1 Deep Inelastic Scattering Kinematics

1.1 Conventions and Basic Relations

We choose the direction of the hadron beam as the $+z$ direction, with the lepton beam traveling in the opposite direction. The initial hadron and electron energies are E_{h} and E_ℓ, respectively. The scattering angle of the lepton is denoted by θ ($\theta = \pi$ for the initial lepton direction), and the angle of the final state hadronic system is denoted by γ ($\gamma = 0$ for the initial hadron direction). We denote the momentum four-vectors of the initial lepton and the inital hadron as k and P, respectively, and the scattered four-momenta as k' and P'. The final state energies of the lepton and hadronic system are denote E'_ℓ and E'_{rmh}, respectively.

The squared center-of-mass energy, s, is given by:

$$s = 4E_\ell E_{\mathrm{h}} \tag{1}$$

The negative squared four-momentum transferred in deep inelastic scattering (DIS) between the lepton and hadron is

$$Q^2 \equiv -q^2 = -(k - k')^2 = sxy \tag{2}$$

where x is the Bjorken scaling variable and y is the fractional energy transfer from the lepton, defined as:

$$x = \frac{Q^2}{2P \cdot q} \tag{3}$$

$$y = \frac{P \cdot q}{P \cdot k} \tag{4}$$

1.2 Lepton Variables

The DIS variables using the scattered lepton variables are given by:

$$Q^2 = 2E_\ell E'_\ell \,(1 + \cos\theta) \tag{5}$$

$$y = 1 - \frac{E'_\ell}{2E_\ell}\,(1 - \cos\theta) \tag{6}$$

The variable x can be found from Eq.(2).

An expression for Q^2 as a function of x based on the scattered lepton energy E'_ℓ can be found by solving for $\cos\theta$ in Eq.(5) and plugging into Eq.(6), then using Eq.(2):

$$Q^2 = \frac{1 - E'_\ell/E_\ell}{\frac{1}{sx} - \frac{1}{4E_\ell^2}} \tag{7}$$

Likewise, an expression for Q^2 as a function of x based on the scattered lepton angle θ can be found by solving for E'_ℓ in Eq.(5) and plugging into plugging into Eq.(6), then using Eq.(2):

$$Q^2 = \frac{s}{\frac{1}{x} + \frac{E_\mathrm{h}}{E_\ell}(\frac{1-\cos\theta}{1+\cos\theta})} = \frac{sx}{1 + x\,\frac{E_\mathrm{h}}{E_\ell}\,\tan^2(\frac{\theta}{2})} \tag{8}$$

We can invert Eq. (7) to find E'_ℓ in terms of Q^2 and x:

$$E'_\ell = E_\ell \left[1 - \frac{Q^2}{4E_\ell}\left(\frac{1}{xE_\mathrm{h}} - \frac{1}{E_\ell}\right)\right] \tag{9}$$

and also invert Eq. (8) to obtain θ (or η) in terms of Q^2 and x:

$$e^{-\eta} = \tan\frac{\theta}{2} = \sqrt{\frac{4E_\ell^2}{Q^2} - \frac{E_\ell}{xE_\mathrm{h}}} \tag{10}$$

When the second term in the radical is negligible (for all but the smallest x values), the scattered lepton angle depends only on Q^2:

$$\eta \approx -\ln\left(\frac{2E_\ell}{\sqrt{Q^2}}\right) \tag{11}$$

1.3 Hadron variables

The DIS variables using the scattered hadronic variables are given by:

$$Q^2 = \frac{E'^2_\mathrm{h}\sin^2\gamma}{1-y} \tag{12}$$

$$y = \frac{E'_\mathrm{h}}{2E_\ell}\,(1-\cos\gamma) \tag{13}$$

These variables can be found from the measured hadronic energy flow via the Jacquet-Blondel method:

$$Q^2 = \frac{(\sum_i p_{\mathrm{T},i})^2}{1-y} \tag{14}$$

$$y = \frac{\sum_i (E_{\mathrm{h},i} - p_{z,i})}{2E_\ell} \tag{15}$$

In other words,

$$\sin\gamma = \frac{\sum_i p_{\mathrm{T},i}}{E'_{\mathrm{h}}} \tag{16}$$

The variable x can be found from Eq.(2).

An expression for Q^2 as a function of x based on the scattered hadron energy E'_{h} can be found by solving for $\cos\gamma$ in Eq.(13) and plugging into Eq.(12), expanding out $\sin^2\gamma = (1+\cos\gamma)(1-\cos\gamma)$, then using Eq.(2):

$$Q^2 = \frac{sx\,(1 - \frac{E'_{\mathrm{h}}}{x\,E_{\mathrm{h}}})}{(1 - \frac{E_\ell}{x\,E_{\mathrm{h}}})} \tag{17}$$

Likewise, an expression for Q^2 as a function of x based on the scattered hadronic angle γ can be found by solving for E'_{h} in Eq.(13) and plugging into plugging into Eq.(12), then using Eq.(2):

$$Q^2 = \frac{sx}{1 + \frac{4E_\ell^2}{sx}\frac{(1+\cos\gamma)}{(1-\cos\gamma)}} = \frac{sx}{1 + \frac{E_\ell}{x\,E_{\mathrm{h}}}\cot^2(\frac{\gamma}{2})} \tag{18}$$

We can invert Eq. (17) to find E'_{h} in terms of Q^2 and x:

$$E'_{\mathrm{h}} = xE_{\mathrm{h}} + \frac{Q^2}{4E_\ell}\left(\frac{E_\ell}{xE_{\mathrm{h}}} - 1\right) \tag{19}$$

which for $x \ll E_\ell/E_{\mathrm{h}}$ is approximated by:

$$E'_{\mathrm{h}} = yE_\ell \tag{20}$$

We can also invert Eq. (18) to obtain γ (or η) in terms of Q^2 and x:

$$e^{-\eta} = \tan\frac{\gamma}{2} = \frac{1}{\sqrt{\frac{4E_{\mathrm{h}}^2x^2}{Q^2} - x\frac{E_{\mathrm{h}}}{E_\ell}}} \tag{21}$$

which again for $x \ll E_\ell/E_{\mathrm{h}}$ can approximated by:

$$\eta \approx -\ln\left(\frac{\sqrt{Q^2}}{2E_{\mathrm{h}}\,x}\right) \tag{22}$$

1.4 Kinematic Resolutions

1.4.1 Q^2, Lepton Variables

In the far backward region, $\theta \approx \pi$, we can express the relation for Q^2 in terms of the lepton angle from the $-z$ axis, where $\alpha = \pi - \theta$:

$$Q^2 = 2E_\ell E'_\ell \,(1 - \cos\alpha) \tag{23}$$

From this relation, the relative resolution on Q^2 in terms of the relative resolution on the measured scattered lepton energy E', assuming negligible uncertainty on the scattered angle, is given by:

$$\frac{dQ^2}{Q^2} = \frac{dE'_\ell}{E'_\ell} \tag{24}$$

The dependence of the relative uncertainty on Q^2 on the uncertainty of the measured scattering angle for small α (θ near π) can be found from $\cos\alpha \approx 1 - \alpha^2/2$, such that:

$$Q^2 \approx E\, E'_\ell\, \alpha^2 \tag{25}$$

This implies:

$$\frac{dQ^2}{Q^2} = 2\frac{d\alpha}{\alpha} \tag{26}$$

1.4.2 y, Lepton Variables

We can express the relation for y in terms of the lepton angle from the $-z$ axis, where $\alpha = \pi - \theta$:

$$y = 1 - \frac{E'_\ell}{2E_\ell}\,(1 + \cos\alpha) \tag{27}$$

We can then express the relative uncertainty on $(1-y)$ in terms of the relative resolution on the measured scattered lepton energy E' as:

$$\frac{d(1-y)}{1-y} = \frac{dE'_\ell}{E'_\ell} \tag{28}$$

Or put another way:

$$dy = \frac{dE'_\ell}{E'_\ell}(1-y) \tag{29}$$

This implies that the reconstruction of y from the scattered lepton energy is good for large y, but degrades and reaches 100% uncertainty for $y \approx dE'_\ell/E'_\ell$.

For the dependence of the uncertainty on $1-y$ from the uncertainty on the measured scattering angle α we get:

$$\frac{d(1-y)}{1-y} \approx \frac{\alpha\, d\alpha}{(2-\alpha^2/2)} \approx \frac{1}{2}\alpha\, d\alpha \tag{30}$$

1.4.3 y, Hadron Variables

From Eq.(13) we can then express the relative uncertainty on y in terms of the relative resolution on the measured scattered hadron energy E'_h as:

$$\frac{dy}{y} = \frac{dE'_\mathrm{h}}{E'_\mathrm{h}} \tag{31}$$

For small angles from the far backward region, $\gamma \approx \pi$, we can express the relation for y in terms of the hadron angle from the $-z$ axis, where $\beta = \pi - \gamma$:

$$y = \frac{E'_\mathrm{h}}{2E_\ell}\,(1+\cos\beta) \approx \frac{E'_\mathrm{h}}{2E_\ell}\,(2-\beta^2/2) \tag{32}$$

The dependence of the relative uncertainty on y on the uncertainty of the measured scattering angle for small β is therefore:

$$\frac{dy}{y} = \frac{\beta\, d\beta}{(2-\beta^2/2)} \tag{33}$$