Newton's Second Law and Momentum

- Newton's Second Law: $\vec{F} = m\vec{a}$

  can also be expressed as: $\vec{F} = \dfrac{d\vec{p}}{dt}$

  where linear momentum $\vec{p} = m\vec{u}$ and velocity $\vec{u} = \dfrac{d\vec{x}}{dt}$

We use ***u*** to denote the velocity of an object in a reference frame so as not to confuse it with the velocity ***v*** between one frame and another

- But *x* and *t* transform differently under Lorentz transformations from one reference frame to another.
  - Therefore the <u>form</u> of the equation for Newton's Law would change from one frame to another. (Would not look the same).
  - But the law of physics should look the same from one inertial frame to another!
  - This is another more subtle aspect of Relativity.
- We need a modified form of Newton's Second Law

Relativistic Momentum

- Let's define momentum as: $\vec{p} = m\dfrac{d\vec{x}}{d\tau}$

  where τ is the proper time in the object's reference frame (aka T_0)

  - Everyone can agree on what is the rest frame of an object, so the denominator of the derivative is the same in all reference frames

- We can rewrite this for another frame that is not the object's rest frame:

  - $\vec{p} = m\dfrac{d\vec{x}}{d\tau} = m\dfrac{d\vec{x}}{dt}\dfrac{dt}{d\tau}$
  - From time dilation, $t = \gamma\tau$, we have $dt/d\tau = \gamma$
  - And $d\vec{x}/dt = \vec{u}$

- So the new definition for momentum is: $\boxed{\vec{p} = \gamma m\vec{u},}$ where $\gamma = \dfrac{1}{\sqrt{1 - u^2/c^2}}$

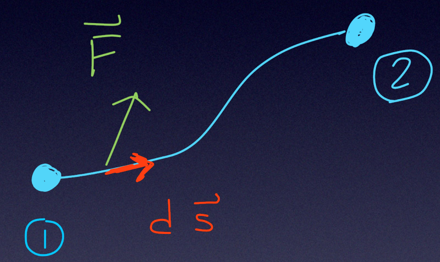
u denotes the velocity of the object in the reference frame

Relativistic Force

- With the previous definition for momentum, we can retain the usual definition for force as per Newton's 2nd Law:
  - $$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\frac{d\vec{x}}{d\tau}\right) = \frac{d}{dt}(\gamma_u m\vec{u})$$
- The component of force perpendicular to the direction of a Lorentz boost is given by
  - $$\vec{F}_\perp = m\frac{d}{dt}\frac{d\vec{x}}{d\tau} = m\frac{d}{d\tau}\vec{u}_\perp = m\frac{dt}{d\tau}\frac{d}{dt}\vec{u}_\perp = \gamma_u m\vec{a}_\perp$$
- Now let's consider how this perpendicular force transforms under a Lorentz transformation.
  - The component of velocity transverse to a boost along *x* transforms as:
    - $$\vec{u}_\perp = \frac{\vec{u}'_\perp}{\gamma_v(1 + vu'_x/c^2)}$$ where $\gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}}$ v = frame speed
  - and force transforms as $\vec{F}_\perp = m\frac{d}{d\tau}\left[\frac{\vec{u}'_\perp}{\gamma_v(1 + vu'_x/c^2)}\right] = \frac{m\frac{d}{d\tau}\vec{u}'_\perp}{\gamma_v(1 + vu'_x/c^2)} = \frac{\vec{F}'_\perp}{\gamma_v(1 + vu'_x/c^2)}$

Relativistic Force, Work, & Kinetic Energy

- Retain usual definition of force: $\vec{F} = \dfrac{d\vec{p}}{dt} = \dfrac{d}{dt}\left(\gamma m \vec{u}\right)$
- Work-Energy theorem: $W = \Delta K = K_2 - K_1$
- Work done to move an object from point 1 to 2:



- $$W = \Delta K = \int_1^2 \vec{F} \cdot d\vec{s}$$ where $d\vec{s} = \vec{u}dt$
- $$\Delta K = \int_1^2 \frac{d}{dt}\left(\gamma m \vec{u}\right) \cdot \vec{u}dt$$
- $$\Delta K = m\int_1^2 dt \frac{d}{dt}\left(\gamma \vec{u}\right) \cdot \vec{u} = m \boxed{\int_1^2 d\left(\gamma \vec{u}\right) \cdot \vec{u}}$$

Relativistic Force, Work, & Kinetic Energy

- Start from rest ($K_1 = 0$, $K_2 = K$), and integrate by parts:
- $$K = m\int_0^{\gamma u_2} d\left(\gamma\vec{u}\right)\cdot\vec{u} = \gamma m u_2^2 - m\int_0^{u_2}\gamma\vec{u}\cdot d\vec{u}$$
- $$K = \gamma m u_2^2 - m\int_0^{u_2}\frac{u\,du}{\sqrt{1-u^2/c^2}} = \gamma m u_2^2 + mc^2\sqrt{1-u^2/c^2}\,\Big|_0^{u_2}$$
- $$K = \gamma m u_2^2 + mc^2\sqrt{1-u_2^2/c^2} - mc^2 = \gamma\left[m u_2^2 + mc^2(1-u_2^2/c^2)\right] - mc^2$$
- $\boxed{K = (\gamma - 1)mc^2}$ where $\gamma = 1/\sqrt{1-u^2/c^2}$

Kinetic Energy

- The kinetic energy $K = (\gamma - 1)mc^2$ doesn't look anything like the nonrelativistic expression $K = \frac{1}{2}mu^2$
- But it <u>should</u> seamlessly match for speeds $u \ll c$.
  - Correspondence principle to classical physics
  - Put another way, it's not that Newton was totally wrong and Einstein was right, it's just that the laws of physics need modification for speeds approaching *c*
- Try a binomial (Taylor) expansion:

$$\gamma = (1 - u^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots$$

$$\Rightarrow K = (\gamma - 1)mc^2 \approx \frac{1}{2}\frac{u^2}{c^2}mc^2 = \frac{1}{2}mu^2 \checkmark$$

Kinetic Energy vs. Speed

- Calculate the kinetic energy in Joules for a 1kg mass using Newtonian and Relativistic mechanics as a function of speed

$\boxed{\beta = \frac{v}{c}}$

Speed	K_{Newton}	K_{Rel}	K_{Rel} / K_{New} $= \frac{2(\gamma - 1)}{\beta^2}$
4×10^{-5} c	6.3×10^{11}	6.3×10^{11}	1.0
0.1c	4.5×10^{14}	4.5×10^{14}	1.008
0.5c	1.1×10^{16}	1.4×10^{16}	1.24
0.9c	3.6×10^{16}	1.2×10^{17}	3.2
0.99c	4.4×10^{16}	5.5×10^{17}	12.4
0.999c	4.5×10^{16}	1.9×10^{18}	42.8
0.9999c	4.5×10^{16}	6.4×10^{18}	141

Escape speed from Earth

By the way, it's best to work with γ in your calculations rather than v, given how fast it changes as $v \to c$

It would take infinite energy to reach the speed of light!

Total Energy and Rest Mass Energy

- Let's rewrite the equation for kinetic energy: $\gamma mc^2 = K + mc^2$
- mc^2 has the same units of energy
- It looks like a potential energy when written this way, such that
- $\boxed{E = \gamma mc^2 = K + E_0}$ is the total energy of an object
- $E_0 = mc^2$ is referred to as the rest-mass energy of an object, and is a constant

Note that a convenient way to determine γ is $\gamma = \dfrac{E}{mc^2}$

  - Energy in 1kg of mass is 9×10^{16} J ! 💥
  - 1 ton TNT = 4.18×10^9 J,
⇒ 1kg equivalent to 21 megatons of TNT 😱

Electron-Volt Energy Unit

- Recall that potential energy of an particle, *U*, calculated from the electric potential, *V*, is given by $\Delta U = q\Delta V$, where q is the electric charge of the particle

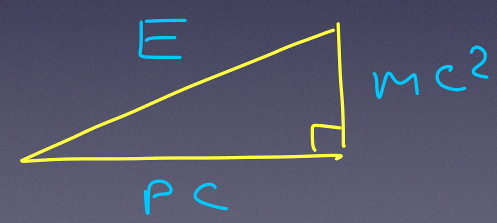
- For an electron, which has an electric charge of magnitude $e = 1.6022 \times 10^{-19}$ C, crossing a potential difference of 1 Volt has a tiny potential energy change of 1.6022×10^{-19} J

- More convenient to define a new unit for **subatomic particle energies**, the electron-volt (eV), such that $1\ \mathrm{eV} = 1.6022 \times 10^{-19}\ \mathrm{J}$
  - And with Greek prefixes: **1 keV =** 10^3 **eV**, **1 MeV =** 10^6 **eV**, **1 GeV =** 10^9 **eV**

- For example, the electron rest-mass energy is
  - $E_0 = mc^2 = (9.11 \times 10^{-31}\,\mathrm{kg})(3.0 \times 10^8\,\mathrm{m/s})^2 = 8.2 \times 10^{-14}\,\mathrm{J}$
  - $E_0 = 511{,}000\,\mathrm{eV} = 0.511\,\mathrm{MeV}$

Relationship Between Energy and Momentum

- In Newtonian physics: $K = \frac{1}{2}mu^2$ and $p = mu$, where $p^2 = p_x^2 + p_y^2 + p_z^2$
- So $K = \frac{p^2}{2m}$ (= E in absence of any other potential energies)
- In Relativity: $E = \gamma mc^2$ and $p = \gamma mu$ (in magnitude)
- $p^2c^2 = \gamma^2 m^2 u^2 c^2 = \gamma^2 m^2 c^4 \frac{u^2}{c^2} = \gamma^2 m^2 c^4 \left(1 - \frac{1}{\gamma^2}\right)$
- $\Rightarrow p^2c^2 = E^2 - m^2c^4$
- $E^2 = p^2c^2 + m^2c^4$
- $\boxed{m^2c^4 = E^2 - p^2c^2}$



Since mass is constant, this is an example of an invariant quantity. Another example of **Lorentz Invariance**

Summary of Relativistic Relationships

- $\vec{p} = \gamma m \vec{u}$
- $K = (\gamma - 1)mc^2$
- $E = \gamma mc^2 = K + E_0$
- $E_0 = mc^2$
- $E^2 = p^2c^2 + m^2c^4$
- $\gamma = \dfrac{E}{mc^2}$
- $\beta = \dfrac{v}{c} = \dfrac{pc}{E}$

Aside: Massless Particles

- For a particle without mass: $0 = E^2 - p^2c^2 \Rightarrow \boxed{E = pc}$
- But separately: $E = \gamma mc^2$ and $pc = \gamma muc$
- These can only equal if $u = c$
  - i.e. massless particles must travel at the speed of light!
  - And vice versa, to travel reach the speed of light requires the object to have no mass. If not, it would take infinite energy:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}} \rightarrow \infty \text{ as } u \rightarrow c$$

Conservation Laws

- Momentum is always converved for interactions with no external force acting, as with Newtonian mechanics
    - $\vec{p}_i = \vec{p}_f$, where $\vec{p}_i = \sum_{k=1}^{N_{\text{initial}}} \vec{p}_k$ and $\vec{p}_f = \sum_{k=1}^{N_{\text{final}}} \vec{p}_k$
    - But in relativity, momentum has the new form $\vec{p} = \gamma m \vec{u}$
- Total energy is also always conserved in relativity, even in inelastic collisions, unlike kinetic energy
    - Where does the excess energy go in an inelastic collision?
    - It goes into mass! (at least at the particle interaction level)
- So mass does not need to be conserved in interactions!

Binding Energy and Mass

- The mass of the proton is 938.28 MeV/c^2, and the mass of the neutron is 939.57 MeV/c^2
- Now consider the mass of the deuteron, which is a bound state of a proton and neutron (i.e. the nucleus of a heavy isotope of hydrogen)
  - It's mass is 1875.63 MeV/c^2
  - But the sum of the masses of a free neutron and proton is 1877.85 MeV/c^2
  - The difference is 2.22 MeV/c^2
- Why? It takes energy to break up the bound deuteron in an interaction, to overcome what is called **binding energy**
  - BE = $\left\{ M(\text{separate}) - M(\text{bound}) \right\} c^2$
- That extra energy goes into the mass of the separate objects (plus kinetic energy if any left over)
- The same applies to the hydrogen atom, for example. The mass of an atom of hydrogen is 13.6 eV less than that of the electron and proton separately

Reaction Energy

- Consider a reaction from an initial set of objects to a final set
- **Reaction energy** is defined as
  - $Q = \left\{M(\text{initial products}) - M(\text{final products})\right\}c^2$
  - The negative of binding energy essentially
- $Q > 0$: **Exothermic reaction**, energy is released in reaction
- $Q < 0$: **Endothermic reaction**, requires energy to proceed
- Example: decay of a neutron $n \rightarrow p + e^- + \bar{\nu}_e$
  - $Q = \left\{M(n) - M(p) - M(e^-) - M(\nu)\right\}c^2$
  - $Q = \left\{939.57 - 938.28 - 0.511 - \approx 0\right\}\ \text{MeV} = 0.78\ \text{MeV}$ (goes into momentum)

Invariant Mass

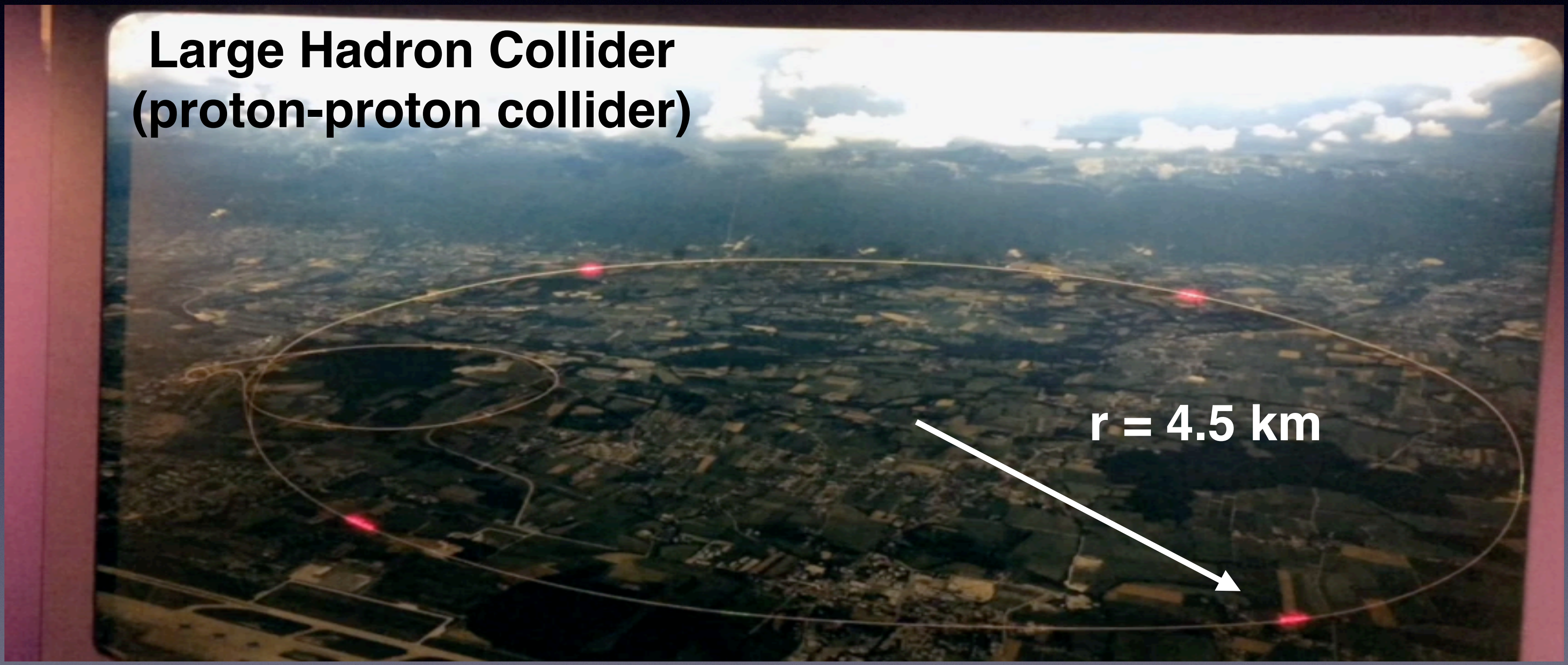
- While mass does not need to conserved in interactions, the invariant mass of a system of objects <u>is</u> conserved since energy and momentum are each conserved:
  - $E_i = E_f$, where $E_i = \sum_{k=1}^{N_{\text{initial}}} E_k$ is the initial energy, and $E_f = \sum_{k=1}^{N_{\text{final}}} E_k$ is final energy
  - $\vec{p}_i = \vec{p}_f$, where $\vec{p}_i = \sum_{k=1}^{N_{\text{initial}}} \vec{p}_k$ for initial momenta, and $\vec{p}_f = \sum_{k=1}^{N_{\text{final}}} \vec{p}_k$ for final
  - $\Rightarrow E_i^2 - p_i^2c^2 = m^2c^4$ = constant
  - = $E_f^2 - p_f^2c^2$
- So whatever this is initially, even if the initial particles disintegrated and new particles were created, is what it is afterward

Invariant Mass Example

An electron and a positron (an anti-electron) annihilate with equal and opposite momentum of magnitude1.55 GeV/*c* (note the new unit of momentum!) The collision produces a new particle called the J/ψ in the reaction $e^- + e^+ \rightarrow J/\psi$. What is the mass of this new particle? [Note that the rest mass energy of the electron is 0.511 MeV]

Accelerators

- Particle accelerators and colliders test Special Relativity every day!



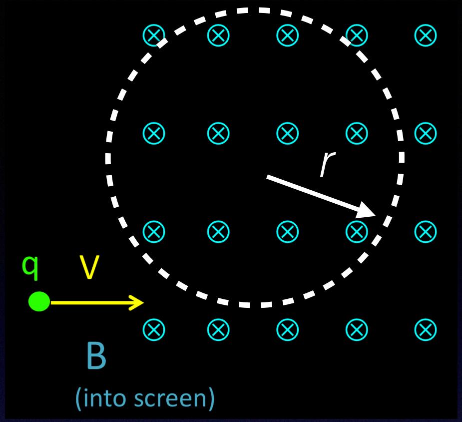
The LHC

Where the Higgs boson was discovered in 2012
(confirms theoretical mechanism for mass)

Inside the tunnel →

Dipole bending magnet, B=8.4T
15m long (1 of 1200!)
Superconducting and cooled to T=1.9°K

Cyclotrons and Orbital Frequency



- **Electric fields** accelerate the charged particles, and **magnetic fields** bend them into a circular orbit
- The **orbital frequency** for uniform circular motion is given by $f = \dfrac{v}{2\pi r}$
- The relativistic momentum is $p = \gamma m v$
- $f = \dfrac{p}{2\pi\gamma m r}$
- Plug in the relationship for circular motion in B field: $p = qBr$
- $f = \dfrac{qB}{2\pi m}\sqrt{1 - \dfrac{v^2}{c^2}}$
- Frequency is constant provided $v \ll c$, but at relativistic speeds the frequency slows down. Speed is limited to *c*, but the circumference increases with r.

High energy machines must synchronize the orbital frequency according to Special Relativity and are known as synchro-cyclotrons, or synchrotrons in short.

Four-Vectors

- Recall these **Lorentz invariant** quantities:

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$

$$m^2c^4 = E^2 - p_x^2c^2 - p_y^2c^2 - p_z^2c^2$$

- Can generalize that these are the **invariant "lengths"** of four-dimensional vectors in time+space and energy+momentum

- "**Four vectors**", ***x*** and ***p***, are indexed by Greek letter μ=0, 1, 2, 3:
  - $x_\mu = (x_0, x_1, x_2, x_3) = (ct, x, y, z)$
  - $p_\mu = (p_0, p_1, p_2, p_3) = (E/c, p_x, p_y, p_z)$

- Define a **4-vector dot product** as:
  - $\boxed{x^2 = x \cdot x = x_0x_0 - x_1x_1 - x_2x_2 - x_3x_3 = s^2}$
  - $\boxed{p^2 = p \cdot p = p_0p_0 - p_1p_1 - p_2p_2 - p_3p_3 = m^2c^2}$

No vector arrow over 4-vector symbols *x* and *p*, just over 3-vectors

Note *ct* and *E/c* to put into same units as *x* and *p*, respectively

"Natural units" set *c* = 1
Can always reinsert *c*'s later to make units correct

Working with 4-Vectors Example

Consider the decay of an excited neutral kaon into a charged pion and a kaon: $K^{0*} \rightarrow K^{+} + \pi^{-}$. What is the energy of the pion in the rest frame of the K^{0*} ?

4-vector energy-momentum conservation: $p_{K*} = p_K + p_\pi \Rightarrow p_K = p_{K*} - p_\pi$

$$p_K^2 = (p_{K*} - p_\pi)^2 = p_{K*}^2 + p_\pi^2 - 2p_{K*} \cdot p_\pi$$

$m_K^2 c^2 = m_{K*}^2 c^2 + m_\pi^2 c^2 - 2p_{K*} \cdot p_\pi$ (from Lorentz invariance)

In K^{0*} rest frame: $p_{K*} = (m_{K*}c, 0, 0, 0)$, $p_\pi = (E_\pi / c, p_x, p_y, p_z)$

so the dot product only picks the 0th component:

$$m_K^2 c^2 = m_{K*}^2 c^2 + m_\pi^2 c^2 - 2(m_{K*}c)(E_\pi / c)$$

$$\Rightarrow \boxed{E_\pi = \frac{m_{K*}^2 c^4 + m_\pi^2 c^4 - m_K^2 c^4}{2m_{K*}c^2}}$$

$m_{K*} = 892\ \text{MeV}/c^2$
$m_K = 494\ \text{MeV}/c^2$
$m_\pi = 140\ \text{MeV}/c^2$
$\Rightarrow E_\pi = 320$ MeV

Lorentz Transformation as Matrix Math

- Consider a Lorentz transformation along the z-axis:
- $z' = \gamma_v(z - \beta_v ct)$, $y' = y$, $x' = x$, $t' = \gamma_v(t - \beta_v z/c)$ $\quad \beta_v = v/c$
- Can write this matrix mathematics:

$$x'_\lambda = \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda^\mu_\lambda x_\mu = \begin{pmatrix} \gamma_v & 0 & 0 & -\beta_v\gamma_v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_v\gamma_v & 0 & 0 & \gamma_v \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma_v(t - \beta_v z) \\ x \\ y \\ \gamma_v(z - \beta_v t) \end{pmatrix}$$

- Matrix representation of Lorentz transformation

Lorentz Transformation as Matrix Math

- Can apply same Lorentz transformation matrix to E and p transformation:

$$p'_\lambda = \begin{pmatrix} \frac{E'}{c} \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \Lambda^\mu_\lambda p_\mu = \begin{pmatrix} \gamma_v & 0 & 0 & -\beta_v\gamma_v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_v\gamma_v & 0 & 0 & \gamma_v \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \gamma_v\left(\frac{E}{c} - \beta_v p_z\right) \\ p_x \\ p_y \\ \gamma_v\left(p_z - \beta_v \frac{E}{c}\right) \end{pmatrix}$$