

# 1 Deep Inelastic Scattering Kinematics

## 1.1 Conventions and Basic Relations

We choose the direction of the hadron beam as the  $+z$  direction, with the lepton beam traveling in the opposite direction. The initial hadron and electron energies are  $E_h$  and  $E_\ell$ , respectively. The scattering angle of the lepton is denoted by  $\theta$  ( $\theta = \pi$  for the initial lepton direction), and the angle of the final state hadronic system is denoted by  $\gamma$  ( $\gamma = 0$  for the initial hadron direction). We denote the momentum four-vectors of the initial lepton and the initial hadron as  $k$  and  $P$ , respectively, and the scattered four-momenta as  $k'$  and  $P'$ . The final state energies of the lepton and hadronic system are denoted  $E'_\ell$  and  $E'_h$ , respectively.

The squared center-of-mass energy,  $s$ , is given by:

$$s = 4E_\ell E_h \quad (1)$$

The negative squared four-momentum transferred in deep inelastic scattering (DIS) between the lepton and hadron is

$$Q^2 \equiv -q^2 = -(k - k')^2 = sxy \quad (2)$$

where  $x$  is the Bjorken scaling variable and  $y$  is the fractional energy transfer from the lepton, defined as:

$$x = \frac{Q^2}{2P \cdot q} \quad (3)$$

$$y = \frac{P \cdot q}{P \cdot k} \quad (4)$$

## 1.2 Lepton Variables

The DIS variables using the scattered lepton variables are given by:

$$Q^2 = 2E_\ell E'_\ell (1 + \cos \theta) \quad (5)$$

$$y = 1 - \frac{E'_\ell}{2E_\ell} (1 - \cos \theta) \quad (6)$$

The variable  $x$  can be found from Eq.(2).

An expression for  $Q^2$  as a function of  $x$  based on the scattered lepton energy  $E'_\ell$  can be found by solving for  $\cos \theta$  in Eq.(5) and plugging into Eq.(6), then using Eq.(2):

$$Q^2 = \frac{1 - E'_\ell/E_\ell}{\frac{1}{sx} - \frac{1}{4E_\ell^2}} \quad (7)$$

Likewise, an expression for  $Q^2$  as a function of  $x$  based on the scattered lepton angle  $\theta$  can be found by solving for  $E'_\ell$  in Eq.(5) and plugging into Eq.(6), then using Eq.(2):

$$Q^2 = \frac{s}{\frac{1}{x} + \frac{E_h}{E_\ell} \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)} = \frac{sx}{1 + x \frac{E_h}{E_\ell} \tan^2(\frac{\theta}{2})} \quad (8)$$

Graphs showing the dependence of  $Q^2$  vs.  $x$  for various choices of the scattered lepton energy and pseudorapidity are shown in the left plots of Figs. 1, 2, and 3 for different machine configurations.

We can invert Eq. (7) to find  $E'_\ell$  in terms of  $Q^2$  and  $x$ :

$$E'_\ell = E_\ell \left[ 1 - \frac{Q^2}{4E_\ell} \left( \frac{1}{xE_h} - \frac{1}{E_\ell} \right) \right] \quad (9)$$

and also invert Eq. (8) to obtain  $\theta$  (or  $\eta$ ) in terms of  $Q^2$  and  $x$ :

$$e^{-\eta} = \tan \frac{\theta}{2} = \sqrt{\frac{4E_\ell^2}{Q^2} - \frac{E_\ell}{xE_h}} \quad (10)$$

When the second term in the radical is negligible (for all but the smallest  $x$  values), the scattered lepton angle depends only on  $Q^2$ :

$$\eta \approx -\ln \left( \frac{2E_\ell}{\sqrt{Q^2}} \right) \quad (11)$$

### 1.3 Hadron variables

The DIS variables using the scattered hadronic variables are given by:

$$Q^2 = \frac{E_h'^2 \sin^2 \gamma}{1 - y} \quad (12)$$

$$y = \frac{E'_h}{2E_\ell} (1 - \cos \gamma) \quad (13)$$

These variables can be found from the measured hadronic energy flow via the Jacquet-Blondel method:

$$Q^2 = \frac{(\sum_i p_{T,i})^2}{1 - y} \quad (14)$$

$$y = \frac{\sum_i (E_{h,i} - p_{z,i})}{2E_\ell} \quad (15)$$

In other words,

$$\sin \gamma = \frac{\sum_i p_{T,i}}{E'_h} \quad (16)$$

The variable  $x$  can be found from Eq.(2).

An expression for  $Q^2$  as a function of  $x$  based on the scattered hadron energy  $E'_h$  can be found by solving for  $\cos \gamma$  in Eq.(13) and plugging into Eq.(12), expanding out  $\sin^2 \gamma = (1 + \cos \gamma)(1 - \cos \gamma)$ , then using Eq.(2):

$$Q^2 = \frac{sx \left(1 - \frac{E'_h}{xE_h}\right)}{\left(1 - \frac{E_\ell}{xE_h}\right)} \quad (17)$$

Likewise, an expression for  $Q^2$  as a function of  $x$  based on the scattered hadronic angle  $\gamma$  can be found by solving for  $E'_h$  in Eq.(13) and plugging into Eq.(12), then using Eq.(2):

$$Q^2 = \frac{sx}{1 + \frac{4E_\ell^2}{sx} \frac{(1+\cos \gamma)}{(1-\cos \gamma)}} = \frac{sx}{1 + \frac{E_\ell}{xE_h} \cot^2\left(\frac{\gamma}{2}\right)} \quad (18)$$

Graphs showing the dependence of  $Q^2$  vs.  $x$  for various choices of the scattered hadron energy and pseudorapidity are shown in the right plots of Figs. 1, 2, and 3 for different machine configurations.

We can invert Eq. (17) to find  $E'_h$  in terms of  $Q^2$  and  $x$ :

$$E'_h = xE_h + \frac{Q^2}{4E_\ell} \left( \frac{E_\ell}{xE_h} - 1 \right) \quad (19)$$

which for  $x \ll E_\ell/E_h$  is approximated by:

$$E'_h = yE_\ell \quad (20)$$

We can also invert Eq. (18) to obtain  $\gamma$  (or  $\eta$ ) in terms of  $Q^2$  and  $x$ :

$$e^{-\eta} = \tan \frac{\gamma}{2} = \frac{1}{\sqrt{\frac{4E_h^2 x^2}{Q^2} - x \frac{E_h}{E_\ell}}} \quad (21)$$

which again for  $x \ll E_\ell/E_h$  can be approximated by:

$$\eta \approx -\ln \left( \frac{\sqrt{Q^2}}{2E_h x} \right) \quad (22)$$

### 1.3.1 $\Sigma$ and $T$ variables

A variation of the hadron method is to define the  $\Sigma$  and  $T$  variables:

$$\Sigma = \sum_i (E_{h,i} - p_{z,i}) = 2E_\ell y \quad (23)$$

$$T = \sqrt{\left(\sum_i p_{x,i}\right)^2 + \left(\sum_i p_{y,i}\right)^2} = E'_h \sin \gamma \quad (24)$$

which again are sums over the hadron energies and momenta. The variable  $T$  is essentially the transverse momentum ( $p_T$ ) of the hadron system (i.e. jet from the struck parton). From these definitions we can find  $Q^2$  vs.  $x$  in terms of  $\Sigma$  and  $T$ :

$$Q^2 = 2E_h x \Sigma \quad (25)$$

and using the quadratic equation:

$$Q^2 = \frac{sx}{2} \left( 1 \pm \sqrt{1 - \frac{4T^2}{sx}} \right) \quad (26)$$

Graphs showing the dependence of  $Q^2$  vs.  $x$  for various choices of  $\Sigma$  and  $T$  are shown in Fig. 4. Since it can be shown that:

$$T^2 = (1 - y) Q^2 \quad (27)$$

$T \approx Q$  for small  $y$ , where  $T$  is also the transverse momentum of the scattered hadron system.

## 1.4 Double Angle Method

Assuming no initial state radiation or other unaccounted energy loss in the measured final state products, only two independent measurements are necessary to determine the kinematic variables  $Q^2$ ,  $y$ , and  $x$  (that can be obtained from  $Q^2 = sxy$ ). These can come from lepton-only measurements or hadron-only measurements, as above, but also mixed methods. Here we discuss the “double angle” method that uses the scattered lepton ( $\theta$ ) and hadron ( $\gamma$ ) angles.

By eliminating the scattered lepton and hadron energies in the formulae given earlier, it can be shown that:

$$Q^2 = 4E_\ell^2 \frac{\sin \gamma (1 + \cos \theta)}{\sin \gamma + \sin \theta - \sin(\gamma + \theta)} \quad (28)$$

$$y = \frac{\sin \theta (1 - \cos \gamma)}{\sin \gamma + \sin \theta - \sin(\gamma + \theta)} \quad (29)$$

## 1.5 $E/\gamma$ Method

Another mixed method is to eliminate the scattered lepton angle, and use just the scattered lepton energy and the angle  $\gamma$  of the scattered hadronic system. This leads to the following equations:

$$Q^2 = 4E_\ell E'_\ell - 4E_\ell^2(1 - y) \quad (30)$$

$$y = \frac{1}{2} \left( 1 - \frac{E'_\ell}{2E_\ell} \right) (1 - \cos \gamma) \pm \frac{E'_\ell}{2E_\ell} \sqrt{\frac{1}{4}(1 - \cos \gamma)^2 + \frac{E_\ell}{E'_\ell} \left( 1 - \frac{E_\ell}{E'_\ell} \right) \sin^2 \gamma} \quad (31)$$

Unfortunately there does seem to be an ambiguity in the value of  $y$  that arises from  $\pm$  in solving the quadratic equation, which perhaps can be broken by using one of the other methods.

## 1.6 Kinematic Constraints

Energy and momentum are always conserved. For collider experiments, since the beam particles collide along the longitudinal ( $z$ ) axis, this means the vector sum of the transverse momentum of all final state particles must be zero.

$$\sum_i p_{x,i} = \sum_i p_{y,i} = 0 \quad (32)$$

Any measured imbalance is attributed to undetected neutrinos or detector resolution/mismeasurement effects.

Another useful constraint for lepton-hadron collisions applies in the longitudinal direction:

$$\sum_i E_i - p_{z,i} = 2 E_\ell \quad (33)$$

where the hadron beam direction is again defined to be the  $+z$  direction, such that  $E - p_z$  is 0 for the incoming hadron beam and  $2E_\ell$  for the incoming lepton beam. This constraint can be used to improve the resolution of the DIS quantities. For example, for leptons scattering at very small angles with respect to the incident direction (e.g.  $\eta < -3$ ), the scattered lepton  $E - p_z \approx 0$ . This can be used to solve for the final state lepton energy using this relation and the hadron measurements:

$$E'_\ell \approx \frac{1}{2}(2E_\ell - \Sigma) \quad (34)$$

## 1.7 Kinematic Resolutions

### 1.7.1 $Q^2$ , Lepton Variables

In the far backward region,  $\theta \approx \pi$ , we can express the relation for  $Q^2$  in terms of the lepton angle from the  $-z$  axis, where  $\alpha = \pi - \theta$ :

$$Q^2 = 2E_\ell E'_\ell (1 - \cos \alpha) \quad (35)$$

From this relation, the relative resolution on  $Q^2$  in terms of the relative resolution on the measured scattered lepton energy  $E'$ , assuming negligible uncertainty on the scattered angle, is given by:

$$\frac{dQ^2}{Q^2} = \frac{dE'_\ell}{E'_\ell} \quad (36)$$

The dependence of the relative uncertainty on  $Q^2$  on the uncertainty of the measured scattering angle for small  $\alpha$  ( $\theta$  near  $\pi$ ) can be found from  $\cos \alpha \approx 1 - \alpha^2/2$ , such that:

$$Q^2 \approx E E'_\ell \alpha^2 \quad (37)$$

This implies:

$$\frac{dQ^2}{Q^2} = 2 \frac{d\alpha}{\alpha} \quad (38)$$

Thus the angular resolution impact on the reconstructed  $Q^2$  becomes significant for scattering angles near the beamline, and is exacerbated by the factor 2 on the relative precision.

### 1.7.2 $y$ , Lepton Variables

We can express the relation for  $y$  in terms of the lepton angle from the  $-z$  axis, where  $\alpha = \pi - \theta$ :

$$y = 1 - \frac{E'_\ell}{2E_\ell} (1 + \cos \alpha) \quad (39)$$

We can then express the relative uncertainty on  $(1-y)$  in terms of the relative resolution on the measured scattered lepton energy  $E'$  as:

$$\frac{d(1-y)}{1-y} = \frac{dE'_\ell}{E'_\ell} \quad (40)$$

Or put another way:

$$dy = \frac{dE'_\ell}{E'_\ell} (1-y) \quad (41)$$

This implies that the reconstruction of  $y$  from the scattered lepton energy is good for large  $y$ , but degrades and reaches 100% uncertainty for  $y \approx dE'_\ell/E'_\ell$ .

For the dependence of the uncertainty on  $1-y$  from the uncertainty on the measured scattering angle  $\alpha$  we get:

$$\frac{d(1-y)}{1-y} \approx \frac{\alpha d\alpha}{(2 - \alpha^2/2)} \approx \frac{1}{2} \alpha d\alpha \quad (42)$$

### 1.7.3 $y$ , Hadron Variables

From Eq.(13) we can then express the relative uncertainty on  $y$  in terms of the relative resolution on the measured scattered hadron energy  $E'_h$  as:

$$\frac{dy}{y} = \frac{dE'_h}{E'_h} \quad (43)$$

For small angles from the far backward region,  $\gamma \approx \pi$ , we can express the relation for  $y$  in terms of the hadron angle from the  $-z$  axis, where  $\beta = \pi - \gamma$ :

$$y = \frac{E'_h}{2E_\ell} (1 + \cos \beta) \approx \frac{E'_h}{2E_\ell} (2 - \beta^2/2) \quad (44)$$

The dependence of the relative uncertainty on  $y$  on the uncertainty of the measured scattering angle for small  $\beta$  is therefore:

$$\frac{dy}{y} = \frac{\beta d\beta}{(2 - \beta^2/2)} \quad (45)$$

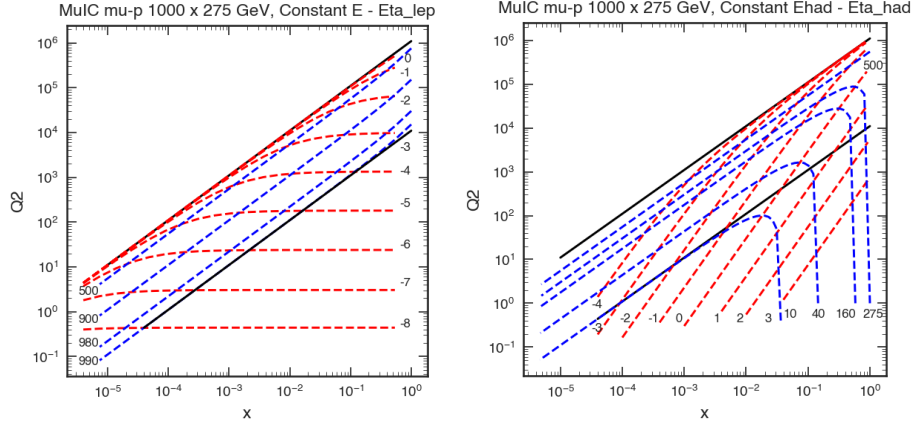


Figure 1: Kinematics of the scattered muon (left) and final-state hadrons (right) in the  $Q^2$ - $x$  plane for muon-proton deep inelastic scattering with a 1000 GeV muon beam colliding with a 275 GeV proton beam. The dashed blue lines correspond to constant energy in GeV and the dashed red lines to constant pseudorapidity, respectively. The lower diagonal solid line corresponds to the inelasticity  $y = 0.01$ .



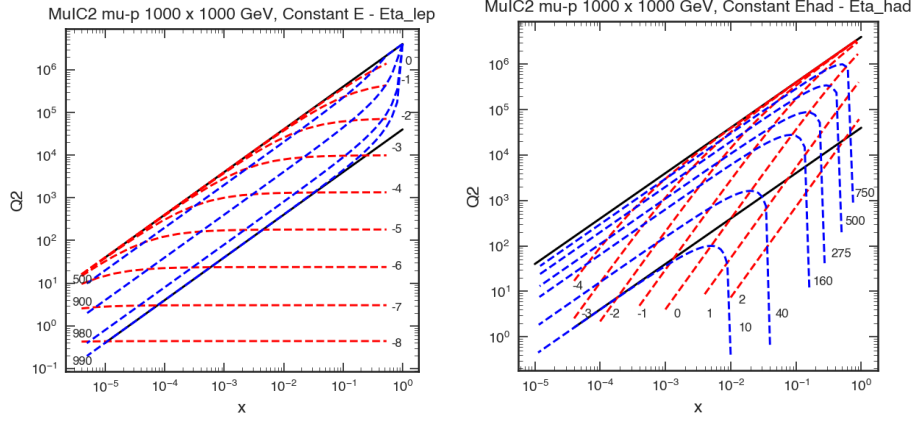


Figure 2: Kinematics of the scattered muon (left) and final-state hadrons (right) in the  $Q^2$ - $x$  plane for muon-proton deep inelastic scattering with a 1000 GeV muon beam colliding with a 1000 GeV proton beam. The dashed blue lines correspond to constant energy in GeV and the dashed red lines to constant pseudorapidity, respectively. The lower diagonal solid line corresponds to the inelasticity  $y = 0.01$ .

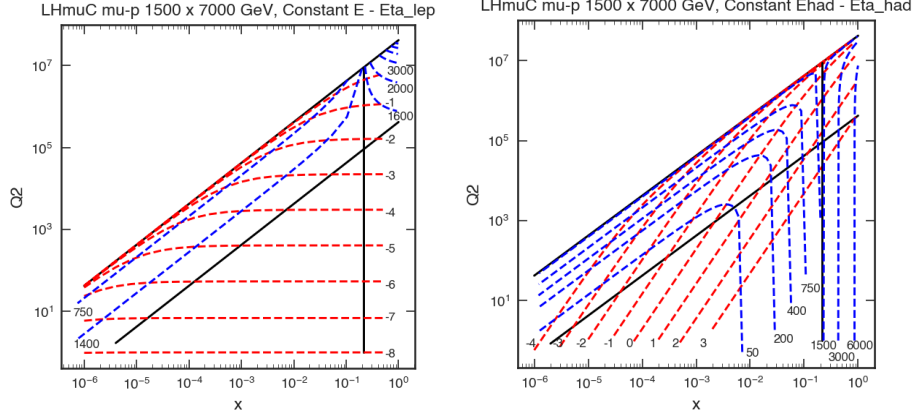


Figure 3: Kinematics of the scattered muon (left) and final-state hadrons (right) in the  $Q^2$ - $x$  plane for muon-proton deep inelastic scattering with a 1500 GeV muon beam colliding with a 7000 GeV proton beam. The dashed blue lines correspond to constant energy in GeV and the dashed red lines to constant pseudorapidity, respectively. The lower diagonal solid line corresponds to the inelasticity  $y = 0.01$ .

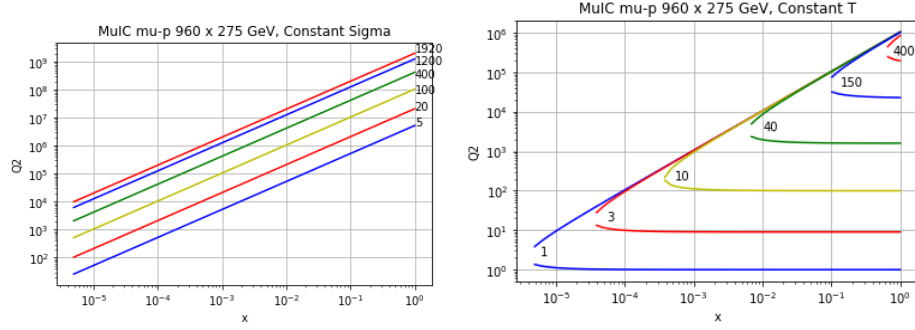


Figure 4:  $\Sigma$  (left) and  $T$  (right) in the  $Q^2$ - $x$  plane for muon-proton deep inelastic scattering with a 1000 GeV muon beam colliding with a 275 GeV proton beam.