with a 1000 GeV muon beam colliding with a 275 GeV proton beam.